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A SHORT SURVEY ON CHAOTIC DYNAMICS IN SOLOW-TYPE GROWTH MODELS

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Abstract:

In this paper we review some Solow-type growth models, framed is discrete time, which are able to generate complex dynamic behaviour. For these models – put forward by Day (1982, 1983); Böhm and Kaas (2000); and Commendatore (2005) – we show that crucial features which could determine the emergence of regular or irregular growth cycles are (i) if the average saving ratio is constant or not; and (ii) the curvature of production function, representing the degree of substitutability between labour and capital. The lower the degree of substitutability, the higher the likelihood of complex behaviour.

Keywords: Logistic Map, Li-York Chaos, Growth Models, Local Stability, Triangle Stability.

1. Introduction

The analysis of the fundamental issues in dynamical macroeconomics usually begins with the study of two (one-sector and one-dimensional) growth models: the Ramsey model [Ramsey, (1928)] and the Solow model [Solow, (1956)]. In the Ramsey model a representative consumer has an infinite life horizon and optimizes his/her utility. In the Solow model consumption is not optimal the representative agent saves a constant fraction of his income. In the next sections we will describe the Solow model and a few models which are very close to the Solow one and are able to generate chaotic dynamics. We note here that researches in several direction have spanned from the Solow model. For example, the Solow model inspired the works of Shinkay (1960), Meade (1961), Uzawa (1961,1963), Kurz (1963) and Srinivasan (1964) on two-sector growth models. Following this line of research, works about two-sector models appeared on the *Review of Economic Studies* in the 1960s [Drandakis (1963), Takajama (1963, 1965), Oniki-Uzawa (1965), Hahn (1965), Stiglitz (1967), among others].

2. The Solow growth model in discrete time

Following Hans-Walter Lorenz (1989) and Costas Aziariadis (1993), we will develop a discrete time variant of the growth model due to Solow (1956). We consider *a single good economy, i.e. an economy in which only one good is produced and consumed*. We assume that time t is discrete, that is t = 0, 1, 2, … . The symbols $Y_t, K_t, C_t, I_t, L_t, S_t$ indicate economywide aggregates respectively equal to *income, capital stock, consume, investment, labor force, saving at time* t. The capital stock K_0 and labor L_0 at time 0 are given. The constant s denotes both the average and the *marginal savings rates* and the constant n denotes the *growth rate of population*. We consider s and n as given exogenously. The map $F: (K_t,L_t) \to F(K_t,L_t)$ is *the production function*. We assume that:

1. $Y_t = C_t + I_t$: *for all time t = 0, 1, …, the economy is in equilibrium, i.e. the supply of income* Y_t *is equal to the demand composed of the quantity* C_t *of good to consume plus the stock* I_t *of capital to invest (closed economy like a Robinson Crusoe economy)*;

2. $I_t = K_{t+1}$: *investment at time t corresponds to all capital available to produce at time t+1 (working capital hypothesis)*;

3. $S_t = Y_t - C_t = s\,Y_t$ $(0 < s < 1)$: *saving is a share of income*;

4. $Y_t = F(K_t, L_t)$, i.e. *at time t all income is equal to the output obtained by the inputs capital and labor*;

5. $L_t = (1+n)^t L_0$ (n > 0): *the labor force grows as a geometric progression at the rate* (n).

From (1.) and (3.) in a short run equilibrium $Y_t = C_t + S_t$ or $I_t = S_t$. Thus, applying (2.) and (3.), we have $K_{t+1} = sY_t$. Finally, from (4.) we obtain $K_{t+1} = s\,F(K_t, L_t)$.

From the latter expression, $K_{t+1}/L_{t+1} = s\,F(K_t,L_t)/L_{t+1}$.

If F is *linear-homogeneous* (or it tells that *F exhibits constant returns to scale)*, i.e.

$F(\lambda K, \lambda L) = \lambda F(K, L)$ (for all $\lambda > 0$),

then we have $K_{t+1}/L_{t+1} = s\,L_t F(K_t/L_t ,1)/L_t(1+n)$.

We set $k_t = K_t/L_t$ (*capital-labor ratio or capital per worker*) and $y_t = Y_t/L_t$.(*output-labor ratio or output per worker*). We derive in this way the production function in the intensive form: $y_t = f(k_t) = f(K_t/L_t ,1)$. Therefore we get the equation of accumulation *for the Solow model in discrete time with the working capital hypothesis*:

$k_{t+1} = s\,f(k_t)/(1+n)$ (2.1)

If we assume that *capital depreciates at the rate* $0 \le \delta \le 1$ (*fixed capital hypothesis*), the capital available at time t +1 corresponds to $K_{t+1} = K_t - \delta K_t + I_t$, from which

$K_{t+1} = s\,F(K_t, L_t) + (1 - \delta) K_t$.

As before we get the following time-map for capital accumulation

$k_{t+1}(1+n) = s\,f(k_t) + (1 - \delta) k_t$ (2.2) or $k_{t+1} = h(k_t)$ (2.3)

where $h(k_t) = \frac{1}{1+n}[s\,f(k_t) + (1 - \delta) k_t]$.

We notice that I_t is the *gross investment* while $K_{t+1} - K_t = I_t - \delta K_t$ is *the net investment*.

Costas Azariadis (1993, p. 4) tells us that *this model captures explicitly a simple idea that is missing in static formulations: there is a trade-off between consumption and investment or between current and future consumption. The implications of this ever-present competition for resources between today and tomorrow are central to macroeconomics and can be explored only in a dynamic framework. Time is clearly of the essence*.

For example, if we use the *Cobb-Douglas* production function $f(k_t) = Bk^{\beta}$ (B > 0, $0 < \beta < 1$, k ≥ 0) – in intensive form - the eq. (2.1) becomes $k_{t+1} = h(k_t) = (sBk_t^{\beta})/(1+n)$ (2.4).

For all k ≥ 0 and B > 0, if we assume (β > 0) we deduce only that $h(k_t)$ is strictly monotonically increasing, instead we need to use both inequalities ((β > 0) and (β < 1) t show the concavity of $h(k_t)$.

As a matter of fact, from above assumptions we get

$$h'(k_t) = \frac{sB\beta k^{\beta-1}}{1+n} > 0, \quad h''(k_t) = \frac{sB\beta(\beta-1)k^{\beta-2}}{1+n} < 0.$$

The h-map has two fixed point at k = 0 (*trivial and repelling fixed point*) and at $k^s = [sB/(1+n)]^{1/(1-\beta)}$ (*interior and asymptotically stable*). *The eq. (2.4) is able only to generate monotonic convergence to a fixed point* (See Figure 1.).

If we use the *Leontief* production function , i.e. $f(k_t) = \min\{ak_t, b\} + c$, a,b,c > 0 [Böhm, Kaas, (2000)], that is only piecewise differentiable, from (2.1) we deduce that the dynamical system is described by two affine-linear maps

$$k_{t+1} = h(k_t) = \begin{cases} (ask_t + cs)/(1+n), & \text{if } k_t \le b/a; \\ (bs + cs)/(1+n), & \text{if } k_t > b/a. \end{cases}$$

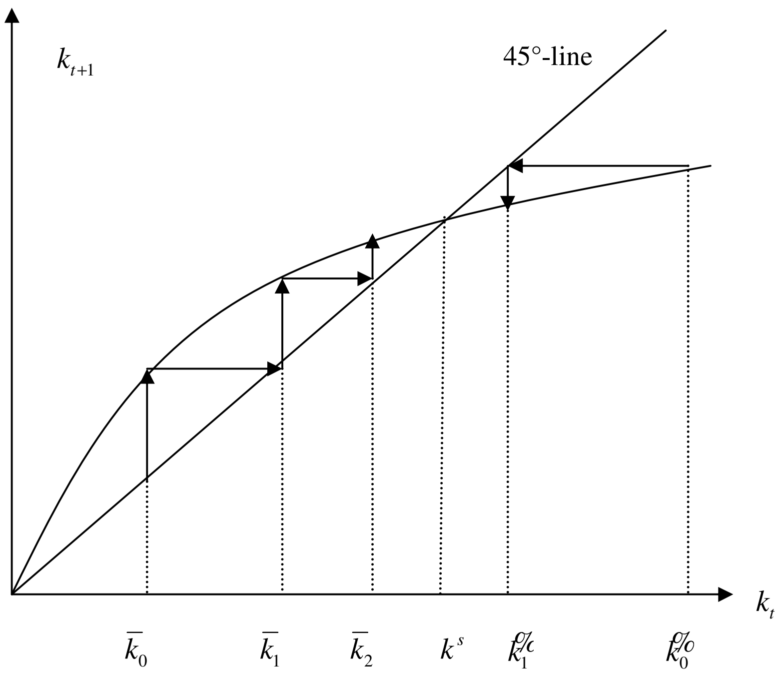


Figure 1: Monotonic Convergence to the fixed point k^s

3. Complex dynamics in the Solow Discrete Time Growth Model

R.H. Day (1982,1983) first has noticed that *complex dynamics can emerge from simple economic structures* as, for example, the neoclassical theory of capital accumulation. Day focuses on the assumptions of the standard Solow growth model and argues that *the kind of nonlinearity of the* $h(k_t)$ *map and the lag present in* (2.1) *are not sufficient to lead to chaos.* Day rewrites (2.1) in a more general form:

$$k_{t+1} = \frac{1}{1+n} s(k_t) f(k_t), \tag{3.1}$$

where $s(k_t)$ is the saving propensity. Then, Day makes changes in (3.1) deriving two alternative models able to generate chaos in the Li-Yorke (1975) sense. In the first model he keeps $s(k_t)$ as an exogenous constant and modifies the production function $f(k_t)$ into a unimodal map, i.e. a concave and one humped shaped map. Instead in the second model he modifies $s(k_t)$ into a unimodal map and he keeps $f(k_t)$ as an neoclassical production function like the Cobb-Douglas, obtaining a *robust* result [Boldrin and Woodford, (1990)].

In particularly, in the former case he sets $s(k_t) \equiv s$ and defines

$$f(k_t) = \begin{cases} B\, k_t^{\beta} (m-k_t)^{\gamma}, & \text{if } k_t < m; \\ 0, & \text{otherwise} \end{cases}$$

where m is a positive constant, $0 < \beta < 1$, $0 < \gamma < 1$ and $B > 0$.

In the latter case he sets $f(k_t) = Bk_t^{\beta}$ $(B \geq 2, 0 < \beta < 1)$ and he replaces the constant s with the saving function $s(k_t) = a\,(1-\frac{b}{r})\frac{k_t}{y_t}$, where $r = f'(k_t) = \beta \frac{y_t}{k_t}$, $a > 0, b > 0$.

Thus from the equation (3.1) we deduce respectively the expressions

$$k_{t+1} = \begin{cases} \frac{1}{1+n} sBk_t^{\beta}(m-k_t)^{\gamma}, & \textit{if } k_t \le m; \\ 0, & \text{otherwise}, \end{cases} \quad (3.2)$$

and

$$k_{t+1} = \frac{a}{1+n} k_t \left[1-\left(\frac{b}{\beta B}\right)k_t^{1-\beta}\right] \quad (3.3)$$

It is very simple to solve the equation (3.2) when $m = \gamma = \beta = 1$. As a matter of fact we can rewrite it like this

$$k_{t+1} = \frac{1}{1+n} s B k_t(1-k_t) \quad (3.4)$$

If we set $\mu = \frac{sB}{1+n}$ then the (3.4) becomes the well-known logistic equation

$k_{t+1} = \mu k_t(1- k_t)$.

To obtain chaos as Li-Yorke (1975) occurs that in the interval J, in which the continuous map $h(k_t)$ is defined, exists a point k_c such that $h^3(k_c) \le k_c < h(k_c) < h^2(k_c)$, where $h^2(k_c) \equiv h(h(k_c))$ and $h^3(k_c) \equiv h(h^2(k_c))$. We restrict our study to eq. (3.2). The steps followed by Day to obtain k_c are: (1) he derives the point k^* that maximizes $h(k_t)$ and calls k^m the maximum of $h(k_t)$; (2) he solves the equation $h(k_t) = k^*$ and indicate with k_c the smallest root; (3) he assumes that $h(k^m) = 0$, $k^m < m$ and $k^* < k^m$; (4) he names k^s the steady-state of $h(k_t)$; (5) he observes that, fixing the parameters β, γ and m, *the graph of* $h(k_t)$ *stretches upwards as B is increased* and at same time the position of k^c does not change because in the expression of k^c the parameter B does not appear while the maximum $h(k^c)$ depends linearly on B (See Figure 2 and Figure 3).

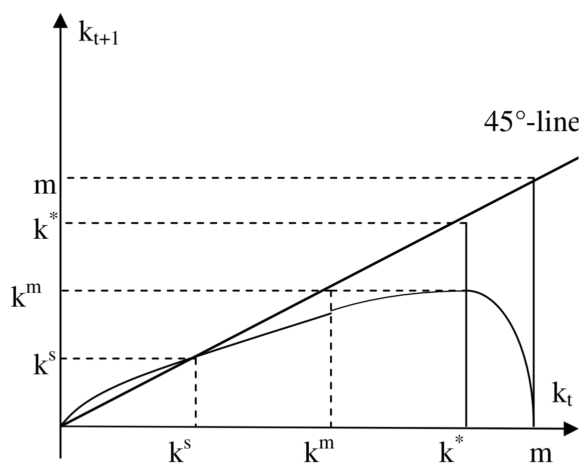


Figure 2: Monotonic Growth or Contraction Chaos

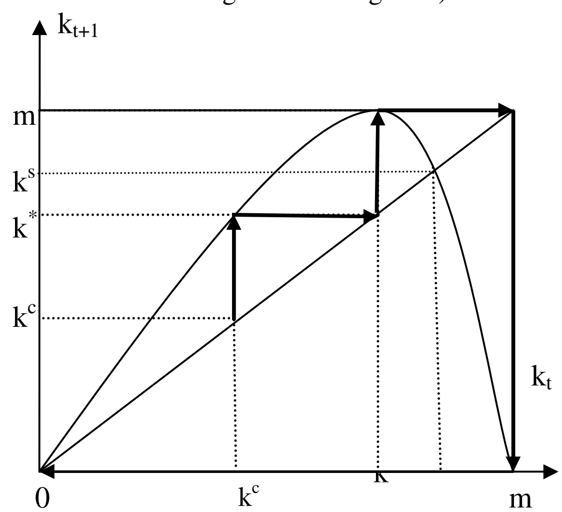


Figure 3: Sufficient Conditions for Li-Yc

4. A Two Class Growth Model: Böhm and Kaas (2000)

4.1 Introduction

The main results of Böhm and Kaas (2000) work are two. The first consists in proving that, slightly changing the standard assumptions of the neoclassical production function (introducing conditions which are slightly weaker than the *Inada conditions*), the dynamics is similar to the one generated by the Solow model. As a matter of fact they define a differentiable, monotonically strictly increasing and strictly concave capital accumulation map that admits a fixed point but not cycles (See below **Proposition 4.1** , **Proposition 4.2**, **Proposition 4.3**).

The second involves the introduction of a Leontief production function, which does not satisfy the weak Inada conditions, in order to construct a piecewise differentiable capital accumulation map. This map can admit zero, one or two steady-states (See **Proposition 4.2.1**, **Proposition 4.2.2**, **Figure 4**. **Figure 5**). Moreover Böhm and Kaas (2000) are also able to derive Li-Yorke chaos [Böhm and Kaas, (2000)].

In particular, in the model of Böhm and Kaas (2000) there are two types of agents (*two class model*), called workers and shareholders, and only one good (or commodity) is produced which is consumed or invested (*one sector model*). Like Kaldor (1956,1957) and Pasinetti (1962), the workers and shareholders have constant savings propensities, denoted respectively with s_w and s_r $(0 \leq s \leq 1$ and $0 \leq s \leq 1)$. The output is produced with two factors: labor and capital. We consider that the capital depreciates at a rate $0 < \delta \leq 1$ and the labor grows at rate $n \geq 0$. We write the production function $f: \Re \rightarrow \Re$ in intensive form (i.e. it is maps capital per worker k into output per worker y), and suppose that f satisfies the following conditions :

- f is C^2;
- $f(\lambda k) = \lambda f(k)$ (*constant returns to capital*);
- f is monotonically increasing and strictly concave (i.e. $f'(k)>0$ and $f''(k)<0$ for all $k > 0$);
- $\lim_{k\to\infty} f(k) = \infty$;
- $\lim_{k\to 0} \frac{f(k)}{k} = \infty$ and (b) $\lim_{k\to\infty} \frac{f(k)}{k} = 0$ (*weak Inada conditions (WIC)*)

If we assume that the market is competitive then the wage rate w(k) is coincident with the marginal product of labor, i.e. $w(k) = f(k) - k f'(k)$, and the interest rate (or investment rate) r is equal to the marginal product of capital, i.e. $r = f'(k)$. We suppose that f(0) generally is not equal to 0. We observe that the total capital income per worker is $kf'(k)$. Moreover from WIC we deduce that:

- $w(k) \geq 0$;
- $w'(k) = - k f''(k) > 0$ (w(k) is strictly monotonically increasing);
- $0 \leq kf'(k) \leq f(k) - f(0)$;
- $\lim_{k\to 0} kf'(k) = 0$.

Similarly to the Solow model we obtain that the time-one map of capital accumulation is

$$k_{t+1} = G(k_t) = \frac{1}{1+n}((1-\delta)\, k_t + s_w w(k_t) + s_r k_t f'(k_t)) \quad (4.1).$$

Proposition 4.1.1 Given $n \geq 0$ and $0 \leq \delta \leq 1$, let f(k) be a production function which satisfies the WIC. If the workers do not save less than shareholders (i.e. $s_w \geq s_r$) or $e_{f'(k)} \geq -1$ then G is monotonically increasing in k.

The following proposition investigates *the existence and the uniqueness of steady states*.

Proposition 4.1.2 Consider n and δ fixed and let f(k) be a production function which satisfies the WIC. The following conditions hold:

- $k = 0$ if and only if $s_w = 0$ or $f(0) = 0$.
- There exists al least one positive steady state if $(s_r > 0$ and $\lim_{k\to 0} f'(k) = 0)$ or if $(s_w > 0$ and $f'(0) < \infty)$.
- There exists at most one positive steady state if $(s_r \geq s_w)$.

Proposition 4.1.3 k^* is a steady state of Pasinetti-Kaldor iff, for given n and δ the pairs (s_r, s_w) of savings rates describe the line $s_r + \frac{1-e_f(k^*)}{e_f(k^*)} s_w = 1$ in the (s_r, s_w)-diagram, where $e_{f(k)} = \frac{kf'(k)}{f(k)}$

- has negative slope;
- goes across the point $(s_r, s_w) = (1,0)$;
- is below or above the $45°$- line $s_w = s_r$ depending on $e_{f(k^*)}$ is less or greater than $1/2$.

The (s_r, s_w) - plane is coincident with the square $[0,1]^2$.

4.2 The dynamics with fixed proportions

We consider the Leontief technology $f_L(k) = \min\{ak, b\} + c, \quad a, b, c > 0$.

Let $k^* = b/a$ be. We have

$$f_L(k) = \begin{cases} ak+c, & \text{if } k \le k^*, \\ b+c, & \text{if } k > k^* \end{cases} \quad \text{and} \quad f_L'(k) = \begin{cases} a, & \text{if } k \le k^*; \\ 0, & \text{if } k > k^*. \end{cases}$$

The map G becomes

$$G_L(k) = \begin{cases} G_1(k) = \frac{1}{1+n}((1 - \delta + s_r a)\, k + s_w c), & \text{if } k \le k^*, \\ G_2(k) = \frac{1}{1+n}((1-\delta)k + (b+c)\, s_w), & \text{if } k > k^*. \end{cases}$$

We may say that:

- G_1 and G_2 are affine-linear maps strictly monotonically increasing;
- $G_1' = \frac{1}{1+n}(1 - \delta + s_r a) > G_2' = \frac{1}{1+n}(1 - \delta)$;
- $G_2' < 1$: the map G_2' has always a fixed point k_2 if we define $G_2(k)$ for all $k \ge 0$;
- G_1 has the fixed point k_1 if and only if $G_1' < 1$, that is $n + \delta - s_r a > 0$;
- $G_1(0) = \frac{1}{1+n} s_w\, c < G_2(0) = \frac{1}{1+n}(b+c)\, s_w$ if we define $G_2(k)$ for all $k \ge 0$.

Let k_1 be the fixed point for G_1. Then k_1 *is a fixed point also for G if and only if* $k_1 < k^*$. Analogously, found the fixed point k_2 for G_2, we have that k_2 *is a fixed point also for* G *if and only if* $k^* < k_2$.

Proposition 4.2.1 Let $G_1' < 1$ be. We obtain that:

(i) the fixed point k_1 for G_1 is equal to $\frac{cs_w}{n+\delta-as_r}$;

(ii) k_1 is a fixed point also for G if and only if $bs_r + cs_w < (n+\delta)\frac{b}{a}$;

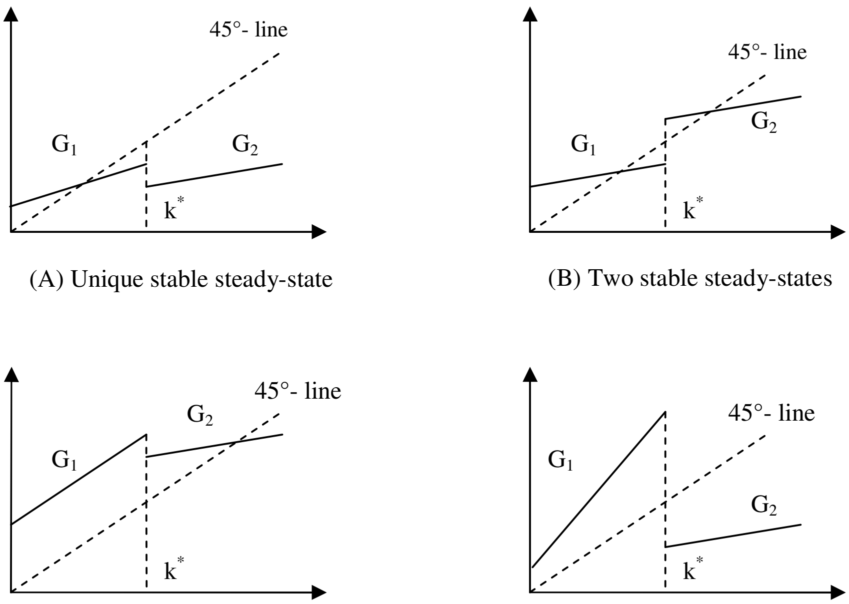
(iii) $G_1(k^*) < k^*$ if and only if $bs_r + cs_w < (n+\delta)\frac{b}{a}$.

Proposition 4.2.2 We get

(i) the fixed point of G_2 is $k_2 = \dfrac{(b+c)s_w}{n+\delta}$;

(ii) k_2 is the fixed point also for G if and only if $s_w > \dfrac{(n+\delta)b}{(b+c)a}$;

(iii) $G_2(k_*) > k^*$ if and only if $s_w > \dfrac{(n+\delta)b}{(b+c)a}$.



(A) Unique stable steady-state

(B) Two stable steady-states

(C) Unique stable steady-state

(A) No stable steady-state

Figure 4 Types of time-one map with Leontief technology

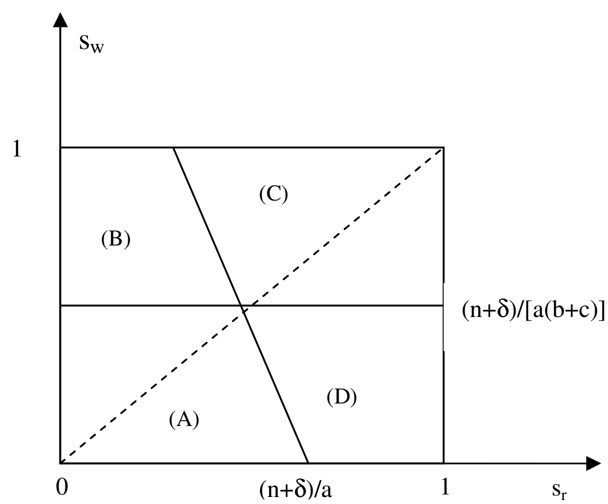


Figure 5 Stability regions for the Leontief technology

5. Complex Dynamics in a Pasinetti-Solow Model of Growth and Distribution: a Model of P.Commendatore

5. 1 Introduction

Similarly to the paper of Böhm and Kaas (2000), the model of Commendatore (2005)

- is a two-class model, that is two distinct group of economic agents (workers and capitalists) exist, with constant propensities to save [Kaldor, (1956)];
- labor and capital markets are perfectly competitive;
- the income sources of workers are wages and profits and the income of capitalists is only profits [Pasinetti, (1962)];
- the time is discrete;
- there is a single good in the economy (one sector model).

Commendatore's model differs from the model of Böhm and Kaas in some assumptions:

- following Chiang (1973), workers not save in same proportions out of labor and income of capital;
- the production function is not with fixed proportions (Leontief technology) but it is a CES production function;
- likewise Samuelson-Modigliani (1966) that, following Pasinetti (1962), extend the Solow growth model (1956) to two-dimensions, the map that describes the accumulation of capital in discrete time is two-dimensional because it considers not only the different saving behaviour of two-classes but also their respective wealth (capital) accumulation.

5.2 The model: the economy, short-run equilibrium, steady growth equilibrium

Let $f(k)=\left[\alpha+(1-\alpha)k^{\rho}\right]^{1/\rho}$ be the CES production function in intensive form, where k is the capital/labor ratio, $0<\alpha<1$ is the distribution coefficient, $-\infty<\rho<1$ ($\rho\neq 0$), $\eta=1/(1-\rho)$ is the constant elasticity of substitution. We consider $f(k)>0$. Therefore $f(k)=\left[\alpha+(1-\alpha)k^{\rho}\right]^{1/\rho}=\left[\alpha k^{-\rho}+(1-\alpha)\right]^{1/\rho}k$. The terms k_w and k_c denote, respectively, workers' and capitalists' capital per worker, where $0\leq k_w\leq k$, $0\leq k_c\leq k$, $k=k_w+k_c$. The workers' saving out of wages are represented by $s_{ww}(f(k)-kf'(k))$ and the workers' saving out of capital revenues consist in $s_{wP}f'(k)k_w$, where $0\leq s_{ww}\leq 1$, $0\leq s_{wP}\leq 1$. Instead the capitalists' savings are $s_cf'(k)k_c$, where $0\leq s_c\leq 1$. We assume $s_c>\max\{s_{ww},s_{wP}\}$. Thus the aggregate savings correspond to $s(k_c,k_w)=s_{ww}(f(k)-f'(k)k)+s_{wP}f'(k)k_w+s_cf'(k_c)$.

Let n be the constant rate of growth of labor force, the following map $G(k_w,k_c)=\frac{1}{1+n}\left[(1-\delta)k+i\right]$ describes the rule of capital accumulation per worker, where i indicates gross investment per worker and $0<\delta<1$ is the constant rate of capital depreciation. In a short-run equilibrium G becomes

$$G(k_w,k_c)=\frac{1}{1+n}\left[(1-\delta)k+s_{ww}(f(k)-f'(k)k)+s_{wP}f'(k)k_w+s_cf'(k_c)\right], \tag{5.2.1}$$

from which we deduce the capitalist' process of capital accumulation

$$G_w(k_w,k_c)=\frac{1}{1+n}\left[(1-\delta)k_w+s_{ww}(f(k)-f'(k)k+s_{wP}f'(k)k_w\right] \tag{5.2.2}$$

and the capitalist's rule of capital accumulation

$$G_c(k_w,k_c)=\frac{1}{1+n}\left[(1-\delta)k_c+s_cf'(k)k_c\right] \tag{5.2.3}$$

In order to obtain the steady states of G_w and G_c, we imposing

$G_w(k_w,k_c)=k_w$ and $G_c(k_w,k_c)=k_c$.

We get $(n+\delta)k_w=s_{ww}(f(k)-f'(k)k)+s_{wP}f'(k)k_w$, $(n+\delta)k_c=s_cf'(k_c)$.

Commendatore (2005) find three types of equilibria: *Pasinetti -equilibrium* (capitalists own positive share of capital), *dual equilibrium* (only workers own capital) and *trivial equilibrium* (the overall capital is zero) and, developing ingeniously a geometrical method used by Meade [Meade, (1966)], describes geometrically the coexistence of them (See Figure 6). To detect above equilibria he relates $\frac{f(k)}{k}$ to $e_{f(k)} = \frac{f'(k)k}{f(k)}$ and he finds that $\frac{f(k)}{k} = \varphi(e_f)$ for Pasinetti-equilibrium and $\frac{f(k)}{k} = \theta(e_f)$, where $\varphi(x) \equiv \left(\frac{1-\alpha}{x}\right)^{1/\rho}$ $(0 < \alpha < 1)$ and $\theta(x) \equiv \frac{n+\delta}{s_{ww}(1-x) + s_{wP}x}$.

For example, if ρ < -1, the curve $\varphi(e_f)$ is monotonically strictly increasing and strictly concave while the curve $\theta(e_f)$ is (a) a horizontal line if $s_{ww} = s_{wP}$, (b) monotonically strictly decreasing and strictly concave if $s_{ww} < s_{wP}$ and (c) monotonically strictly increasing and strictly convex if $s_{ww} > s_{wP}$. Look at the Figure 6: the intersection between $\varphi(e_f)$ and $\theta(e_f)$ represents a Dual-equilibrium, the point $(1, \varphi(1))$ gives a trivial equilibrium and the intersection between the vertical line $e_f(k^P) = (1-\alpha)^{\frac{1}{1-\rho}}\left(\frac{n+\delta}{s_c}\right)^{\frac{\rho}{\rho-1}}$ and $\varphi(e_f)$ identifies the Pasinetti -equilibrium. To derive the remaining cases we observe that the other diagrams of $\varphi(x)$ are showed in Figure 7.

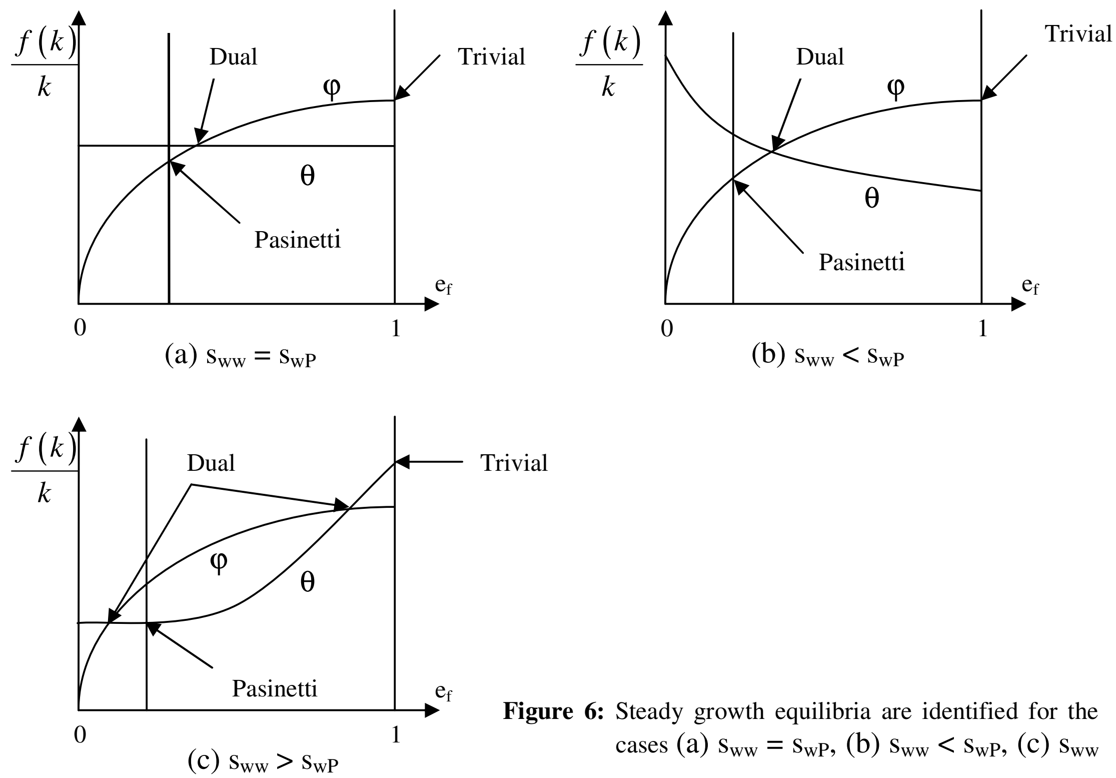


Figure 6: Steady growth equilibria are identified for the cases (a) $s_{ww} = s_{wP}$, (b) $s_{ww} < s_{wP}$, (c) s_{ww}

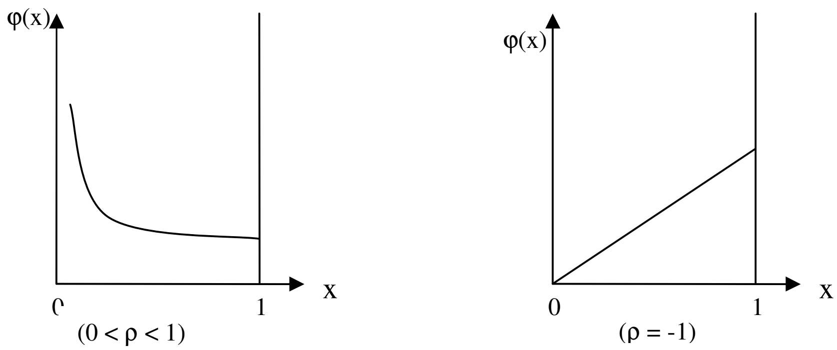


Figure 7 The diagram of $\varphi(x)$ for (0 < ρ < 1) and (ρ = -1).

The author analyses the *local stability* and the *global stability* of the nonlinear dynamical system given by (5.2.2) and (5.2.3). To study the local stability, he starts from Pasinetti equilibria and he finds the trace **T** and the determinant **D** of the Jacobian matrix **J** evaluated at a Pasinetti-equilibrium fixed. Then he applies the *conditions of stability* of dynamical system [Azariadis, C., (1993)] and identifies the stability region (*Triangle Stability*) in TD-plane. Moreover, with the Theory of Local Bifurcations, he studies which bifurcation appears when a given Pasinetti-equilibrium loses a stability. Using the *characteristic equation* he derives the eigenvalues of **J** and analyses the global stability.

6. Conclusions

We observe that Commendatore's model generalizes Böhm and Kaas (2000) model and Solow (1956) model. As a matter of fact

- setting $s_{ww} = s_{wP}$ and $k = k_w = k_c$ in (5.2.1)

$$G(k_w, k_c) = \frac{1}{1+n}\left[(1-\delta)k + s_{ww}(f(k) - f'(k)k) + s_{wP}f'(k)k_w + s_c f'(k_c)\right],$$

we have the (4.1), i.e. from Commendatore's model we deduce Böhm and Kaas (2000) model;

- setting $s_w = s_r$ in (4.1)

$$k_{t+1} = G(k_t) = \frac{1}{1+n}((1-\delta)\, k_t + s_w w(k_t) + s_r\, k_t\, f'(k_t)),$$

we obtain the (2.2), i.e. from Böhm and Kaas (2000) model we deduce the Solow (1956) model.

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