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ROBUST TWO–STAGE LEAST SQUARES: SOME MONTE CARLO EXPERIMENTS

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Abstract

The Two–Stage Least Squares (2–SLS) is a well known econometric technique used to estimate the parameters of a multi–equation econometric model when errors across the equations are not correlated and the equation(s) concerned is (are) over–identified or exactly identified. However, in presence of outliers in the data matrix, the classical 2–SLS has a very poor performance. In this study a method has been proposed to generalize the 2–SLS to the Weighted Two–Stage Least Squares (W2–SLS), which is robust to the effects of outliers and perturbations. Monte Carlo experiments have been conducted to demonstrate the performance of the proposed method. It has been found that robustness of the proposed method is not much destabilized by the magnitude of outliers. The breakdown point of the method is quite high, somewhere between 45 to 50 percent of the number of points in the data matrix.

Keywords: Two–Stage Least Squares, multi–equation econometric model, simultaneous equations, outliers, robust, weighted least squares, Monte Carlo experiments, unbiasedness, efficiency, breakdown point, perturbation, structural parameters, reduced form

JEL Classification: **C13, C14, C63, C15, C01**

1. Introduction:

The Two–Stage Least Squares (2–SLS) is a well known econometric technique used to estimate the parameters of a multi–equation (or simultaneous equations) econometric model when errors across the equations are not correlated and the equation(s) concerned is (are) over–identified or exactly identified. It is one of the members of the family of k–class estimators. Unlike the Three–Stage Least Squares, it does not estimate the parameters of all the equations of the model in one go. The 2–SLS estimates the parameters of an econometric model equation by equation, that is, one equation at a time.

Let a multi–equation econometric model be described by the system of its structural equations $YA+XB+U=0$, where Y is an $n\times m$ data matrix of m endogenous variables in n observations, X is an $n\times k$ data matrix of k exogenous or pre–determined variables in n observations, A is an $m\times m$ full rank matrix of unknown parameters or coefficients associated with Y, B is a $k\times m$ matrix of unknown parameters or coefficients associated with X and U is an $n\times m$ matrix of (unobserved) errors. The elements of A and B are called the structural parameters. Since U is often correlated with Y which is itself stochastic, the parameters in the columns of A and B cannot be estimated by means of the Ordinary Least Squares (OLS) in view of the violation of the Gauss–Markov assumptions for the applicability of the OLS. Instead of using the OLS directly, the system of equations $YA+XB+U=0$ is first transformed into the reduced form equations. The reduced form equations describe Y in terms of X only. Indeed if we post–multiply the system of equations $YA+XB+U=0$ by A^{-1}, we have $YAA^{-1}+XBA^{-1}+UA^{-1}=0$ or $Y=XP+E$, where $P=-BA^{-1}$ and $E=-UA^{-1}$. Now since X is fixed (non–stochastic) and it cannot be correlated with E, the system of reduced form equations $Y=XP+E$ is amenable to estimation by the OLS. Therefore, P (which is the matrix of the reduced form coefficients) is estimated by the OLS as $\hat{P}=[X'X]^{-1}X'Y$ and used to obtain $\hat{Y}=X\hat{P}$. Then in each equation where any endogenous variable $Y_j \subset Y$ appears as an explanatory variable, Y_j is replaced by $\hat{Y}_j$. Due to this replacement, the explanatory variables are no longer stochastic or correlated with the error term in the equation concerned, and so the equation is amenable to estimation by the OLS. Application of the OLS (once again) on this transformed equation readily gives the estimates of the parameters in that equation.

2. Implications of the Presence of Outliers in the Data Matrices:

Now suppose there are some outliers in X, Y or both the data matrices. This would affect $\hat{P} = [X'X]^{-1}X'Y$ and consequently $\hat{Y} = X\hat{P}$. At the second stage since $\hat{Y}_j \subset \hat{Y}$ appear as explanatory variables, all the estimated parameters would be affected. As a matter of fact, the effects of outliers will pervade through all the equations and the estimated structural parameters in them. These effects are so intricately pervasive that it is very difficult to assess the influence of outliers on the estimated structural parameters.

A number of methods have been proposed to obtain robust estimators of regression parameters but most of them are limited to single equation models. Their adaptation to estimation of the structural parameters of multi–equation models is not only operationally inconvenient, it is also theoretically unconvincing. Moreover, generalization of those methods to multi–equation cases has scarcely been either successful or popular.

3. The Objectives of the Present Study:

In this study a method has been proposed to conveniently generalize the 2–SLS to the weighted 2–SLS (W2–SLS) so that $\hat{P} = [(wX)'(wX)]^{-1}(wX)'(wY)$, where w is the weight matrix applied to Y and X. Accordingly, we have $\hat{Y} = X\hat{P}$. At the 2nd stage, for the i^{th} equation we have $g_i = [(\omega_i Z_i)'(\omega_i Z_i)]^{-1}(\omega_i Z_i)'(\omega_i y_i)$, where $g_i = [a_i \mid b_i]'$; $Z_i = [\hat{Y}_i \mid X_i]$; $y_i \subset Y$; $\hat{Y}_i \subset \hat{Y}$; $\hat{y}_i \not\in \hat{Y}_i$; $X_i \subset X$; y_i is the observed endogenous variable appearing in the i^{th} structural equation as the dependent variable, $\hat{Y}_i$ is the set of estimated endogenous variables appearing in the i^{th} equation as the explanatory variables and X_i is the set of exogenous (or predetermined) variables appearing in the i^{th} equation as the explanatory variables. It may be noted that at the second stage of the proposed W2–SLS we use different weights (ω) for different equations. These weights (w and ω_i) are obtained in a particular manner as described latter in this paper. We also conduct some Monte Carlo experiments to demonstrate that our proposed method performs very well in estimating the structural parameters of multi–equation econometric models while the data matrices are containing numerous large outliers.

4. Determination of Weights in the Weighted Two–Stage Least Squares

Using the Mahalanobis distance as a measure of deviation from center, Campbell (1980) obtained a robust covariance matrix that is almost free from the influence of outliers. Campbell's method is an iterative method. Given an observed data matrix, Z, in n observations (rows) and v variables (columns) it obtains a v-elements vector of weighted (arithmetic) mean, $\bar{z}$, and weighted variance–covariance matrix, $S(v,v)$, in the following manner. Initially, all weights, $\varpi_\ell; \ell = 1, n$ are considered to be equal, $1/n$, and the sum of weights, $\sum_{\ell=1}^{n} \varpi_\ell = 1$. Defining $d_0 = \sqrt{v} + \beta_1/\sqrt{2}$; $\beta_1 = 2$, $\beta_2 = 1.25$, we obtain

$$\bar{z} = \sum_{\ell=1}^{n} \varpi_\ell z_\ell / \sum_{\ell=1}^{n} \varpi_\ell \;;\; S = \sum_{\ell=1}^{n} \varpi_\ell^2 (z_\ell - \bar{z})'(z_\ell - \bar{z}) / \left[\sum_{\ell=1}^{n} \varpi_\ell^2 - 1 \right] \;;$$

$$d_\ell = \left\{ (z_\ell - \bar{z}) S^{-1} (z_\ell - \bar{z})' \right\}^{1/2} ; \ell = 1, n \;;$$

$$\varpi_\ell = \varpi(d_\ell) / d_\ell ; \ell = 1, n :$$

$$\varpi(d_\ell) = d_\ell \text{ if } d_\ell \le d_0 \text{ else } \varpi(d_\ell) = d_0 \exp[-0.5(d_\ell - d_0)^2 / \beta_2^2].$$

If $d_\ell \cong 0$ then $\varpi_\ell = 1$. We will call it the original Campbell procedure to obtain a robust covariance matrix. However, our experience with this procedure to obtain a robust covariance matrix is not very encouraging in this study as well as elsewhere (Mishra, 2008). We will use the acronym OCP for this original Campbell procedure.

Hampel *et al*. (1986) defined the median of absolute deviations (from median) as a measure of scale, $s_H(z_a) = \underset{\ell}{median} | z_{\ell a} - \underset{\ell}{median}(z_{\ell a}) | / 0.6745$ which is a very robust measure of deviation. Using this measure of deviation also, we may assign weights to different data points. If we choose to heuristically assign the weight $\varpi_\ell = 1$ for $d_\ell - s_H(d) \le d_\ell < d_\ell + s_H(d)$, $\varpi_\ell = (1/2)^2$ for $d_\ell - 2s_H(d) \le d_\ell < d_\ell - s_H(d)$ as well as $d_\ell + 2s_H(d) \ge d_\ell > d_\ell + s_H(d)$ and so on, and use Campbell's iterative method incorporating these weights, we may obtain a robust covariance matrix and weights. Our experience with this procedure has been highly rewarding in this study as well as elsewhere [Mishra, (2008)]. We will call it the Modified Campbell Procedure (MCP) to obtain a robust covariance matrix and weights to different data points.

The weights (ϖ) obtained through the MCP (or OCP, as the case may be) are used as w in $\hat{P} = [(wX)'(wX)]^{-1}(wX)'(wY)$ at the first stage of the W2–SLS to obtain the robust estimates of the matrix of reduced form coefficients. In this procedure of obtaining $\hat{P}$, X contains the unitary vector to take care of the intercept term, although weights $(w = \varpi)$ are obtained with Z^* that contains Y and all the variables in X, sans the unitary vector relating to the intercept term. Similarly, at the second stage, the MCP/OCP weights $(\omega_i = \varpi_i)$ are obtained from $Z^* = [y_i \,|\, \hat{Y}_i \,|\, X_i^*]$, where X_i^* contains all exogenous (predetermined) variables appearing in the i^{th} structural equations, sans the unitary vector related to the intercept term. However, in obtaining $g_i = [a_i \,|\, b_i]'$, the matrix $Z_i = [\hat{Y}_i \,|\, X_i]$ is used wherein X_i contains all exogenous (predetermined) variables, including the one related to the intercept term in the i^{th} equation.

5. Some Monte Caro Experiments

In order to assess the performance of our proposed method and compare it with the 2–SLS when data matrices (Y and X) contain outliers, we have conducted some Monte Carlo experiments. Using the random number generator seed = 1111, we have generated X containing five exogenous variables in 100 observations and appended to it the 6th column of unitary vector to take care of the intercept term. Thus, in all, we have X in 100 rows and 6 columns. All values of X lie between 0 and 20 such that $0 < x_{ij} < 20$. Then the data matrix for endogenous variables, Y, has been generated with the parameter matrices, A and B and adding a very small normally distributed random error, $U \sim N(0, 0.001)$ directly, without going into the subtleties of obtaining $U = -EA$. The magnitude of error has been kept at a very low level since our objective is not to mingle the effects of errors with those of outliers on the estimated parameters. If the magnitude of errors is large, it would affect the estimated values of parameters and it would be difficult to disentangle the effects of outliers from those of the errors. The computer program GENDAT (in

FORTRAN 77) to generate data is appended. As already mentioned, the program was run with the random number generator seed = 1111. The following are the matrices of structural parameters used in our experiments.

$$A' = \begin{bmatrix} -1 & 7 & 0 & -6 & 0 \\ 3 & -1 & 5 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 6 & 0 & 0 & -1 & -3 \\ -11 & 0 & 9 & 0 & -1 \end{bmatrix}; \qquad B' = \begin{bmatrix} 0 & 5 & 0 & -7 & 0 & 60 \\ 3 & 0 & -5 & 0 & 0 & 20 \\ 0 & 2 & 0 & 0 & 0 & 9 \\ 0 & 4 & 0 & 0 & -3 & -8 \\ 0 & 0 & 0 & 6 & 0 & -11 \end{bmatrix}$$

The data (Y and X) thus generated are used as the base data to which different number and different sizes of perturbation quantities are added in different experiments. For every experiment we have limited the number of replicates (NR) to 100, although this number could have been larger or smaller. For each experiment the mean, standard deviation and RMS (Root–Mean–Square) of expected parameters ($\hat{A}$ and $\hat{B}$) have been computed over the 100 replicates. The following formulas are used for computing these statistics.

$$Mean(\hat{a}_{ij}) = (1/NR)\sum_{\ell=1}^{NR} \hat{a}_{\ell ij};\ i, j = 1, m \quad ; \quad Mean(\hat{b}_{ij}) = (1/NR)\sum_{\ell=1}^{NR} \hat{b}_{\ell ij} \ ; i = 1, k \ ; \ j = 1, m$$

$$SD(\hat{a}_{ij}) = \left[\frac{1}{NR}\sum_{\ell=1}^{NR} (\hat{a}_{\ell ij})^2 - Mean^2(\hat{a}_{ij})\right]^{0.5} ; i, j = 1, m \quad ; \quad SD(\hat{b}_{ij}) = \left[\frac{1}{NR}\sum_{\ell=1}^{NR} (\hat{b}_{\ell ij})^2 - Mean^2(\hat{b}_{ij})\right]^{0.5} ; i = 1, k \ ; \ j = 1, m$$

$$RMS(\hat{a}_{ij}) = \left[\frac{1}{NR}\sum_{\ell=1}^{NR} (\hat{a}_{\ell ij} - a_{ij})^2\right]^{0.5} ; i, j = 1, m \quad ; \quad RMS(\hat{b}_{ij}) = \left[\frac{1}{NR}\sum_{\ell=1}^{NR} (\hat{b}_{\ell ij} - b_{ij})^2\right]^{0.5} ; i = 1, k \ ; \ j = 1, m$$

A distance between RMS and SD entails bias of the estimation formula and a larger SD entails inefficiency of the estimation formula. Reduction in SD as a response to increase in the number of replicates entails consistency of the estimator formula. In the present exercise we have not looked into the consistency aspect by fixing the number of replicates (NR) to 100, although it could have been done without much effort by increasing NR from (say) 20 to 200 (or more) by an increment of 20 or so.

Experiment–1: In this experiment we have set the number of perturbations at 10 (i.e. NOUT=10) and the size of perturbation (OL) in the range of 10 ± 25 or between –15 to 35. In this range the size of perturbation quantities is randomly chosen and those quantities are added to the data at equiprobable random locations. Accordingly, in the program ROB2SLS the parameters are set at OMIN=10, OMAX=50 such that OL=OMIN+(OMAX–OMIN)*(RAND–0.5). The random number RAND lays between zero and unity (exclusive of limits). To generate the random numbers seed = 2211 has been used (in this as well as subsequent experiments). With this design, we have estimated the structural parameters by 2–SLS, OCP and MCP. The results are presented in tables 1.1 through 3.3. A perusal of these table immediately reveals that the 2–SLS and the W2–SLS(OCP) perform very poorly. Of the two, the 2–SLS appears to perform somewhat better. However, the performance of the W2–SLS(MCP) is excellent.

Table–1.1. Mean of Estimates of Structural Parameters: Method –2–SLS

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	1.493 5	0	0.348 3	0	0	2.078 9	0	– 1.146	0	17.937 1
Eq–2	2.894	–1	4.978	0	0	2.925	0	–	0	0	22.126

	5		5			6		4.9462			2
Eq–3	0	0	–1	2.8951	0	0	1.9202	0	0	0	9.5226
Eq–4	1.99	0	0	–1	–0.8833	0	0.7576	0	0	–0.9063	–5.6702
Eq–5	–9.9835	0	8.1432	0	–1	0	0	0	5.5251	0	–10.194

Table–1.2. Standard Deviation of Estimates of Structural Parameters: Method –2–SLS

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	2.1017	0	2.591	0	0	1.1407	0	2.2643	0	14.9058
Eq–2	0.4527	0	1.6893	0	0	0.7541	0	1.3289	0	0	13.289
Eq–3	0	0	0	0.3976	0	0	0.3328	0	0	0	2.0584
Eq–4	6.1134	0	0	0	3.2882	0	4.9767	0	0	3.227	3.1004
Eq–5	3.8894	0	2.7011	0	0	0	0	0	2.076	0	9.5898

Table–1.3. Root Mean Square of Estimates of Structural Parameters: Method –2–SLS

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	5.894	0	6.8566	0	0	3.1359	0	6.2767	0	44.6259
Eq–2	0.4648	0	1.6894	0	0	0.7577	0	1.33	0	0	13.458
Eq–3	0	0	0	0.4112	0	0	0.3422	0	0	0	2.1237
Eq–4	7.3112	0	0	0	3.9105	0	5.9397	0	0	3.8467	3.8782
Eq–5	4.02	0	2.8337	0	0	0	0	0	2.1296	0	9.6236

Table–2.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (OCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	4.5418	0	–3.2473	0	0	3.6247	0	–4.271	0	36.9997
Eq–2	2.0327	–1	5.0332	0	0	3.1394	0	–4.0161	0	0	12.6559
Eq–3	0	0	–1	2.654	0	0	1.6841	0	0	0	9.0044
Eq–4	2.402	0	0	–1	–1.1363	0	2.1191	0	0	–2.2551	–1.4636
Eq–5	–8.5972	0	7.1151	0	–1	0	0	0	4.68	0	–7.7846

Table–2.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (OCP)

Variables / Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	3.3486	0	3.7184	0	0	2.0598	0	3.7719	0	48.4423
Eq–2	4.0175	0	9.9488	0	0	6.521	0	3.7596	0	0	64.2183
Eq–3	0	0	0	1.1314	0	0	1.2864	0	0	0	12.064
Eq–4	10.0494	0	0	0	4.2537	0	7.802	0	0	6.6725	53.6402

Eq–5	13.1535	0	10.9899	0	0	0	0	0	8.5956	0	24.2266

Table–2.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (OCP)

Variables / Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	4.154	0	4.6265	0	0	2.4767	0	4.6556	0	53.6253
Eq–2	4.1323	0	9.9489	0	0	6.5225	0	3.8862	0	0	64.6369
Eq–3	0	0	0	1.1831	0	0	1.3246	0	0	0	12.064
Eq–4	10.674	0	0	0	4.6441	0	8.0255	0	0	6.7139	54.037
Eq–5	13.3711	0	11.1504	0	0	0	0	0	8.6964	0	24.439

Table–3.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	7.0498	0	–6.058	0	0	5.0261	0	–7.0532	0	60.3819
Eq–2	3.0002	–1	5.0011	0	0	3.0004	0	–5.0011	0	0	20.01
Eq–3	0	0	–1	2.9999	0	0	1.9999	0	0	0	8.9995
Eq–4	5.9973	0	0	–1	–2.9984	0	3.9975	0	0	–2.9986	–7.9969
Eq–5	–11.0005	0	9.0001	0	–1	0	0	0	5.9999	0	–10.9989

Table–3.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0067	0	0.0078	0	0	0.0035	0	0.0071	0	0.051
Eq–2	0.0001	0	0.0004	0	0	0.0002	0	0.0004	0	0	0.0035
Eq–3	0	0	0	0.0001	0	0	0.0001	0	0	0	0.0005
Eq–4	0.002	0	0	0	0.001	0	0.0016	0	0	0.0011	0.0014
Eq–5	0.0013	0	0.0009	0	0	0	0	0	0.0006	0	0.002

Table–3.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0503	0	0.0585	0	0	0.0263	0	0.0536	0	0.3852
Eq–2	0.0002	0	0.0012	0	0	0.0004	0	0.0012	0	0	0.0106
Eq–3	0	0	0	0.0002	0	0	0.0001	0	0	0	0.0007
Eq–4	0.0033	0	0	0	0.0019	0	0.003	0	0	0.0017	0.0034
Eq–5	0.0014	0	0.0009	0	0	0	0	0	0.0006	0	0.0022

Experiment–2: In this experiment we have set the number of perturbations at 10 (i.e. NOUT=10) and the size of perturbation (OL) in the range of 10 ± 50 or between –40 to 60. The parameters in the program are set at OMIN=10, OMAX=100 and hence OL=OMIN+(OMAX–OMIN)*(RAND–0.5). The dismal performance of 2–SLS and W2–SLS(OCP) observed in experiment–1 has been further aggravated and therefore we do not consider it necessary to report the mean, SD and RMS of estimated structural parameters for those estimators. However, once again the W2–SLS(MCP) has performed exceedingly well and the results have been presented in Tables 4.1 through 4.3.

A comparison of Tables 3.1 through 3.3 with the Tables 4.1 through 4.3 reveals that increase in the magnitude of perturbation has hardly affected the results.

Table–4.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	7.0498	0	–6.0579	0	0	5.0261	0	–7.0531	0	60.3817
Eq–2	3.0002	–1	5.0011	0	0	3.0004	0	–5.0011	0	0	20.0097
Eq–3	0	0	–1	2.9999	0	0	2	0	0	0	8.9996
Eq–4	5.9973	0	0	–1	–2.9984	0	3.9974	0	0	–2.9986	–7.9969
Eq–5	–11.0005	0	9.0001	0	–1	0	0	0	5.9999	0	–10.9989

Table–4.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Deviation of Estimated A Matrix					Standard Deviation of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0065	0	0.0076	0	0	0.0034	0	0.0069	0	0.0492
Eq–2	0.0001	0	0.0005	0	0	0.0002	0	0.0004	0	0	0.0038
Eq–3	0	0	0	0.0001	0	0	0.0001	0	0	0	0.0005
Eq–4	0.0018	0	0	0	0.001	0	0.0015	0	0	0.001	0.0015
Eq–5	0.0014	0	0.0009	0	0	0	0	0	0.0006	0	0.002

Table–4.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0502	0	0.0584	0	0	0.0263	0	0.0536	0	0.3848
Eq–2	0.0002	0	0.0012	0	0	0.0004	0	0.0012	0	0	0.0104
Eq–3	0	0	0	0.0002	0	0	0.0001	0	0	0	0.0007
Eq–4	0.0033	0	0	0	0.0019	0	0.003	0	0	0.0017	0.0034
Eq–5	0.0014	0	0.0009	0	0	0	0	0	0.0006	0	0.0023

Experiment–3: In this experiment we have once again set the number of perturbations at 10 (i.e. NOUT=10) and the size of perturbation (OL) in the range of 10 ± 150 or between –140 to 160. The parameters in the program are set at OMIN=10, OMAX=300 and hence OL=OMIN+(OMAX–OMIN)*(RAND–0.5). The results are presented in Tables 5.1 through 5.3. The findings are that increase in the magnitude of perturbation has not affected the W2–SLS(MCP) estimates in any significant manner.

Table–5.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	7.0501	0	–6.0583	0	0	5.0262	0	–7.0534	0	60.3836
Eq–2	3.0002	–1	5.0011	0	0	3.0004	0	–5.0011	0	0	20.0095
Eq–3	0	0	–1	2.9999	0	0	1.9999	0	0	0	8.9996
Eq–4	5.9973	0	0	–1	–2.9984	0	3.9975	0	0	–2.9986	–7.997
Eq–5	–	0	9	0	–1	0	0	0	5.9998	0	–10.9989

	11.0004

Table–5.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0071	0	0.0083	0	0	0.0036	0	0.0075	0	0.0539
Eq–2	0.0001	0	0.0004	0	0	0.0002	0	0.0004	0	0	0.0036
Eq–3	0	0	0	0.0001	0	0	0.0001	0	0	0	0.0005
Eq–4	0.0019	0	0	0	0.001	0	0.0015	0	0	0.001	0.0014
Eq–5	0.0012	0	0.0009	0	0	0	0	0	0.0006	0	0.002

Table–5.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0506	0	0.0589	0	0	0.0264	0	0.054	0	0.3874
Eq–2	0.0002	0	0.0012	0	0	0.0004	0	0.0012	0	0	0.0101
Eq–3	0	0	0	0.0002	0	0	0.0001	0	0	0	0.0006
Eq–4	0.0032	0	0	0	0.0019	0	0.0029	0	0	0.0017	0.0033
Eq–5	0.0013	0	0.0009	0	0	0	0	0	0.0006	0	0.0023

Experiment–4: In this experiment we have set the number of perturbations at 30 (i.e. NOUT=30) and the size of perturbation (OL) in the range of 10 ± 25 or between –15 to 35 as in the experiment–1. We want to look into the effects of increasing the number of perturbations in the data matrix. A perusal of the results (presented in Tables 6.1 through 6.3) reveals that the W2–SLS estimator continues to be robust.

Table–6.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	7.0367	0	–6.0423	0	0	5.0194	0	–7.0391	0	60.2825
Eq–2	3.0002	–1	5.0009	0	0	3.0003	0	–5.0009	0	0	20.0077
Eq–3	0	0	–1	2.9998	0	0	1.9999	0	0	0	8.9992
Eq–4	5.9981	0	0	–1	–2.9988	0	3.9981	0	0	–2.9989	–7.9988
Eq–5	–11.0017	0	9.001	0	–1	0	0	0	6.0005	0	–11.0009

Table–6.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.012	0	0.014	0	0	0.0063	0	0.0129	0	0.0915
Eq–2	0.0001	0	0.0007	0	0	0.0003	0	0.0006	0	0	0.0055
Eq–3	0	0	0	0.0001	0	0	0.0001	0	0	0	0.0007
Eq–4	0.0026	0	0	0	0.0014	0	0.0021	0	0	0.0014	0.0025

Eq–5	0.0014	0	0.001	0	0	0	0	0	0.0007	0	0.0025

Table–6.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0386	0	0.0446	0	0	0.0204	0	0.0412	0	0.2969
Eq–2	0.0002	0	0.0011	0	0	0.0004	0	0.0011	0	0	0.0095
Eq–3	0	0	0	0.0002	0	0	0.0001	0	0	0	0.001
Eq–4	0.0032	0	0	0	0.0018	0	0.0028	0	0	0.0017	0.0027
Eq–5	0.0022	0	0.0014	0	0	0	0	0	0.0008	0	0.0026

Experiment–5: In this experiment we set NOUT=30 as in experiment–4, but increase the size of perturbations (OL) in the range of 10 ± 150 or between –140 to 160 (as in experiment–3). The results are presented in the Tables 7.1 through 7.3. It is observed that the increase in the size of perturbation has not affected the robustness of W2–SLS(MCP) in any significant manner.

Table–7.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	7.0359	0	–6.0416	0	0	5.0189	0	–7.0383	0	60.277
Eq–2	3.0002	–1	5.0009	0	0	3.0003	0	–5.0009	0	0	20.0075
Eq–3	0	0	–1	2.9999	0	0	1.9999	0	0	0	8.9992
Eq–4	5.9982	0	0	–1	–2.9989	0	3.9982	0	0	–2.999	–7.9989
Eq–5	–11.0016	0	9.0009	0	–1	0	0	0	6.0005	0	–11.0009

Table–7.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.0124	0	0.0145	0	0	0.0066	0	0.0133	0	0.0948
Eq–2	0.0001	0	0.0007	0	0	0.0003	0	0.0006	0	0	0.0055
Eq–3	0	0	0	0.0002	0	0	0.0001	0	0	0	0.0008
Eq–4	0.0025	0	0	0	0.0013	0	0.002	0	0	0.0013	0.0024
Eq–5	0.0016	0	0.0011	0	0	0	0	0	0.0007	0	0.0026

Table–7.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.038	0	0.044	0	0	0.02	0	0.0406	0	0.2928
Eq–2	0.0002	0	0.0011	0	0	0.0004	0	0.001	0	0	0.0094
Eq–3	0	0	0	0.0002	0	0	0.0001	0	0	0	0.0011
Eq–4	0.0031	0	0	0	0.0017	0	0.0027	0	0	0.0017	0.0026
Eq–5	0.0022	0	0.0015	0	0	0	0	0	0.0009	0	0.0028

Experiment–6: Now we increase the number of perturbations (NOUT=60) but keep the size as in experiment–1 (between –15 to 35). The results are presented in the Tables 8.1 through 8.3. We observe an increase in the RMS of estimated parameters. Yet, the SD and the RMS values are quite close to each other and the mean coefficients are not far from the true values. These findings indicate that even now the robustness of W2–SLS has not been much affected.

Table–8.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	6.8745	0	–5.8587	0	0	4.9293	0	–6.8653	0	59.0309
Eq–2	2.993	–1	5.0427	0	0	3.017	0	–5.0316	0	0	20.3147
Eq–3	0	0	–1	2.9998	0	0	1.9999	0	0	0	8.9992
Eq–4	5.9328	0	0	–1	–2.9627	0	3.9389	0	0	–2.9597	–7.9645
Eq–5	–11.0289	0	9.0121	0	–1	0	0	0	6.0301	0	–11.2607

Table–8.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.8672	0	0.9883	0	0	0.477	0	0.9275	0	6.6729
Eq–2	0.1023	0	0.2672	0	0	0.0961	0	0.1807	0	0	2.0419
Eq–3	0	0	0	0.0002	0	0	0.0001	0	0	0	0.0009
Eq–4	0.9662	0	0	0	0.5063	0	0.7721	0	0	0.4964	0.5506
Eq–5	1.1557	0	0.7296	0	0	0	0	0	0.5238	0	2.1615

Table–8.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	0.8762	0	0.9984	0	0	0.4822	0	0.9372	0	6.7429
Eq–2	0.1025	0	0.2706	0	0	0.0976	0	0.1834	0	0	2.066
Eq–3	0	0	0	0.0002	0	0	0.0002	0	0	0	0.0012
Eq–4	0.9685	0	0	0	0.5077	0	0.7746	0	0	0.498	0.5517
Eq–5	1.1561	0	0.7297	0	0	0	0	0	0.5247	0	2.1772

Experiment–7: Now we keep NOUT=60 but increase the size of perturbations to –140 to 160 (as in experiment–3). The results are presented in the Tables 9.1 through 9.3. We observe that the mean estimated structural parameters are as yet quite close to the true values, SDs are quite close to the RMS values, much smaller than the magnitude of the mean estimates in most cases. Hence, we may hold that the W2–SLS continues to be robust to outliers/perturbations.

Table–9.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	6.4972	0	–5.4211	0	0	4.7328	0	–6.4631	0	56.1148
Eq–2	2.9728	–1	4.7654	0	0	2.9132	0	–4.8139	0	0	18.0302
Eq–3	0	0	–1	2.9348	0	0	1.9504	0	0	0	8.7548
Eq–4	5.6354	0	0	–1	–2.8069	0	3.7122	0	0	–2.8101	–7.8982
Eq–5	–9.9977	0	8.1903	0	–1	0	0	0	5.4531	0	–9.2529

Table–9.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6

Eq–1	0	1.8062	0	2.0791	0	0	0.961	0	1.9283	0	13.9632
Eq–2	0.5312	0	2.0574	0	0	0.8135	0	1.489	0	0	15.7156
Eq–3	0	0	0	0.4834	0	0	0.3779	0	0	0	1.8151
Eq–4	1.4475	0	0	0	0.7716	0	1.1431	0	0	0.759	1.6985
Eq–5	3.5302	0	2.6157	0	0	0	0	0	1.7315	0	5.5525

Table–9.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	1.8749	0	2.1582	0	0	0.9974	0	2.0016	0	14.4936
Eq–2	0.5319	0	2.0708	0	0	0.8181	0	1.5006	0	0	15.8385
Eq–3	0	0	0	0.4878	0	0	0.3811	0	0	0	1.8316
Eq–4	1.4927	0	0	0	0.7954	0	1.1787	0	0	0.7823	1.7016
Eq–5	3.6697	0	2.7381	0	0	0	0	0	1.8158	0	5.8209

Experiment–8: Next, we increase the number of perturbations to set NOUT=75 and set the size of perturbations in the range of –15 to 35. The results are presented in the Tables 10.1 through 10.3. We observe that the unbiasedness of W2–SLS is not much disturbed since the SDs and the RMS values are close to each other. However, many of the mean estimated structural parameters are now quite far from the true values and many SDs are not much smaller than the mean estimated structural parameters. These observations suggest that the W2–SLS is no longer robust to perturbations and it has surpassed its breakdown point. It may be noted that the data matrix has 100 points. When NOUT=60, on an average about 45 of the points are perturbed. Some points are perturbed more than once. For NOUT= 75 about 52 of the points are perturbed; some points are perturbed more than once. Hence we may conclude that W2–SLS has a breakdown point somewhere between 45 to 50 percent. When more than 45 percent of points are perturbed, the estimator may break down and hence may not be reliable.

Table–10.1. Mean of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Mean of Estimated A Matrix					Mean of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	–1	3.3231	0	–1.7563	0	0	3.0309	0	–3.0828	0	31.9744
Eq–2	2.9671	–1	4.985	0	0	3.007	0	–4.9555	0	0	19.8781
Eq–3	0	0	–1	2.9232	0	0	1.9397	0	0	0	9.1577
Eq–4	4.7061	0	0	–1	–2.319	0	2.9417	0	0	–2.335	–7.0549
Eq–5	–10.0395	0	8.2862	0	–1	0	0	0	5.5304	0	–9.8211

Table–10.2. Standard Deviation of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	Standard Dev of Estimated A Matrix					Standard Dev of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	3.5138	0	4.0812	0	0	1.8679	0	3.7447	0	26.6848
Eq–2	0.3998	0	1.8369	0	0	0.7822	0	1.3523	0	0	12.602
Eq–3	0	0	0	0.3961	0	0	0.3163	0	0	0	1.3959
Eq–4	4.1717	0	0	0	2.2287	0	3.5142	0	0	2.1507	4.343

Eq–5	2.7754	0	2.1354	0	0	0	0	0	1.387	0	5.734

Table–10.3. Root Mean Square of Estimates of Structural Parameters: Method –W2–SLS (MCP)

Variables/ Equations	RMS of Estimated A Matrix					RMS of Estimated B Matrix					
	y_1	y_2	y_3	y_4	y_5	x_1	x_2	x_3	x_4	x_5	x_6
Eq–1	0	5.0859	0	5.8877	0	0	2.7141	0	5.4191	0	38.6977
Eq–2	0.4011	0	1.837	0	0	0.7822	0	1.353	0	0	12.6026
Eq–3	0	0	0	0.4035	0	0	0.322	0	0	0	1.4047
Eq–4	4.3678	0	0	0	2.3304	0	3.6701	0	0	2.2512	4.4446
Eq–5	2.9369	0	2.2516	0	0	0	0	0	1.4644	0	5.8539

6. Conclusion

In this paper we have proposed a robust 2–Stage Weighted Least Squares estimator for estimating the parameters of a multi–equation econometric model when data contain outliers. The estimator is based on the procedure developed by Norm Campbell which has been modified by using the measure of robust median deviation suggested by Hampel et al. The estimation method based on the original Campbell procedure performs poorly, while the method based on the modified Campbell procedure shows appreciable robustness. Robustness of the proposed method is not much destabilized by the magnitude of outliers, but it is sensitive to the number of outliers/perturbations in the data matrix. The breakdown point of the method, is somewhere between 45 to 50 percent of the number of points in the data matrix.

7. References:

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