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Abstract: Genuine Saving under Stochastic Growth by Chuan Zhong Li and Karl-Gustaf Löfgren

The concept of genuine saving appeared for the first time in a proof of a now well known theorem in Weitzman (1976). It was reinvented and used as a local welfare indicator by Pearce and Atkinson (1993). The purpose of this paper is to generalize this welfare measure to a stochastic Brownian motion context. We will use a stochastic version of a growth model introduced by Ramsey (1928). The particular model was developed by Merton (1975). Although the model is simple, it is enough to understand what its welfare results will look like in a general case.

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Genuine Saving under Stochastic Growth

By Chuan-Zhong Li, Karl-Gustaf Löfgren [1]

It has been known for quite a while that in a comprehensive deterministic dynamic growth model of the Ramsey type Genuine Saving (GS) is a welfare indicator. More precisely, the sum of net investment of all capital goods is a perfect local welfare indicator in the sense that positive GS means that welfare is increasing. Who first derived this result is not quite clear to us[2], but it shows up indirectly in Weitzman (1976). He derives the result as a step in the proof of the main theorem; the proportionality between the Hamiltonian and the present value of future utility. Important theoretical observations on its implications for sustainability are found in Pearce and Atkinson (1993), Asheim (1994) and Pezzey (1995).The measure has been popularized by Hamilton (1994), and used in practice by among others, Hamilton and Lutz (1996) and Hamilton and Clemens (1999). The purpose of this paper is to generalize this welfare measure to a stochastic context. We will use a simple version of a stochastic dynamic growth model introduced by Ramsey (1928). The model was developed by Merton (1975). Here we use a slightly modified version, where the model is optimized by choosing

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[2] It is almost like the Folk Theorem in cooperative game theory; known to many, first proved by Aumann (1959), and understood by only a few.

consumption rather than the rate of saving. Although the model is simple, the derivations are enough to understand how the result generalizes to a multi-sector version of the model.

The model

Merton treats the asymptotic properties of both the neoclassical growth model developed by Solow (1956) and Swan (1956), as well as the Ramsey (1928) optimal growth model, when the growth of the labour force follows a geometric Brownian motion process. We will concentrate on the Ramsey model and mainly deal with the one dimensional case.

Let $F(K(s),L(s)) \in C^2(R^{2+})$ be a net production function (i.e., depreciation has been accounted for) of degree one, where $K(s)$ denotes units of capital input, and $L(s)$ denotes units of labour input at time *s*. The capital stock evolves according to

$$\dot{K}(s) = F(K(s),L(s)) - C(s) = L(s)F(k(s),1) - C(s) \qquad (1)$$

where $\dot{K}(s) = \frac{dK}{dt}$, and $C(s)$ denotes consumption. The last equality follows from homogeneity of the production function. Now let $k(s) = K(s)/L(s)$, $c(s) = C(s)/L(s)$, and assume that $L(t) = L(0)e^{nt}$, $L(0) > 0$, $0 < n < 1$, and differentiate totally with respect to time. Again, using the degree one homogeneity of the production function and putting $F(K,1) = f(k)$, it follows that

$$\dot{k}(s) = f(k(s)) - c(s) - nk(s) \qquad (2)$$

Equation (2) is a variation of the Solow-Swan neoclassical differential equation of capital stock growth under certainty. Now suppose that growth of the labour force follows a geometric Brownian motion[3] of the following shape

$$dL(s) = nL(s)ds + \sigma L(s)dB(s) \qquad (3)$$

[3] Geometric Brownian motion is used to guarantee that the labour force remains positive. Note, however, that this does not result in an equation for the capital stock per capita that is Geometric Brownian motion.

The increment $dB(s)$ is the stochastic part. $B(s)$ is a Brownian motion process defined on some probability space with values in R. The drift of the process is governed by the expected rate of labor growth, *n*. In other words, over a short interval of time, ds the proportionate change of the labor force (dL/L) is normally distributed with mean nds and variance

$\sigma^2 dB^2(s) = \sigma^2 ds$

.

We can now use Ito's lemma to transform the uncertainty of growth in the labour force into uncertainty about the growth of the capital labour ratio. Non-standard calculations[4] yield

$$dk(s) = [f(k(s)) - c(s) - (n - \sigma^2)k(s)]ds - k(s)\sigma dB(s) \quad (4)$$

In other words, we have translated uncertainty with respect to the growth rate of the labour force into uncertainty with respect to capital per unit of labour and, indirectly, to uncertainty with respect to output per unit of labour, $y(s) = f(k(s)) = F(k(s),1)$ .The optimization problem is to find an optimal consumption policy, and the stochastic Ramsey problem is typically written

$$V(s,k_s) = \max_{c(\tau)} E_s \left\{ \int_s^T u\left(c(\tau)\right) e^{-\theta\tau} d\tau \right\} \quad (5)$$

subject to

$$dk(\tau) = [f(k(\tau)) - c(\tau) - (n - \sigma^2)k(\tau)]d\tau - \sigma k(\tau)dB(\tau)$$

$$(6)$$

$$k(s) = k_s$$

$$c(\tau) \geq 0 \quad \forall \tau$$

where E_s denotes the mathematical expectation taken at time s. The function $u(c(\tau))$ is the instantaneous utility function which is assumed to be twice continuously differentiable. $e^{-\theta\tau}$ is the discount factor, and θ is the utility discount rate. T is the first exit time from the solvency set[5] $G = \{k_\tau(\omega); k_\tau > 0\}$, i.e. $T = \inf\{\tau > s; k_\tau(\omega) \notin G\} \leq \infty$. In other words, the process is stopped when the capital stock per capita becomes non-positive (when bankruptcy

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[4] Ito calculus means that the first order differential contains second order terms. The details are available in the appendix.

[5] *G* is simply the real positive line $(0, \infty)$.

occurs). The stochastic differential equation in (6) is not Geometric Brownian motion and we cannot guarantee that $k(\tau)$ stays non-negative, i.e. that bankruptcy does not occur[6].

Genuine saving under growth

In most contexts it is realistic to assume that the control process $c(\tau)$ is allowed to be conditioned solely on past observed values of the state process $k(\tau)$. In such a case, mathematicians would say that the control process is adapted to the state process. Here, it is assumed that the optimal control function is a time autonomous Markov control of the following type

$$c(\tau) = c(k(\tau)) \tag{7}$$

Equation (7) means that the control at time τ only depends on the state of the system at this time. In particular, it does not depend on the starting point or time as a separate argument.

Starting from the value function in present value

$$J\left(s,k,c\right) = E_s\left\{\int_s^T u\left[c(\tau)\right]e^{-\theta\tau}d\tau\right\}; \tag{8}$$

which is maximized with respect to $c(\tau)$ and subject to equation (6), the stochastic differential equation for the capital stock, and $k(s) = k$, we obtain

$$e^{\theta s}V(s,k) = E_s\{\int_s^T u(c^*(k(\tau)))e^{-\theta(\tau-s)}d\tau\} = W(s,k) \tag{9}$$

Where $c^*(k(\tau))$ is the optimal control, $V(s,k)$ is the optimal present value function, and $W(s,k)$ is the optimal current value function. We now prove that

Observation 1: $V(s,k) = V(0,k)e^{-\theta s} = W(k)e^{-\theta s}$*, where the endogenous time spent in the solvency set* $\tau_G = \inf\{t > 0; k(t) \notin G\} = T - s$*.*

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[6] A hard question is whether it occurs with probability one.

Proof: *The optimal control is a Markov control, i.e., it depends only on the capita stock* $k(t)$ *at time t.*

Putting $\tau = s+t$ *means that the optimal control*

$c^*(k(\tau)) = c^*(k(s+t)),\ \ \tau = s \Leftrightarrow t = 0,\ \ \tau = \tilde{T} \Leftrightarrow t = \tilde{T} - s$ *and* $k(s) = k(0) = k.$

The time spent in the solvency set, τ_G *=T-s, is, for a given experiment* $\omega_\tau = \omega_{s+t}$*, endogenous, and the solvency set does not depend on the rescaling, i.e ,*

$G = \{k_\tau(\omega); k_\tau > 0\} = \{k_{s+t}(\omega); k_{s+t} > 0\}$. Therefore

$$V(s,k) = e^{-\theta s} \max_c E_s[\int_s^T u(c(k(\tau))e^{-\theta(\tau-s)}d\tau] = e^{-\theta s} \max_c E_0[\int_0^{\tau_G} u(c(k(s+t)))e^{-\theta(s+t-s)}dt] =$$
$$= e^{-\theta s}V(0,k) = e^{-\theta s}W(k)$$

The second equality follows since substituting $k_\tau = k_{s+t}$ *into the time autonomous stochastic differential equation (6), we get a process that starts at* $x_0 = (0,k)$*, has the same probability law on an equivalent solvency set as the process that starts at* $x_s = (s,k)$*, and the optimal control is Markov. The remaining two equalities follow from definitions.*

Observation 1 means that the current value function, $W(k) = e^{\theta s}V(s,k) = V(0,k)$, does not depend on the starting time. This implies that

$$V_s(s,k) = \frac{d}{ds}\left[e^{-\theta s}W\right] = -\theta e^{-\theta s}W(k)$$

and the Hamilton-Jacobi-Bellman, HJB, equation

$$-V_s(s,k)) = -\frac{\partial V}{\partial s} = \underset{c}{Max}\left[u(c(s))e^{-\theta s} + V_k(s,k))h(k,c;\sigma^2,n) + \frac{1}{2}\sigma^2 k^2 V_{kk}(s,k))\right] \qquad (10)$$

can be rewritten in the following manner

$$\theta W(k)) = \underset{c}{Max}\left[u(c(s)) + W_k(k))h(k,c;\sigma^2,n) + \frac{1}{2}\sigma^2 k^2 W_{kk}(k)\right] \qquad (10a)$$

.

Here $h(k,c;\sigma^2,n) = f(k) - c - (n-\sigma^2)k$, $W_k = dW/dk$ and $W_{kk} = d^2W/dk^2$. We can now define a co-state variable $p(s)$ as

$$p(s) = W_k(k)) \tag{11}$$

and its derivative

$$\frac{dp(s)}{dk} = W_{kk}(k) \tag{12}$$

Given the optimal consumption policy, equation (10a) can be written (neglecting the time index to save notational clutter) as

$$\theta W\left(k\right) = u(c^{*}\ ) + ph(k,c^{*};\sigma^{2},n) + \frac{1}{2}\frac{dp}{dk}\sigma^{2}k^{2} = H^{c^{*}}(k,p,\frac{dp}{dk}) \tag{13}$$

i.e., the total expected future welfare is proportional to a stochastic version of the deterministic Hamiltonian[7].

To derive a local welfare measure like GS we start the optimal value function

$$W(s,k) = \underset{c}{Max}\ E_s\{\int_s^T u[c(\tau)]e^{-\theta(\tau-s)}d\tau\} = E_s\{\int_s^T u(c^{*}(\tau))e^{-\theta(\tau-s)}d\tau\} \tag{14}$$

Differentiating with respect to time (the lower integration level) yields

$$\frac{\partial W}{\partial s} = \dot{W}(k)) = -u(c^{*}(k)) + \theta W(k) \tag{15}$$

Now, using equation (13), i.e., the HJB-equation for the time autonomous problem, we get after substituting for $\theta W(k)$

$$\begin{aligned} \dot{W}(s) &= p(s))h(c^{*}(k(s),k(s);\sigma^{2},n) + \frac{1}{2}\frac{dp(s)}{dk}\sigma^{2}k^{2} = \\ &= W_k(k)[f(k(s)) - c^{*}(k(s)) - (n-\sigma^{2})] + \frac{1}{2}W_{kk}(k))\sigma^{2}k^{2} \end{aligned} \tag{16}$$

The interpretation of the co-state variable $p(s)$ is the derivative of the optimal value function with respect to the initial capital stock, and $h(\Box)$ is the drift in net investment along the optimal path. The second term in the expression originates from Ito calculus and the sign of this

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[7] This may have been known by many but was pointed out by Aronsson and Löfgren (1995). Hence, Weitzman's theorem from 1976 follows directly from the HJB equation.

second order derivative of the value function with respect to the capital stock, W_{kk}, or, what amounts to the same, the derivative of the co-state variable (the shadow utility value of net investment) from an increase in the capital stock. For a "well behaved" maximization problem this entity should be negative. For $\sigma = 0$ equation (16) collapses to the static GS measure[8].This means that we would under a stochastic Ramsey problem expect that a positive net investment value would not be enough to indicate a local welfare improvement. Net investment has to be large enough to compensate for the variance component. In the variance component we interpret $-W_{kk}(k(t))$ as the price of risk, and $\sigma^2 k^2$ as the "quantity of risk".

The reason why this particular component appears is that an Ito integral is constructed from forward increments. An alternative, well known, way of constructing a stochastic integral is the Stratonovich integral[9], which picks the middle of the increment to weigh the components of the sums that approximates the integral. For a whole economy, where risk cannot be diversified away the Ito integral seems reasonable. However, if risk can be diversified a stochastic integral which leaves out the risk component in expressions like (16) is more relevant. In Weitzman[10] (2003) it is shown that under a Stratonovich integral the variance component disappears.

A general conclusion

To find the solution in the general time autonomous case with *n* consumption goods and *m* capital goods the above procedure can be generalized. We will only have to change into a general HJB-equation. The derivative of the value function will look like the one in equation (15). In other words, we are left with the following result.

Observation 2: *In a stochastic time autonomous Ramsey problem with n consumption goods and m capital goods the derivative of the value function with respect to time is given by the following expression* $\dot{W}(k(s)) = HJB - u(\mathbf{c}^*)$.

If the problem is not time autonomous extra first order terms will be added in the HJB equation and change the time derivative accordingly. An example would be exogenous

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[8] Equation (13) collapses into Weitzman's theorem on the proportionality between the Hamiltonian and future welfare

[9] The seminal reference is Stratonovich (1966).

[10] The Theorem is found in Chapter 7 page 321,

technological progress, which would add net value to the GS component. Another example would be negative externalities that would deduct net value from the GS component. Under a Stratonovich integral the variance component is exactly netted out also in the general case.

Finally, a Markov control may seem overly specific. A more general control would be to allow the control at time *t* to be conditioned on the whole process from start up to *t*, i.e., the control function is $F_t - adapted$. Such controls are called closed loop or feed back controls. Under an integrability condition and a smoothness condition on the set *G* it is possible to show that the optimal value function for the Markov control coincides with the optimal control for the open loop control for any starting point in *G*. Hence, the Markov control is not particular restricted[11].

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[11] See e.g. Öksendal (2003) Theorem 11.2.3..

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Appendix: Derivation of equation (4) in the main text

We start from:

$$dK = [F(K,L) - C]dt = [Lf(k) - C]dt$$
$$dL = nLdt + \sigma LdB$$

Letting $k = K/L$, Ito's lemma yields

$$dk = \frac{\partial k}{\partial t}dt + \frac{\partial k}{\partial L}dL + \frac{\partial k}{\partial K}dK + \frac{1}{2}[\frac{\partial^2 k}{\partial K^2}(dK)^2 + 2\frac{\partial^2 k}{\partial K \partial L}dLdK + \frac{\partial^2 k}{\partial L^2}(dL)^2 =$$
$$= -\frac{K}{L^2}(nLdt + \sigma LdB) + \frac{1}{L}(Lf(k) - C) + \frac{1}{2}(\frac{2}{L^3}K\sigma^2 L^2 dt) = [f(k(s)) - c(s) - (n - \sigma^2)k(s)]ds - k(s)\sigma dB(s)$$