Beliefs and rationalizability in games with complementarities

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BELIEFS AND RATIONALIZABILITY IN GAMES WITH COMPLEMENTARITIES

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ABSTRACT. We propose two characteristics of players' beliefs and study their role in shaping the set of rationalizable strategy profiles in games with incomplete information. The first characteristic, *type-sensitivity*, is related to how informative a player thinks his type is. The second characteristic, *optimism*, is related to how "favorable" a player expects the outcome of the game to be. The paper has two main results: the first result provides an upper bound on the *size* of the set of rationalizable strategy profiles, the second gives a lower bound on the change of *location* of this set. These bounds have explicit and relatively simple expressions that feature type-sensitivity, optimism, and properties of the payoffs. Our results generalize and clarify the well-known uniqueness result of global games (Carlsson and van Damme (1993)). They imply new uniqueness results and allow to study rationalizability in new environments. We provide applications to supermodular mechanism design (Mathevet (2010)) and non-Bayesian updating (Epstein (2006)).

Keywords: Complementarities, rationalizability, beliefs, type-sensitivity, optimism, global games, equilibrium uniqueness.

JEL Classification: C72, D82, D83.

1. Introduction

In all social or economic interactions, the beliefs of the actors contribute to shape the set of outcomes. In game-theoretical models, the richness of outcomes is captured by the set of rationalizable strategy profiles. The global game literature (e.g. Carlsson and van Damme (1993), Frankel et al. (2003), and Morris and Shin (2003)) suggests a perturbation of complete information that delivers a unique rationalizable equilibrium. This perturbation gives players' beliefs the right properties to obtain uniqueness. What are these properties? How do they act with the payoffs to determine the rationalizable outcome? The standard global game method does not cover games with heterogeneous beliefs (non-common prior type spaces), games with general information structures, games played by non-Bayesian players, and Bayesian mechanism design. In these cases, our understanding of rationalizability requires an answer to the above questions.

In this paper, we study some properties of type spaces that explain the size and the location of the set of rationalizable strategy profiles, where rationalizability corresponds to the definition of interim correlated rationalizability in Dekel et al. (2007). These properties are characteristics of players' beliefs that do not require to specify the origin of the beliefs. They are compatible with general belief formation and apply to all the aforementioned cases.

This paper deals with games with incomplete information and complementarities. A flexible framework for modeling beliefs is that of type spaces. Players have a payoff type, called the state of nature, and an informational type. The state of nature may represent the physical reality, such as the weakness of a currency. Conditional on his informational type, a player formulates beliefs about the state of nature and about others' (informational) types. Players are assumed to care exclusively about an aggregate of others' actions, such as their average action. The games under consideration have strategic complementarities and dominance regions, that is, "tail regions" of the state space for which the extremal actions are strictly dominant. The model incorporates many classic problems such as investment games, currency crisis, search models, etc.

The first characteristic that we study is the type-sensitivity of players' beliefs. The notion has two dimensions, one for the beliefs about the state and one for the beliefs about others'

types. The first dimension answers the question: when the player's type increases, by how much does he think the state will increase on average? This is related to how informative the player thinks his type is. A large answer denotes high sensitivity. The second dimension applies to the beliefs about others' types and it is determined by the question: when a player's type increases, does he think others' types will increase more or less than his type and by how much? Since the games under study are aggregative, we will actually ask the more operational yet equivalent question: assuming that other players decrease their strategies, by how much does a player think the aggregate will decrease on average if his type increases? The answer is the second dimension of type-sensitivity. The player is asked to consider counterfactual information: his opponents decrease their strategies but simultaneously his type increases.

The second characteristic that we study is the optimism of players' beliefs. This characteristic also has two dimensions and it aims to measure how favorable a player expects the outcome to be. By convention, an outcome is more favorable if it is larger, which happens when the aggregate and the state are larger. A player becomes more optimistic if, for each informational type, he now believes larger states and larger aggregates are more likely.

Let us discuss our contribution in more details. Recall that, in games with complementarities, there exist a largest and a smallest equilibrium and the distance between them gives the size of the set of rationalizable strategy profiles (Milgrom and Roberts (1990)).

Our first result provides an explicit upper bound on the size of the set of rationalizable strategy profiles. The second result provides an explicit lower bound on the movement of the rationalizable set after a change of optimism. Both bounds condense type-sensitivity, optimism, and the characteristics of the payoffs in expressions that determine the size and the (change of) location of the rationalizable strategies. These expressions are easy to compute in comparison to applying iterative dominance and computing the rationalizable outcomes directly. Examples will illustrate this practical advantage.

Our main contribution is to provide the tools to study rationalizability in general environments. The global game method suggests a specific perturbation of complete information that delivers a unique rationalizable equilibrium: a payoff parameter — that was common

knowledge — is drawn from a common prior and players receive a noisy additive signal of its realization. As the noise vanishes, a unique Bayesian equilibrium survives. Many scenarios do not fit the global game description: games with heterogeneous beliefs (non-common prior type spaces), games with general (non-additive) signal structures, games played by non-Bayesian players, and Bayesian mechanism design. To study these cases, it is important to understand the properties of type spaces inherited from the global game perturbation and how they interact with the payoffs to form the rationalizable strategies. Type-sensitivity and optimism are such properties and they exist regardless of the specification that produces the posterior beliefs — with or without a common prior, additive signals, Bayesian updating, etc. This has several important implications described next.

Our results imply new uniqueness results and promote a better understanding of existing ones. The upper bound provided by the first result subsumes the global game uniqueness result. The bound shows that if type-sensitivity is high compared to the strategic complementarities, then there is a unique equilibrium. In global games, type-sensitivity becomes high as the noise vanishes, because the type becomes a perfect predictor of the state and of other's types. The expression of the bound shows *explicitly* that the global game information structure dampens the complementarities to the point where a unique equilibrium survives. This generalizes and formalizes arguments presented by Vives (2004) and Mathevet (2007). But equilibrium uniqueness holds much more generally than in standard global games. We illustrate this fact in Section 2 with a simple investment game where uniqueness obtains in a non-common prior type space and in an asymmetric signaling function specification.

Our results allow a general analysis of equilibrium multiplicity. While the literature has focused on uniqueness, it is important to understand and quantify equilibrium multiplicity. In supermodular mechanism design, for example, knowing the size of the equilibrium set allows to compute the welfare loss that may be caused by bounded rationality (Mathevet (2010)). Our results show that a larger type-sensitivity is conducive to tighter equilibrium sets. Moreover, certain characteristics of equilibrium multiplicity are interesting. For example, which players decide to dramatically change their equilibrium strategy in response to changes in the equilibrium strategies of others? This type of questions evoke a form of

influence in games. In Section 6.3, we claim that players with higher type-sensitivity are more influential than others in the sense that they "stick" to their strategies across equilibria instead of dramatically changing them.

Finally, our results give rise to new applications. An economist may be interested in studying a phenomenon, a currency crisis or a bank run, with players having different priors (Varian (1986)), asymmetric non-additive signals, or updating biases. In Section 6.1, we apply our results to Bayesian mechanism design. Mathevet (2010) introduces supermodular mechanism design. The main idea is to design direct mechanisms that are robust to certain forms of bounded rationality. The author suggests to design mechanisms that induce super-modular games but he warns that excessive complementarities may produce new equilibria and disrupt learning.[1] This justifies his construction of the optimal (or minimal) supermod-ular mechanism, one that gives the smallest equilibrium set in the class of supermodular mechanisms. But what is the size of the smallest equilibrium set? In certain applications, as in Section 6.1, our first result provides an answer and helps the designer in his choice of the mechanism parameters. In Section 6.2, we deal with games played by non-Bayesian players. Consider a standard global game setting but assume that players literally make updating mistakes. The results clarify the strategic implications of certain updating biases.

The importance of understanding rationalizability beyond global games is emphasized by Morris and Shin (2009). They characterize the hierarchies of beliefs that imply dominance-solvability in binary-action games with incomplete information. Our paper formulates al-ternative but related conditions in games with finitely many actions. We will discuss the relationship of type-sensitivity to the notion of decreasing rank beliefs suggested by Morris and Shin (2009). Izmalkov and Yildiz (2010) is another paper close to ours. The authors introduce sentiments into the study of global games. They define notions of optimism and analyze partnerships and currency crises. Our second result is a generalization of their re-sults in the partnership game. Other papers, e.g. Weinstein and Yildiz (2007) and Oyama and Tercieux (2011), study rationalizability in general environments but their objective is

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[1] This is because Mathevet (2010) studies weak implementation and truthtelling is the only equilibrium known to be desirable.

different from ours. Weinstein and Yildiz (2007) show that for any rationalizable action of any type, the beliefs of the type can be perturbed in a way that this action is uniquely rationalizable for the new type. As a result, the beliefs may satisfy the conditions for dominance-solvability — high type-sensitivity for example — yet the unique equilibrium may vary with other properties of the beliefs — optimism for example.[2]

The remainder of the paper is organized as follows. The next section gives a motivating example. Section 3 presents the model and the assumptions. Sections 4 and 5 contain the main definitions and results. In Section 6, we provide two applications. The last section concludes.

2. An Investment Game

Consider a standard investment game. Two players are deciding whether to invest. The net profits are given by the following matrix where $\theta \in \mathbb{R}$ is the fundamental of the economy:

	1	0
1	θ, θ	$\theta - 1, 0$
0	$0, \theta - 1$	$0, 0$

We want to model the effect of strategic uncertainty on investment decisions. The state θ is drawn randomly and players receive a signal t_i of its realization. Several versions of that scenario are possible:

(i) Standard global games. State θ is drawn from a common prior. Each investor receives a signal $t_i = \theta + \nu \epsilon_i$ where $\nu > 0$ and ϵ_i is a random variable. The analyst studies the case $\nu \to 0$ where signals become infinitely precise.

(ii) Non-common priors. Each player i formulates beliefs about θ and t_j given his signal. Assume i's beliefs about θ given t_i are a normal distribution with mean $\frac{4t_i}{5}$ and variance σ^2. Conditionally on θ, assume i's beliefs about t_j assigns probability one to $t_j = \frac{3\theta}{2}$. These beliefs do not come from a common prior type space: each player i believes that j's signal is a perfect predictor of the state, while each j believes his own signal to be an imperfect signal

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[2]Thus the analyst may know that there is a unique equilibrium but without further knowledge of players' beliefs, such as their optimism level, she may be unable to pin it down, which is a form of multiplicity.

of the state. An alternative way of obtaining heterogeneous beliefs, proposed by Izmalkov and Yildiz (2010), is to start with the global game formulation but assume that each player has his own subjective beliefs about (ϵ_1, ϵ_2).

(iii) Subjective signaling functions. Suppose θ is drawn from a uniform prior, but the signaling functions are subjective. Each player i uses $t_i^i = \alpha_i^i \theta + \nu\epsilon$ and $t_i^j = \alpha_i^j \theta + \nu\epsilon$ with $\alpha_i^i < \alpha_i^j$ when formulating his posterior beliefs. This information structure models players who think that their signals do not carry fundamental shocks like their opponent's signal. That is, players think that they obtain their private information from a different channel than their opponent. When i's signal increases, i believes j's signal will increase more. This scenario also produces heterogeneous beliefs.

(iv) Non-vanishing noise. Consider the standard global game setup with $t_i = \theta + \nu_i\epsilon$ but let ν_i be fixed, strictly positive, and different across players (see Section 6.3).

(v) Non-Bayesian Updating. Consider the standard global game setup with non-Bayesian players. Players have updating biases. For example, players overreact and amplifies the information contained in their signal (see Section 6.2).

The beliefs generated by scenarios (ii) and (iii) cannot be the product of a global game formulation. Likewise, the analysis of scenarios (iv) and (v) requires new concepts.

Our main concept is *type-sensitivity*. This concept has two dimensions. Let $T_i = \mathbb{R}$ be player i's type set. The first dimension is the answer to the question: if i's type increases by $v > 0$, by how much does i think the state will increase on average? In the above non-common prior example, the answer would simply be $4v/5$. The second dimension applies to i's beliefs about t_j. We want to know how much i thinks j's type increases after t_i increases. Suppose an event E occurs if $\{t_j > s_j\}$ and i's type is t_i, or it occurs if $\{t_j > s_j + v\}$ and i's type is $t_i + v$. In which case does i believe E is more likely? If i believes that j's type increases at least as much as his, a case referred to as *highly type-sensitive* beliefs, then the event is more likely in the second case. Thus, beliefs are highly type-sensitive in our non-common prior and subjective signals examples. In the global game specification, E is

more likely in the first case but the difference in the probabilities of the event in the two cases vanishes as $\nu \to 0$.

If beliefs are highly type-sensitive, then there is a unique equilibrium. Let $\mu_i(\theta|t_i)$ be the cdf representing the beliefs about θ given t_i. In this game, a strategy for i is characterized by a cutoff s_i. Player i invests if and only if his type is above s_i, where s_i is the type at which i is indifferent between investing and not investing:

$$\int_{\theta \in \mathbb{R}} (\theta - 1 + \text{Prob}(t_j > s_j|\theta, s_i))d\mu_i(\theta|s_i) = 0. \tag{2.1}$$

By way of contradiction, suppose there exist two symmetric equilibria, characterized by cutoffs $\bar{s}$ and $\underline{s}$ ($\underline{s} < \bar{s}$), whose interval in between contains all rationalizable strategy profiles. High type-sensitivity says that i expects j to invest at least as often under strategy $\bar{s}$ given $t_i = \bar{s}$ as what i expects under strategy $\underline{s}$ given $t_i = \underline{s}$. Note that a larger type leads i to expect a larger state: when $t_i = \bar{s}$, i believes the state is larger by at least $\sigma_1^i > 0$ than when $t_i = \underline{s}$.[3] Therefore, high type-sensitivity means $\text{Prob}(t_j > \bar{s}|\theta + \sigma_1^i, \bar{s}) \geq \text{Prob}(t_j > \underline{s}|\theta, \underline{s})$. If (2.1) holds at $(s_1, s_2) = (\underline{s}, \underline{s})$ — which is the case by definition of an equilibrium — then

$$\int_{\theta \in \mathbb{R}} (\theta + \sigma_1^i - 1 + \text{Prob}(t_j > \bar{s}|\theta + \sigma_1, \bar{s}))d\mu_i(\theta|\underline{s}) > 0. \tag{2.2}$$

The lhs of (2.2) is weakly smaller than the lhs of (2.1) evaluated at $(s_1, s_2) = (\bar{s}, \bar{s})$. Thus (2.1) does not hold for $(s_1, s_2) = (\bar{s}, \bar{s})$, which contradicts the optimality of $\bar{s}$.

3. The Model

We study games with incomplete information. The set and the number of players are $N < \infty$.[4] Player i's action set is a finite and linearly ordered set $A_i = \{a_{i,1}, \ldots, a_{i,M_i}\}$ where actions are indexed in increasing order. Let $A_{-i} = \prod_{i \neq i} A_j$ be the set of action profiles of players other than i. Let $\theta \in \Theta \equiv \mathbb{R}$ be the state of nature.

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[3]We will call this condition strict first-order stochastic dominance.

[4]The paper and its results can be extended to $N = \infty$ if players are placed into finitely many groups.

3.1. **The Payoffs.** Each player i only cares about an aggregate Γ_i of his opponents' actions. This aggregate is an increasing function that maps action profiles and states from $A_{-i} \times \mathbb{R}$ *onto* a linearly ordered set $\mathcal{G}_i$. For example, a player may care about the average of his opponents' actions or the proportion of them playing an action.[5] Our payoff structure allows for common values, $u_i(a_i, \Gamma_i(a_{-i}, \theta), \theta)$, or private values, $u_i(a_i, \Gamma_i(a_{-i}), t_i)$, but no mixture of the two. A player's utility does not depend on the state and his type.

The Assumptions. Let X and T be two ordered sets. A function $f : X \times T \to \mathbb{R}$ has increasing differences in (x, t) if for all $x' > x$ and $t' > t$, $f(x', t') - f(x, t') \geq f(x', t) - f(x, t)$. The function has strictly increasing differences if the previous inequality holds strictly. The assumptions are given in the common value formulation but the same ones — replacing the state by the type — must hold under private values:

(A1) For each i, u_i has increasing differences in (a_i, a_{-i}) for each θ.

(A2) For each i, u_i has strictly increasing differences in (a_i, θ) for each a_{-i}.

(A3) For each a, u_i is bounded on all compact sets of θ.

(A4) There exist states $\overline{\theta}$ and $\underline{\theta}$ such that for $\theta > \overline{\theta}$, the largest action is strictly dominant, and for $\theta < \underline{\theta}$, the smallest action is strictly dominant.

The first assumption introduces strategic complementarities, by which a player wants to increase his action when others do so as well. The second assumption introduces state monotonicity, by which a player wants to increase his action when the state is larger. The third is a technical condition, and the last one imposes dominance regions.

All these assumptions are standard in the global game literature. The currency crisis model of Morris and Shin (1998), the bank run model of Morris and Shin (2000), and the model of merger waves of Toxvaerd (2008) are examples where these assumptions hold. We refer the reader to Morris and Shin (2003) for further examples.

3.2. **The Beliefs.** The state of nature is randomly drawn. Players are uncertain about its realization. They receive some private information about the realized state. Then they formulate beliefs about the state and others' information. A flexible framework for modeling

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[5]In these cases $\Gamma_i(a_{-i}, \theta) = \sum_{j \neq i} a_j$ and $\Gamma_i(a_{-i}, \theta) = (\sum_{j \neq i} 1_{a_j \geq a^*(\theta)})/(N-1)$.

beliefs is that of type spaces. A type space is a collection $\mathcal{T} = (T_i, \mu_i)_{i \in N}$. Let $T_i = \mathbb{R}$ for each $i \in N$ and denote $T_{-i} = \prod_{j \neq i} T_j$. Let $\Delta(Z)$ be the space of probability measures on Z. Player i's beliefs are a function

$$\mu_i : T_i \to \Delta(\Theta \times T_{-i})$$

where $\mu_i(t_i)$ is i's beliefs about the state and others' types when his type is t_i. For practical reasons, we decompose $\mu_i(t_i)$ into two beliefs: $\mu_i(\theta|t_i)$ is (the cdf of) the marginal distribution of θ and $\mu_i(\cdot|\theta, t_i)$ is the conditional measure on T_{-i} given θ. For any subsets of states and types, $\hat{\Theta}$ and $\hat{T}_{-i}$, $\mu_i(t_i)[\hat{\Theta} \times \hat{T}_{-i}] = \int_{\hat{\Theta}} \mu_i(\hat{T}_{-i}|\theta, t_i) d\mu_i(\theta|t_i)$. Under private values, there is no state of nature, but this is technically equivalent to a common values case where $\mu_i(\theta|t_i)$ is derived from the Dirac measure.[6]

The Assumptions. Let $>_{st}$ stand for the (strict) first-order stochastic dominance ordering.[7] Let $\geq_{st}$ be the multidimensional first-order stochastic ordering (Shaked and Shanthikumar (1994)). We impose the following assumptions on beliefs:

(A1) For each i, if $t_i' > t_i$, then $\mu_i(\cdot|t_i') >_{st} \mu_i(\cdot|t_i)$.

(A2) For each i, if $(t_i', \theta') \geq (t_i, \theta)$, then $\mu_i(\cdot|\theta', t_i') \geq_{st} \mu_i(\cdot|\theta, t_i)$.

(A3) For each i, there is $D_i > 0$ such that $|t_i - \theta| > D_i$ implies $\mu_i(\theta + \epsilon|t_i) - \mu_i(\theta - \epsilon|t_i) = 0$ for all $\epsilon > 0$ small enough.

(A4) For each i, $\int \mu_i(\{t_j > s_j\}_{j \neq i}|\theta, t_i) d\mu_i(\theta|t_i)$ is continuous in t_i and s_{-i}.

The first assumption says that a player believes that larger states are more likely when his type increases. The second assumption says that a player believes that the other players are more likely to have larger types when his type and the state increase. According to the third assumption, the likelihood of states that are excessively far from a player's type is null. Under private values, (A1) and (A3) are automatically satisfied.

These assumptions are satisfied by the global game information structure, and therefore, by most applications of global games (see e.g. Morris and Shin (2003)). There are no further

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[6]The Dirac measure gives measure 1 to every set that contains t_i and 0 to others. It implies that all expected terms of the form $\int_{\mathbb{R}} u(\theta) d\mu_i(\theta|t_i)$ are simply equal to $u(t_i)$ for every function u.

[7]It means that for every strictly increasing function u on $\mathbb{R}$, $\int_{\mathbb{R}} u(\theta) d\mu_i(\cdot|t_i') > \int_{\mathbb{R}} u(\theta) d\mu_i(\cdot|t_i)$.

requirement. Belief formation can be rather general. Players may not share the same prior distribution. Players need not be Bayesian, as they need not be to form posterior beliefs (see e.g. Epstein (2006)).

3.3. **Strategies and Aggregate Distribution.** A strategy for player i is a function $s_i : T_i \to A_i$. Under our assumptions, only strategies that are monotone in a player's type will be relevant. The argument relies on Van Zandt and Vives (2007) and it is developed in the appendix. Given the finite number of actions, i's relevant strategies are step functions, represented by a vector of cutoffs in $\mathbb{R}^{M_i-1}$. The games under consideration are aggregative. Therefore, player i ultimately cares about the probability distribution of the state and of the aggregate Γ_i. Conditionally on type t_i, state θ, and others' strategies s_{-i}, i can construct the probability distribution of the aggregate values. The derivation is relegated to the appendix. Let $g_i(\gamma|\tau_i)$ where $\tau_i = (\theta, s_{-i}, t_i)$ be the probability of $\{\Gamma_i = \gamma\}$. Let G_i be the corresponding cdf, i.e. $G_i(\gamma|\tau_i)$ is the probability of $\{\Gamma_i < \gamma\}$ given τ_i.

3.4. **Rationalizability.** Our solution concept corresponds to interim correlated rationalizability (Dekel et al. (2007)). Morris and Shin (2009) note that there is no difference between ex-ante and interim rationalizability in this environment due to the supermodularity assumptions. Best-response dynamics starting from the largest strategy profile converges to the largest equilibrium in an incomplete information game with supermodular payoffs (Vives (1990)) and the largest equilibrium correspond to the largest rationalizable strategy profile (Milgrom and Roberts (1990)).

4. Type-sensitivity and Rationalizability

This section defines type-sensitivity and investigates its role in determining the size of the set of rationalizable strategy profiles. Since strategies are vectors $s_i = (s_{i,\ell})_{\ell=1}^{M_i-1}$, we let the distance between profiles s and s' be the sup norm $d(s, s') = \max_i \max_\ell |s'_{i,\ell} - s_{i,\ell}|$.

4.1. **Type-sensitivity.** The basic ingredients of our definition are the average state and the average aggregate. Let $\Gamma_i^e[G_i(\tau_i)]$ be the average aggregate value obtained from $G_i(\tau_i)$. Let $\mu_i^\sigma(\theta|t_i) = \mu_i(\theta - \sigma|t_i)$ denote i's beliefs after a rightward shift by an amount $\sigma \geq 0$.

Since a player produces marginal beliefs about the state and conditional beliefs about others' types, type-sensitivity has two dimensions. Let $v > 0$.

Definition 1. *The type-sensitivity of the marginal beliefs is given by function σ_1^i where $\sigma_1^i(v)$ is the supremum of all σ such that $\mu_i(\cdot|t_i + v) \geq_{st} \mu_i^\sigma(\cdot|t_i)$ for all t_i.*

This definition describes the minimal shift in player i's beliefs after an increase in type. If beliefs $\mu_i(\cdot|t_i)$ belong to a location-scale family,[8] such as the normal or logistic distribution, then type-sensitivity is simply the answer to the question: when a player's type increases by v, by how much does he think the state will increase on average?

The second dimension of type-sensitivity applies to the conditional beliefs $\mu_i(\cdot|\theta, t_i)$. The basic idea is to know whether i thinks that others' types increase more than his after his own type increases. Suppose i's type increases by v. One immediate consequence is that i believes the state increases by at least $\sigma_1^i(v)$ on average. Consider the two distributions $\mu_i(\cdot|\theta, t_i)$ and $\mu_i(\cdot|\theta + \sigma_1^i(v), t_i + v)$. Let us compare the likelihood of the event $\{t_j > s_j\}_{j \neq i}$ under the first distribution (i.e. before t_i increases), and the likelihood of the event $\{t_j > s_j + v\}_{j \neq i}$ under the second distribution (i.e. after t_i increases). If the event after increase is more likely, then i believes that others' types increase at least as much as his. Another way to proceed, which we adopt, is to ask the similar question: if every $j \neq i$ decreases his strategy, i.e. j increases each cutoff in his strategy from $s_{j,\ell}$ to $s_{j,\ell} + v$,[9] while t_i increases by v, by how much does i believe the aggregate will decrease on average? This question forces the player to consider counterfactual information. The first piece of information indicates that the aggregate should decrease, while the second indicates that it should increase. The next definition formalizes the answer.

Let $c(v) = (\sigma_1^i(v), \boldsymbol{v}, v)$ where $\boldsymbol{v}$ is a vector with identical entries v. The vector $\tau_i + c(v) = (\theta + \sigma_1^i(v), s_{-i} + \boldsymbol{v}, t_i + v)$ represents the counterfactual information: i's opponents each decrease their strategies while i's type increases.

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[8]Let $f(x)$ be a pdf. For $k \in \mathbb{R}$ and $\eta > 0$, the family of pdfs $(1/\eta)f((x - k)/\eta)$ indexed by (k, η) is called the location-scale family with standard pdf f. For example, $\mu_i(\cdot|t_i)$ could be the cdf of a normal distribution with mean $t_i/2$ and variance σ^2.

[9]Increasing the cutoffs delays the play of larger actions, and thus, it corresponds to decreasing a strategy.

Definition 2. *The type sensitivity of the conditional beliefs is given by any function σ_2^i such that $\sigma_2^i(v) \geq \Gamma_i^e[G_i(\tau_i) \vee G_i(\tau_i + c(v))] - \Gamma_i^e[G_i(\tau_i + c(v))]$ for all v and τ_i.*[10]

Under private values, this is the only definition of type-sensitivity. If the conditional beliefs are highly type-sensitive, then the player believes that larger aggregates are at least as likely despite the counterfactual information: $G_i(\tau_i + c(v)) \geq_{st} G_i(\tau_i)$. In this case, if asked by how much Γ_i should decrease, the player would answer zero, $\sigma_2^i(v) = 0$.

Type-sensitivity is related to the decreasing rank beliefs condition used by Morris and Shin (2009) to prove dominance solvability in binary-action games. For each k, they define rank beliefs as the probability that a player assigns to there being k players whose signals are lower than his signal. The condition requires that as a player's signal increases he believes that his rank in the population decreases. They take the example of a student whose test score increases. If grading is on a curve, is it good news or bad news? Under decreasing rank beliefs, it is bad news, because the student believes the test was easy, hence others' scores must have increased more than his. Therefore, such a player has highly type-sensitive beliefs and his $\sigma_2^i(v)$ should be small.

4.2. **The First Theorem.** Let $\Delta_m^n u_i(\gamma, \theta) = u_i(a_n, \gamma, \theta) - u_i(a_m, \gamma, \theta)$ be the difference in utility between actions a_n and a_m. Define

$$C_i^*(\theta) = \max_{\gamma} \frac{\Delta_1^{M_i} u_i(S(\gamma), \theta) - \Delta_1^{M_i} u_i(\gamma, \theta)}{S(\gamma) - \gamma}, \tag{4.1}$$

where $S(\gamma)$ is the successor of γ[11] and for $x \geq 0$,

$$M_*^i(x, \theta) = \min_{(\gamma, n, m)} \Delta_m^n u_i(\gamma, \theta + x) - \Delta_m^n u_i(\gamma, \theta). \tag{4.2}$$

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[10]$\vee$ stands for the supremum between two distributions w.r.t. first-order stochastic dominance. The supremum of two cdfs is the pointwise minimum between them. In the main theorem, it will be important to choose the smallest $\sigma_2^i(v)$ satisfying the condition.

[11]$S(\gamma) = \min\{\gamma' \in \mathcal{G}_i : \gamma' > \gamma\}$ is the value that comes right after γ in $\mathcal{G}_i$. If $A_i = \{0, 1, 2, \ldots, n\}$ for all i and the aggregate is the sum, then $S(\gamma) = \gamma + 1$.

The first assumption on payoffs (A1) defines strategic complementarities. Function C^* measures the maximal amount of strategic complementarities in the game. The second assumption on payoffs (A2) describes strict increasing differences in action and state. Function M_* measures the minimal amount of monotonicity between the action and the state. Denote by $M_*(x, t_i)$ and $C^*(t_i)$ the expected value of these functions under $\mu_i(\theta|t_i)$.

The main result features function ε

$$\varepsilon(\mu, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow M_*^i(\sigma_1^i(v), t_i) - \sigma_2^i(v)C_i^*(t_i) > 0$$

$$\text{for all } t_i \in [\underline{\theta} - D_i, \overline{\theta} + D_i - v] \text{ and all } i\}. \quad (4.3)$$

Among all the $\underline{v}$'s above which the inequality within (4.3) is always satisfied, ε chooses the infimum value. The monotonicity properties of $\varepsilon(\cdot)$ will be important: (i) if M_* and σ_1^i increase uniformly, then ε decreases; (ii) if C^* and σ_2^i increase uniformly, then ε increases.

Theorem 1. *In the game of incomplete information, the distance between any two profiles of rationalizable strategies is less than $\varepsilon(\mu, \boldsymbol{u})$.*

The proof is relegated to the appendix.

The theorem suggests a nice interpretation. A type-sensitive player acts *as if* he were not affected much by the complementarities (the term $\sigma_2^i(v)C_i^*(t_i)$ in function ε). Such a player merely follows his type, which "disconnects" him from others. Therefore, type-sensitivity dampens the strategic complementarities and this favors uniqueness. To the contrary, if the beliefs are not sensitive to one's type, then they can easily be swayed by others' strategies. This gives bite to the complementarities and favors multiplicity.

Two main comparative statics lessons can be learned. The first one is that state monotonicity tends to shrink the set of rationalizable strategy profiles, whereas strategic complementarities tend to enlarge it. Function ε is, indeed, decreasing in M_* and increasing in C^*. The explanation is intuitive. State sensitivity disconnects a player from the others by making his action very sensitive to his own information, while strategic complementarities connect players together. Interestingly, strategic complementarities not only favor multiplicity but may also enlarge the equilibrium set.

The second lesson is that type-sensitivity tends to shrink the set of rationalizable strategy profiles. This fact is strong because it holds across belief structures. Say that beliefs μ are more type-sensitive than μ' if for all $i \in N$ and v, $\sigma_1^i(v) \geq \sigma_1'^i(v)$ and $\sigma_2^i(v) \leq \sigma_2'^i(v)$.

Corollary 1. *If beliefs μ are more type-sensitive than μ', then $\varepsilon(\mu, \boldsymbol{u}) \leq \varepsilon(\mu', \boldsymbol{u})$.*

As type-sensitivity becomes very high, the strategic complementarities have no impact. Thus, high type-sensitivity implies uniqueness. Beliefs μ are highly type-sensitive if $\sigma_1^i(v) > 0$ and $\sigma_2^i(v) = 0$ for all v and $i \in N$.

Corollary 2. *If beliefs are highly type-sensitive, then there is a unique equilibrium.*

Proof. If $\sigma_2^i = 0$ and $\sigma_1^i(\cdot) > 0$, then $\varepsilon(\mu, \boldsymbol{u}) = 0$, because $M_*(\sigma_1^i(v), t_i) > 0$ for all $v > 0$. $\square$

4.3. **Examples.**

4.3.1. *Investment Game.* Consider the game from Section 2. It is easy to compute $C_i^*(\theta) = 1$ and $M_*^i(x, \theta) = x$ for all i. By Theorem 1, the size of the equilibrium set is bounded by

$$\varepsilon(\mu, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow \sigma_1^i(v) - \sigma_2^i(v) > 0 \text{ for all } i\}. \tag{4.4}$$

In a two-player game, the aggregate is the other player's action. Therefore, $\Gamma_i^e[G_i(\tau_i)] = \text{Prob}(\Gamma_i = 1|\tau_i) = \text{Prob}(t_j > s_{j1}|\theta, t_i)$. In Section 2, we suggested a non-common prior and a subjective signaling function scenarios. In both cases, we argued that $\sigma_i(v) > 0$ and $\sigma_2^i(v) = 0$ for all v and i, which implies equilibrium uniqueness since $\varepsilon(\mu, \boldsymbol{u}) = 0$.[12] Consider an alternative specification. Take a global game structure where $\theta \sim N(1/2, \tau)$, $\nu = 1$, and ϵ_i has a (truncated) normal distribution with mean 0, variance η^2, and support $[-4\eta, 4\eta]$. Choose $\tau = .1$ and $\eta = .01$. The beliefs $\mu_i(\theta|t_i)$ and $\mu_i(t_j|\theta)$ are approximately (truncated) normal distributions: the first has mean $0.99t_i + .005$ and the second has mean θ and variance η^2. Therefore, $\Gamma_i^e[G_i(\tau_i + c(v))]$ is equal to (i) $1 - \Phi\left(\frac{s_{j1} + v - \theta - \sigma_1^i(v)}{\eta}\right)$ if $s_{j1} + v \in [\theta + \sigma_1^i(v) - 4\eta, \theta + \sigma_1^i(v) + 4\eta]$, (ii) 1 if $s_{j1} + v$ is below this interval, (iii) 0 otherwise. Computations give $\sigma_1^i(v) \approx .99v$ and $\sigma_2^i(v) = \max \Gamma_i^e[G_i(\tau_i + c(v))] - \Gamma_i^e[G_i(\tau_i + c(0))] \approx .4$. In conclusion, $\varepsilon(\mu, \boldsymbol{u}) \approx 0.4$.

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[12] The non-common prior example satisfies all our assumptions on beliefs. The subjective signaling scenario requires some conditions on the distribution of ϵ to satisfy these assumptions.

4.3.2. *Global Games.* In global games, players have a common prior over θ, $t_i = \theta + \nu\epsilon_i$ is common knowledge, and $\nu \to 0$. The main result is uniqueness. As $\nu \to 0$, the signal becomes a perfect predictor hence $\lim_{\nu \to 0} \sigma_1^i(v) = v$ and $\lim_{\nu \to 0} \sigma_2^i(v) = 0$ for all v.[13] Corollary 2 implies uniqueness. (4.3) describes how the global game information structure dampens the complementarities to the point where a unique equilibrium survives. This generalizes and formalizes arguments presented by Vives (2004) and Mathevet (2007). Moreover, when the prior is uniform, there is a unique equilibrium for all $\nu > 0$. Since the prior provides no information, posterior beliefs are highly type-sensitive. Corollary 2 implies uniqueness.

5. OPTIMISM AND RATIONALIZABILITY

This section studies the role of optimism and type-sensitivity in locating the rationalizable outcomes. In the investment game of Section 2, equilibrium uniqueness does not say whether the unique equilibrium cutoff $s = 1/2$ or $3/4$ or else. Theorem 1 does not give the value, or the position, of the rationalizable strategy profiles within the whole set of strategy profiles. This section addresses the question: when optimism changes, across two groups of players or two periods, how do the extremal rationalizable strategies change? The answer enables us to compute the change of likelihood of an event, such as a currency attack or a bank run.

First we define optimism. Then we measure its change across belief structures. Finally, we present the main result and apply it to the model of Izmalkov and Yildiz (2010).

5.1. **Optimism.** We compare two sets of players, or the same players at two different dates, whose beliefs are $\{\mu_i\}$ and $\{\mu_i'\}$. Let G_i and G_i' be the corresponding aggregate distributions.

Player i's beliefs become *more optimistic* if $\mu_i'(\cdot|t_i) \geq_{st} \mu_i(\cdot|t_i)$ and $G_i'(\cdot|\tau_i) \geq_{st} G_i(\cdot|\tau_i)$ for all τ_i, i.e. if i believes larger states and larger aggregates are more likely. This definition generalizes the notion of optimism defined by Izmalkov and Yildiz (2010) (see Section 5.4).

5.2. **Measuring Changes in Optimism.** Our objective is to measure the shift of the rationalizable outcomes. This shift depends on the magnitude of the shift in optimism.

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[13]It is not trivial to show this because convergence has to be uniform in type and strategies.

Definition 3. *The change of optimism of the marginal beliefs, denoted ω_1^i, is the supremum of all ω such that $\mu_i'(\cdot|t_i) \geq_{st} \mu_i^\omega(\cdot|t_i)$ for all t_i.*[14]

The change of optimism of the conditional beliefs is measured in a slightly different way. Take two aggregate distributions G and H. If H is more optimistic than G (i.e. $H \geq_{st} G$), then the difference in optimism is the difference in the expectations. If neither H nor G is more optimistic, then a worst-case analysis is used: if H is not more optimistic than G, then at least G does not dominate H more than $G \vee H$ does. The next definition formalizes these ideas. Let $\chi(H, G, \tau_i)$ be equal to $G(\tau_i)$ if $H(\tau_i) \geq_{st} G(\tau_i)$, and $H(\tau_i) \vee G(\tau_i)$ otherwise.

Definition 4. *The change of optimism from aggregate distribution G_i to H is any number $\omega_2^i \leq \Gamma_i^e[H_i(\tau_i)] - \Gamma_i^e[\chi(H, G_i, \tau_i)]$ for all τ_i.*[15]

5.3. **The Second Theorem.** There is another effect to understand before measuring the change of the rationalizable outcomes. To illustrate it, suppose players become more optimistic but their optimism is "fragile." Although they are more optimistic, a slight decrease in type (say $t_i - \epsilon$) leads them to have the same outlook on the state and Γ_i as under their original beliefs (at type t_i). In this case, it is intuitive that the set of rationalizable outcomes should not change much. Therefore, the result must account for the change of optimism in response to a change in type. Let us introduce first another notion of type-sensitivity.

Definition 5. *The type-sensitivity of the marginal beliefs is the function ψ_1^i, where $\psi_1^i(v)$ is the infimum of all ψ such that $\mu_i(\theta + \psi|t_i + v) \geq \mu_i(\theta|t_i)$ for all θ and all t_i.*

This alternative definition is the amount by which a stochastically dominant distribution should be shifted to the left to become dominated. This notion is always larger than the notion from Definition 1. The two notions only give different values when the shape of the beliefs change after a change in type. For location-scale families, both notions coincide.

Let $o(v) = (\psi_1^i(v) - \omega_1^i, \mathbf{0}, v)$ where $\mathbf{0}$ is a vector of zeroes. A player with an optimistic view on the state (ω_1^i) who receives a negative news v, thereby decreasing the state by at

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[14] Recall $\mu_i^\omega(\theta|t_i) = \mu_i(\theta - \omega|t_i)$ for $\omega > 0$.

[15] One obvious choice is the largest ω_2^i satisfying the inequality.

most $\psi_1^i(v)$, is represented by vector $\tau_i - o(v) = (\theta - \psi_1^i(v) + \omega_1^i, s_{-i}, t_i - v)$. The main result features function δ:

$$\delta(\mu, \mu', \boldsymbol{u}) = \sup\{v : M_*(\omega_1^i - \psi_1^i(v), t_i) + \min\{\omega_2^i(v)C_*(t_i), \omega_2^i(v)C^*(t_i)\} \geq 0$$

$$\text{for all } t_i \in [\underline{\theta} - D_i + v, \overline{\theta} + D_i] \text{ and all } i\} \quad (5.1)$$

where $\omega_2^i(v)$ is the change of optimism from distribution G_i to $H_v : \tau_i \mapsto G_i'(\tau_i - o(v))$, and C_* measures the minimal amount of strategic complementarities.[16] Among all the v's that satisfy the inequality in (5.1), δ picks the supremum value. The monotonicity properties of $\delta(\cdot)$ are important: (i) if M_* and ω_1^i increase uniformly, then δ increases; (ii) if ψ_1^i increases uniformly, then δ decreases.

Theorem 2. *In the game of incomplete information, if each player $i \in N$ becomes more optimistic from μ_i to μ_i', then the largest and the smallest rationalizable strategy profiles increase by at least $\delta(\mu, \mu', \boldsymbol{u})$.*

The theorem has several important implications.

The more optimistic players become, the larger the increase of the rationalizable strategy profiles tends to be. This result is intuitive and holds across belief structures.

Interestingly, type-sensitivity is involved in locating the rationalizable strategy profiles and its role is intuitive. If a player's beliefs are not type-sensitive, then as he becomes more optimistic, it takes a lot of negative information to convince him that his optimism was unfounded. Thus, larger actions can be supported at much lower types and the rationalizable outcomes change a lot. This is the next corollary.

Corollary 3. *Everything else equal, if beliefs become less type-sensitive and more optimistic, then the minimal amount by which the extremal rationalizable profiles must rise increases.*

Lastly, state monotonicity is conducive to larger shifts in the rationalizable outcomes via M_*. The role of strategic complementarities is ambiguous. On the one hand, when a player

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[16]The definition is omitted because it is similar to C^* but with a minimum instead of a maximum.

becomes more optimistic, he foresees larger aggregate values and the strategic complementarities determine his reaction to it. On the other hand, when a player receives a bad news, the effect of strong complementarities is reversed. Bad news become worse news.

5.4. **Example.** Izmalkov and Yildiz (2010) study the investment game of Section 2. Players have a uniform common prior and types $t_i = \theta + \nu\epsilon_i$. But each i has his own beliefs about (ϵ_1, ϵ_2) given by Pr_i. They define optimism as $\mathrm{Pr}_i(\epsilon_j > \epsilon_i)$, the probability with which a player believes his type is lower than his opponent's. The aggregate distribution is $G_i(\tau_i) = \mathrm{Prob}_i(t_j > s_j | t_i, \theta)$, but in symmetric two-action games, the only relevant types t_i in equilibrium are equal to s_j. Hence $G_i(\tau_i) = \mathrm{Pr}_i(\epsilon_j > \epsilon_i)$ and $\omega_2^i(v) \equiv \omega_2^i = \Delta\mathrm{Pr}_i(\epsilon_j > \epsilon_i)$. A player becomes more optimistic according to our definition iff he becomes more optimistic in the sense of Izmalkov and Yildiz (2010). This notion is related to second-order beliefs. We already know $C^*(t_i) = C_*(t_i) = 1$ and $M_*(x, t_i) = x$ for all t_i. Given the uniform prior, the marginal beliefs are highly type-sensitive, $\psi_1^i(v) = v$. The marginal beliefs do not change, $\omega_1^i = 0$. It follows from theorem 2 that

$$\delta(\mu, \mu', \boldsymbol{u}) = \sup\{v : -v + \omega_2^i(v) \geq 0, \forall t_i, \, i\} = \omega_2^i = \min_i \Delta\mathrm{Pr}_i(\epsilon_j > \epsilon_i), \qquad (5.2)$$

which is conform to their finding. In their model, there is a unique rationalizable profile and it co-varies perfectly with optimism, as shown by (5.2).

6. Applications

6.1. **Supermodular Mechanism Design.** Consider an adaptation of Mathevet (2010)'s motivating example. A principal needs to decide the level of a public good $x \in [0, 2]$. There are two agents, 1 and 2, whose type spaces are $T_1 = T_2 = [-.3, 1.3]$. Types are independently and uniformly distributed. Preferences are quasilinear, $u_i(x, t_i) = V_i(x, t_i) + m_i$ with $V_1(x, t_1) = t_1 x - x^2$ and $V_2(x, t_2) = t_2 x + \frac{x^2}{2}$. The principal wishes to make the efficient decision $x^*(t) = t_1 + t_2$, because it maximizes the sum $V_1 + V_2$. She asks each agent to report his type. Denote i's reported type by a_i. Given the reports $a = (a_1, a_2)$, the principal chooses public good level $x^*(a)$ and money transfers $m_i(a)$. If the reports are truthful, the

decision is efficient. Let $a_i \in A = \{0, \delta, 2\delta, \ldots, 1\}$ where $\delta > 0$.[17] Mathevet (2010) suggests using the following transfers:

$$m_1(a) = \frac{13}{12} + a_1 + \frac{a_1^2}{2} + \rho_1 a_1(a_2 - 1/2)$$

and

$$m_2(a) = -\frac{7}{6} - \frac{a_2}{2} - a_2^2 + \rho_2 a_2(a_1 - 1/2)$$

where ρ_1 and ρ_2 have to be chosen. The utility functions $V_i(x^*(a), t_i) + m_i(a)$, $i = 1, 2$, define a private value environment. There are values of ρ_1 and ρ_2, including resp. 2 and -1, for which the assumptions of Section 3 are satisfied. In particular, the utility functions have strategic complementarities and for each i, $a_i = 1$ is strictly dominant for $t_i > \bar{t} = 1\frac{1}{4}$ and $a_i = 0$ is strictly dominant for $t_i < \underline{t} = -\frac{1}{4}$. By Theorem 1, this mechanism induces a game whose size of the equilibrium set is less than

$$\varepsilon(\mu, \boldsymbol{u}) = \inf\{\underline{v} > 0 : v > \underline{v} \Rightarrow \delta v - \sigma_2^1(v)(\rho_1 - 2) > 0 \text{ and } \delta v - \sigma_2^2(v)(1 + \rho_2) > 0\}. \quad (6.1)$$

Thus, the equilibrium set may enlarge as ρ_1 and ρ_2 increase, an observation at the heart of optimal supermodular implementation (Mathevet (2010)). (6.1) also shows that the mechanism has a unique equilibrium for $\rho_1 = 2$ and $\rho_2 = -1$. For these values, the unique equilibrium is essentially truthful: if his type falls in A a player reports truthfully, otherwise he chooses the report closest to his type. Our conclusions hold for any $\delta > 0$.

6.2. **The Effect of Updating Biases.** This section studies the strategic implications of some updating biases. Our framework relies on posterior beliefs. Epstein (2006) axiomatizes posterior beliefs that are the product of non-Bayesian updating. Although our framework does not capture every kind of non-Bayesian settings, it applies to some situations axiomatized by Epstein (2006). We consider two of them: the prior and the overreaction bias.

Players have a common prior about θ with cdf P and they each receive a signal $t_i = \theta + \epsilon_i$ of the realized state. Conditional on signal t_i, player i has marginal beliefs $\mu_i(\cdot|t_i)$ that may be different from the beliefs $BU_i(\cdot|t_i)$ that a Bayesian player would have.

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[17]Reports are finite to satisfy our framework. Moreover, the largest and smallest types that an agent can report are 0 and 1. This will imply some lying for extreme true types but it guarantees the existence of dominance regions.

The beliefs formulated by the non-Bayesian players under P and the linear signaling functions might be the same as the beliefs formulated by Bayesian players with some priors $\{\hat{P}_i\}$ and other signaling functions. Therefore, we must be cautious when talking about non-Bayesian updating. However, from an applied perspective, the analyst may believe that players indeed make mistake when processing information, hence she may take the above specification seriously. Besides, the analyst may be unwilling to recover the priors $\{\hat{P}_i\}$, or the signaling functions, that correspond to the same type space under Bayesian updating. In effect, then, the players in her model are not Bayesian (see Epstein (2006)).

6.2.1. *Prior Bias.* A player who has a prior bias gives "too little" weight to observation. Given Epstein (2006), this can be modeled as

$$\mu_i(\cdot|t_i) = \alpha P(\cdot) + (1-\alpha)BU_i(\cdot|t_i), \tag{6.2}$$

where $\alpha \in [0,1]$ measures the magnitude of the bias. Since P gives no weight to the type, it is clear $BU_i(\cdot|t_i)$ is more type-sensitive than μ_i. Therefore $\sigma_1^i(\cdot)$ is smaller than that of a Bayesian player. What about $\sigma_2^i(\cdot)$? Player i constructs the same aggregate distribution as the Bayesian player, because t_i plays no role in G_i conditionally on θ. Since $\sigma_1^i(\cdot)$ is smaller than for a Bayesian player, so is $\sigma_2^i(\cdot)$ by definition. By Corollary 1, the prior bbias tends to favor multiplicity and wider sets of rationalizable strategy profiles.

6.2.2. *Overreaction.* A player who is subject to overreaction gives "too much" weight to observation. Let θ^* be the expected state under P. This bias can be modeled as

$$Q_i(\cdot|t_i) = BU_i(\cdot|t_i + \alpha(t_i - \theta^*)), \tag{6.3}$$

where $\alpha \in [0,1]$ measures the magnitude of the bias. A biased player believes at t_i what a Bayesian player would believe at $t_i + \alpha(t_i - \theta^*)$. Hence, after receiving $t_i > (<)\theta^*$, i interprets his information as a better (worse) news than what it is. Assume BU_i belongs to a location-scale family. Because $t_i + \alpha t_i - \theta^* = (1 + \alpha)t_i - \alpha\theta^*$, overreaction leads to larger type-sensitivity. The overreaction bias promotes tighter rationalizable sets.

6.3. **Type-sensitivity and Influence.** This section investigates the relationship between type-sensitivity and a notion of influence in games. Players whose beliefs are more type-sensitive are more influential. This relationship is particularly interesting when type sensitivity is viewed as confidence in one's information. Behavioral economics provides many definitions of confidence, some related to the perceived precision of one's information (Odean (1999), Healy and Moore (2009)). More confident players are more influential.

Consider binary-action games. Theorem 1 says that, unless type-sensitivity is high for all players, there should be multiple equilibria. Assume there are many equilibria. Take any two of them, s^* and s^{**}, such that $s^* < s^{**}$. One way of measuring the influence of a player is via $s_i^{**} - s_i^*$. This is the amount by which a player changes his equilibrium strategy in response to changes in others' equilibrium strategies. For example, if $s_1^{**} - s_1^* < \max_{j \neq 1} s_j^{**} - s_j^*$, then any player j changes his strategy more than 1, although j responds to a smaller change in his opponents' strategies than 1.[18] 1 is said to be more *influential*.

Proposition 1. *For any player i, any subset $\mathcal{N} \subset N \backslash \{i\}$, and any two equilibria s^{**} and s^*, there exist $\overline{\sigma}_1(\cdot)$ and $\overline{\sigma}_2(\cdot)$ such that $\sigma_1^i(v) \geq \overline{\sigma}_1(v)$ and $\sigma_2^i(v) \leq \overline{\sigma}_2(v)$ for all $v > 0$ imply that i is more influential than any $j \in \mathcal{N}$: $s_i^{**} - s_i^* < \max_{j \in \mathcal{N}} s_j^{**} - s_j^*$.*

The proof uses arguments from the proof of Theorem 1 and is omitted.

7. CONCLUSION

In this paper, we have introduced type-sensitivity and generalized optimism, two notions that capture essential features of the beliefs involved in shaping the set of rationalizable strategy profiles. The main advantage of the approach is twofold. First, it does not specify the origin of the beliefs, and thus it covers new scenarios. Second, it synthesizes properties of beliefs and payoffs into explicit expressions that give insightful comparative statics.

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[18]Player 1's opponents change their strategies more than j's opponents because $s_{-1}^{**} - s_{-1}^* > s_{-j}^{**} - s_{-j}^*$ in the product order.

Appendix A. Aggregate Distribution

Consider the set of vectors of types that are lower than t'_{-i}:

$$L(t'_{-i}) = \{t_{-i} \in T_{-i} : t_j \leq t'_j \text{ for all } j \neq i\}.$$

Let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_{i-1}, \ell_{i+1}, \ldots, \ell_N) \in \mathbb{N}^{N-1}$ and denote by $a_{-i,\boldsymbol{\ell}}$ the vector of actions such that each $j \neq i$ plays action a_{j,ℓ_j}. Define $A_{-i}(\gamma, \theta) = \{\boldsymbol{\ell} \in \mathbb{N}^{N-1} : \Gamma_i(a_{-i,\boldsymbol{\ell}}, \theta) = \gamma\}$ to be the set of combinations of actions that yield aggregate value γ at state θ. Recall that player j plays action a_{j,ℓ_j} if and only if his type is in $[s_{j,\ell_j-1}, s_{j,\ell_j}]$. The aggregate distribution is described by the following probability mass function

$$g_i(\gamma|\tau_i) = \mu_i(\theta, t_i) \left[\bigcup_{\boldsymbol{\ell} \in A_{-i}(\gamma,\theta)} \left\{ L((s_{j,\ell_j})_{j \neq i}) \bigcap \overline{L((s_{j,\ell_j-1})_{j \neq i})} \right\} \right].$$

Let $G_i(\cdot|\tau_i)$ be the cumulative distribution function obtained from g_i.

Appendix B. Proofs

The argument of the first result goes as follows:

(1) The games under consideration have strategic complements (GSC). This implies the existence of a largest and a smallest equilibrium (Milgrom and Roberts (1990) and Vives (1990)).

(2) Furthermore, the payoffs display some monotonicity between actions and states, and the beliefs display monotonicity in type. By Van Zandt and Vives (2007), (a) best-responses to monotone (in-type) strategies are monotone and (b) the extremal equilibria are in monotone strategies.

(3) We prove that the best-reply mapping, restricted to monotone strategies, is a contraction for all pairs of profiles that are distant enough. Since the extremal equilibria are in monotone strategies, they can be no further apart than this distance.

(4) Since extremal equilibria bound the set of profiles in rationalizable strategies in GSC, this gives a distance between any pair of rationalizable profiles.

In view of (2), we restrict attention to monotone (in-type) strategies. Any such strategy can be represented as a finite sequence of cutoff points. Call these cutoff points *real cutoffs*

as opposed to the *fictitious cutoffs* defined later. Player i's strategy is $s_i = (s_{i,\ell})_{\ell=1}^{M_i-1}$ where each $s_{i,\ell}$ is the threshold type below which i plays a_ℓ and above which he plays $a_{\ell+1}$.

Definition 6. *For each i, the fictitious cutoff between a_n and a_m, denoted $c_{n,m}$ is defined, if it exists, as the (only) type t_i such that $Eu_i(a_n, s_{-i}, t_i) - Eu_i(a_m, s_{-i}, t_i) = 0$.*

The notation $\Delta_m^n u_i(\gamma, \theta)$ gives the difference in utility between a_n and a_m given aggregate value γ and state θ. Define the expected utility as

$$Eu_i(a_i, s_{-i}, t_i) = \int_\mathbb{R} \sum_{\gamma \geq \underline{\gamma}} u_i(a_i, \gamma, \theta) g_i(\gamma|\theta, s_{-i}, t_i) d\mu_i(\theta|t_i). \tag{B.1}$$

Similarly, $\Delta_m^n Eu_i(s_{-i}, t_i) = Eu_i(a_{i,n}, s_{-i}, t_i) - Eu_i(a_{i,m}, s_{-i}, t_i)$. We often write Δ instead of Δ_m^n.

B.1. **Proposition 2.**

Proposition 2. *If $v > \varepsilon(\mu, \boldsymbol{u})$, then for all pairs of actions (a_n, a_m), types t_i, strategies s_{-i}, and $i \in N$ such that*

$$Eu_i(a_n, s_{-i}, t_i) - Eu_i(a_m, s_{-i}, t_i) \geq 0 \tag{B.2}$$

the following inequality holds

$$Eu_i(a_n, s_{-i} + \boldsymbol{v}, t_i + v) - Eu_i(a_m, s_{-i} + \boldsymbol{v}, t_i + v) > 0 \tag{B.3}$$

Proof. Suppose (B.2) is satisfied. From the definition of type-sensitivity, $\mu_i(\cdot|t_i + v) \geq_{st} \mu_i^{\sigma_1^i(v)}(\cdot|t_i)$. Therefore,

$$\int_\mathbb{R} \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta) g_i(\gamma|\theta, s_{-i} + \boldsymbol{v}, t_i + v) d\mu_i^{\sigma_1^i(v)}(\theta|t_i) > 0 \tag{B.4}$$

would imply (B.3), because $\sum \Delta u_i(\gamma, \theta) g_i(\gamma|\theta, s_{-i}, t_i + v)$ is increasing in θ. After a change of variables, (B.4) becomes

$$E_{\theta|t_i} \left[\sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta + \sigma_1^i(v)) g_i(\gamma|\tau_i + c(v)) \right] > 0. \tag{B.5}$$

Since (B.2) holds by assumption,

$$E_{\theta|t_i}\left[\sum_{\gamma\geq\underline{\gamma}}(\Delta u_i(\gamma,\theta+\sigma_1^i(v))-\Delta u_i(\gamma,\theta))g_i(\gamma|\tau_i+c(v))\right]$$

$$+E_{\theta|t_i}\left[\sum_{\gamma\geq\underline{\gamma}}\Delta u_i(\gamma,\theta)(g_i(\gamma|\tau_i+c(v))-g_i(\gamma|\tau_i))\right]>0 \quad \text{(B.6)}$$

would imply (B.5). The first member of (B.6) is strictly positive, because Δu_i is strictly increasing in θ. Although the second member is not always positive, it admits a lower bound to be constructed next. For any $\gamma\in\mathcal{G}_i$, define $S(\gamma)=\min\{\gamma'\in\mathcal{G}_i:\gamma'>\gamma\}$ to be the successor of γ. By convention, let $G_i(S(\overline{\gamma})|\cdot)=1$. Define

$$C^*(\theta)=\max_{\gamma}\frac{\Delta u_i(a_{i,M_i},a_{i,1},S(\gamma),\theta)-\Delta u_i(a_{i,M_i},a_{i,1},\gamma,\theta)}{S(\gamma)-\gamma} \quad \text{(B.7)}$$

to be the largest amount of complementarities in i's payoffs. Let G_i^* be the cdf of distribution $G_i(\tau_i)\vee G_i(\tau_i+c(v))$ and let g_i^* be its probability mass function. Note that

$$\sum_{\gamma\geq\underline{\gamma}}\Delta u_i(\gamma,\theta)(g_i(\gamma|\tau_i+c(v))-g_i(\gamma|\tau_i))=$$

$$\sum_{\gamma\geq\underline{\gamma}}(G_i(S(\gamma)|\tau_i+c(v))-G_i(S(\gamma)|\tau_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(S(\gamma),\theta)). \quad \text{(B.8)}$$

Since Δu_i is increasing in γ, it follows from the definition of type-sensitivity that

$$\sum_{\gamma\geq\underline{\gamma}}(G_i(S(\gamma)|\tau_i+c(v))-G_i(S(\gamma)|\tau_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(S(\gamma),\theta))$$

$$\geq\ \sum_{\gamma\geq\underline{\gamma}}(G_i(S(\gamma)|\tau_i+c(v))-G_i^*(S(\gamma)|\tau_i))(\Delta u_i(\gamma,\theta)-\Delta u_i(S(\gamma),\theta))$$

$$\geq\ \sum_{\gamma\geq\underline{\gamma}}(G_i(S(\gamma)|\tau_i+c(v))-G_i^*(S(\gamma)|\tau_i))(\gamma-S(\gamma))C^*(\theta)$$

$$=\ C^*(\theta)\sum_{\gamma}\gamma(g_i(\gamma|\tau_i+c(v))-g_i^*(\gamma|\tau_i))$$

$$\geq\ -C^*(\theta)\sigma_2^i(v)$$

The last expression provides a lower bound on the second member of (B.6). Let us now bound the first member of (B.6). For $x \in \mathbb{R}$, let

$$M_*(\theta, x) = \min_{(\gamma, n, m)} \Delta u_i(a_n, a_m, \gamma, \theta + x) - \Delta u_i(a_n, a_m, \gamma, \theta) \tag{B.9}$$

be the smallest amount of complementarities between action and state. Note

$$\sum_{\gamma \geq \underline{\gamma}} (\Delta u_i(\gamma, \theta + \sigma_1^i(v)) - \Delta u_i(\gamma, \theta)) g_i(\gamma | \tau_i + c(v)) \geq M_*(\theta, \sigma_1^i(v)).$$

Therefore, if inequality

$$E_{\theta|t_i}[M_*(\theta, \sigma_1^i(v))] - \sigma_2^i(v) E_{\theta|t_i}[C^*(\theta)] > 0 \tag{B.10}$$

holds, then it implies (B.6). By definition of $\varepsilon(\mu, \boldsymbol{u})$, if $v > \varepsilon(\mu, \boldsymbol{u})$, then (B.10) holds for all pairs of actions a_n and a_m, types t_i, strategies s_{-i}, and $i \in N$. Therefore (B.6) holds, hence (B.5) and (B.3) are satisfied for all these parameters. $\qquad \square$

B.2. **Real vs. Fictitious Cutoffs and Proposition 4.** The real cutoffs are the threshold types that separate an action from its successor. They are sufficient to represent any increasing strategy. How to recover the real cutoffs from the fictitious cutoffs?

Example 1. Consider a game with two players. Let $A_1 = A_2 = \{0, 1, 2\}$. There are three fictitious cutoffs, $c_{1,0}$, $c_{2,0}$ and $c_{2,1}$, but only two are needed to represent a player's best-response. Which ones? For instance, suppose strategy $(0.2, 0.8)$ is a best-response for i to some strategy s_j of player j. It consists in playing 0 for types below 0.2, 2 for types above 0.8, and 1 in between. In this case, the first real cutoff, $s_{i,1}$, that separates 0 and 1 is $0.2 = c_{1,0}$. The second real cutoff, $s_{i,2}$, that separates 1 and 2 is $0.8 = c_{2,1}$. Now, consider the following best-response $(0.4, 0.4)$ to s'_j. In this case, the player never plays 1 except possibly on a set of measure zero (when receiving exactly type 0.4). The first real cutoff, $s'_{i,1}$, that separates 0 and 1 is $0.4 = c'_{2,0}$, but the second real cutoff, $s'_{i,2}$, is also $c'_{2,0}$, because 1 is not played. This shows that a real cutoff is not always the same fictitious cutoff. It can take on the value of different fictitious cutoffs.

This leads to the following definition where the real cutoffs are defined recursively from the fictitious cutoffs.[19]

Definition 7. *The largest real cutoff, s_{i,M_i-1}, is the fictitious cutoff $c_{M_i,\alpha}$ such that (i) for any $t_i > c_{M_i,\alpha}$, $\Delta_k^{M_i} Eu_i(s_{-i}, t_i) > 0$ for all $k \neq M_i$, (ii) for some $\epsilon > 0$ and any $t_i \in (c_{M_i,\alpha} - \epsilon, c_{M_i,\alpha})$, $\Delta_k^{\alpha} Eu_i(s_{-i}, t_i) > 0$ for all $k \neq \alpha$. Suppose $s_{i,\ell} = c_{n,m}$. Then define $s_{i,\ell-1}$ as follows. If $\ell > m$, then the real cutoff $s_{i,\ell-1} = c_{n,m}$. If $\ell = m$, then $s_{i,\ell-1} = c_{m,\beta}$ such that (i) for any $t_i > c_{m,\beta}$, $\Delta_k^m Eu_i(s_{-i}, t_i) > 0$ for all $k \neq m$ and (ii) for some $\epsilon > 0$ and any $t_i \in (c_{m_i,\beta} - \epsilon, c_{m_i,\beta})$, $\Delta_k^{\beta} Eu_i(s_{-i}, t_i) > 0$ for all $k \neq \beta$.*

The dominance regions imply that a_{i,M_i} will be played, so the largest real cutoff is the fictitious cutoff between a_{i,M_i} and the action $a_{i,\alpha}$ played before it. All actions in between are not played, hence they receive the same real cutoff. We proceed in a downward fashion to find the action that was played before $a_{i,\alpha}$ and so on.

The next proposition shows that if an action is strictly dominated by another action for all types against some opposing profile, then it must be strictly dominated by that same action for all types and against all opposing profiles. As a result, the same set of fictitious cutoffs will exist across opposing strategy profiles.

Proposition 3. *Let $\varepsilon(\mu, \boldsymbol{u}) < \overline{\theta} - \underline{\theta} + 2 \max_i D_i$.[20] For any $a_i, a_i' \in A_i$, if there is $s_{-i}' \in \mathbb{R}$ such that $Eu_i(a_i', s_{-i}', t_i) > Eu_i(a_i, s_{-i}', t_i)$ for all $t_i \in \mathbb{R}$, then $Eu_i(a_i', s_{-i}, t_i) > Eu_i(a_i, s_{-i}, t_i)$ for all s_{-i} and $t_i \in \mathbb{R}$.*

Proof. Let $\varepsilon(\mu, \boldsymbol{u}) < \overline{\theta} - \underline{\theta} + 2 \max_i D_i$. Suppose first $a_i' > a_i$. If there is s_{-i}' such that $Eu_i(a_i', s_{-i}', t_i) > Eu_i(a_i, s_{-i}', t_i)$ for all t_i, then it follows from Proposition 2 that

$$Eu_i(a_i', s_{-i}' + \boldsymbol{v}, t_i + v) - Eu_i(a_i, s_{-i}' + \boldsymbol{v}, t_i + v) > 0, \tag{B.11}$$

for all $v > \varepsilon(\mu, \boldsymbol{u})$ and t_i. For any s_{-i}, choose $v > \varepsilon(\mu, \boldsymbol{u})$ such that $s_{-i}' + \boldsymbol{v} \geq s_{-i}$ (so s_{-i} is a larger strategy). Larger strategies lead to larger aggregates, hence (B.11) and the strategic

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[19]Existence of the fictitious cutoffs poses no problem in the definition, for if a real cutoff takes on the value of a fictitious cutoff, that fictitious cutoff must exist.

[20]If $\varepsilon(\mu, \boldsymbol{u}) = \overline{\theta} - \underline{\theta} + 2D_i$, then the main result says that the size of the equilibrium set is the whole space. The result is interesting for $\varepsilon(\mu, \boldsymbol{u}) < \overline{\theta} - \underline{\theta} + 2 \max_i D_i$.

complementarities imply

$$Eu_i(a_i', s_{-i}, t_i + v) - Eu_i(a_i, s_{-i}, t_i + v) > 0$$

for all t_i. This is equivalent to $Eu_i(a_i', s_{-i}, t_i) - Eu_i(a_i, s_{-i}, t_i) > 0$ for all t_i. Since s_{-i} was arbitrary, the claim is proved. Suppose now that $a_i' < a_i$. If there is s_{-i}' such that $Eu_i(a_i', s_{-i}', t_i) > Eu_i(a_i, s_{-i}', t_i)$ for all t_i, then Proposition 2 implies

$$Eu_i(a_i', s_{-i}' - \boldsymbol{v}, t_i - v) - Eu_i(a_i, s_{-i}' - \boldsymbol{v}, t_i - v) > 0 \tag{B.12}$$

for all $v > \varepsilon(\mu, \boldsymbol{u})$ and t_i. For any s_{-i}, choose $v > \varepsilon(\mu, \boldsymbol{u})$ such that $s_{-i} \geq s_{-i}' - \boldsymbol{v}$. By (B.12) and the strategic complementarities, we have

$$Eu_i(a_i', s_{-i}, t_i - v) - Eu_i(a_i, s_{-i}, t_i - v) > 0$$

for all t_i, which is equivalent to $Eu_i(a_i', s_{-i}, t_i) - Eu_i(a_i, s_{-i}, t_i) > 0$ for all t_i. $\qquad \square$

The next proposition is an important piece of the main theorem. If all of i's fictitious cutoffs contract in response to a variation of s_{-i}, then so do all of i's real cutoffs. That is, i's best-reponse contracts as well.

Proposition 4. *Suppose $\varepsilon(\mu, \boldsymbol{u}) < \overline{\theta} - \underline{\theta} + 2\max_i D_i$. If, for some $v > 0$, we have $|c_{n,m}' - c_{n,m}| < v$ for all n and m for which $c_{n,m}'$ and $c_{n,m}$ exist, then $|s_{i,\ell} - s_{i,\ell}'| < v$ for all $\ell = 1, \ldots, M_i - 1$.*

Proof. The result is proved by induction. Suppose that, for some $v > 0$, $|c_{n,m}' - c_{n,m}| < v$ for all n and m for which both $c_{n,m}'$ and $c_{n,m}$ exist.

We first prove that the result holds for the largest real cutoff and then extend it to other cutoffs by induction. Let the largest real cutoff $s_{i,M_i-1} = c_{n,m}$ and $s_{i,M_i}' = c_{k,\ell}'$.

The largest action a_{i,M_i} is always played for large enough types. So the largest real cutoff always takes on the value of the fictitious cutoff between a_{i,M_i} and some other action. Suppose that $s_{i,M_i-1} = c_{M_i,w}$ and $s_{i,M_i-1}' = c_{M_i,z}'$. Proposition 3 implies that $c_{M_i,z}$ must exist. To see why, suppose $c_{M_i,z}$ did not exist. Since a_{M_i} must be played, it would mean that a_{M_i} strictly dominates a_z for all t_i against s_{-i}. Proposition 3 would then imply that a_{M_i} strictly dominates a_z for all t_i and all opposing strategies, s_{-i}' in particular, contradicting the

existence of $c'_{M_i,z}$. Therefore, $s'_{i,M_i-1} - s_{i,M_i-1} = c'_{M_i,z} - c_{M_i,w} = c'_{M_i,z} - c_{M_i,z} + c_{M_i,z} - c_{M_i,w}$. Note that $c_{M_i,z} - c_{M_i,w} \leq 0$. Indeed, $s_{i,M_i-1} = c_{M_i,w}$ implies that a_{i,M_i} is played right after $a_{i,w}$ in the best-response, hence a_{i,M_i} became preferable to $a_{i,z}$ before $c_{M_i,w}$. Since $c'_{M_i,z} - c_{M_i,z} < v$, then $s'_{i,M_i-1} - s_{i,M_i-1} < v$. The proof is similar for $s_{i,M_i-1} - s'_{i,M_i-1}$, hence $|s'_{i,M_i-1} - s_{i,M_i-1}| < v$.

For the other real cutoffs, the situation is more difficult, because the corresponding action may not be played. By induction hypothesis, suppose that $|s'_{i,\ell+1} - s_{i,\ell+1}| < v$. The objective is to show that it implies $|s'_{i,\ell} - s_{i,\ell}| < v$. There are several cases:

Case 1: Action $a_{i,\ell}$ is played both under s_i and s'_i. This case is similar to the case of the largest real cutoff, and the proof is identical.

Case 2: Action $a_{i,\ell}$ is played neither under s_i nor s'_i. By definition (7), $s_{i,\ell} = s_{i,\ell+1}$ and $s'_{i,\ell} = s'_{i,\ell+1}$. By induction hypothesis, $|s'_{i,\ell} - s_{i,\ell}| = |s'_{i,\ell+1} - s_{i,\ell+1}| < v$.

Case 3: Action $a_{i,\ell}$ is not played in s_i but it is in s'_i. Then, $s_{i,\ell} = c_{w,z}$ for some actions $a_{i,w}$ and $a_{i,z}$ such that $z < \ell < w$, and $s'_{i,\ell} = c'_{\ell,x}$ for some $a_{i,x}$. Write $s'_{i,\ell} - s_{i,\ell} = c'_{\ell,x} - c_{w,z}$.

First, we establish that both $c_{w,\ell}$ and $c'_{w,\ell}$ exist. Action $a_{i,w}$ is played (under s_i) against s_{-i} but it cannot strictly dominate $a_{i,\ell}$ for all types t_i, because if it did, then Proposition 3 would imply that it is also the case (under s'_i) against s'_{-i} (thus $a_{i,\ell}$ could not be played under s'_i, yet it is). Therefore, $c_{w,\ell}$ must exist. This implies that for all $t_i \geq c_{w,\ell}$,

$$Eu_i(a_{i,w}, s_{-i}, t_i) > Eu_i(a_{i,\ell}, s_{-i}, t_i). \tag{B.13}$$

Let $\boldsymbol{h} = (h, \ldots, h)$ where $h > \varepsilon(\mu, \boldsymbol{u})$ is large enough such that $s_{-i} + \boldsymbol{h} \geq s'_{-i}$. It follows from Proposition 2 and (B.13) that for all $t_i \geq c_{w,\ell}$,

$$Eu_i(a_{i,w}, s_{-i} + \boldsymbol{h}, t_i + h) > Eu_i(a_{i,\ell}, s_{-i} + \boldsymbol{h}, t_i + h)$$

and thus by strategic complementarities,

$$Eu_i(a_{i,w}, s'_{-i}, t_i + h) > Eu_i(a_{i,\ell}, s'_{-i}, t_i + h),$$

for all $t_i \geq c_{w,\ell}$. We know $a_{i,\ell}$ is played (under s'_i) against s'_{-i}, so the last inequality implies that $c'_{w,\ell}$ exists.

Second, we prove that real cutoff contracts. The following inequality must hold, $c'_{w,\ell} \geq c'_{\ell,x}$, because $a_{i,\ell}$ is played under s'_i in an open set of types above $c'_{\ell,x}$ (so it is only for types larger than $c'_{\ell,x}$ that $a_{i,w}$ can be preferred to $a_{i,\ell}$). Similarly, $c_{w,\ell} \leq c_{w,z}$, because $a_{i,w}$ is played under s_i in an open set of types above $c_{w,z}$, hence $a_{i,w}$ started to be preferred to $a_{i,\ell}$ for smaller types. As a result,

$$s'_{i,\ell} - s_{i,\ell} = c'_{\ell,x} - c_{w,z} \leq c'_{w,\ell} - c_{w,\ell},$$

so $s'_{i,\ell} - s_{i,\ell} < v$. By a similar reasoning, $s_{i,\ell} - s'_{i,\ell} \leq c'_{\ell,z} - c_{\ell,z}$, and so $s_{i,\ell} - s'_{i,\ell} < v$. Putting everything together, $|s'_{i,\ell} - s_{i,\ell}| < v$.

Case 4: Action a_ℓ is played in s_i but it is not in s'_i. The argument is similar to case 3. □

B.3. **Proof of Theorem 1.** The theorem relies on the concept of a q-contraction.

Definition 8. *Let (X, d) be a metric space. If $\xi : X \to X$ satisfies the condition $d(\xi(x), \xi(y)) < d(x, y)$ for all $x, y \in X$ such that $d(x, y) > q$, then ξ is called a q-contraction.*

A traditional contraction mapping "shrinks" the image of all points. A q-contraction only "shrinks" the image of points that are sufficiently far apart (further apart than q). Naturally, a q-contraction cannot have fixed points that are too far apart.

Proof of Theorem 1. Recall that i's expected utility of playing a_i when his type is t_i and the other players play s_{-i} is given by (B.1). Now pick $n, m \in \{1, \ldots, M_i\}$ such that $n > m$. If it exists, the fictitious cutoff between $a_{i,n}$ and $a_{i,m}$ is defined as the type t_i such that

$$Eu_i(a_{i,m}, s_{-i}, t_i) = Eu_i(a_{i,n}, s_{-i}, t_i),$$

that is,

$$\int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta) g_i(\gamma | \theta, s_{-i}, c_{n,m}) d\mu_i(\theta | c_{n,m}) = 0. \tag{B.14}$$

By state monotonicity, Δu_i is strictly increasing in θ and increasing in γ. Since μ_i is strictly increasing in t_i w.r.t. first-order stochastic dominance, and since G_i is increasing in (θ, t_i) w.r.t. to first-order stochastic dominance, there can be only one type t_i that satisfies (B.14). As a result, the best-replies (which are cutoff strategies) are almost everywhere functions, and not correspondences. Consider two profiles of strategies for players $-i$, $s_{-i} = (s_{j,\ell})$ and $s'_{-i} =$

$(s'_{j,\ell})$. Denote $v_{j,\ell} = |s'_{j,\ell} - s_{j,\ell}|$ for $\ell = 1, \ldots, M_j - 1$. Let $v = \max_{j \neq i} \max_{\ell \in \{1, \ldots, M_j - 1\}} v_{j,\ell}$. Player i's cutoff between $a_{i,n}$ and $a_{i,m}$ against s_{-i}, denoted $c_{n,m}$, satisfies (B.14). The cutoff between $a_{i,n}$ and $a_{i,m}$ against s'_{-i} is $c'_{n,m}$. By way of contradiction, assume $c'_{n,m} = c_{n,m} + v$. Hence

$$\int_{\mathbb{R}} \sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta) g_i(\gamma | \theta, s'_{-i}, c_{n,m} + v) d\mu_i(\theta | c_{n,m} + v) = 0. \tag{B.15}$$

If $v > \varepsilon(\mu, \boldsymbol{u})$, Proposition 2 says that (B.14) and (B.15) cannot hold simultaneously. That is, $c'_{n,m} = c_{n,m} + v$ cannot be the cutoff against s'_{-i} if $c_{n,m}$ is the cutoff against s_{-i}. Clearly, this claim holds for $c'_{n,m} \geq c_{n,m} + v$. Therefore, $c'_{n,m} - c_{n,m} < v$. If $c'_{n,m}$ is the cutoff against s'_{-i}, the same argument shows that whenever $c_{n,m}$ is larger than $c'_{n,m} + v$, both cannot cutoffs. In conclusion, if $v > \varepsilon(\mu, \boldsymbol{u})$, then for all players, $|c'_{n,m} - c_{n,m}| < v$ for all n, m such that both cutoffs exist. Proposition 4 implies that each i's best-reply is an $\varepsilon(\mu, \boldsymbol{u})$-contraction. From Milgrom and Roberts (1990), it follows that there exist two extremal equilibria, $\overline{s}$ and $\underline{s}$, that correspond to the extremal profiles of rationalizable strategies. We abuse notation and use d as the sup-norm on different metric spaces. Let $\boldsymbol{e} = (1, 1, \ldots, 1)$ be the vector of ones. Since br_i is an $\varepsilon(\mu, \boldsymbol{u})$-contraction, if $d(\overline{s}, \underline{s}) > \varepsilon(\mu, \boldsymbol{u})$, then we have $d(br_i(\underline{s}_{-i} - d(\overline{s}, \underline{s})\boldsymbol{e}), br_i(\underline{s}_{-i})) < d(\overline{s}, \underline{s})$. Thus,

$$
\begin{aligned}
d(\overline{s}, \underline{s}) &= d(br(\overline{s}), br(\underline{s})) \\
&= \max_{i \in N} d(br_i(\overline{s}_{-i}), br_i(\underline{s}_{-i})) \\
&\leq \max_{i \in N} d(br_i(\underline{s}_{-i} - d(\overline{s}, \underline{s})\boldsymbol{e}), br_i(\underline{s}_{-i}))) \\
&< d(\overline{s}, \underline{s}),
\end{aligned}
$$

where the first inequality holds because best-replies are increasing.[21] This string of inequalities leads to a contradiction, and thus $d(\overline{s}, \underline{s}) \leq \varepsilon(\mu, \boldsymbol{u})$. $\square$

B.4. **Theorem 2.** We first state a proposition that will be used in the proof.

Proposition 5. *Let $\{c_{n,m}\}$ be the set of fictitious cutoffs under μ, and let $\{c_{n,m}\}$ be the set of fictitious cutoffs under μ', where μ'_i is more optimistic than μ_i for each i. If, for some*

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[21]Notice $\underline{s}_{-i} - d(\overline{s}, \underline{s})$ is a larger strategy than $\overline{s}_{-i}$.

$v > 0$, $c_{n,m} - c'_{n,m} \geq v$ *for all n and m such that both fictitious cutoffs exist, then $s_{i,\ell} - s'_{i,\ell} \geq v$*
for all $\ell = 1, \ldots, M_i - 1$.

The proof is similar to that of Proposition 4, hence it is omitted.

Proof of Theorem 2. In supermodular games, the largest (smallest) equilibrium coincide with the largest (smallest) profile of rationalizable strategies. Consider the largest (smallest) equilibrium, denoted by $\bar{s}$ ($\underline{s}$), under beliefs μ_i, $i = 1, \ldots, n$. Against $\bar{s}_{-i}$, i's fictitious cutoff between a_n and a_m satisfies

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g_i(\gamma | \theta, \bar{s}_{-i}, c_{n,m}) d\mu_i(\theta | c_{n,m}) = 0. \tag{B.16}$$

Since beliefs μ'_i are more optimistic than μ_i,

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g'_i(\gamma | \theta, \bar{s}_{-i}, c_{n,m}) d\mu'_i(\theta | c_{n,m}) d\theta \geq 0, \tag{B.17}$$

because Δu_i is increasing in θ and γ. Thus, the fictitious cutoff between a_n and a_m must be smaller under μ'_i than μ_i. Consider any s_{-i} and t_i such that

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g_i(\gamma | \theta, s_{-i}, t_i) d\mu_i(\theta | t_i) = 0 \tag{B.18}$$

and if for $v \geq 0$

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g'_i(\gamma | \theta, \bar{s}_{-i}, t_i - v) d\mu'_i(\theta | t_i - v) d\theta > 0, \tag{B.19}$$

then $t_i - v$ cannot be the fictitious cutoff under μ'_i, because $t_i - v$ is too large. This means that v should be increased. It follows from type-sensitivity and optimism that

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta) g'_i(\gamma | \theta, \bar{s}_{-i}, t_i - v) d\mu_i(\theta - \omega_1 + \psi_1^i(v) | t_i) > 0 \tag{B.20}$$

implies (B.19). After a change of variables, (B.20) is equivalent to

$$\int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta + \omega_1 - \psi_1^i(v)) g'_i(\gamma | \theta + \omega_1 - \psi_1^i(v), \bar{s}_{-i}, t_i - v) d\mu_i(\theta | t_i) > 0. \tag{B.21}$$

Assuming that (B.16) holds, (B.21) is equivalent to

$$
\int_{\mathbb{R}} \sum_{\gamma} (\Delta u_i(\gamma, \theta + \omega_1 - \psi_1^i(v)) - \Delta u_i(\gamma, \theta)) g_i'(\gamma | \theta + \omega_1 - \psi_1^i(v), \overline{s}_{-i}, t_i - v) d\mu_i(\theta | t_i)
$$

$$
+ \int_{\mathbb{R}} \sum_{\gamma} \Delta u_i(\gamma, \theta)(g_i'(\gamma | \theta + \omega_1 - \psi_1^i(v), \overline{s}_{-i}, t_i - v) - g_i(\gamma | \theta, \overline{s}_{-i}, t_i)) d\mu_i(\theta | t_i) > 0. \quad \text{(B.22)}
$$

For each of the two expressions forming (B.22), we find a lower bound. Consider the first expression. By definition of M_*,

$$
\int_{\mathbb{R}} \sum_{\gamma} (\Delta u_i(\gamma, \theta + \omega_1 - \psi_1^i(v)) - \Delta u_i(\gamma, \theta)) g_i'(\gamma | \theta + \omega_1 - \psi_1^i(v), \overline{s}_{-i}, t_i - v) d\mu_i(\theta | t_i) \geq
$$

$$
\int_{\mathbb{R}} M_*(\omega_1 - \psi_1^i(v), t_i) d\mu_i(\theta | t_i) \quad \text{(B.23)}
$$

Consider the second expression in (B.22). Note that

$$
\sum_{\gamma \geq \underline{\gamma}} \Delta u_i(\gamma, \theta)(g_i'(\gamma | \theta + \omega_1 - \psi_1^i(v), \overline{s}_{-i}, t_i - v) - g_i(\gamma | \theta, \overline{s}_{-i}, t_i)) =
$$

$$
\sum_{\gamma \geq \underline{\gamma}} (G_i'(S(\gamma) | \theta + \omega_1 - \psi_1^i(v), \overline{s}_{-i}, t_i - v) - G_i(S(\gamma) | \theta, s_{-i}, t_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(S(\gamma), \theta))
$$

$$
\text{(B.24)}
$$

Let $G_i^*(\tau_i)$ be the cdf of distribution $G_i(\tau_i) \vee G_i'(\tau_i - o(v))$. Let $G_{*,i}(\tau_i)$ be the cdf of $G_i(\tau_i) \wedge G_i'(\tau_i - o(v))$. Use the same notation for the probability mass functions. Define

$$
C_*(\theta) = \min_{(\gamma, n, m)} \frac{\Delta u_i(a_n, a_m, S(\gamma), \theta) - \Delta u_i(a_n, a_m, \gamma, \theta)}{S(\gamma) - \gamma} \quad \text{(B.25)}
$$

to be the minimal amount of strategic complementarities at state θ (recall $n > m$). Suppose first that $G_i'(\tau_i - o(v)) \geq_{st} G_i(\tau_i)$ for all τ_i. Then $w_2^i(v) \leq \Gamma_i^e[G_i'(\tau_i - o(v))] - \Gamma_i^e[G_i(\tau_i)]$ for

all τ_i. Since Δu_i is increasing in γ, the definition optimism gives

$$\sum_{\gamma \geq \underline{\gamma}} (G'_i(S(\gamma)|\tau_i - o(v)) - G_i(S(\gamma)|\tau_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(S(\gamma), \theta))$$

$$= \sum_{\gamma \geq \underline{\gamma}} (G'_i(S(\gamma)|\tau_i - o(v)) - G_{*,i}(S(\gamma)|\tau_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(S(\gamma), \theta))$$

$$\geq \sum_{\gamma \geq \underline{\gamma}} (G'_i(S(\gamma)|\tau_i - o(v)) - G_{*,i}(S(\gamma)|\tau_i))(\gamma - S(\gamma))C_*(\theta)$$

$$= C_*(\theta) \sum_{\gamma} \gamma(g'_i(\gamma|\tau_i - o(v)) - g_{*,i}(\gamma|\tau_i))$$

$$\geq C_*(\theta)w_2^i(v)$$

Suppose now that $G'_i(\tau_i - o(v)) \not\geq_{st} G_i(\tau_i)$ for some τ_i. Then $w_2^i(v) \leq \Gamma_i^e[G'_i(\tau_i - o(v))] - \Gamma_i^e[G'_i(\tau_i - o(v)) \vee G_i(\tau_i)]$ for all τ_i. For all τ_i,

$$\sum_{\gamma \geq \underline{\gamma}} (G'_i(S(\gamma)|\tau_i - o(v)) - G_i(S(\gamma)|\tau_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(S(\gamma), \theta))$$

$$\geq \sum_{\gamma \geq \underline{\gamma}} (G'_i(S(\gamma)|\tau_i - o(v)) - G_i^*(S(\gamma)|\tau_i))(\Delta u_i(\gamma, \theta) - \Delta u_i(S(\gamma), \theta))$$

$$\geq \sum_{\gamma \geq \underline{\gamma}} (G'_i(S(\gamma)|\tau_i - o(v)) - G_i^*(S(\gamma)|\tau_i))(\gamma - S(\gamma))C^*(\theta)$$

$$= C^*(\theta) \sum_{\gamma} \gamma(g'_i(\gamma|\tau_i - o(v)) - g_i^*(\gamma|\tau_i))$$

$$\geq C^*(\theta)w_2^i(v)$$

Putting this together with (B.23), if (B.16) holds, then

$$\int_{\mathbb{R}} M_*(\theta, \omega_1 - \psi_1^i(v))d\mu_i(\theta|t_i) - \min\left\{ \int_{\mathbb{R}} \omega_2^i(v)C_*(\theta)d\mu_i(\theta|t_i), \int_{\mathbb{R}} \omega_2(v)C^*(\theta)d\mu_i(\theta|t_i) \right\} > 0 \tag{B.26}$$

implies (B.19). Let $M_*(\omega_1 - \psi_1^i(v), t_i) = E_{\theta|t_i}[M_*(\theta, \omega_1 - \psi_1^i(v))]$, $C^*(t_i) = E_{\theta|t_i}[C^*(\theta)]$ and $C_*(t_i) = E_{\theta|t_i}[C_*(\theta)]$. Let us summarize. Assume that beliefs become more optimistic from μ to μ'. Since the real cutoffs associated with some extremal equilibrium under μ satisfy (B.18) (see e.g. (B.16)), the transition to μ' must lead each of them to increase by at least

$\delta(\mu, \mu', \boldsymbol{u})$, for otherwise a smaller increase would imply (B.19), a contradiction of optimality. Formally, $\delta(\mu, \mu', \boldsymbol{u})$ gives the infimum value of v such that (B.26) is satisfied for all pair of actions, strategies of players $-i$, and player i. This means that $c_{n,m} - c'_{n,m} \geq \delta(\mu, \mu', \boldsymbol{u})$ for all n and m. Proposition 5 completes the proof. $\qquad\qquad\square$

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