Semi-Infinite Assignment Problems and Related Games

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Abstract

In 1972 Shapley and Shubik introduced assignment games associated to finite assignment problems in which two types of agents were involved and they proved that these games have a non-empty core. In this paper we look at the situation where the set of one type is infinite and investigate when the core of the associated game is non-empty. Two infinite programming problems arise here, which we tackle with the aid of finite approximations. We prove that there is no duality gap and we show that the core of the corresponding game is non-empty. Finally, the existence of optimal assignments is discussed.

Keywords: Infinite programs, assignment, cooperative games, balancedness.

1 Introduction

Nowadays many markets and transactions are bilateral, so *'two-sided'* market models have become widely used in economic theory.

Since 1972, when Shapley and Shubik ([9]) introduced finite assignment games, much work related to these games has been developed. We point out the book of Roth and Sotomayor ([7]) as an important monograph on two-sided matching. Curiel ([1]) provides a thorough analysis of assignment games. In their work, Shapley and Shubik proved that the core of an assignment game is the non-empty set of solutions of the dual problem corresponding to the assignment problem. In ([8]), Sasaki gives axiomatic characterizations of the core of assignment games. Some generalizations and extensions of these models are presented in Kaneko and Wooders ([5],[6]).

In this paper, we look at semi-infinite assignment problems where the number of one of the two types of agents involved is finite and the other is countable infinite and we prove that semi-infinite bounded assignment games are balanced. Recently, Fragnelli et al. ([2]) and Timmer et al. ([11]) have studied some kinds of semi-infinite balanced games arising from

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different linear programming situations, where one of the factors involved in the problem is countable infinite but the number of players is finite. However, here we tackle semi-infinite assignment games with the aid of some tools that are related to Tijs ([10]).

This paper consists of four sections. In the next section we present the most relevant definitions and results for the assignment problem with two finite sets of agents. We extend these problems in section 3 to semi-infinite bounded assignment problems where one of the sets of agents is countable infinite and the set of values of matched pairs of agents is upper bounded. We show that the corresponding primal and dual program have no duality gap and that there exist optimal solutions to the dual program, which is equivalent to the non-emptiness of the core of the corresponding game. Finally, in section 4 we introduce the critical number and the existence of optimal assignments is discussed.

2 Finite Assignment Problems

An assignment problem describes a situation in which there are two types of agents, for example, sellers and buyers or firms and workers. Denote by M and W respectively these two finite and disjoint sets of agents. Let m be the number of agents in M, i.e., $m = |M|$, and $n = |W|$. Assume without loss of generality that $m \leq n$. When agent $i \in M$ is matched to agent $j \in W$ then this gives the couple a value of $a_{ij} \geq 0$. An assignment problem is thus described by the triple (M, W, A) with $A = [a_{ij}]_{i \in M, j \in W}$.

The maximal total value of paired agents, where each agent $i \in M$ is coupled to at most one agent $j \in W$ and vice versa, can be determined by the following linear program.

$$\begin{aligned} \max \quad & \sum_{i \in M} \sum_{j \in W} a_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i \in M} x_{ij} \leq 1, \text{ for all } j \in W \\ & \sum_{j \in W} x_{ij} \leq 1, \text{ for all } i \in M \\ & x_{ij} \in \{0, 1\}, \text{ for all } i \in M,\ j \in W. \end{aligned} \tag{1}$$

The assignment matrix $X \in \{0,1\}^{M \times W}$, $X = [x_{ij}]_{i \in M, j \in W}$, corresponds to the situation in which the agents $i \in M$ and $j \in W$ are matched if and only if $x_{ij} = 1$.

We will distinguish between two types of assignments or matchings. An M-assignment is an injective function $\pi : M' \to W$, where $M' \subset M$, and a W-assignment is an injective function $\sigma : W' \to M$ where $W' \subset W$. A complete M-assignment is an M-assignment $\pi : M \to W$, thus $M' = M$, which is only possible if $m \leq n$. To an assignment matrix X there corresponds the M-assignment $\pi_x : M_x \to W$ and the W-assignment $\sigma_x : W_x \to M$ where $M_x = \left\{ i \in M \mid \sum_{j \in W} x_{ij} = 1 \right\}$, $W_x = \{ j \in W \mid \sum_{i \in M} x_{ij} = 1 \}$ and $\pi_x(i) = j$ if $x_{ij} = 1$, for all $i \in M_x$, and $\sigma_x(j) = i$ if $x_{ij} = 1$, for all $j \in W_x$. Conversely, corresponding to an M-assignment $\pi : M' \to W$ is the assignment matrix X with $x_{ij} = 1$ if $i \in M'$ and $j = \pi(i)$, otherwise $x_{ij} = 0$.

Given an assignment problem (M, W, A), the corresponding assignment game (N, w) is a game with player set $N = M \cup W$. Let $S \subset N$ be a coalition of players. Then the worth $w(S)$ is defined to be the maximal value this coalition can obtain by matching its members. Define $M_S = S \cap M$ and $W_S = S \cap W$. If $M_S = \emptyset$ or $W_S = \emptyset$ then $w(S) = 0$ since no matchings can be made. Otherwise, if $M_S \neq \emptyset$ and $W_S \neq \emptyset$ then $w(S) = \text{val}(M_S, W_S, A)$

where

$$\mathrm{val}(M', W', A) = \max \left\{ \sum_{(i,j)\in M'\times W'} a_{ij}x_{ij} \,\middle|\, \begin{array}{l} X' = [x_{ij}]_{i\in M', j\in W'} \text{ is an} \\ M' \times W' - \text{assignment matrix} \end{array} \right\}$$

for all $M' \subset M$, $W' \subset W$. Since we assumed that $m \leq n$, it holds that

$$\mathrm{val}(M, W, A) = \max \left\{ \sum_{i\in M} a_{i\pi(i)} \,\middle|\, \begin{array}{l} \pi \text{ is a complete} \\ M-\text{assignment} \end{array} \right\}.$$

An optimal matching is a complete M-assignment such that $\sum_{i\in M} a_{i\pi(i)} \geq \sum_{i\in M} a_{i\pi'(i)}$ for all complete M-assignments π'. Let $O_p(A)$ be the set of these optimal matchings.

The vector (u, v), $u \in \mathbb{R}^M_+$ and $v \in \mathbb{R}^W_+$, is called a *feasible payoff* for the assignment problem (M, W, A) if there is a complete M-assignment π such that $\sum_{i\in M} u_i + \sum_{j\in W} v_j = \sum_{i\in M} a_{i\pi(i)}$. In this case, we say $((u, v), \pi)$ is a *feasible outcome* and it is *stable* if (u, v) is an element of the core $C(w)$ of the corresponding assignment game, where

$$C(w) = \left\{ (u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^W_+ \,\middle|\, \begin{array}{l} \sum\limits_{i\in M_S} u_i + \sum\limits_{j\in W_S} v_j \geq w(S) \ \forall S \subset N \\ \text{and} \sum\limits_{i\in M} u_i + \sum\limits_{j\in W} v_j = w(N) \end{array} \right\}.$$

If $(u, v) \in C(w)$ is proposed as payoff to the players, then each coalition $S \subset N$ gets at least as much as it can obtain on its own since $\sum_{i\in M_S} u_i + \sum_{j\in W_S} v_j \geq w(S)$. Thus no coalition has an incentive to break up with the grand coalition N. The following lemma by Roth and Sotomayor ([7]) tells something more about stable outcomes.

Lemma 2.1 (Roth and Sotomayor) *Let* $((u, v), \pi)$ *be a stable outcome for* (M, W, A). *Then*
(a) $u_i + v_j = a_{ij}$ *if* $\pi(i) = j$
(b) $u_i = 0$ *and* $v_j = 0$ *for all unassigned* i *and* j.

This result implies that at a stable outcome, the only utility transfers occur between agents in M and W who are matched to each other. It also shows that those players who remain unmatched in some optimal solution receive a zero payoff.

It is well known that if we replace the integer condition $x_{ij} \in \{0, 1\}$ in the linear program (1) by $x_{ij} \geq 0$ for all $i \in M$, $j \in W$, then all the optimal solutions will still have $x_{ij} \in \{0, 1\}$. Thus the dual problem (D) equals

$$\begin{array}{ll} \min & \sum\limits_{i\in M} u_i + \sum\limits_{j\in W} v_j \\ \text{s.t.} & u_i + v_j \geq a_{ij}, \text{ for all } i \in M,\ j \in W \\ & u_i, v_j \geq 0, \text{ for all } i \in M,\ j \in W. \end{array}$$

Because the primal problem has a solution, we know that also (D) must have a solution and the fundamental duality theorem asserts that these programs attain the same value. We denote by $O_d(A)$ and $R_d(A)$ the set of optimal dual solutions and the set of feasible dual solutions, respectively.

By definition of $w(S)$ it holds that if (u, v) is an optimal solution of the dual program then $\sum_{i\in M_S} u_i + \sum_{j\in W_S} v_j \geq w(S)$ for any coalition S, which ensures that this coalition cannot improve by splitting off from N when (u, v) is proposed as payoff. The following theorem says that these conditions are exactly the conditions that determine the core of an assignment game.

Theorem 2.2 (Shapley and Shubik) *Let (M, W, A) be an assignment problem. Then the core of the corresponding assignment game is the non-empty set of solutions of the dual LP for the grand coalition N, i.e., $C(w) = O_d(A)$.*

Moreover, if π is an optimal assignment then $((u, v), \pi)$ is a stable outcome for all core-elements (u, v). Vice versa, if $((u, v), \pi)$ is a stable outcome then π is an optimal assignment (see [7] for the proofs). So, we can concentrate on the payoffs to the agents rather than on the underlying assignment.

Example 2.3 Let $m = 2$, $n = 3$ and

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then the maximization problem of $N = M \cup W$ equals

$$\begin{array}{rl} \max & x_{11} + 2x_{12} + x_{21} + x_{23} \\ \text{s.t.} & \sum_{i \in M} x_{ij} \leq 1, \text{ for all } j \in W \\ & \sum_{j \in W} x_{ij} \leq 1, \text{ for all } i \in M \\ & x_{ij} \in \{0, 1\}, \text{ for all } i \in M, \ j \in W. \end{array}$$

One of the optimal solutions is: $x_{12} = x_{21} = 1$ and $x_{ij} = 0$ otherwise. Thus the third agent of W is not matched. The corresponding optimal assignment $\pi : M \to W$ is: $\pi(1) = 2$ and $\pi(2) = 1$ and the value of this program equals $v_p(A) = w(N) = 3$.

The dual problem reads

$$\begin{array}{rl} \min & u_1 + u_2 + v_1 + v_2 + v_3 \\ \text{s.t.} & u_i + v_j \geq a_{ij}, \text{ for all } i \in M, \ j \in W \\ & u_i, v_j \geq 0, \text{ for all } i \in M, \ j \in W. \end{array}$$

One of the dual solutions is: $u_1 = u_2 = 1$, $v_1 = 0$, $v_2 = 1$ and $v_3 = 0$. It is easy to check that $(1, 1, 0, 1, 0)$ is a core-element of the corresponding 5-person assignment game. Note that since agent $3 \in W$ is not matched, he should receive $v_3 = 0$.

Let (M, W, A) be an assignment problem and let $j \in W$. By $B_i(j, A)$ we denote the set of agents in $W \setminus \{j\}$ who are at least as good as j for agent $i \in M$, so,

$$B_i(j, A) = \{k \in W | \ k \neq j, \ a_{ik} \geq a_{ij}\}.$$

The following proposition tells us that an agent $j \in W$ gets zero in each core-element if for each $i \in M$ there are at least m (weakly) better agents in W than j.

Proposition 2.4 *Let (M, W, A) be an assignment problem and let $j \in W$. If $|B_i(j, A)| \geq m$ for all $i \in M$ then $v_j = 0$ for all $(u, v) \in O_d(A)$.*

Proof. Take an optimal assignment $\pi \in O_p(A)$. If $j \notin \{\pi(i) | i \in M\}$, then $v_j = 0$ by lemma 2.1.

If $j = \pi(i^*)$ for some $i^* \in M$ then there is a $k \in W \backslash \{j\}$ such that $k \in B_{i^*}(j, A) \setminus \{\pi(i) | i \in M \backslash \{i^*\}\}$ because $|B_{i^*}(j, A)| \geq m$ and $|\{\pi(i) | i \in M \backslash \{i^*\}\}| = m - 1$. But k is not matched, implying $v_k = 0$ by lemma 2.1. Since $k \in B_{i^*}(j, A)$ we have that $u_{i^*} = u_{i^*} + v_k \geq a_{i^*k} \geq a_{i^*j} = u_{i^*} + v_j$ where the last equality follows from $\pi(i^*) = j$. Thus $v_j \leq 0$ and since $v_j \geq 0$ by the dual program we conclude that $v_j = 0$.□

3 Semi-Infinite Bounded Assignment Problems

In this section we introduce semi-infinite bounded assignment problems (M, W, A), where $M = \{1, 2, ..., m\}$, a finite set, $W = \mathbb{N}$, the countable infinite set of natural numbers, and $0 \le a_{ij} \le b$ for some $b \in \mathbb{R}$, for all $i \in M$, $j \in W$. We analyze the corresponding semi-infinite bounded assignment games by *finite approximation matrices* $A_n \in \mathbb{R}^{m\times n}$ where $A_n = [a_{ij}]_{i\in M, j=1,2,...,n}$, and by means of the so-called *hard-choice number* of the matrix A, to be introduced later. Since $m < \infty$ we will talk, from now on, about assignments instead of (complete) M-assignments.

We start by defining two types of agents in M. An agent $i \in M$ is of *type 1* if this agent can choose one-by-one m best elements $j \in \mathbb{N}$ with respect to the largest reward a_{ij}. We denote by M_1 the set of agents of type 1. If $i \in M \setminus M_1$ then the agent is of *type 2* and M_2 denotes the set of all these agents.

The *choice set* C_i of an agent i of type 2 is the set of all his chosen best elements in W. Since this agent cannot choose m best elements (otherwise he is of type 1), we have $0 \le |C_i| < m$. The choice set C_i of an agent $i \in M_1$ consists of those m agents in W obtained in m steps by taking in each step that agent $j \in W$ not yet chosen by him and which gives him the maximal value a_{ij} over all non-chosen $j \in W$. In case there are more agents $j \in W$ that give the same maximal value a_{ij} then we choose that agent j with the smallest ranking number. The following example illustrates these concepts.

Example 3.1 Let $M = \{1, 2, 3\}$, $W = \mathbb{N}$ and

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \end{bmatrix}.$$

Agent $1 \in M$ attains his maximal value of 3 if he is assigned to agent $1 \in W$. The second largest value he can obtain is $a_{12} = 2$ and $a_{13} = 1$ is the third largest value he can get. This agent has no problems with choosing his three best agents from W and therefore he is of type 1. His choice set thus equals $C_1 = \{1, 2, 3\}$.

The largest value that agent $2 \in M$ can attain is $a_{22} = 1$. However, there is no second largest value because a_{2n} reaches the value 1 from below when n goes to infinity. This agent can only choose one best agent from W and therefore he is of type 2. His choice set equals $C_2 = \{2\}$.

Finally, agent $3 \in M$ has an easy job, since for all $j \in W$ he gets the value $a_{3j} = 1$. All agents in W are best elements for him. We will choose those three agents with the smallest ranking number, thus $C_3 = \{1, 2, 3\}$. This agent is of type 1. We conclude that $M_1 = \{1, 3\}$ and $M_2 = \{2\}$.

We will now introduce the hard-choice number.

Definition 3.2 *The* hard-choice number $n^*(A)$ *is the smallest number in* $\mathbb{N} \cup \{0\}$ *such that* $\bigcup_{i=1}^{m} C_i \subset \{1, 2, .., n^*(A)\}$.

Lemma 3.3 *Let* (M, W, A) *be a semi-infinite bounded assignment problem. If* $j > n^*(A)$, $j \in W$, *then there is an agent* $n(j) \ge j$, $n(j) \in W$, *such that* $\left|B_i(j, A_{n(j)})\right| \ge m$ *for each* $i \in M$.

Proof. Note that $j > n^*(A)$ implies that $j \notin C_i$ for all $i \in M$. If $i \in M_1$ then $B_i(j,A) \cap \{1,2,...,n^*(A)\} \supset C_i$ thus $|B_i(j,A) \cap \{1,2,...,n^*(A)\}| \geq |C_i| = m$ and we define $n_i(j) = j$. If $i \in M_2$ then $|C_i| < m$ and there are an infinite number of agents in $W \setminus \{1,2,...,n^*(A)\}$ strictly better than j. So, for n sufficiently large, say $n_i(j) \geq j$, there are (at least) m agents in $\{1,2,...,n_i(j)\}$ better than j. Take $n(j) = \max\{n_i(j)|\ i \in M\}$. Then $\left|B_i(j, A_{n(j)})\right| \geq m$ for all $i \in M$. □

Remark 3.4 *From lemma 3.3 and from proposition 2.4 it follows that for all $j > n^*(A)$ and for each $(u,v) \in O_d(A_n)$, $n \geq n(j)$, we have $v_j = 0$.*

The games corresponding to these semi-infinite bounded assignment problems are defined as follows. The player set $N = M \cup W$ consists of an infinite number of players. The value of coalition S, $w(S)$, equals 0 if $S \subset M$ or $S \subset W$ and

$$w(S) = \sup \left\{ \sum_{(i,j)\in M_S \times W_S} a_{ij}x_{ij} \;\middle|\; \begin{array}{l} X(S) = [x_{ij}]_{i\in M_S, j \in W_S} \text{ is an} \\ M_S \times W_S\text{−assignment matrix} \end{array} \right\},$$

otherwise. Just as in the previous section, the value $w(N) = v_p(A)$ of the grand coalition N can be determined by the linear program (1), replacing the maximum by the supremum since the set W is countable infinite. The following problem is the dual when we replace the integer condition by nonnegativity in the primal problem.

$$\begin{array}{rl} v_d(A) = \inf & \sum_{i\in M} u_i + \sum_{j \in W} v_j \\ \text{s.t.} & u_i + v_j \geq a_{ij}, \text{ for all } i \in M,\ j \in W \\ & u_i, v_j \geq 0, \text{ for all } i \in M,\ j \in W. \end{array}$$

Notice that both the primal and the dual program have an infinite number of variables and an infinite number of restrictions. In general, $\infty \times \infty$-programs show a gap between the optimal primal and dual value. There is a large literature on the existence or absence of so-called duality gaps in (semi-)infinite programs. See e.g. the books by Glashoff and Gustafson ([3]) and Goberna and López ([4]). Our goal is to prove that here the primal and the dual problem have the same value and that there exist optimal solutions of the dual problem. We achieve this result in some steps starting with a limit process in the finite space $\mathbb{R}^m \times \mathbb{R}^{n^*}$, where for the sake of brevity we will write n^* instead of $n^*(A)$ in a subscript or a superscript.

We take for each $n \in \mathbb{N}$ with $n > n^*(A)$, an element (u^n, v^n) of $O_d(A_n)$. Then we remove all coordinates of v^n with index larger than $n^*(A)$ and obtain $(u^n, s^{n^*}(v^n)) \in \mathbb{R}^m \times \mathbb{R}^{n^*}$, where $s^{n^*} : \mathbb{R}^n \to \mathbb{R}^{n^*}$ is the map $s^{n^*}(v_1^n, .., v_{n^*}^n, .., v_n^n) = (v_1^n, .., v_{n^*}^n)$, $\forall n > n^*(A)$. Note that $\{(u^n, s^{n^*}(v^n))|n \in \{n^*(A)+1, n^*(A)+2, ...\}\}$ is a bounded set in the finite dimensional space $\mathbb{R}^m \times \mathbb{R}^{n^*}$ since A is a bounded matrix and $(u^n, v^n) \in O_d(A_n)$. So,

$$u_i^n \leq \max\{a_{ij}|i \in M, j \in \{1,2,...,n\}\} \leq \sup\{a_{ij}|i \in M, j \in \mathbb{N}\}$$

and similarly we get $v_j^n \leq \sup\{a_{ij}|i \in M, j \in \mathbb{N}\}$.

Without loss of generality, we suppose that $\lim_{n\to\infty}(u^n, s^{n^*}(v^n))$ exists (otherwise take a subsequence) and we denote this limit by $(\overline{u}, \overline{v}) \in \mathbb{R}^m \times \mathbb{R}^{n^*}$. With the aid of $(\overline{u}, \overline{v})$ we construct the vector $(\widehat{u}, \widehat{v}) \in \mathbb{R}^m \times \mathbb{R}^\infty$ by taking $\widehat{u} = \overline{u}$ and $\widehat{v} = \alpha_{n^*}(\overline{v})$, where $\alpha_k : \mathbb{R}^k \to \mathbb{R}^\infty$

is the map defined by $\alpha_k(x) = (x_1, .., x_k, 0, 0..)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}^k$. So, $\widehat{v}$ is obtained from $\overline{v}$ by adding an infinite number of zeros. Later we will see that $(\widehat{u}, \widehat{v})$ is a core-element of the corresponding semi-infinite bounded assignment game but we start with showing that $(\widehat{u}, \widehat{v})$ is feasible in the dual problem.

Lemma 3.5 *Let (M, W, A) be a semi-infinite bounded assignment problem and let $(\widehat{u}, \widehat{v})$ be as defined above. Then $(\widehat{u}, \widehat{v}) \in R_d(A)$.*

Proof. By definition of $(\widehat{u}, \widehat{v})$ it holds that all its coordinates are non-negative. Furthermore, $\widehat{u}_i + \widehat{v}_j \geq a_{ij}$ for all $i \in M$, $j \in \{1, 2, ..., n^*(A)\}$ since $u_i^n + v_j^n \geq a_{ij}$ for all $i \in M$, $j \in \{1, 2, ..., n^*(A)\}$. For $i \in M$, $j > n^*(A)$, we know from remark 3.4 that $\lim\limits_{n\to\infty} v_j^n = 0$. Together with $u_i^n + v_j^n \geq a_{ij}$ for all $j \in \{1, 2, ..., n\}$ it follows by taking the limit for $n \to \infty$ that $\widehat{u}_i + \widehat{v}_j \geq a_{ij}$. So $(\widehat{u}, \widehat{v})$ is a feasible solution of the dual problem.□

The next three lemmas deal with the relations between the values of the finite subproblems and the infinite problems and with weak duality.

Lemma 3.6 $v_d(A) \leq \lim\limits_{n\to\infty} v_d(A_n)$

Proof. For $n > n^*(A)$ and $(u^n, v^n) \in O_d(A_n)$ we have $\sum_{i=1}^m u_i^n + \sum_{j=1}^n v_j^n = v_d(A_n)$. We construct $(\widehat{u}, \widehat{v})$ as we did before and so it follows that $\sum_{i=1}^m \widehat{u}_i + \sum_{j=1}^n \widehat{v}_j = \lim\limits_{n\to\infty} v_d(A_n)$. Then, from lemma 3.5 we obtain that $v_d(A) \leq \sum_{i=1}^m \widehat{u}_i + \sum_{j=1}^\infty \widehat{v}_j = \lim\limits_{n\to\infty} v_d(A_n)$.□

Lemma 3.7 $v_p(A) = \lim\limits_{n\to\infty} v_p(A_n)$

Proof. Clearly $v_p(A_n) \leq v_p(A)$ because each matching $\pi : M \to \{1, 2, ..., n\}$ in the finite problem is also feasible in the infinite problem. Furthermore, $\{v_p(A_n) \mid n > n^*(A)\}$ is an increasing sequence. So, $\lim\limits_{n\to\infty} v_p(A_n)$ exists and $\lim\limits_{n\to\infty} v_p(A_n) \leq v_p(A)$.

For the converse inequality, take $\varepsilon > 0$ and a matching $\pi^\varepsilon : M \to \mathbb{N}$ such that $\sum_{i=1}^m a_{i\pi^\varepsilon(i)} \geq v_p(A) - \varepsilon$. Let $k \in \mathbb{N}$ be such that $\{\pi^\varepsilon(i) \,|i \in M\} \subset \{1, 2, ..., k\}$. Then for all $n \geq k : v_p(A_n) \geq \sum_{i=1}^m a_{i\pi^\varepsilon(i)} \geq v_p(A) - \varepsilon$. This implies that $\lim\limits_{n\to\infty} v_p(A_n) \geq v_p(A)$.□

Lemma 3.8 *Weak duality, $v_p(A) \leq v_d(A)$, holds.*

Proof. Note that $R_d(A) \neq \emptyset$ because $(u', v') \in R_d(A)$, where $v' = 0$ and $u'_i = \sup_{j\in\mathbb{N}} a_{ij}$ for all $i \in M$. Take an assignment $\pi : M \to \mathbb{N}$ and a payoff vector $(u, v) \in R_d(A)$. Then

$$\sum_{i=1}^m a_{i\pi(i)} \leq \sum_{i=1}^m \left(u_i + v_{\pi(i)}\right) \leq \sum_{i=1}^m u_i + \sum_{j=1}^\infty v_j$$

and therefore

$$\begin{aligned} v_p(A) &= \sup\left\{\sum_{i=1}^m a_{i\pi(i)} \middle| \pi \text{ is an assignment}\right\} \\ &\leq \inf\left\{\sum_{i=1}^m u_i + \sum_{j=1}^\infty v_j \middle| u_i + v_j \geq a_{ij},\ u_i,\ v_j \geq 0, \text{ for all } i \in M,\ j \in W\right\} \\ &= v_d(A).\square \end{aligned}$$

Now we formulate the main result in this section, which tells us that there is no duality gap and that the set of optimal dual solutions is non-empty.

Theorem 3.9 *Let (M, W, A) be a semi-infinite bounded assignment problem. Then $v_p(A) = v_d(A)$ and $O_d(A) \neq \emptyset$.*

Proof. First, we prove that there is no duality gap using the fact that finite problems have no duality gap. From lemmas 3.6 and 3.7 follows,

$$v_d(A) \leq \lim_{n\to\infty} v_d(A_n) = \lim_{n\to\infty} v_p(A_n) = v_p(A).$$

Conversely, lemma 3.8 shows that $v_p(A) \leq v_d(A)$. So $v_p(A) = v_d(A) = \lim_{n\to\infty} v_d(A_n)$.

Second, we prove that $(\widehat{u}, \widehat{v}) \in O_d(A)$. It follows from the proof of lemma 3.6 and the first part of this proof that $\sum_{i=1}^{m} \widehat{u}_i + \sum_{j=1}^{\infty} \widehat{v}_j = \lim_{n\to\infty} v_d(A_n) = v_d(A)$. Furthermore, by lemma 3.5, $(\widehat{u}, \widehat{v}) \in R_d(A)$. So, $(\widehat{u}, \widehat{v}) \in O_d(A)$.□

Since $O_d(A)$ equals the core of the corresponding assignment game, it follows from this theorem that semi-infinite bounded assignment games have a non-empty core.

4 The Critical Number and Related Concepts

In this section, we present the *critical number* of a semi-infinite bounded assignment game. It turns out to be a key concept because, as we will show, it is related to the hard-choice number, introduced in section 3, and to the finite approximations.

Definition 4.1 *The* critical number $c(A)$ *equals* $\min\{n \in \mathbb{N} \mid v_p(A_n) = v_p(A)\}$, *if there exists an $n \in \mathbb{N}$ with $v_p(A_n) = v_p(A)$. Otherwise, $c(A) = \infty$.*

First, we present some results for finite critical numbers. The next proposition shows a relation between the hard-choice number and the critical number.

Proposition 4.2 *Let (M, W, A) be a semi-infinite bounded assignment problem. If $c(A) < \infty$ then $c(A) \leq n^*(A)$.*

Proof. Let $\pi \in O_p(A)$. If $\pi(i) \notin C_i$, for $i \in M_1$, then $C_i \setminus \{\pi(i^*) \,| i^* \in M \setminus \{i\}\} \neq \emptyset$ since $|C_i| = m$ and $|\{\pi(i^*) \,| i^* \in M \setminus \{i\}\}| = m - 1$. Thus there is a $j \in C_i$ such that $j \neq \pi(i)$ for all $i \in M_1$. If we redefine $\pi(i) = j$ then the matching remains optimal and agent i restricts his choice to C_i.

For $i \in M_2$ there is no optimal matching π with $\pi(i) \notin C_i$. This follows immediately from the definition of C_i. We conclude that $\pi \in O_p(A)$ but also $\pi \in O_p(A_{n^*})$. Thus $c(A) \leq n^*(A)$.□

As the next example shows, an optimal assignment can use agents $j \in W$ for which $j > n^*(A)$.

Example 4.3 Let $M = \{1, 2, 3\}$, $W = \mathbb{N}$, and

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 1 & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \end{bmatrix}.$$

We have seen in example 3.1 that $C_1 = \{1, 2, 3\}$, $C_2 = \{2\}$, $C_3 = \{1, 2, 3\}$, $M_1 = \{1, 3\}$ and $M_2 = \{2\}$. Also, $n^*(A) = 3$, $v_p(A) = 5$ and each π_k, with $k \geq 3$, defined by $\pi_k(1) = 1$, $\pi_k(2) = 2$, $\pi_k(3) = k$, is optimal. For $k > 3$ we have optimal matchings with $\pi_k(3) \notin C_3$, but the assignment π_3 is optimal and uses only elements in A_{n^*}. So, $c(A) = n^*(A) = 3$.

The next example shows that we may have $c(A) < n^*(A)$.

Example 4.4 Let $M = \{1, 2\}$, $W = \mathbb{N}$ and

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \end{bmatrix}.$$

Then $C_1 = \{1, 2\}$, $C_2 = \{1, 3\}$, $n^*(A) = 3$ and $v_p(A) = 3$. An optimal assignment is $\pi(1) = 2$ and $\pi(2) = 1$ and so $c(A) = 2 < 3 = n^*(A)$.

In the next theorem we characterize the structure of the sets of optimal primal and dual solutions when the critical number is finite.

Theorem 4.5 *Let (M, W, A) be a semi-infinite bounded assignment problem. If $c(A) < \infty$ then*

(i) $O_p(A) = \bigcup_{n \geq c(A)} O_p(A_n)$

(ii) $O_d(A) = \bigcap_{n \geq n^*(A)} \alpha_n(O_d(A_n))$.

Proof. *(i)* First, we prove that $O_p(A) \supset \cup_{n \geq c(A)} O_p(A_n)$. If $n \geq c(A)$ and $\pi \in O_p(A_n)$, then $\sum_{i=1}^m a_{i\pi(i)} = v_p(A_n) = v_p(A)$. So, $\pi \in O_p(A)$.

Next, we prove that $O_p(A) \subset \cup_{n \geq c(A)} O_p(A_n)$. Let $\pi \in O_p(A)$. Take $n \geq c(A)$ such that $\{\pi(1), ..., \pi(m)\} \subset \{1, .., n\}$. Then, $\pi \in R_p(A_n)$ and $\sum_{i=1}^m a_{i\pi(i)} = v_p(A) = v_p(A_n)$. So, $\pi \in O_p(A_n)$.

(ii) Suppose $(u, v) \in O_d(A)$. Then, it follows from remark 3.4 that $v_j = 0$ for $j > n^*(A)$. So, for $n > n^*(A)$ we have $(u, s^n(v)) \in O_d(A_n)$ and $(u, v) \in \alpha_n(O_d(A_n))$. Conversely, take an element in $\alpha_n(O_d(A_n))$ for all $n \geq n^*(A)$. Then it is of the form $(u, \alpha_n(v))$ where $(\alpha_n(v))_j = 0$ for all $j > n^*(A)$. For $n = n^*(A)$ there exists an optimal assignment π. This π is also optimal in A because $v_p(A) = v_p(A_n)$. On the other hand, $\sum_{i=1}^m a_{i\pi(i)} = v_p(A) = \sum_{i=1}^m u_i + \sum_{j=1}^\infty (\alpha_n(v))_j$. So $(u, \alpha_n(v)) \in O_d(A)$. $\square$

In case $c(A) = \infty$, we construct an auxiliary matrix H corresponding to the matrix A. This $m \times (n^*(A) + |M_2|)$-matrix H is defined by $H = [A_{n^*} \; T]$ where for each $i \in M_2$ we have a column $t_i e^i$ in T with $t_i = \sup\{a_{ij} | \; j \in \mathbb{N} \setminus C_i\}$ and $e^i_k = 1$ if $k = i$ and $e^i_k = 0$ otherwise. We will show that there are no optimal assignments if $c(A) = \infty$, but $v_p(A)$ and ε-optimal assignments can be obtained with the corresponding auxiliary matrix H. We illustrate these facts in the next example.

Example 4.6 Let $M = \{1,2,3\}$, $W = \mathbb{N}$ and

$$A = \begin{bmatrix} 3 & 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{5}{6} & \dots \\ 0 & 2 & 1\frac{2}{3} & 1\frac{3}{4} & 1\frac{4}{5} & 1\frac{5}{6} & \dots \end{bmatrix}.$$

Then $C_1 = \{1,2,3\}$, $C_2 = \{1\}$, $C_3 = \{2\}$, $M_1 = \{1\}$, $M_2 = \{2,3\}$ and $n^*(A) = 3$. The feasible matching π with $\pi(1) = 3,\ \pi(2) = 1, \pi(3) = 2$ has the property $\pi(i) \in C_i$ for each $i \in M$. But this assignment is not optimal since $\sum_{i=1}^{m} a_{i\pi(i)} = 4 < 6 = v(A)$. In this example we have that $c(A) = \infty$, so, no optimal assignment exists. But, we can use the auxiliary matrix H,

$$H = \begin{bmatrix} \mathbf{3} & 2 & 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{2}{3} & \mathbf{1} & 0 \\ 0 & \mathbf{2} & 1\frac{2}{3} & 0 & 2 \end{bmatrix},$$

where $v_p(H) = 6$ and now the matching π', with $\pi'(1) = 1,\ \pi'(2) = n\ \ (n \geq 3)$, $\pi'(3) = 2$, is an $\frac{1}{n}$−optimal assignment in A.

Theorem 4.7 *Let (M, W, A) be a semi-infinite bounded assignment problem with $c(A) = \infty$ and let H be the corresponding auxiliary matrix. Then*

(i) $O_p(A) = \emptyset$*;*

(ii) $v_p(A) = v_p(H)$*;*

(iii) For each optimal $\pi \in O_p(H)$ and each $\varepsilon > 0$ there is a matching $\pi^\varepsilon \in O_p(A)$ such that $\pi^\varepsilon(i) = \pi(i)$ for all $i \in M_1$ and $\pi^\varepsilon(i) \in \{n^*(A)+1, n^*(A)+2, ...\}$ such that $a_{i\pi^\varepsilon(i)} \geq t_i - \varepsilon/m$, if $i \in M_2$.

Proof. *(i)* For all assignments $\pi : M \to \mathbb{N}$ it holds for n large enough that $\{\pi(i)|\ i \in M\} \subset \{1, 2, \dots, n\}$ and thus is π a matching for the assignment problem $(M, \{1, 2, \dots, n\}, A_n)$. Together with $c(A) = \infty$ this gives

$$\sum_{i=1}^{m} a_{i\pi(i)} \leq v_p(A_n) < v_p(A).$$

Hence, $O_p(A) = \emptyset$.

To prove *(ii)* and *(iii)* it is sufficient to show that

1. $v_p(H) \geq v_p(A)$. Let $\pi \in R_p(A)$. Construct $\pi^* \in R_p(H)$ as follows. Let $i \in M$. If $\pi(i) \in C_i$ then $\pi^*(i) = \pi(i)$. If $\pi(i) \notin C_i$ and $i \in M_1$ then we can choose a partner $\pi^*(i) = j^* \in C_i$ because C_i is large enough. (See the proof of proposition 4.2.) If $\pi(i) \notin C_i$ and $i \in M_2$ then define $\pi^*(i) = j^*$, where j^* corresponds to column $t_i e^i$ in T. Thus for all $i \in M$ we have $h_{i\pi^*(i)} \geq a_{i\pi(i)}$, so, $v_p(H) \geq v_p(A)$.

2. $v_p(A) \geq v_p(H) - \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon > 0$ and $\pi \in R_p(H)$. We will construct a matching $\pi^\varepsilon \in R_p(A)$ as follows. Take one-by-one elements $i \in M$. Note that $\pi(i) \notin \{1, 2, \dots, n^*(A)\} \setminus C_i$ since otherwise player i can improve by choosing t_i. If $\pi(i) \in C_i$ then define $\pi^\varepsilon(i) = \pi(i)$. If $\pi(i) \in T$ then take $j^* > n^*(A)$ such that

$a_{ij^*} \geq t_i - \varepsilon/m$ and $j^* \neq \pi(i')$ for all $i' \neq i$ and define $\pi^\varepsilon(i) = j^*$. This can be done such that all $i \in M$ are matched to m different elements in W. Then

$$\begin{aligned}\sum_{i\in M} a_{i\pi^\varepsilon(i)} &= \sum_{i\in M:\pi^\varepsilon(i)\in C_i} a_{i\pi^\varepsilon(i)} + \sum_{i\in M:\pi^\varepsilon(i)\notin C_i} a_{i\pi^\varepsilon(i)} \\ &\geq \sum_{i\in M:\pi(i)\in C_i} h_{i\pi(i)} + \sum_{i\in M:\pi(i)\in T} (t_i - \varepsilon/m) \\ &\geq \sum_{i\in M} h_{i\pi(i)} - \varepsilon,\end{aligned}$$

where the last inequality holds because $|\{i \in M \mid \pi(i) \in T\}| \leq m$. Thus $v_p(A) \geq v_p(H) - \varepsilon$.$\square$

Given a semi-infinite bounded assignment problem (M, W, A) consider the sequence $(u^1, \alpha_1(v^1))$, $(u^2, \alpha_2(v^2))$, $(u^3, \alpha_3(v^3)), \ldots$, where $(u^n, v^n) \in O_d(A_n)$ for all $n \in \mathbb{N}$. Denote by $L(A)$ the set of points that can be obtained as a limit of a subsequence as above. Then we have the following result.

Theorem 4.8 *$L(A) \neq \emptyset$ and $L(A) \subset O_d(A)$.*

Proof. Analyzing the proofs in section 3 and the construction of $(\widehat{u}, \widehat{v})$, we conclude that $(\widehat{u}, \widehat{v}) \in L(A)$ since $\widehat{v}_j = 0$ for all $j > n^*(A)$. Hence it may be clear that $L(A) \subset O_d(A)$.$\square$

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