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SEQUENCING INTERVAL SITUATIONS AND RELATED GAMES

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Abstract

In this paper we consider one-machine sequencing situations with interval data. We present different possible scenarioes and extend classical results on well known rules and on sequencing games to the interval setting.

Keywords: cooperative games, interval data, sequencing situations, convex games

JEL Classification: C71

1 Introduction

Sequencing situations arise in several instances of real life. Here, we refer to the classical scheduling of a sequence of jobs and the waiting line in front of a counter. The use of an optimal ordering may reduce the cost connected with the time spent in the system and is particularly interesting in sequencing situations where several agents are involved. In such situations, the optimal order is good for the agents as a whole (because it increases the efficiency of the system), but since agents are basically interesting in their individual benefit, an agreement is equally important. The agreement includes how to compensate those agents that are required to spend more time in the system and how to share the joint cost savings. In the classical approach to the problem, the processing time of each job and the cost per unit of time associated with it are supposed to be known with certainty. It should be clear that the optimality of an ordering may be affected when the actual processing times and/or unitary costs are different from the forecasted ones. In this paper we simply require an estimation of intervals of values for the processing times and/or unitary costs, avoiding the difficulties of associating a reasonable probability distribution.

In this setting the optimal order may be difficult to reach, but the agents may accept to switch their position in the queue in change of an adequate compensation.

Depending on the agents' attitude towards risk, various possibilities could be considered to settle the agreement, both for improving the ordering (with more switches) and for sharing the joint cost savings.

To handle sequencing situations with interval data, the theory of cooperative interval games is helpful. In this paper, we use some notions and results (Alparslan Gök, Miquel and Tijs 2008; Alparslan Gök, Branzei and Tijs 2008a, b). The reader is referred to Branzei, Tijs and Alparslan Gök (2008a, b) for a brief survey on cooperative interval games and interval solution concepts, and for a guide for using interval solution when uncertainty on data is removed.

The paper is organized as follows. We recall in Section 2 basic notions and results from interval calculus, theory of cooperative interval games, one-machine sequencing situations and classical sequencing games. Section 3 is devoted to one-machine sequencing situation with interval uncertainty. In Section 4 we introduce the class of cooperative sequencing interval games and show that they are convex interval games. Furthermore, we extend the classical equal gain splitting rule to the interval setting, give an explicit formula to compute the interval equal gain splitting allocation for a sequencing situation with interval data, and prove that this allocation belongs to the interval core of the related sequencing interval game. Section 5 concludes.

2 Preliminaries and notations

2.1 Interval games

In this section some preliminaries from interval calculus and some useful results from the theory of cooperative interval games are given (Alparslan Gök, Branzei and Tijs 2008a, b).

We denote by $I(\mathbb{R})$ the set of all closed intervals in $\mathbb{R}$, by $I(\mathbb{R}_+)$ the set of all closed intervals in $\mathbb{R}_+$, and by $I(\mathbb{R}_+)^N$ the set of all n-dimensional vectors with components in $I(\mathbb{R}_+)$.

Let $a, b \in I(\mathbb{R})$ with $a = [\underline{a}, \overline{a}]$, $b = [\underline{b}, \overline{b}]$, $|a| = \overline{a} - \underline{a}$ and $\beta \in \mathbb{R}_+$. Then, $a + b = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$; $\beta a = [\beta\underline{a}, \beta\overline{a}]$. The subtraction operator $a - b$ is defined, only if $|a| \geq |b|$, by $a - b = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$. Let $a, b \in I(\mathbb{R}_+)$. Then, $a \cdot b = [\underline{a}\ \underline{b}, \overline{a}\ \overline{b}]$. The division operator $\frac{a}{b}$ is defined, only if $\underline{a}\overline{b} \leq \underline{b}\overline{a}$ and $\underline{b}, \overline{b} \neq 0$, by $\frac{a}{b} = [\frac{\underline{a}}{\underline{b}}, \frac{\overline{a}}{\overline{b}}]$. We say that a is weakly better than b, which we denote by $a \succcurlyeq b$, if and only if $\underline{a} \geq \underline{b}$ and $\overline{a} \geq \overline{b}$. We also use the reverse notation $b \preccurlyeq a$, if and only if $\underline{b} \leq \underline{a}$ and $\overline{b} \leq \overline{a}$.

A cooperative interval game in coalitional form (Alparslan Gök, Miquel and Tijs 2008) is an ordered pair $< N, w >$ where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \rightarrow I(\mathbb{R})$ is the characteristic function such that $w(\emptyset) = [0, 0]$. For each $S \in 2^N$, the worth $w(S)$ of the coalition S in the interval game $< N, w >$ is of the form $[\underline{w}(S), \overline{w}(S)]$, where $\underline{w}(S)$ is the lower bound and $\overline{w}(S)$ is the upper bound of $w(S)$. We denote by IG^N the family of all cooperative interval games with player set N. Some classical cooperative games associated with an interval game $w \in IG^N$ play a key role, namely the border games $< N, \underline{w} >$, $< N, \overline{w} >$ and the length game $< N, |w| >$, where $|w|(S) = \overline{w}(S) - \underline{w}(S)$ for each $S \in 2^N$. Note that $\overline{w} = \underline{w} + |w|$.

For $w_1, w_2 \in IG^N$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $< N, w_1 - w_2 >$ is defined by $(w_1 - w_2)(S) = w_1(S) - w_2(S)$.

Let $w \in IG^N$. The interval core $\mathcal{C}(w)$ is defined by

$$\mathcal{C}(w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N | \sum_{i \in N} I_i = w(N), \sum_{i \in S} I_i \succcurlyeq w(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

We call a game $< N, w >$ convex if $w(S) + w(T) \preccurlyeq w(S \cup T) + w(S \cap T)$ for all $S, T \in 2^N$ and $< N, |w| >$ is convex (in the classical sense, i.e. $|w|(S) + |w|(T) \leq |w|(S \cup T) + |w|(S \cap T)$).

We denote by CIG^N the class of convex interval games with player set N.

A game $< N, w >$ is size monotonic if $< N, |w| >$ is monotonic, i.e. $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. We denote by $SMIG^N$ the class of size monotonic interval games with player set N.

Let $\sigma : N \rightarrow N$ be a permutation of the set N. The interval marginal vector of $w \in SMIG^N$ with respect to σ, $m^\sigma(w)$, is the vector whose component i is defined by $m_i^\sigma(w) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i))$ for each $i \in N$, where $P_\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$ is the set of predecessors of i in σ.

2.2 Sequencing situations and related games

A one-machine sequencing situation arises when a set of ordered jobs has to be processed sequentially on a machine. The basic issue is to determine the optimal order of the jobs to be processed taking into account the individual processing times and the costs per unit of time. Formally, a sequencing situation is a 4-tuple (N, σ_0, α, p) where:

- $N = \{1, ..., n\}$ is the set of jobs;
- $\sigma_0 : N \to \{1, ..., n\}$ is a permutation that defines the initial order of the jobs;
- $\alpha = (\alpha_i)_{i\in N} \in \mathbb{R}^n_+$ is a non-negative real vector, where α_i is the cost per unit of time of job i;
- $p = (p_i)_{i\in N} \in \mathbb{R}^n_+$ is a positive real vector, where p_i is the processing time of job i.

Given a sequencing situation and an ordering σ of the jobs, we can associate to it the cost C_σ defined by the sum of the costs of the jobs, where the cost of job $i \in N$ is given by the product of its unitary cost α_i and the time that it spends in the system, i.e. its processing time p_i plus the waiting time for completing all the jobs preceding i in the queue. In formula $C_\sigma = \sum_{i\in N} \alpha_i \left(\sum_{j\in P(\sigma,i)} p_j + p_i\right)$, where $P(\sigma, i)$ is the set of jobs preceding i, according to the order σ.

The optimal order of the jobs σ^* produces the minimum cost $C_{\sigma^*} = \sum_{i\in N} \alpha_i \left(\sum_{j\in P(\sigma^*,i)} p_j + p_i\right)$ or the maximum cost saving $C_{\sigma_0} - C_{\sigma^*}$. Smith (1956) proved that an optimal order can be obtained reordering the jobs according to decreasing urgency indices, where the urgency index of job $i \in N$ is defined as $u_i = \dfrac{\alpha_i}{p_i}$ (clearly, if this condition holds for the initial order no reordering of jobs is necessary).

If the jobs belong to the same agent he will agree to reorder them optimally, according to Smith's result. The situation is completely different when each job belongs to a different agent. In this case, a reordering requires that at least the agents that change their position agree on the new order. So, we can say that a switch among two jobs is always possible if they are consecutive in the current order or if all the agents that own one of the jobs in between the two that are switched agree.

The following question arises: Is it possible to share this cost savings $C_{\sigma_0} - C_{\sigma^*}$ among the agents in such a way that the new order results to be stable? In other words we want to find fair shares of the overall cost savings to be given to the different agents, in such a way that all of them agree on the optimal order and have no incentive to recede from the agreement. This question finds its natural habitat in cooperative game theory.

In 1989 Curiel, Pederzoli and Tijs introduced the class of sequencing games. An updated survey on these games can be found in Curiel, Hamers and Klijn (2002). See also the survey on Operation Research Games (Borm, Hamers and Hendricks 2001). A sequencing game is a pair $< N, v >$ where N is the set of players, that coincides with the set of jobs, and the characteristic function v assigns to coalition S the maximal cost savings that the members of S can obtain by reordering only their jobs. We say that a set of jobs T is *connected according to an order* σ if for all $i, j \in T$ and $k \in N, \sigma(i) < \sigma(k) < \sigma(j)$ implies $k \in T$.

Switching two connected jobs i, j the cost associated to the ordering varies of $\alpha_j p_i - \alpha_i p_j$. The variation is positive if and only if the urgency indices verify $u_i < u_j$. Clearly, if $\alpha_j p_i - \alpha_i p_j$ is negative it is not beneficial for i and j to switch their positions. We denote the gain of the switch as

$$g_{ij} = (\alpha_j p_i - \alpha_i p_j)_+ = max\{0, \alpha_j p_i - \alpha_i p_j\}$$

and, consequently, the gain of a connected coalition T according to an order σ is defined by $v(T) = \sum\limits_{j \in T} \sum\limits_{i \in P(\sigma,j) \cap T} g_{ij}$.

If S is not a connected coalition, the order σ induces a partition in connected components, denoted by S/σ. In view of this, the characteristic function v of the sequencing game can be defined as $v(S) = \sum\limits_{T \in S/\sigma} v(T)$ for each $S \subset N$, or equivalently as $v = \sum\limits_{i,j \in N : i<j} g_{ij} u_{[i,j]}$, where $u_{[i,j]}$ is the unanimity game defined as:

$$u_{[i,j]}(S) = \begin{cases} 1 & \text{if } \{i, i+1, ..., j-1, j\} \subset S \\ 0 & \text{otherwise} \end{cases} .$$

Curiel, Pederzoli and Tijs (1989) show that sequencing games are convex games and, consequently, their core is nonempty. Moreover, it is possible to determine a core allocation without computing the characteristic function of the game. They propose to share equally between the players i, j the gain g_{ij} produced by the switch and call this rule the *Equal Gain Splitting EGS* rule. It can be computed by $EGS_i = \frac{1}{2} \sum_{k \in P(\sigma,i)} g_{ki} +$

$\frac{1}{2}\sum_{j:i\in P(\sigma,j)} g_{ij}$ for each $i \in N$. There exist two other simple allocation rules, denoted respectively by $\mathcal{P}$ and $\mathcal{S}$. According to the first rule the gain of each switch is assigned to the predecessor in the initial order, while the second rule assigns the gain to the successor. We can write $\mathcal{P}_i = \sum_{j:i\in P(\sigma,j)} g_{ij}$ and $\mathcal{S}_i = \sum_{j\in P(\sigma,i)} g_{ji}$ for each $i \in N$ and it is easy to see that $EGS = \frac{1}{2}(\mathcal{P}+\mathcal{S})$.

In a similar way, we can define the EGS^{ε} solution for each $\varepsilon \in [0,1]$ as $EGS^{\varepsilon} = \varepsilon\mathcal{P} + (1-\varepsilon)\mathcal{S}$. Clearly, for $\varepsilon = 0$ we get $\mathcal{S}$, for $\varepsilon = \frac{1}{2}$ we get EGS, and for $\varepsilon = 1$ we get $\mathcal{P}$.

3 Sequencing interval situations

In this section we drop the hypothesis of complete knowledge of the parameters of a sequencing situation, in order to better fit the real-world situations. In particular, we suppose that the processing time and/or the cost per unit of time of each job are represented by intervals. In fact each agent may have some difficulties in evaluating the actual duration of his/her job and the unitary cost. On the other hand, it is often possible to assign minimal and maximal values for both elements. We consider three scenarioes: in the first one the processing time of each job is a positive real number but its unitary cost is an interval of positive real values; in the second one the unitary costs are positive real numbers and the processing times are intervals of positive real values; in the last one both elements are intervals of positive real values.

3.1 The first scenario

A one-machine sequencing situation with interval-uncertain costs per unit of time can be described as a 4-tuple (N, σ_0, α, p), where N, σ_0 and p are the same as in the classical case and $\alpha = ([\underline{\alpha}_i, \overline{\alpha}_i])_{i\in N} \in I(\mathbb{R}_+)^N$ is the vector of intervals where $\underline{\alpha}_i$ is the minimal unitary cost and $\overline{\alpha}_i$ is the maximal unitary cost of job i.

In this situation, the arithmetic of intervals allows us to compute the urgency index of the jobs, $u_i = \frac{\alpha_i}{p_i} = \left[\frac{\underline{\alpha}_i}{p_i}, \frac{\overline{\alpha}_i}{p_i}\right], i \in N$.

To use Smith's result for finding the optimal order we need not only to compare u_i and u_j to check if $u_i \preccurlyeq u_j$ for any two possible candidates i and j to a neighbor switch, but also that these intervals are disjoint, i.e. $\overline{u}_i \leq \underline{u}_j$. This setting corresponds to the maximal risk aversion of the agents that agree on a switch of their job only if it is surely profitable.

EXAMPLE 3.1. *Consider the sequencing interval situation with* $N = \{1, 2\}$*,* $\sigma_0 = \{1, 2\}$*,* $p = (2, 3)$ *and* $\alpha = ([2, 4], [12, 21])$*. The urgency indices are* $u_1 = [1, 2]$ *and* $u_2 = [4, 7]$*, so the two jobs may be switched.*

Now, the question is how to share among the switching agents i and j the gain arising from their switch. We consider two possible approaches.

First, the agents i and j may agree on the dictatorial solution for agent i, i.e. the compensation corresponds to the upper bound $\overline{\alpha}_i p_j$; this means that agent i asks to be fully compensated referring to his maximal unitary cost, plus the possibility of an extra gain if the actual cost per unit of time is lower.

Second, the agents i and j could determine the individual compensation when the jobs are performed and realizations of the unitary costs are available. This leads to a classical sequencing situation and the agents may agree on one of the existing allocation rules, e.g. the EGS-rule.

EXAMPLE 3.2. *Referring to the situation in Example 3.1 the dictatorial approach assigns to agent 1 a compensation* $\overline{\alpha}_2 p_1 = 21 \times 2 = 42$ *and* 0 *to agent 2. The realization approach may be performed only when the two jobs are processed. Suppose that the realization of the unitary cost is 4 for agent 1 and 16 for agent 2. The EGS-rule for the resulting classical sequencing situation assigns to both agents a compensation of* 4.

3.2 The second scenario

We describe a one-machine sequencing situation with interval-uncertain processing time as a 4-tuple (N, σ_0, α, p), where N, σ_0 and α are as in the classical case and $p = ([\underline{p}_i, \overline{p}_i])_{i \in N} \in I(\mathbb{R}_+)^N$ is the vector of intervals where $\underline{p}_i$ is the minimal processing time and $\overline{p}_i$ is the maximal processing time of job i.

In this situation, the arithmetic of intervals does not allow us to compute the urgency index of a job, as we cannot divide a real number by an interval, so we introduce the notion of *relaxation index of job i* defined by $r_i = \alpha_i^{-1} p_i = \left[\frac{\underline{p}_i}{\alpha_i}, \frac{\overline{p}_i}{\alpha_i}\right]$ for all $i \in N$.

We notice that the relaxation index is the inverse of the urgency index in the classical case, so we may reformulate for this scenario the rule of Smith saying that to obtain an optimal order the jobs have to be ordered according to increasing relaxation indices. Two jobs $i, j \in N$ may be switched only if $r_i \succcurlyeq r_j$ and the intervals are disjoint, i.e. $\underline{r}_i \geq \overline{r}_j$.

We can consider the same sharing approaches of the first scenario, with suitable modification.

3.3 The third scenario

Here a one-machine sequencing interval situation is described as a 4-tuple (N, σ_0, α, p), where N and σ_0 are as usual, whereas $\alpha = ([\underline{\alpha}_i, \overline{\alpha}_i])_{i \in N} \in I(\mathbb{R}_+)^N$ and $p = ([\underline{p}_i, \overline{p}_i])_{i \in N} \in I(\mathbb{R}_+)^N$ are the vectors of intervals with $\underline{\alpha}_i$, $\overline{\alpha}_i$ representing the minimal and maximal unitary cost of job i, respectively, and $\underline{p}_i$, $\overline{p}_i$ representing the minimal and maximal processing time of job i, respectively.

To handle such sequencing situations we propose to use either the approach based on urgency indices or the approach based on relaxation indices. This requires to be able to compute either $u_i = \left[\frac{\underline{\alpha}_i}{\underline{p}_i}, \frac{\overline{\alpha}_i}{\overline{p}_i}\right]$ for all $i \in N$ or $r_i = \left[\frac{\underline{p}_i}{\underline{\alpha}_i}, \frac{\overline{p}_i}{\overline{\alpha}_i}\right]$ for all $i \in N$, i.e. for each such index the lower bound has to be less than or equal to the upper bound. Example 3.5 shows that this could be impossible. When all indices of a certain type can be calculated, they are useful to find an optimal order only in case they can be ordered properly and are also disjoint. Example 3.3 illustrates a successful use of the urgency indices, while Example 3.4 shows that although the relaxation indices can be computed and compared they are not useful to find an optimal order because they are not disjoint.

Example 3.3. *Consider the two-agent situation with* $p_1 = [1, 4], p_2 = [6, 8], \alpha_1 = [5, 25], \alpha_2 = [10, 30]$. *We can compute* $u_1 = \left[5, \frac{25}{4}\right], u_2 = \left[\frac{5}{3}, \frac{15}{4}\right]$ *and use them to reorder the jobs as the intervals are disjoint.*

Example 3.4. *Consider the two-agent situation with* $p_1 = [1, 3], p_2 = [4, 6], \alpha_1 = [5, 6], \alpha_2 = [11, 12]$. *Here, we can compute* $r_1 = \left[\frac{1}{5}, \frac{1}{2}\right], r_2 = \left[\frac{4}{11}, \frac{1}{2}\right]$, *but we cannot reorder the jobs as the intervals are not disjoint.*

EXAMPLE 3.5. *Consider the two-agent situation with* $p_1 = [1, 3], p_2 = [5, 8], \alpha_1 = [5, 6], \alpha_2 = [10, 30]$. *Now,* r_1 *is defined but* r_2 *is undefined; on the other hand* u_1 *is undefined and* u_2 *is defined, so no comparison is possible and, consequently, the reordering cannot take place.*

If two jobs may be switched, we can use the sharing approaches introduced above. In particular, we may have not a total order, as some pairs of jobs cannot be compared, but we may reach just a partial optimal order and share the associated gains.

REMARK 3.1. *Allowing degenerate intervals* $[a, a] \in I(\mathbb{R}_+)$ *leads to the possibility of unique game-theoretic treatment of all three scenarios of sequencing situations with interval data, based on the third scenario. In fact in the first scenario we may consider the vector of real numbers* $p = (p_i)_{i \in N}$ *as a vector of degenerate intervals* $p = ([p_i, p_i])_{i \in N}$. *Analogously, in the second scenario we may consider the vector of real numbers* $\alpha = (\alpha_i)_{i \in N}$ *as a vector of degenerate intervals* $\alpha = ([\alpha_i, \alpha_i])_{i \in N}$.

4 Cooperative interval games

In this section we introduce the class of cooperative sequencing interval games. In view of Remark 3.1 we refer to the general situation presented in the third scenario.

Let $i, j \in N$. We define the interval gain of the switch of jobs i and j by Let the *interval gain of a switch* be:

$$G_{ij} = \begin{cases} \alpha_j p_i - \alpha_i p_j & \text{if jobs } i \text{ and } j \text{ switch} \\ [0,0] & \text{otherwise} \end{cases}.$$

The sequencing interval game associated to a one-machine sequencing situation (N, σ_0, α, p) is defined by

$$w = \sum_{i,j \in N : i<j} G_{ij} u_{[i,j]}.$$

provided that $G_{ij} \in I(\mathbb{R})$ for all switching jobs $i, j \in N$.

REMARK 4.1. *The condition* $G_{ij} I(\mathbb{R})$ *is equivalent to* $\underline{G}_{ij} \leq \overline{G}_{ij}$. *Note that for the first two scenarioes this condition may be written as* $\frac{|\alpha_i|}{p_i} \leq \frac{|\alpha_j|}{p_j}$ *and* $\frac{|p_i|}{\alpha_i} \geq \frac{|p_j|}{\alpha_j}$, *respectively,* *and such conditions may be not satisfied. Consider the sequencing interval situation with $N = \{1,2\}$, $\sigma_0 = \{1,2\}$, $p = ([2,2],[3,3])$ and $\alpha = ([2,4],[12,13])$. The urgency indices are $u_1 = [1,2]$ and $u_2 = \left[4, \frac{13}{3}\right]$, so the switch is profitable, as u_2 is larger than $u_1 = [1,2]$; moreover the intervals are disjoint but $\dfrac{|\alpha_1|}{p_1} = 1 \geq \dfrac{|\alpha_2|}{p_2} = \dfrac{1}{3}$, implying that $G_{12} = [18,14]$ that is not an interval.*

In the following we show that each sequencing interval game is convex.

Proposition 4.1. *Let $< N, w >$ be a sequencing interval game. Then, $< N, w >$ is convex.*

Proof. By definition $G_{ij} \succcurlyeq [0,0]$. So, $\underline{G}_{ij} \geq 0$ and $|G_{ij}| \geq 0$ for all (i,j). It is well know that classical unanimity games are convex. Then, $\underline{w} = \sum_{i,j \in N: i<j} \underline{G}_{ij} u_{[i,j]}$ and $|w| = \sum_{i,j \in N: i<j} |G_{ij}| \, u_{[i,j]}$ are convex games, in the classical sense. So, $w = \sum_{i,j \in N: i<j} G_{ij} u_{[i,j]}$ is convex (see Proposition 3.2 (iii) Alparslan Gök, Branzei and Tijs 2008b). □

The interval equal gain splitting rule is defined by $IEGS_i = \frac{1}{2} \sum_{i,j \in N: i<j} G_{ij} + \frac{1}{2} \sum_{i,j \in N: i>j} G_{ij}$ for each $i \in N$.

Proposition 4.2. *Let $< N, w >$ be a sequencing interval game. Then,*
i) $IEGS(w) = \frac{1}{2}(m^{(1,2\ldots,n)}(w) + m^{(n,n-1,\ldots,1)}(w))$.
ii) $IEGS(w) \in \mathcal{C}(w)$.

Proof.

i) If $\sigma = (1,2,\ldots,n)$, then

$$m^{(1,2\ldots,n)}(w) = ([0,0], G_{12}, G_{13} + G_{23}, G_{14} + G_{24} + G_{34}, \ldots, G_{1n} + \ldots + G_{n-1,n}).$$

If $\sigma = (n, n-1, \ldots, 1)$, then

$$m^{(n,n-1,\ldots,1)}(w) = (G_{12} + \ldots + G_{1,n}, \ldots, G_{n-1,n}, [0,0]).$$

ii) It is proved (Alparslan Gök, Branzei and Tijs 2008b) that the interval marginal vectors are interval core elements for convex interval games. The proof follows immediately as the sequencing interval games are convex by Proposition 4.1 and the interval core is a convex set (see Proposition 3.3 Alparslan Gök, Branzei and Tijs 2008a).

□

EXAMPLE 4.1. *Referring to the situation in Example 3.1, the interval gain is* $G_{12} = [18, 30]$*, the sequencing interval game* $< N, w >$ *is* $w(1) = w(2) = [0, 0]$*,* $w(1, 2) = [18, 30]$ *and* $IEGS(w) = ([9, 15], [9, 15])$*.*

5 Concluding remarks

In this paper we introduced and studied sequencing situations with interval data and introduced the related class of interval games. Our approach to find an optimal order was to try to use either urgency indices u_i for all $i \in N$ or relaxation indices r_i for all $i \in N$.

However, as we already saw, for some sequencing interval situations we may have difficulties in ordering the jobs using only the urgency indices or the relaxation indices. In such situations, we can (partially) reorder the jobs using a mixed approach: We can consider actually adjacent pairs of jobs i and j for which both u_i and u_j or both r_i and r_j are defined and decide if they may be switched, i.e. if all the required conditions are satisfied. Consider the sequencing interval situation with $N = \{1, 2, 3, 4\}$, $\sigma_0 = \{1, 2, 3, 4\}$, $p = ([1, 6], [8, 15], [2, 3], [2, 7])$ and $\alpha = ([1, 3], [2, 3], [6, 12], [6, 8])$. We may compute $u_1 = [1, 2]$, $u_2 = [4, 5]$, $r_3 = [3, 4]$ and $r_4 = \left[\frac{1}{3}, \frac{1}{2}\right]$, while the other indices are undefined. We can observe that jobs 1 and 2 and jobs 3 and 4 may be switched, but we can say nothing about jobs 1 and 4, that become adjacent after the first two switches, as we have no common index. But we can go further in our analysis. In fact it is easy to realize that the urgency of job 1 is a number in the interval $[1, 2]$ while the relaxation of job 4 is a number in the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$ so, in any realization the urgency of job 4 is a number in the interval $[2, 3]$ and apparently the switch is surely profitable. The approach using both urgency indices and relaxation indices when dealing with sequencing interval situation is a topic for further research.

Other approaches for sharing the gain generated by a switch may be investigated. For example, it is possible to assign to each job its minimal compensation obtained supposing that its unitary cost and the processing time of the jobs involved in the switches coincide with the lower bound; after a realization, the difference between the actual cost savings and the sum of shares already distributed over the switched jobs, can be allocated according to a fair division procedure or a bankruptcy rule.

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