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ON THE UNIQUE D1 EQUILIBRIUM IN THE STACKELBERG MODEL WITH ASYMMETRIC INFORMATION

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November 1997

Abstract

This note studies a version of the Stackelberg model in which the Leader has more information about demand than the Follower. We show that there exists a unique D1 equilibrium and that this equilibrium is perfectly revealing. We also give a full characterization of the equilibrium in terms of the posterior beliefs of the Follower and show under which condition there is first mover disadvantage.

Keywords: Separating equilibria, signalling games, Stackelberg competition.
JEL Classification: C72, C73, D82, L10

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* We thank Eric van Damme and Santanu Roy for helpful comments on earlier versions of the paper.

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1. Introduction

The Stackelberg model in which the Leader has some information about demand that the Follower does not have has been first studied by Gal-Or (1987). She showed that the model has many perfect Bayes-Nash equilibria, depending on the specification of the out-of-equilibrium beliefs. Moreover, she demonstrated that unlike the Stackelberg model with perfect information, there are equilibria such that the Leader makes less profit than the Follower. She obtains this result by making specific assumptions about the nature of uncertainty.

In this note we assume that the out-of-equilibrium beliefs satisfy the D1 requirement as introduced by Banks and Sobel (1987) and Cho and Kreps (1987). We will show that there exists a unique D1 equilibrium which is perfectly revealing. We will also characterize the equilibrium strategies and provide a necessary and sufficient condition under which the Leader's profit is smaller than the Follower's. Our result is completely independent of any distributional assumptions concerning the type of uncertainty.

The note makes use of results obtained by Malaith (1987) and Ramey (1996). Ramey (1996) extends the analysis of Cho and Sobel (1990) to signalling games with a continuum of types. Under some appropriate assumptions, he shows that any D1 equilibrium must be separating. The Stackelberg model with asymmetric information is a signalling model with a continuum of types. As some of the assumptions made by Ramey (1996), in particular Assumption 1, do not hold in our case we provide a more intuitive proof of the fact that any D1 equilibrium must be separating in the special case of our model. Malaith (1987) shows that under a set of regularity conditions, there exists a unique separating equilibrium in signalling games with a continuum of types. We basically show that the Stackelberg model with asymmetric information satisfies the assumptions imposed by him.

Section 2 briefly presents the model. The sequential equilibrium concept and the D1 criterion are defined in Section 3. Results and their proofs are given in Section 4. Section 5 concludes with some comments.

2. The Model

We consider a Stackelberg model with two firms, a Leader and a Follower. Demand is given by the linear inverse demand function $p = a - b(q_L+q_F)$, where p is the price and q_L and q_F are, respectively, the output chosen by the Leader and the Follower. The Leader and the Follower choose their output level so as to maximize profits. The profit functions of the Leader and the Follower are given by $\pi_L(q_L,q_F,a)$ and $\pi_F(q_L,q_F,a)$, respectively. The output choice of the Leader is observed by the Follower before it makes an output choice itself. The value of the intercept a is known to the Leader, but unknown to the Follower. Before observing the output choice of the Leader, the Follower thinks that a is drawn from some continuous probability distribution with support $[a_L,a_H]$, where $a_L > 0$. Hence, the strategies of the Leader and Follower can be written as $q_L(a)$, respectively, $q_F(q_L)$. As the results are independent of the particular shape of the probability distribution, we do not make any further assumptions about it. Without loss of generality we assume b to be equal to 1. The above model is assumed to be common knowledge among the players.

3. Sequential Equilibrium and the D1 Criterion

Let the players' equilibrium strategies be given by $q_L^*(a)$ and $q_F^*(q_L)$, and let the Follower's belief about a conditional on observing q_L be denoted by the distribution function $\mu(a|q_L)$. Moreover, let $\hat{a}(q_L)$ be the expectation of this distribution. If $q_L^*(a)$ is strictly monotonic, then the strategy is revealing and $\hat{a}(q_L)$ is the inverse of $q_L^*(a)$ on the relevant domain. The triple $\{q_L^*(a),q_F^*(q_L),\mu^*(a|q_L)\}$ is a sequential equilibrium if the following three conditions hold:

(1) $q_L^*(a) \in \underset{q_L \in \mathbb{R}_+}{\operatorname{argmax}}\ \pi_L(q_L, q_F^*(q_L), a)$ for all $a \in [a_L, a_H]$;

(2) $q_F^*(q_L) \in \underset{q_F \in \mathbb{R}_+}{\operatorname{argmax}} \int_{a_L}^{a_H} \pi_F(q_L, q_F, a) \mathrm{d}\mu^*(a | q_L)$ for all $q_L \in \mathbb{R}_+$;

(3) a) if $q_L \in$ range q_L^* and $\int_{\{a | q_L^*(a) = q_L\}} \mathrm{d}\mu(a) > 0$, then

$\mu^*(a|q_L)$ is calculated using Bayes′ Rule;

b) if $q_L \in$ range q_L^* and $\int_{\{a | q_L^*(a) = q_L\}} \mathrm{d}\mu(a) = 0$, then

$\mu^*(a|q_L)$ is any distribution with the property that

supp $\mu^*(a|q_L) \in$ cl $\{a | q_L^*(a) = q_L\}$;

c) if $q_L \notin$ range q_L^*, then $\mu^*(a|q_L)$ is unrestricted.

The above definition is the standard notion of sequential equilibrium applied to the present context. The full-support assumption in (3) b) is invoked only in establishing that sequential equilibria satisfying the D1 criterion must be separating.

The D1 criterion imposes restrictions on the out-of-equilibrium beliefs of the Follower. In particular, let $\pi_L^*(a) \equiv \pi_L(q_L^*(a), q_F^*(q_L), a)$ be the equilibrium profit of the Leader observing a. Fix $q_L \notin$ range q_L^*. Suppose there is a nonempty set $A \subset [a_L, a_H]$ such that the following holds: for all $\tilde{a} \notin A$, there is an $a \in A$ such that $\pi_L(q_L, q_F, \tilde{a}) \geq \pi_L^*(\tilde{a})$ implies that $\pi_L(q_L, q_F, a) > \pi_L^*(a)$. A sequential equilibrium satisfies the D1 criterion if, and only if, supp $\mu^*(a|q_L) \subset A$ for all $q_L \notin$ range q_L^*. Intuitively, the Follower observing an out-of-equilibrium quantity q_L is restricted to place zero posterior weight on a type $\tilde{a}$ whenever there is another type a that has a stronger incentive to deviate from the equilibrium, in the sense that type a would strictly prefer to deviate for all q_F that would give type $\tilde{a}$ a weak incentive to deviate.

4. Analysis

In this section we show that there exists a unique D1 equilibrium. Without imposing restrictions on the reaction of the Follower we first, however, show that a profit maximizing strategy for the Leader is non-decreasing.

Lemma. $\forall q_F(q_L)$: *if $q_L(a)$ is a best response to $q_F(q_L)$, then $q_L(a)$ is non-decreasing.*

Proof. Suppose the statement in the lemma is not true and that there exist two points *a'* and *a''* where *a'<a''*, such that $q_L^{\prime} > q_L^{\prime\prime}$. Since $q_L^{\prime}$ is a profit maximizing choice given *a'* and $q_L^{\prime\prime}$ is a profit maximizing choice given *a''*, it follows that:[1]

$$\pi_L(q_L^{\prime}|a^{\prime}) = q_L^{\prime}(a^{\prime}-q_L^{\prime}-q_F(q_L^{\prime})) \geq q_L^{\prime\prime}(a^{\prime}-q_L^{\prime\prime}-q_F(q_L^{\prime\prime})) = \pi_L(q_L^{\prime\prime}|a^{\prime});$$

$$\pi_L(q_L^{\prime\prime}|a^{\prime\prime}) = q_L^{\prime\prime}(a^{\prime\prime}-q_L^{\prime\prime}-q_F(q_L^{\prime\prime})) \geq q_L^{\prime}(a^{\prime\prime}-q_L^{\prime}-q_F(q_L^{\prime})) = \pi_L(q_L^{\prime}|a^{\prime\prime}).$$

Multiplying the first inequality by -1 and adding the two inequalities up yields

$$(q_L^{\prime\prime}-q_L^{\prime})(a^{\prime\prime}-a^{\prime}) \geq 0.$$

As $q_L^{\prime} > q_L^{\prime\prime}$ and *a'' > a'*, this leads to a contradiction. So we conclude that $q_L(a)$ is non-

[1] If $q_F(q_L)$ would be a mixed strategy, the same argument applies when substituting $Eq_F(q_L)$ for $q_F(q_L)$ and $E\pi_L$ for π_L.

decreasing.

Q.E.D.

It is clear that we cannot get any additional results without imposing some restrictions on the out-of-equilibrium beliefs of the Follower. We believe, however, that some specifications of the out-of-equilibrium beliefs are not very reasonable. In particular, when the model is common knowledge, the Follower can also infer the content of the Lemma, namely that a rational Leader's strategy is non-decreasing. It seems reasonable that the out-of-equilibrium beliefs should be consistent with this fact in the following sense: if in equilibrium the Leader produces some $\bar{q}_L$ for all values of a in some interval [a',a''], then the Follower should infer from observing $\bar{q}_L-\varepsilon$ that the intercept is smaller than or equal to a'. From the proof of Proposition 1 below it becomes clear that this is essentially what the D1 requirement amounts to in the context of our model.

Proposition 1. *In any D1 equilibrium the strategy* $q_L^*(a)$ *is strictly monotonic increasing.*

Proof. The proof consists of two parts. First, we show that if $q_L^*(a)$ is not strictly monotonic increasing on [$\underline{a}$,a_H] for some $\underline{a} > a_L$, then it cannot be continuous. Second, we show that when the out-of-equilibrium beliefs satisfy Criterion D1 $q_L^*(a)$ can neither be discontinuous nor be constant on an interval starting from a_L. Hence, the equilibrium strategy must be strictly increasing.

(1) Suppose $q_L^*(a)$ is continuous and not strictly increasing on [$\underline{a}$,a_H] for some $\underline{a} > a_L$. Then there exist $a' > a_L$, a'' and $\bar{q}_L$ such that $q_L(a)=\bar{q}_L$ for all $a \in [a',a'']$. We show that a firm that observes a' is strictly better off by producing slightly less.

$$\pi_L(\bar{q}_L|a') = (a' - \frac{\bar{q}_L}{2} - \frac{\hat{a}(\bar{q}_L)}{2})\bar{q}_L \qquad (*)$$

$$\pi_L(\bar{q}_L-\varepsilon|a') = (a'-\frac{\bar{q}_L-\varepsilon}{2}-\frac{\hat{a}(\bar{q}_L-\varepsilon)}{2})(\bar{q}_L-\varepsilon) \qquad (**)$$

As $\hat{a}(\bar{q}_L)>a'>\hat{a}(\bar{q}_L-\varepsilon)$, there exist ε such that (**) is strictly larger than (*). Hence, a firm observing *a’* will deviate.

(2) Suppose then that $q_L^*(a)$ is discontinuous and not strictly increasing. Then there exist *a’*, *a’’* and $\bar{q}_L$ such that $q_L(a) = \bar{q}_L$ for all $a \in (a',a'')$ and $\forall\varepsilon>0$: $q_L^*(a'-\varepsilon)<\bar{q}_L$. If $q_L^*(a)$ is continuous at *a’* we can use the argument under (1) to show that the Leader observing *a’* will deviate. Let us then consider the case that $q_L^*(a)$ is discontinuous at *a’*. For small enough ε $\bar{q}_L-\varepsilon$ is not on the equilibrium path. We first show that for any δ>0, the D1 criterion implies that after observing such an $\bar{q}_L-\varepsilon$ the Follower should place zero posterior weight on any *a* strictly larger than *a’*+δ. For any $a \in (a',a'')$, the equilibrium pay-offs are given by

$$\pi_L^*(a) = (a-\bar{q}_L-\frac{\hat{a}(\bar{q}_L)-\bar{q}_L}{2})\bar{q}_L.$$

The pay-offs when deviating depend on the reaction of the Follower and are given by

$$\pi_L(\bar{q}_L-\varepsilon|a) = (a-\bar{q}_L-\varepsilon-q_F)(\bar{q}_L-\varepsilon).$$

It is beneficial to deviate if, and only if, the first expression is smaller than the second, i.e., if and only if

$$\frac{\hat{a}(\bar{q}_L)-\bar{q}_L}{2}\bar{q}_L > q_F(\bar{q}_L-\varepsilon) - 2\bar{q}_L\varepsilon - \varepsilon(a-\varepsilon).$$

The equation reveals that for any ε > 0 the larger *a*, the smaller the maximum value of q_F for which it is beneficial to deviate. Hence, if we fix in the definition of the D1 criterion

A to be equal to $(a',a'+\delta]$ for some small positive δ, then it is clear from the above that the following holds: for all $\tilde{a} > a'+\delta$, there is an a in $(a',a'+\delta]$ such that $\pi_L(q_L,q_F,\tilde{a}) \geq \pi_L^*(\tilde{a})$ implies that $\pi_L(q_L,q_F,a) > \pi_L^*(a)$. Hence, the D1 criterion requires that supp $\mu^*(a|q_L) \leq a'+\delta$.

Next, we focus on the Leader who observes $a'+\delta$, where δ is small. Using the above argument, we know that for small enough δ and ε, $\hat{a}(\bar{q}_L) > a' - \delta \geq \hat{a}(\bar{q}_L - \varepsilon)$. Hence, there exist ε and δ such that

$$\pi(\bar{q}_L-\varepsilon|a'-\delta) = (a'-\delta-\frac{\bar{q}_L-\varepsilon}{2}-\frac{\hat{a}(\bar{q}_L-\varepsilon)}{2})(\bar{q}_L-\varepsilon) > \pi(\bar{q}_L|a'-\delta) = (a'-\delta-\frac{\bar{q}_L}{2}-\frac{\hat{a}(\bar{q}_L)}{2})\bar{q}_L.$$

Thus, given the D1 requirement it is beneficial for the Leader who observes $a'+\delta$ to deviate. By substituting a_L for a' a similar argument can be made to show that $q_L^*(a)$ cannot be constant on an interval starting from a_L. This concludes the proof of Proposition 1.

Q.E.D.

In the next proposition we demonstrate that there exists a unique revealing equilibrium of the Stackelberg model with asymmetric information. We then characterize it in terms of the Follower's conditional expectation of the intercept. Unfortunately, an analytical expression for $q_L(a)$ does not exist.

Proposition 2. *The Stackelberg model with asymmetric information has a unique D1 equilibrium.*

Proof. Proposition 1 shows that a D1 equilibrium must be separating. Here, we will show that there is also a unique separating equilibrium. Mailath (1987) shows that in a class of models that satisfy certain regularity conditions and an initial value condition, there is a unique separating equilibrium. It is easily shown that the model analyzed here satisfies the regularity conditions (1) - (5) of Malaith (1987). The initial value condition holds in our model if $q_L(a_H)=a_H/2$. We demonstrate that this is the case as $q_L(a_H)$ cannot be smaller or larger than $a_H/2$.

(1) Let the strategy of the Leader be such that $q_L(a_H)<a_H/2$. Then the Follower will respond to $q_L(a_H)$ by producing $(a_H-q_L(a_H))/2$. If the Leader deviates and sets $q_L=a_H/2$, the posterior expectation of the Follower will be such that $\hat{a}(a_H/2)\leq a_H$. It follows that producing $q_L=a_H/2$ results in a higher profit when $a=a_H$. Hence, $q_L(a_H)$ cannot be smaller than $a_H/2$.

(2) Let the strategy of the Leader be such that $q_L(a_H)>a_H/2$. Then the Follower will respond by producing $(a_H-q_L(a_H))/2$. The profit to the Leader in this case is equal to $\pi_L=(a_H-q_L-(a_H-q_L)/2)q_L<a_H^2/8$. If the Leader deviates to $a_H/2$, the expectation of the Follower will be $\hat{a}(a_H/2)<a_H$, yielding a best response of $q_F=(\hat{a}(a_H/2)-a_H/2)/2$. This implies that the profit to the Leader by deviating is equal to $\pi_L=(a_H-a_H/2-(\hat{a}(a_H/2)-a_H/2)/2)\ a_H/2$, which due to the fact that $\hat{a}(a_H/2)<a_H$ is larger than $a_H^2/8$. Hence, $q_L(a_H)$ cannot be larger than $a_H/2$.

Q.E.D.

We are now in the position to characterize the unique D1 equilibrium. We first determine the Follower's best response. This is easily seen to be

$$q_F(q_L) = \frac{\hat{a}(q_L)-q_L}{2},$$

provided $q_L \leq \hat{a}(q_L)$. Hence, the profit of the Leader who observes a is equal to

$$(a-q_L-\frac{\hat{a}(q_L)-q_L}{2})q_L = (a-\frac{\hat{a}(q_L)}{2}-\frac{q_L}{2})q_L.$$

The first-order condition for profit maximization by the Leader is, therefore,

$$a - \frac{1}{2}\hat{a}'(q_L(a))q_L(a) - \frac{\hat{a}(q_L(a))}{2} - q_L(a) = 0.$$

In a separating equilibrium, $\hat{a}(q_L(a))=a$ and, hence, the first-order condition simplifies to

$$\frac{a(q_L)}{2} - \frac{1}{2}a'(q_L)q_L - q_L = 0.$$

Solving this differential equation yields

$$a(q_L) = -2q_L \ln q_L - cq_L,$$

where c is a constant.

As $q_L(a_H)=a_H/2$, it follows that $c=2+2\ln(a_H/2)$. Substituting this expression for c yields

$$a(q_L) = 2q_L(1-\ln\frac{a_H}{2q_L}). \qquad (1)$$

The increasing function $a(q_L)$ is concave on the interval $[q_L(a_L),a_H/2]$, where $q_L(a_L)>0$. Consequently, the inverse function $q_L(a)$, which is the decision rule of the Leader, is convex on $[a_L,a_H]$; see Figure 1. As $q_L(a)$ is convex and $\lim_{a_L\to 0} q_L(a_L)=0$, it is easily seen that $q_L(a)<a/2$ for $a\in[a_L,a_H)$. Hence, on the interval $[a_L,a_H)$ $q_L^*(a)$ is strictly smaller than the equilibrium quantity in the full information model and the equilibrium quantity of the Follower is strictly larger.

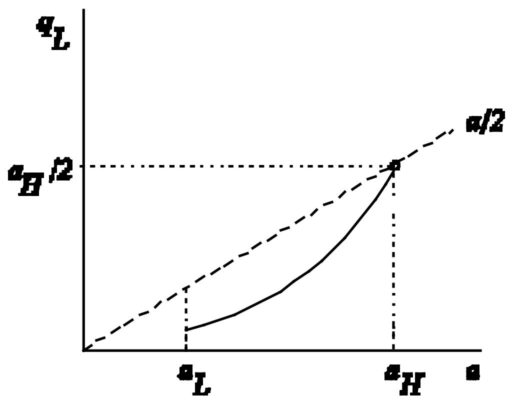

Figure 1

Now that we have characterized the equilibrium strategy of the Leader in terms of its inverse function, it is interesting to investigate the conditions under which there is a first mover disadvantage. We will say that there is an *ex ante* first mover disadvantage if given the distribution of a, the expected profit of the Leader is smaller than the expected profit of the Follower. We will say that there is an *ex post* first mover disadvantage if for a given realization of a, the profit of the Leader is smaller than the profit of the Follower. As we have not made any assumptions regarding the distribution of a on the interval

$[a_L, a_H]$, we investigate the scope for ex post first mover disadvantages.

Proposition 3. *There is an ex post first mover disadvantage if, and only if,* $a<3a_H/(2e^{½})$.

Proof. It is clear that $\pi_L < \pi_F$ if, and only if, $q_L < q_F$. From equation (1) it is clear that this is the case if, and only if, $q_L < ½(q_L+2q_L\ln(a_H/(2q_L)))$, or $½ < \ln(a_H/(2q_L))$. From (1) it follows that $\ln(a_H/(2q_L))=(a-2q_L)/(2q_L)$, so that $q_L < q_F$ if, and only if, $3q_L(a) < a$. Any $a \leq a_H$ can be written as $(1+\gamma)a_H/e^{\gamma}$ for some $\gamma \geq 0$. From (1) again it then follows that $q_L = a_H/(2e^{\gamma})$ if, and only if, $a = (1+\gamma)a_H/e^{\gamma}$. Hence, $3q_L(a) < a$ if, and only if, $3a_H/(2e^{\gamma}) < (1+\gamma)a_H/e^{\gamma} \Leftrightarrow \gamma > ½$. As a is decreasing in γ, there is an ex post first mover disadvantage, if $a < 3a_H/(2e^{½})$.

Q.E.D.

It is clear that ex post first mover disadvantage for some values of a is a necessary condition for ex ante first mover disadvantage. From the above it can be inferred that if there is enough probability mass on small enough values of a, there will also be ex ante first mover disadvantage. Indeed, simulation results in which a is uniformly distributed on $[a_L, a_H]$ show that there is indeed scope for ex ante first mover disadvantage if the ratio of a_H/a_L is large enough.

5. Concluding Remarks

The analysis of this note shows that there exists a unique D1 equilibrium in the Stackelberg model with asymmetric information. This result is, among other things, of interest to the literature on role choice (see, e.g., Mailath, 1993 and Daugethy and Reinganum, 1994). In that literature it is frequently assumed that there are just two or three possible states of demand that are sufficiently distinct from each other. This assumption is made in order to guarantee a unique equilibrium in the subgame in which the informed player moves first. This note basically argues that such an assumption is not needed, because by restricting the out-of-equilibrium beliefs in an appropriate way, there exists a unique D1 equilibrium even if the uncertainty about demand follows a continuous distribution.

We concentrated on uncertainty about the intercept of the inverse demand function. However, it turns out that an analogous analysis can be made for the case of uncertainty about the slope. Of course, in the latter case, the equilibrium strategy of the Leader is strictly decreasing (instead of increasing) in the value of the slope parameter.

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