Discussion Paper

No. 2007–100

OPTIMAL FRAGILE FINANCIAL NETWORKS

By Fabio Castiglionesi, Noemí Navarro

December 2007

ISSN 0924-7815

Optimal Fragile Financial Networks*

Fabio Castiglionesi† Noemí Navarro‡

December 2007

Abstract

We study a financial network characterized by the presence of depositors, banks and their shareholders. Belonging to a financial network is beneficial for both the depositors and banks' shareholders since the return to investment increases with the number of banks connected. However, the network is fragile since banks, which invest on behalf of the depositors, can gamble with depositors' money (making an investment that is dominated in expected terms) when not sufficiently capitalized. The bankruptcy of a bank negatively affects the banks connected to it in the network. First, we compute the social planner solution and the efficient financial network is characterized by a core-periphery structure. Second, we analyze the decentralized solution showing under which conditions participating in a fragile financial network is ex-ante optimal. In particular, we show that this is optimal when the probability of bankruptcy is sufficiently low giving rationale of financial fragility as a rare phenomenon. Finally, we analyze the efficiency of the decentralized financial network. Again, if the probability of bankruptcy is sufficiently low the structure of the decentralized financial network is equal to the efficient one, yielding an expected payoff arbitrarily close to the efficient one. However, the investment decision is not the same. That is, in the decentralized network some banks will gamble as compared to the socially preferred outcome.

JEL Classification: D85, G21.
Keywords: Financial Network, Moral Hazard, Financial Fragility.

*We would like to thank Sandro Brusco for useful comments. Part of this work was developed while Navarro was visiting the HEC Lausanne, Switzerland. She is grateful for their hospitality. Navarro also thanks financial support from Junta de Andalucía through grant SEJ-552. The usual disclaimer applies.

†CentER, Department of Finance, Tilburg University, e-mail: fabio.castiglionesi@uvt.nl.
‡Universidad de Malaga, Departamento de Historia y Teoría Económica, e-mail: navarro.prada@gmail.com.

1 Introduction

It is sometimes observed that financial systems turn out to be fragile. With this economists mean that an adverse shock is able to cause the collapse of the entire financial system, implying large consumption losses for investors. As a consequence, it is usually claimed that fragility should be avoided altogether. This paper challenges this view showing that such fragility could be indeed an optimal feature of financial networks.

We characterize a financial network with the presence of banks, consumers, and banks' shareholders. The economy is made of several regions (countries, sectors) each with its own representative bank. Consumers need to deposit their endowment into banks to take advantage of the investment opportunities in the economy. Shareholders, which are different agents from depositors, provide bank capital and decide the type of investment the bank will choose. Banks have two types of asset in which they can invest. One asset is safe and the other asset is risky. The second asset is risky since it delivers the same return as the safe asset if it succeeds and nothing if it fails, however it gives private benefits to banks' shareholders. The second asset is clearly dominated for depositors and it is a pure gambling asset. Once the type of investment is chosen, both the deposits and the bank capital will be invested in that type of asset. It turns out that poorly capitalized banks find it convenient to gamble. This mimics the usual risk-shifting problem due to limited liability.

Banks also choose whether to join or not the financial network. Participating in the network is beneficial for both the depositors and banks' shareholders since the return to both types of investment increases with the number of banks connected. This captures the idea that more connections give access to more investment opportunities. However, the decision of belonging to the network entails a trade-off. The possible bankruptcy of a bank that gambles affects the connected banks in the network (we make the assumption that when a bank fails all the directly linked banks will be bankrupt as well).

We also allow for the possibility of bank capital transfers among banks. These transfers can be interpreted as direct investment in the bank capital of other banks, or exchange of (asymmetric) cross holding of bank capital. The only aim of these bank capital transfers is to solve the moral hazard problem.

The first-best (efficient) solution is achieved by avoiding the moral hazard problem. A social planner that allocates bank capital in each bank, can guarantee this allocation under the assumption that a sufficient amount of aggregate bank capital is available in the economy. If not, a constrained first-best allocation will be achieved (i.e., allowing some bank to gamble). In this case, the lower the aggregate bank capital, the lower is the expected payoff from the (now fragile) financial network.

Since the transmission of a crisis depends on the links established by the failing bank, contagion is greater the larger is the number of links. Then the optimal network structure is not necessarily the fully connected one. In particular, the efficient structure is characterized by a core-periphery shape. The core includes all banks that (i) invest in the safe asset and (ii) form a complete network structure among them. The periphery includes all the banks that gamble. Gambling banks could be connected among themselves and/or with the core. Only when the risk of bankruptcy is sufficiently low the efficient structure becomes the fully connected one.

In the decentralized environment the bank capital is allocated randomly across the different banks. Same banks now have too few capital and will gamble. In the decentralized network the first-best (efficient) allocation cannot be reached unless capital endowments are high. However, for a probability of bankruptcy arbitrarily low we can assure that (i) joining a fragile financial network is ex-ante optimal, (ii) the structure of the decentralized financial network is the same as the efficient one, and (iii) the fragile network delivers a total payoff arbitrarily close to the first-best (efficient) one. The last two results are obtained even considering bank capital transfers. The intuition is the following. When the probability of failure is arbitrarily low, the cost of financial fragility becomes lower than the cost of bank capital transfer, and this will make depositors and shareholders willing to take the risk of financial instability.

The main result is that financial fragility (that is, the collapse of the entire system) is a rare event since banks and depositors will not enter such network, unless it is ex-ante convenient. Consequently, the fragile financial networks can be 'optimal' since they are ex-ante Pareto-improving with respect to the autarky situation. However, even when the decentralized network has the same structure of the efficient one and it is delivering almost the same total payoff, it could be that some banks invest in the gambling asset in the decentralized network while they would choose the safe asset in the efficient one. That is, the core of the efficient network can be larger than the core of the decentralized network.

The theory of networks has recently been successfully applied in several economics fields. However, very few attempts have been made to use such theory to understand the working of financial systems. One exception is represented by Leitner's [10] model, which gives a rationale of financial networks that are able to spread contagion. We share with Leitner the same goal, extending however his approach in two directions. On the one hand, we model a decentralized formation of the financial network specifying the incentives of the agents involved (i.e., depositors and banks' shareholders). On the other hand, we provide a new rationale for fragile financial network formation. Financial networks in Leitner's [10] model induce private bailouts because of the threat of contagion. With this and given that formal commitments are impossible, networks may be ex ante optimal because by them banks obtain mutual insurance. The bailouts are implemented by means of money transfers between banks. The idea behind Leitner's model is that banks (and depositors) can be surprised by an unexpected liquidity shock which is able to make bankrupt at least one bank in the system (see Allen and Gale, [2]). The possibility that this original failure can spread to the entire system give the rationale of belonging to a financial system. We show that we do not need necessarily the presence of lack of commitment or, which is the same, money transfers among banks, to give rationale of an ex-ante optimal financial network.

Nier et al. [11] consider the link between network models and financial stability. However, their approach is to take the network structure as given and study how an exogenous shock is transmitted through the network. Gale and Kariv [8] show that trade that is restricted to happen on a network generates an efficient outcome as far as the agents are sufficiently connected in the network. Babus [4] studies financial network finding that preventing the collapse of the system is a sufficient (but not necessary) condition to guarantee the stability of the network.

Outside the network literature, various contributions have analyzed financial fragility and contagion. Our approach shares the same scope with the strand of literature that model contagion as the outcome due to the presence of financial links among banks. In particular, banks are connected through interbank deposit markets that are desirable ex-ante, but during a crisis the failure of one institution can have negative payoff effects on the institutions to which it is linked (see Rochet and Tirole, [12]; Allen and Gale, [2]; Aghion, Bolton and Dewatripoint, [1]; Freixas, Parigi and Rochet, [7]). A common feature of these models is the reliance on some exogenous unexpected shock that causes a financial crisis to spill over into other financial institutions. In Allen and Gale [2] and Aghion, Bolton and Dewatripont [1] financial contagion is due to an unexpected aggregate liquidity shortage. Allen and Gale also find that the more connected the interbank deposit market is the more resilient is the system to contagion. Freixas, Parigi and Rochet [7] model financial contagion as a solvency shock to a particular bank finding that, similarly to Allen and Gale [2], the degree of interbank connections enhance the resiliency of the banking system to withstand the insolvency problem.

More recently, Brusco and Castiglionesi [5] have attempted to model contagion in the banking system without relying on unexpected shocks. All events are anticipated and contractible by the agents. The present model captures many features of the Brusco and Castiglionesi's approach. In particular, the possibility of banks bankruptcy comes from the banks' gambling behavior, which occurs when banks are not sufficiently capitalized. The liquidity coinsurance mechanism (built as in Allen and Gale to face idiosyncratic liquidity shocks) implies that a bank's default will cause the linked banks to fail. Thus, the more connected is the banking system the larger the extent of contagion. If this is the correct reason that more links mean more contagion, one has to wonder how robust this result is when endogenous links are considered. The present paper directly addresses this issue.

On general ground, our model is in line with the results of Allen and Gale [3]. They show that financial default is not always best avoided. Their result holds also with incomplete contract, but it crucially depends on the presence of complete markets for aggregate uncertainty. We do not consider aggregate uncertainty, but we extend the analysis to financial fragility. In our model there are only idiosyncratic shocks (i.e., the gambling behavior of the banks) and contracts are incomplete since it is not possible to contract out bank's moral hazard behavior. When a bank defaults all the linked banks will be bankrupt as well, however this does not represent necessarily a market failure because the incidence of financial fragility can be constrained efficient.

The paper is organized as follows. Section 2 sets up the model. Section 3 presents an example of the network formation. Section 4 analyzes the planner problem, characterizing the first-best solution. Section 5 shows the financial network formation and studies its optimality. Section 6 characterizes the efficiency of the decentralized financial network. Section 7 presents a discussion of the modeling choices. Section 8 contains the conclusions, and the Appendix contains the proofs.

2 The Model

There are three dates $t = 0, 1, 2$ and one divisible good called 'dollars' (\$). The economy is divided into n regions, each with its own representative bank. Let $N = \{1, 2, ..., n\}$ be the set of the regions and banks. Each region is populated by a continuum of consumers endowed with 1 \$

at $t = 0$. However, they consume at $t = 2$. In order to access the investment opportunities of the economy, each consumer has to deposit his endowment in the representative bank of the region he belong to.

Each representative bank i randomly receives an endowment $e_i \in [0, 2]$ of dollars, which represents the bank capital and it is owned by banks' shareholders (or investors). Consumers and investors are different type of agents in the economy. The pair (N, e), with $N = \{1, 2, ..., n\}$ and $e = (e_1, e_2, ..., e_n)$ is called an **economy**.

We also allow for transfer of bank capital across banks. Let $x_i = e_i + t_i$ be the bank capital for a bank $i \in N$ after transfers have been made (i.e., t_i is the transfer and can be positive or negative). Then each bank expects to have $1 + x_i$ dollars to invest in $t = 1$. A vector of bank capitals $x = (x_1, x_2, ..., x_n)$ is called feasible for a given economy (N, e) if (i) $x_i \geq 0$ for all i, and (ii) $\sum_{i \in N} x_i = \sum_{i \in N} e_i$. Let $\mathcal{X}$ denote the set of all feasible vectors of bank capital for a given economy (N, e). The sequence of events is reported in Table 1.

Table 1. Sequence of events

Time	Events
$t = 0$	1. Bank's capital is realized; 2. Financial network is chosen.
$t = 1$	1. Bank's capital transfers are made; 2. Projects are chosen and investments are undertaken.
$t = 2$	1. Projects cash flows are realized; 2. Depositors are paid.

Banks have access to two types of project (the investment can be made either in one or the other type of project):

1. The safe project b yields $R > 1$ dollars in $t = 2$ per dollar invested in $t = 1$.
2. The 'gambling' project g yields $R > 1$ dollars with probability η (we assume $\eta R > 1$), and 0 dollars with probability $(1 - \eta)$ in $t = 2$ per dollar invested in $t = 1$. This project yields also a private benefit $B > 0$ to bank's shareholders.

Private benefits are realized by banks' shareholders at the moment of the investment (so they do not have dollar value, consider them as perks or investment in family business). Then the gambling asset becomes a simple device to mimic the risk shifting problem that characterizes financial institutions protected by limited liability. Conditional on the bank choosing the gambling asset, notice that as $\eta \to 1$ the moral hazard problem vanishes. This is true since as $\eta \to 1$ the probability of the bank going bankrupt becomes negligible.

Let $K_i \subseteq N$ be the set of banks to whom bank i is directly linked, then the number of banks connected to bank i is $k_i \in \{0, 1, ..., n - 1\}$. The vector $K = (K_1, K_2, ...K_n)$ captures the interdependence among the banks, and it represents the financial network. We restrict ourselves to undirected networks, i.e., the links in the networks are bidirectional: bank i is related to bank j only if bank j is related to bank i. Formally, $i \in K_j$ if and only if $j \in K_i$. Let $\mathcal{K}$ denote the set of all possible financial networks for a given economy (N, e).

We assume that the per unit return of the investment (both safe and gambling) is increasing in the number of banks linked with the investing bank i. We indicate the increase of return for each unit of investment with the function $f(k_i)$ with $f'(k_i) > 0$, and $f''(k_i) < 0$ for all $k_i \in [0, n-1]$. We assume that $f(0) = 1$ and $f(n-1) = \rho$, with $\rho > 1$, that is $f(k_i) \in [1, \rho]$.

The return of bank i for each unit invested in $t = 0$ is then equal to $f(k_i)R$ in $t = 2$. Consequently a bank that makes the investment in autarky ($k_i = 0$) will obtain the lowest return R, while a bank that is connected with all the other banks ($k_i = n-1$) will achieve the highest return ρR. This assumption has the natural interpretation that more connections give access to more investment opportunities. Since depositors are assumed to be risk averse, then $f(k)$ represents the gain in utility coming from diversification.

Let $s_i \in \{b, g\}$ be the choice of project of a bank i. The vector s denotes the investment strategy profile, that is $s = \{s_i\}_{i \in N}$. Let $\mathcal{S}$ denote the set of all possible investment profiles for a given economy (N, e). For given network $K \in \mathcal{K}$ and strategy profile $s \in \mathcal{S}$, let $p_i(K, s)$ be the probability that the project chosen by agent i succeeds. This probability needs to take into account the possible use of the gambling asset in the financial network K:

$$p_i(K, s) = \prod_{j \in K_i \cup \{i\}} \pi_j(s_j), \tag{1}$$

where $\pi_j(s_j)$ is defined as

$$\pi_j(s_j) = \begin{cases} 1 & \text{if } s_j = b, \\ \eta & \text{otherwise.} \end{cases}$$

Note that, when bank i and all its neighbors are investing in the safe project b, then the probability of success is 1. However, if bank i or one of its neighbors is investing in the gambling asset, while the rest are investing in the safe asset, the system will not collapse with probability η. This happens since the safe projects work and the gambling asset realizes on positive payoffs. If there are exactly two neighbors who are investing in the gambling asset, then the probability of success is η^2, and so on and so forth. Again, as $\eta \to 1$, the moral hazard problem vanishes and the collapse of the financial system becomes a rare event.

We are implicitly assuming that once the project of a bank fails, its neighbors will also be bankrupt with probability equal to 1. This implies that we are going to analyze the properties of a network with a very strong form of fragility. Consequently, if we find conditions under which it can be optimal or even efficient to joint such network, this implies that the same conditions still apply for networks characterized by less strong fragility. In other words, we put ourselves in the worst possible scenario to show our results.[1]

Finally, notice that among the strategies of the banks there is no possibility of avoiding the investment. Since the banks can choose to be connected or not to the network, it is always possible for them to be disconnected and invest in autarky. Given that $R > \eta R > 1$ banks and depositors will always choose to invest.

[1] Clearly, a more attractive and realistic assumption is that the performance of each bank depends on its investment decisions. This would generate a more complicated probability distribution over returns. However, with a less fragile financial system, the main results of the paper would carry over.

The expected investors payoff of bank i choosing a strategy s_i is defined as

$$m_i(K,x,s) = \begin{cases} p_i(K,s)f(k_i)Rx_i & \text{if } s_i = b, \\ p_i(K,s)f(k_i)Rx_i + B & \text{otherwise,} \end{cases} \tag{2}$$

given (i) the bank capital after transfers $x_i = e_i + t_i$, (ii) the financial network K chosen at date 0 with the corresponding return to investment $f(k_i)R$ to bank i, and (iii) the strategies of all the other banks s_{-i}, with $s = (s_i, s_{-i})$. Note that equation (2) means that investors in bank i get the profit from their corresponding investment x_i times the probability that the project succeeds. Furthermore when investors in bank i gamble, they obtain private benefits B independently of the gambling asset being successful or not. The expected amount of dollars for depositors in bank i is given by

$$M_i(K,x,s) = p_i(K,s)f(k_i)R. \tag{3}$$

For the given network K and strategy profile $s = (s_i, s_{-i})$, let $g_i(K, s_{-i})$ denote the number of neighbors of an agent i in the network K choosing the gambling project. To avoid abuse of notation, we will simply make use of g_i instead of $g_i(K, s_{-i})$, unless this simplification would lead to confusion. By definition, $g_i \in [0, k_i]$. Then the probability of success $p_i(K,s)$ can be written as

$$p_i(K,s) = \begin{cases} \eta^{g_i} & \text{if } s_i = b, \\ \eta^{g_i+1} & \text{otherwise.} \end{cases}$$

Let us analyze the incentives that investors in bank i have to choose the safe asset or to gamble, for a given financial network K chosen at $t = 0$. Investors will place the bank's resources in the safe asset whenever the possible capital loss incurred in gambling is higher than the private benefits. Then for given $f(k_i)$ and s_{-i} investors in bank i will invest in the safe asset if and only if

$$\eta^{g_i} f(k_i)Rx_i \geq \eta^{g_i+1} f(k_i)Rx_i + B.$$

This implies that bank i will invest in the safe asset if and only if

$$x_i \geq \frac{B}{(1-\eta)\eta^{g_i} f(k_i)R},$$

that is, banks with relatively low level of bank capital have incentive to invest in the gambling asset while relatively high capitalized banks do not have this incentive.

In order to normalize the parameters of the model so to have a cut off value of x_i inside the possible realization of bank capital, we assume $B = 1 - x_i$.[2] Then the former condition becomes

$$x_i \geq \frac{1}{1 + (1-\eta)\eta^{g_i} f(k_i)R} \equiv I^*[k_i, \eta, g_i],$$

which is less than 1. This means that some banks find it optimal to invest in the safe asset even if they are not well capitalized. Note that the risk of capital loss vanishes when $\eta \to 1$, and in this

[2] The normalization makes clear that when the bank capital is sufficiently high with respect the amount of deposits (i.e., 1) the private benefits do not compensate the risk of loosing bank capital. However, if bank capital is low with respect the amount of deposits then the private benefits can become higher than the risk of losing bank's capital (banks are protected by limited liability).

case all the banks with a capital less than 1 will gamble. Then if $x_i \in [I^*(k, \eta, g_i), 2]$ depositors in bank i will get $\eta^{g_i} f(k_i)R$. Otherwise, if $x_i \in [0, I^*(k_i, \eta, g_i))$, depositors in bank i get $\eta^{g_i+1} f(k_i)R$.

The cut off value $I^*(k_i, \eta, g_i)$ is decreasing in k_i, increasing in g_i, and increasing in η if and only if $\eta > \frac{g_i}{1+g_i}$. Recall that investors' payoffs from investing safe and gambling are given in (2). The level of capital $I^*[k_i, \eta, g_i]$ makes investors in bank i indifferent between investing safe or gamble. As far as g_i is different from zero, both payoffs will increase due to an increase in η. However, to establish the effect of η on $I^*[k_i, \eta, g_i]$ we have to look at the marginal effect of an increase in η on investing safe or gambling. For example, if the increase in the payoff from investing safe is larger than the increase in the payoff of gambling, then $I^*[k_i, \eta, g_i]$ decreases. The intuition is that the bank needs less bank capital in order to be indifferent between investing safe and gambling. Formally, when η increases, the increase in the investors' payoff for investing safe is $g_i \eta^{g_i-1} f(k_i)Rx_i$ and for gambling is $(g_i + 1)\eta^{g_i} f(k_i)Rx_i$. Thus, when the ratio $\frac{g_i}{(g_i+1)\eta}$ is smaller than 1 the cut off value $I^*[k_i, \eta, g_i]$ is increasing in η. Otherwise, when the ratio $\frac{g_i}{(g_i+1)\eta}$ is greater than 1, it is decreasing in η.

Finally, we assume that depositors are risk-averse and investors are risk-neutral. Depositors' utility function is defined on the possible outcomes, i.e. the utility of a strategy profile s that depositors in bank i are getting is denoted by $u_i(s)$, where $u_i(s) = u_i(M_i(s))$. We next turn our attention on an example to show the intuition behind the financial network formation. In this way, we can highlight the different forces that are behind the network formation in the Leitner's [10] model and ours.

3 Network Formation: An Example

In Leitner's model there is one (safe) asset with return $R > 1$ that has a cost of one dollar. In order to realize the return R, all the linked agents in the network need to invest the one dollar. Assume there are two agents: Agent 1 (randomly chosen) has an endowment $e_1 = 2$ and agent 2 has an endowment equal to $e_2 = 0$. Let $u_1(.)$ and $u_2(.)$ the utility of agent 1 and agent 2, respectively.

First-best (efficient) allocation. The social planner reaches efficiency by transferring one unit of endowment from agent 1 to agent 2 getting a level of utility $u_1(R) = u_2(R)$ for both. In this case the sum of utilities of the two agents is higher than not doing the transfer, that is $u_1(1 + R) + u_2(0) < u_1(R) + u_2(R)$ (which is always satisfied assuming the same functional utility for the two agents, i.e. $u_1(.) = u_2(.) = u$ with u quasiconcave). In other words, the loss in utility of the first agent is compensated by the gain in utility of the second agent, that is $u_1(1 + R) - u_1(R) < u_2(R) - u_2(0)$.

No linked agents. In this case agent 1 decides whether to make a transfer or not. Without transfer she gets $u_1(1 + R)$ and agent 2 gets zero. If she makes the transfer she will get $u_1(R)$ and also agent 2 will get $u_2(R)$. So no transfer will be made, and $u_2(0) = 0$.

Linked agents. In this case agent 1 needs agent 2 to invest. Without the transfer we have $u_1(2)$ and $u_2(0)$, as agents can always choose autarky, and in a Nash equilibrium agent 1 will see that agent 2 cannot invest. If the transfer is made we have that $u_1(R) = u_2(R)$, as the social planner coordinates both agents to invest (note that, no one investing is also a Nash equilibrium and the agents get $u_1(1)$ and $u_2(1)$, respectively). Then, when $R \geq 2$, agent 1 bails out agent 2 through the transfer and being linked achieves efficiency. When $R < 2$ there is no transfer in case of the network, or, in other words, agent 1 does not bail out agent 2.

Network formation. Let u_i^{nl} and u_i^l be the utility levels achieved by agent i when she is not linked and when she is linked, respectively. When the agents are not linked, agent 1 will get $u_1^{nl}(1+R)$. When the agents are linked, the payoff of agent 1 depends on making or not the transfer. We have $u_1^l(R)$ when the transfer is made, and $u_1^l(2)$ when the transfer is not made. Accordingly, in both cases we have $u_1^{nl} > u_1^l$. Consequently, agent 1 would not enter in the network if she knew her endowment. The assumption in Leitner's model is that agents may agree to join the network *before* they know the realization of the endowments. This uncertainty makes the agents willing to be connected.

Assume there are only two realizations (equally likely) of (e_1, e_2): $(2, 0)$ and $(0, 2)$. Then if the agents choose to be connected they will get a total expected utility of

$$\frac{1}{2}[u_1(R) + u_2(R)] + \frac{1}{2}[u_1(R) + u_2(R)] \tag{4}$$

if $R > 2$, and an expected utility of

$$\frac{1}{2}[u_1(2) + u_2(0)] + \frac{1}{2}[u_1(0) + u_2(2)], \tag{5}$$

otherwise. If they decide not to be connected they would get a total expected utility of

$$\frac{1}{2}[u_1(1+R) + u_2(0)] + \frac{1}{2}[u_1(0) + u_2(1+R)]. \tag{6}$$

Note that, under regular conditions, (4) is greater than both (5) and (6), so that it is always first-best (or Pareto) efficient to join the network, make the transfer, and have both agents invest one unit. The implementability of the first-best depends on the value of R. If $R < 2$ the first-best is not implementable (or attainable). The agent that turns out to be highly endowed will never bail out the other agent and this results in the empty network. If $R \geq 2$ the network is implementable (or, according to Leitner, it is second-best). Although participating in the network and bailing out is Pareto efficient, the network is implementable (or second-best) only for a range of values of R. In this example, being unlinked is better than being linked when $R < 2$. However, in a completely decentralized framework, agents will never build a network if they know their endowment in advance, even if it is implementable or second-best (that is, even if $R \geq 2$).

The financial network in Leitner [10] plays the role of a coordinating device for agents affected by lack of commitment. This can be achieved by means of transfers, which in a decentralized economy could be implemented by private bailouts. The network allows ex-post bailout in case some of the agents (randomly chosen) do not have enough resources. Events that cause such agents being short of resources in financial systems can be rationalized by aggregate unexpected liquidity shocks (see Allen and Gale, [2]). Therefore, agents have to agree on bailing out each other before their endowments are realized.

We show that the uncertainty on endowments, and consequently the ex-post transfers, is not a necessary condition to give rationale of entering into a fragile financial network. Depositors and investors can find it optimal to belong to a fragile financial network even knowing their endowment and without making ex-post bail out.

Consider an economy with two banks. Since $n = 2$, we have $k_i \in \{0, 1\}$ for $i = 1, 2$. The return to investment in autarky is then $f(0)R = R$ and, when the two banks are connected the return is $f(1)R = \rho R > R$. The first bank is endowed with a level of bank capital equal to $e_1 = I^*(0, \eta, 0) + \varepsilon$ and the second bank has a bank capital equal to $e_2 = I^*(1, \eta, 0) - \varepsilon$. Here, $\varepsilon > 0$ and recall that $I^*(0, \eta, 0) = \frac{1}{1+(1-\eta)R}$ and $I^*(1, \eta, 0) = \frac{1}{1+(1-\eta)\rho R}$. In other words, the first bank has enough bank capital to invest in the safe asset even when in autarky, while the second bank has a level of bank capital not large enough to invest safe even when the banks are connected and the first bank invests in the safe asset. Since $I^*(1, \eta, 1) = \frac{1}{1+(1-\eta)\eta\rho R}$, and assuming $\eta\rho \geq 1$, we have that

$$I^*(1, \eta, 0) < I^*(1, \eta, 1) < I^*(0, \eta, 0) \leq 1. \tag{7}$$

We assume that the values of the parameters are such that $e_1 + e_2 = I^*(1, \eta, 0) + I^*(0, \eta, 0) = 2I^*(1, \eta, 1)$. The two banks have a total amount of dollars to invest equal to $1 + e_1$ and $1 + e_2$, respectively. That is, the amount of deposits (equal to 1 dollar) plus the bank capital. Recall that as η tends to 1, the cut off values $I^*(1, \eta, 0)$, $I^*(1, \eta, 1)$ and $I^*(0, \eta, 0)$ all tend to 1.

Social Planner. The first-best is given by both banks choosing the safe asset, since the social planner does not value private benefits. Given this, the project will yield the highest profit if the banks are connected as $\rho > 1$. In order to achieve this outcome, the social planner is assumed to choose the bank capital, the network and, if necessary, to coordinate the banks in the safe investment. In this case the planner makes a transfer of ε dollars from bank 1 to bank 2. We have to clarify that the social planner acts as a substitute of the random draw of bank capital. The planner pools the overall bank capital endowment, equal to $2I^*(1, \eta, 1)$, and he assigns $I^*(1, \eta, 1)$ to each bank (so the transfer is fictitious) connecting them in the financial network. In this way banks' investors find it optimal to invest (without profitable individual deviations) in the safe asset getting a return $\rho RI^*(1, \eta, 1)$ and depositors in both banks get $u(\rho R)$. The total expected money that the social planner can achieve is then $2\rho RI^*(1, \eta, 1) + 2\rho R = \rho R(2 + e_1 + e_2)$.[3]

Not Linked. In this case the return of investment is equal to R. First note that investors in bank 1 will never make a transfer to bank 2 since the success of their project is independent of the investment decision of bank 2. Investors in bank 1 will invest in the safe project. Hence, investors in bank 1 get $[I^*(0, \eta, 0) + \varepsilon]R$ and depositors expect $u_1(R)$. Investors in bank 2 gamble getting $[I^*(1, \eta, 0) - \varepsilon]\eta R + B$, consequently depositors in bank 2 get $u_2(\eta R)$. Note that depositors in

[3] In this example the planner does not need to coordinate the banks investment decision since the total amount of bank capital $2I^*(1, \eta, 1)$ is sufficient to make the two bank willing to invest safe even if the other is gambling. So that investing safe is a dominant strategy. However, for lower level of total amount of bank capital this outcome is not guaranteed. For example, if the total amount of bank capital is $2I^*(1, \eta, 0)$, and the planner equally splits it among the two banks, then the two banks can invest in the safe asset if they expect the other bank to do the same. However, if they expect the other bank to gamble they will gamble as well since the individual bank capital is less than $I^*(1, \eta, 1)$. Under the assumption that the planner can coordinate the investment decision, he would obtain the same allocation as in the text. For lower levels of total bank capital the planner cannot reach the first-best even with coordination. In this case we have a constrained first-best. Assume for example that total bank capital is $I^*(0, \eta, 0) < 2I^*(1, \eta, 0)$. In this case the planner achieves the highest allocation putting all the bank capital in one bank and nothing in the other. The former invests safe while the latter gambles. If the planner does not link together the two banks, the total return is $R[1 + \eta + I^*(0, \eta, 0)]$. If the planner chooses to link the two banks then the total return is $\eta\rho R[2 + I^*(0, \eta, 0)]$. It is efficient to choose the fragile financial system as long as $\eta\rho > \frac{1+\eta+I^*(0,\eta,0)}{2+I^*(0,\eta,0)}$. This condition is always satisfied since $\eta\rho > 1$.

bank 2 are willing to deposit their dollar in the bank since $\eta R > 1$.

Linked. In this case the return to investment is equal to $\rho R > R$. However, when the two banks belong to the financial network the investment decision of one bank affects the probability of success of the other bank. Since bank 2 always gambles, this behavior reduces the possibility of success for the depositors and investors in bank 1. A way to avoid this event is to make a bank capital transfer to bank 2. So we have four cases that bank 1 faces: a) investing safe without transfer; b) gambling without transfer; c) investing safe with transfer; d) gambling with transfer. Let us examine the four cases in turn.

Case a). Depositors in bank 1 expect to get $u_1(\eta\rho R)$ and investors in bank 1 will get $[I^*(0,\eta,0)+\varepsilon]\eta\rho R$. Depositors in bank 2 will receive $u_2(\eta\rho R)$, while the payoff for investors in bank 2 is $[I^*(1,\eta,0)-\varepsilon]\eta\rho R+B$.

Case b). Depositors in bank 1 will get $u_1(\eta^2\rho R)$ and investors in bank 1 would get $[I^*(0,\eta,0)+\varepsilon]\eta^2\rho R+B$. Since $\eta > \eta^2$, depositors in bank 1 prefer to invest in the safe asset. Also investors in bank 1 prefer to invest in the safe asset since, by definition, we have $x_1 = e_1 > I^*(1,\eta,1)$. Then, case b) cannot be an equilibrium.

When a transfer is made from bank 1 to bank 2 (in order to solve its moral hazard problem) we have to check if investors in bank 2 still have incentive to gamble (no commitment to use the transfer properly). We assume that when a bank is indifferent between gambling or safe, it will choose safe.

Case c). Since investors in bank 1 are planning to transfer and play safe, they will choose the minimal transfer that will make bank 2 change from choosing the gambling asset to choosing the safe asset. This will result in $x_2 = I^*(1,\eta,0)$, and, consequently, the transfer has to be equal to $t_1 = x_2 - e_2 = I^*(1,\eta,0) - [I^*(1,\eta,0)-\varepsilon] = \varepsilon$. Capital in bank 1 is then $x_1 = e_1 - t_1 = I^*(0,\eta,0)$. Depositors in bank 1 get $u_1(\rho R)$ and investors in bank 1 get $\rho R I^*(0,\eta,0)$. Depositors in bank 2 then get $u_2(\rho R)$, and investors in bank 2 get $\rho R I^*(1,\eta,0)$.

Case d). Since bank 1 is planning to gamble, a transfer equal to ε will not induce investors in bank 2 switching from gambling to invest in the safe asset, as by definition, x_2 would be equal to $I^*(1,\eta,0)$, which is smaller than $I^*(1,\eta,1)$. In this case investors in bank 1 will have to make a transfer resulting in a level of capital for bank 2 of at least $I^*(1,\eta,1)$. Since the total amount of capital in the economy is $2I^*(1,\eta,1)$, this will result in bank 1 keeping at most $I^*(1,\eta,1)$ as its own capital. Then the transfer has to be at least

$$I^*(0,\eta,0) - I^*(1,\eta,1) + \varepsilon = I^*(1,\eta,1) - I^*(1,\eta,0) + \varepsilon$$

given that $I^*(1,\eta,0) + I^*(0,\eta,0) = 2I^*(1,\eta,1)$. Take $t_1 = I^*(1,\eta,1) - I^*(1,\eta,0) + \varepsilon + \delta$, with δ very close to zero (the reader may check that the bigger the δ the smaller the payoff for investors in bank 1). Depositors in bank 1 would get $u_1(\eta\rho R)$ and investors in bank 1 would get $\eta\rho R[I^*(1,\eta,1)-\delta]+B$. Thus, depositors prefer investing in the safe asset than gambling when the transfer is made. However, it may not be necessary preferred for investors in bank 1.

Let us compare strategies a), c) and d). The one yielding the highest payoff for investors will be the continuation equilibrium after banks 1 and 2 have joined the network. Investors in bank 1 prefer a) to c) if and only if

$$\eta[I^*(0,\eta,0)+\varepsilon] \geq I^*(0,\eta,0). \tag{8}$$

As η is arbitrarily high, this expression converges to $\varepsilon \geq 0$. On the other hand, they will prefer a) to d) if and only if

$$\eta\rho R[I^*(0,\eta,0)+\varepsilon-I^*(1,\eta,1)+\delta] > 1-I^*(1,\eta,1)+\delta.$$

Again, as η is arbitrarily high, this expression converges to $\rho R[\varepsilon+\delta] \geq \delta$ that always holds true for $\varepsilon \geq 0$. So if investors in bank 1 bear the decision of making the bank transfer, they will never make it (ex-ante) for sufficiently high η.[4]

Network formation. In order to see what are the incentives of being in a network, we have to confront the two possible equilibrium situations: the autarky (that is, when banks are not linked) and equilibrium a).

Given that bank 1 depositors get $u_1(R)$ in autarky and $u_1(\eta\rho R)$ when the banks are linked, they will prefer to join the network if and only if

$$\eta\rho > 1,$$

which is true. Investors in bank 1 will prefer to be in the network with respect being in autarky if and only if

$$\eta\rho R[I^*(0,\eta,0)+\varepsilon] > R[I^*(0,\eta,0)+\varepsilon],$$

which always hold for $\eta\rho > 1$. Therefore, both depositors and investors in bank 1 find it optimal to enter the network. Depositors in region 2 always prefer the network given that they obtain $\eta\rho R$ greater than ηR obtained in autarky. Investors in region 2 also prefer to join the network since

$$\eta\rho R[I^*(1,\eta,0)-\varepsilon]+B > \eta R[I^*(1,\eta,0)-\varepsilon]+B.$$

To sum up, equilibrium a) is optimal for all the agents in the economy and it will be preferred to autarky or equilibrium c) if η is sufficiently high.

Finally, let us consider the efficiency of the equilibria. The total amount of dollars that the social planner would achieve is $\rho R(2+e_1+e_2) = 2\rho R[1+I^*(1,\eta,1)]$. Note that this is the same amount of total dollars that equilibrium c) delivers. Then equilibrium c) is able to deliver the efficient *total* amount of money. However, according to (8), we know that when η is high enough the bank capital transfer becomes relatively more costly than the risk of financial instability, and equilibrium c) would not be played in a decentralized environment.

In autarky the financial system would deliver a total amount of dollars equal to $R+\eta R+[I^*(0,\eta,0)+\varepsilon]R+[I^*(1,\eta,0)-\varepsilon]\eta R$. Notice that the last expression is necessarily strictly less than the first-best outcome. Indeed, the maximum amount of dollars that the financial system can achieve when both banks are in autarky is when both of them invest in the safe asset. In this

[4] The reader could note that depositors in bank 1 would prefer instead equilibrium c) to a). This is true under the assumption that the investors pay the tranfer ε. If the transfer would be paid by depositors they would prefer equilibrium a) to c) if and only if

$$(1-\eta)\rho R \leq \varepsilon.$$

Also this expression converge to $\varepsilon \geq 0$ as $\eta \to 1$, and depositors would oppose the tranfer for sufficiently high η. What is crucial for equilibrium a) to be preferred to equilibrium c) is that the transfer has to be relatively more costly than the risk of financial instability.

case, the financial system gets $R(2 + e_1 + e_2) = 2R[1 + I^*(1, \eta, 1)]$ dollars, which is strictly less than the efficient amount.

Under equilibrium a) instead the total amount of dollars that the financial network achieves is

$$2\eta\rho R + [I^*(0, \eta, 0) + \varepsilon]\eta\rho R + [I^*(1, \eta, 0) - \varepsilon]\eta\rho R = 2\eta\rho R[1 + I^*(1, \eta, 1)].$$

If $\eta = 1$, the total dollars available would be exactly the efficient amount of dollars that a social planner would deliver. For $\eta \to 1$, the financial network characterized by equilibrium a) is able to deliver an amount of dollars close to the efficient amount. That is, as $\eta \to 1$ equilibrium a) gets arbitrarily close to the first-best. Furthermore, the equilibrium network structure is equal to the efficient one (both banks being linked).

The difference between equilibrium a) and the efficient one lies in the investment strategies chosen by the two banks. While in the efficient network both banks invest in the safe asset, in the decentralized network one of the banks is gambling. However, for very small probability of bankruptcy, the structure of the network becomes more relevant in determining the total payoff of the system.

This example shows that a fragile financial network can deliver the first-best outcome when moral hazard problem is nearly negligible. In other words, joining a fragile financial network, that is a network characterized by a small probability event that is potentially able to cause the meltdown of the entire system, not only can be an ex-ante optimal decision but it could allow to achieve a payoff arbitrarily close to the first-best solution (i.e., unconstrained efficient). We are going to check if this result survives in a more general setting.

4 First-best Solution

In order to characterize the results of the model, we need to introduce some definitions. Let (N, e) be a given economy, as defined in Section 2. An allocation is a vector (K, x, s), where: (i) $x \in \mathcal{X}$, (ii) $K \in \mathcal{K}$, and (iii) $s \in \mathcal{S}$. An allocation thus specifies a reallocation of initial endowments of capital x, a network K and an investment decision s for each bank.

Let the function $m(K, x, s)$ be as defined in equation (2). An allocation (K, x, s) is an *Investment Nash Equilibrium* (INE) for a given economy (N, e) if

$$m_i(K, x, s) \geq m_i(K, x, (s_{-i}, \tilde{s}_i)) \qquad \text{for all } i \text{ in } N,$$

with $\tilde{s}_i \in \{b, g\}$. In other words, an allocation is an INE for a given economy if taking the financial network and capital as given there are no unilateral profitable deviations in the choice of the investment asset, with no possibility for further transfers of bank capital. Note that, if an allocation is an INE for a given economy it has to hold that

$$x_i \geq I^*(k_i, \eta, g_i) \text{ if } s_i = b \qquad \text{for all } i \in N.$$

The first-best solution is characterized by the social planner problem, which is defined as follows.

Definition 1 *Let $p_i(K,s)$ be defined as in equation (1) for any given (K,s). Given an economy (N,e), an allocation (K^*,x^*,s^*) is a (constraint) first-best (CFB) if it maximizes*

$$\sum_{i\in N} p_i(K,s)f(k_i)R\left(x_i+1\right), \tag{9}$$

subject to

$$x_i \geq 0, \text{ for all } i \in N \tag{10}$$
$$\sum_{i\in N} x_i = \sum_{i\in N} e_i, \tag{11}$$
$$x_i \geq I^*(k_i,\eta,g_i) \text{ if } s_i = b \tag{12}$$

We will refer to x^* as the optimal capital allocation, K^* as the optimal financial network and s^* as the optimal investment decision.

Note that we are assuming that the planner is able to perfectly transfer the initial endowments of capital across banks, fix a financial network and suggest investment plans to the banks. We allow banks to unilaterally deviate from investment decisions (see constraint (12)). This restricts the social planner problem in a way that moral hazard has to be taken into account. Finally, note that the social planner does not care about private benefits B, and only maximizes the total amount of money that is generated in the economy. Indeed, the expression

$$p_i(K,s)f(k_i)R\left(x_i+1\right)$$

does not include B even when $s_i = g$. On the contrary, the payoff $m_i(K,x,s) + M_i(x,K,x)$ as defined in equations (2) and (3) includes B when $s_i = g$.

We first characterize the optimal distribution of bank capital made by the social planner in the CFB allocation.

Proposition 1 *Let (K^*,x^*,s^*) be a CFB for a given economy (N,e). Then,*

1. *For every bank i such that $s_i^* = g$: if there exists another bank j with $k_j^* \geq k_i^*$ such that either $s_j^* = b$ and $g_j \leq g_i$, or $s_j^* = g$ and $g_j < g_i$, then $x_i^* = 0$.*

2. *For every bank i such that $s_i^* = b$ and $g_i > 0$: if there exist another bank j with $k_j^* \geq k_i^*$ such that either $s_j^* = b$ and $g_j < g_i$ or $s_j^* = g$ with $g_j < g_i - 1$, then $x_i^* = I^*(k_i^*,\eta,g_i)$.*

Proposition 1 states that once the minimal capital that induces to choose the safe asset is met, the planner will distribute the extra amount of bank capital into the nodes that yield a higher return, either because they are better connected or because they face a smaller risk of bankruptcy.

We then establish the shape of the efficient financial network. The following proposition states that the CFB allocation consists of a core-periphery network structure. In this structure, the *core* banks choose the safe asset and are all connected to each other. The *peripheral* banks choose the gambling asset and can be eventually connected to some core banks and some peripheral banks depending on the value of the parameters. Note that the structure where all banks play safe, is a special core-periphery structure where the periphery consists of no banks, that is all banks are in the core.

Proposition 2 *Let (K^*, x^*, s^*) be a CFB for a given economy (N, e). Then, for every pair of banks i and j such that $s_i^* = s_j^* = b$ we have that $i \in K_j^*$ and $j \in K_i^*$.*

The intuition is as follows. When two banks are choosing the safe asset, it is always better to have them connected than unconnected. This is true since one more neighbor always increases investment opportunities. Given that both banks are choosing the safe asset, linking them together does not impose any additional risk. Indeed, if a bank has enough bank capital to choose the safe asset in a given financial network, then the same capital will be sufficient to avoid gambling if the bank has one more neighbor that invests in the safe asset.

Note that the total amount of capital available restricts the number of banks that will invest in the safe asset in the CFB allocation. It is easy to see that, as far as

$$\sum_{i \in N} e_i \geq nI^*(n-1, \eta, 0)$$

the CFB consists of $x_i \geq I^*(n-1, \eta, 0)$, for all i, $K_i^* = N \backslash \{i\}$, and $s_i^* = b$. In other words, as far as the planner has enough bank capital, the CFB allocation is equal to the (unconstrained) first-best where moral hazard is completely avoided and all banks are connected investing in the safe asset. When the planner does not have enough bank capital, i.e. when

$$\sum_{i \in N} e_i < nI^*(n-1, \eta, 0),$$

the structure of the CFB allocation changes depending on the relative magnitude between the risk of gambling and the benefits of the financial network.

On the one hand, the higher η and the lower is the risk of contagion, so the CFB yields structures where the periphery is more and more connected, until in the limit the CFB financial network is the complete network. On the other hand, the lower η and the higher is the the risk of contagion, and then the CFB allocations yield structures where the periphery is less and less connected, until in the limit the CFB financial network only connects banks that invest in the safe asset. That is, the CFB is characterized by the complete network on the core and the empty network on the periphery. This is formally stated in the following two propositions.

Proposition 3 *Let (K^*, x^*, s^*) be a CFB for a given economy (N, e) and assume that $\eta \geq \dfrac{f(n-2)}{f(n-1)}$. Then*

1. *$K_i^* = N \backslash \{i\}$ for all $i \in N$.*

2. *Let c^* be the biggest number in $\{1, 2, ..., n\}$ such that*

$$\sum_{i \in N} e_i \geq c^* I^*(n-1, \eta, n-c^*),$$

then the number of banks investing in the safe asset in the CFB is equal to c^*.

The intuition is straightforward. When η is high enough, the risk of bankruptcy is sufficiently low such that it cannot outweigh the advantages of portfolio diversification represented by $f(k)$. Therefore, connecting two banks always yields more money than leaving them unconnected. Note that it is convenient to add a gambling bank into the network as long as

$$\eta f(k_i+1) > f(k_i).$$

Consequently, if the condition

$$\eta \geq \frac{f(n-2)}{f(n-1)}$$

is satisfied, then it is optimal to add the gambling banks into the network until the $(n-1)th$ bank (i.e., the last bank). Thus, the structure that maximizes the (total) expected amount of money generated in the economy is the complete network structure.

Proposition 4 *Let (K^*, x^*, s^*) be a CFB for a given economy (N, e) and assume that $0 < \eta < \frac{1}{f(1)}$. Then*

1. *$g_i^* = 0$ for all i such that $s_i^* = g$.*
2. *Assume further that $\eta < \frac{1}{\rho + 2(n-1)(\rho-1)}$. Then*
   (a) *$K_i^* = \emptyset$ for all i such that $s_i^* = g$.*
   (b) *Let c^* be the biggest number in $\{1, 2, ..., n\}$ such that*

$$\sum_{i \in N} e_i \geq c^* I^*(c^*-1, \eta, 0),$$

then the number of banks investing in the safe asset in the CFB is equal to c^*.

As before, the same intuition applies. When η is very low, the risk of bankruptcy can outweigh the advantages of financial diversification and it becomes optimal not to connect a bank investing in the gambling asset to another bank. However, we have to differentiate between linking a gambling bank to another gambling bank or to a safe one.

Note that it is not optimal to connect two gambling banks whenever

$$\eta^{g_i} f(k_i) > \eta^{g_i+1} f(k_i+1) \Longrightarrow f(k_i) > \eta f(k_i+1).$$

Since $f(k_i)$ is quasiconcave, then if the condition

$$\eta < \frac{f(0)}{f(1)} = \frac{1}{f(1)}$$

holds, then it is never optimal to connect two gambling banks.

However, the planner could find it optimal to link a gambling bank to a safe bank if the expected gain of the former are higher than the expected loss of the latter. Proposition 4 establishes a sufficient condition under which this possibility does not occur. Note that the condition

$$\eta < \frac{1}{\rho + 2(n-1)(\rho-1)}$$

can be written as $\eta\rho - 1 + 2(n-1)\eta(\rho-1) < 0$. Consequently, the condition implies that $\eta\rho < 1$.

On the one hand, note that when the planner links a gambling bank to a safe bank the expected gain is strictly less than $2(n-1)\eta(\rho-1)R$. To see why, notice that a gambling bank can gain at most

$$\eta\left[f(k_i) - 1\right](e_i + 1)R$$

when it is connected to k_i safe banks. Since $f(.)$ is increasing and $e_i < 1$, that gain is strictly smaller than $2\eta(\rho-1)R$. The number of banks that are in the periphery can be at most $(n-1)$ since one bank has to invest in the safe asset. Then, considering any possible way of joining any number of gambling banks to safe banks, the total expected gain cannot be larger than $2(n-1)\eta(\rho-1)R$. On the other hand, the loss for a safe bank to be connected to g_i gambling banks is

$$\left[\eta^{g_i} f(k_i + g_i) - f(k_i)\right](e_i + 1)R.$$

Given that $f(.)$ is increasing and $\eta < 1$, the minimum loss is given by $[\eta\rho - 1](e_i + 1)R$, which is strictly less than $[\eta\rho - 1]R$. If the minimum loss plus the maximum gain is still negative, that is $(\eta\rho - 1)R + 2(n-1)\eta(\rho-1)R < 0$, then it cannot be optimal to connect a gambling bank with a safe bank.

Consider again the two-banks case discussed in Section 3. Since $n = 2$, we have

$$\frac{f(n-2)}{f(n-1)} = \frac{f(0)}{f(1)} = \frac{1}{\rho}.$$

Assume first that η is high, such that $\eta\rho \geq 1$. The CFB allocation will depend on the aggregate amount of bank capital. When $e_1 + e_2 \geq 2I^*(1,\eta,0)$ the two banks are linked investing in the safe asset. In this case the CFB allocation is an unconstrained first-best. When $I^*(1,\eta,1) < e_1 + e_2 \leq 2I^*(1,\eta,0)$ the two banks being linked is still an optimal financial network, but there is not enough capital for both banks to invest safe. One of the two banks invests optimally in the gambling asset, imposing the risk of contagion on the other bank. Finally, when $0 < e_1 + e_2 \leq I^*(1,\eta,1)$, the CFB allocation still recommends both banks to be connected investing in the gambling asset.

Now assume that η is low, such that $\eta\rho < 1$. Also in this case the CFB allocation will depend on the amount of aggregate bank capital. When $e_1 + e_2 \geq 2I^*(1,\eta,0)$ the two banks will still be linked investing in the safe asset. However, when $I^*(1,\eta,1) \leq e_1 + e_2 < 2I^*(1,\eta,0)$ the two banks being linked might not be an optimal financial network anymore since one bank invests in the gambling asset. It would be optimal to connect them if and only if

$$\eta \geq \frac{e_1 + e_2 + 1}{\rho(e_1 + e_2 + 2) - 1} \equiv \overline{\eta}.$$

Otherwise, if $\eta < \overline{\eta}$, it is optimal to be disconnected. Note that

$$\overline{\eta} > \frac{1}{\rho + 2(\rho - 1)},$$

where $\eta < \dfrac{1}{\rho + 2(\rho-1)}$ is the sufficient (but not necessary) condition for an empty periphery in Proposition 4. Finally, when $0 \leq e_1 + e_2 < I^*(1,\eta,1)$, there is not enough aggregate bank capital to guarantee at least one bank investing safe, and then it is optimal to disconnect the two banks.

4.1 Examples of Efficient Network Structures

In this Section we provide examples of efficient network structures focusing on the case of four banks. We consider the four banks case since it has been widely used in previous banking literature (see Allen and Gale [2]; Brusco and Castiglionesi, [5]).

Let

$$\sum_i e_i = E$$

the total bank capital available. With four banks, when the aggregate bank capital satisfies the condition $E \geq 4I^*(3, \eta, 0)$, then the efficient network structure is the complete one with everybody linked with everybody else and investing in the safe asset. That is, the core of the financial network is made of four banks.

When the aggregate bank capital is such that $E < 4I^*(3, \eta, 0)$, then avoiding moral hazard in the complete network structure is no longer possible. Consequently, the core of the financial network will be made of three (or less) banks.

Figure 1 shows four core-periphery network structures under the assumption that the core is made of three banks. In particular, the two banks represented at the top and the one at the bottom right are investing in the safe asset while the one at the bottom left is investing in the gambling asset.

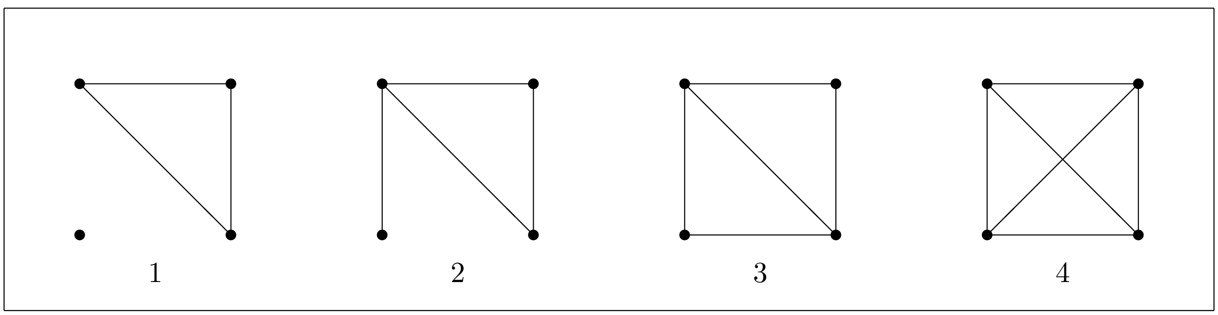

FIGURE 1

Clearly, the efficient structure depends both on the total capital available and on how η relates to the different values of the function $f(k)$. Let us now be more specific about the total capital available and the values of η and $f(k)$. Table 2 reports the minimal capital requirements for network structures 1, 2, 3 and 4 represented in Figure 1.

Table 2. Minimum aggregate capital for network structures 1, 2, 3 and 4

Structure	Minimum Capital
1	$3I^*(2, \eta, 0) = E_1$
2	$2I^*(2, \eta, 0) + I^*(3, \eta, 1) = E_2$
3	$I^*(2, \eta, 0) + 2I^*(3, \eta, 1) = E_3$
4	$3I^*(3, \eta, 1) = E_4$.

In order for structure 1 to be feasible (i.e., to be an INE), the aggregate bank capital has to be at least E_1. Otherwise, it would not be feasible. The same reasoning applies to the other structures. For each feasible network structure with three banks in the core, there are different ways of allocating the total capital endowment. The optimal choice depends on how the values of $\eta f(3)$ and $f(2)$ are related.

Recall that the social planner has to meet the minimal individual capital requirement since the CFB has to be an INE. Then he has to allocate the remaining capital, if any, to the banks with the highest return (given the network). Assume that $\eta f(3) > f(2)$. This implies that a bank with three neighbors, with one of them investing in the gambling asset, obtains higher returns of a bank with two neighbors investing in the safe asset.

Under the assumption that $\eta f(3) > f(2)$ we have $E_1 > E_2 > E_3 > E_4$. The social planner will choose then the structure that maximizes the total amount of money generated by the network. Table 3 specifies the total amount of money generated by each structure in Picture 1 under the assumption that $\eta f(3) > f(2)$.

Table 3. Total amount of money generated by structures 1, 2, 3 and 4

Structure	Total Amount
1	$f(2)[E+3]+\eta$
2	$\eta f(3)[E+1-2I^*(2,\eta,0)]+$ $+2f(2)[I^*(2,\eta,0)+1]+\eta f(1)$
3	$\eta f(3)[E+2-I^*(2,\eta,0)]+$ $+f(2)[I^*(2,\eta,0)+1]+\eta f(2)$
4	$\eta f(3)[E+4]$

From Table 3 it is clear that the total amount of money generated by structure 4 is larger than the one generated by structure 3, which is larger than the one generated by structure 2, which is larger than the one generated by structure 1. Consequently, whenever feasible, the planner would choose structure 4. This is the case when $E_4 < E < 4I^*(3,\eta,0)$.

Under the conditions $\eta f(3) > f(2)$ and $E_4 < E < 4I^*(3,\eta,0)$, the planner chooses network structure 4 with the corresponding optimal capital allocation. Intuitively, when η is high enough the optimal network structure is the complete one. Note the difference with the previous case, where the core of the complete network was made of four banks. Now the complete network is fragile, while before it was safe. The same complete structure can be characterized by different degree of financial fragility. If $E_4 > E$ then the core of the networks has to be made of two banks (or less).

The same reasoning would apply if we have $f(2) > \eta f(3)$. In this case the inequality between the minimum capitals in Table 2 becomes $E_1 < E_2 < E_3 < E_4$. If $f(2) > \eta f(3)$ and $E_1 < E < 4I^*(3,\eta,0)$ then the planner chooses network structure 1 for $\eta < \frac{1}{f(3)+6(f(1)-1)}$, by Proposition 4. In such a case, the optimal network structure implies to link only banks that are not investing in the gambling asset. Also in this case, if $E_1 > E$ then the core of the networks has to be made of two banks (or less).

Figure 2 represents core-periphery structures for four banks where the core is made of two banks. In particular, the two banks represented at the top are the core ones, while the two banks at the bottom are the periphery ones. The latter banks could be disconnected (as in structures 5 until 11) or connected (as in structures 12 until 18). Moreover, the periphery banks could be not connected to the core (as in structures 5 and 12) or connected to one or both the core banks.

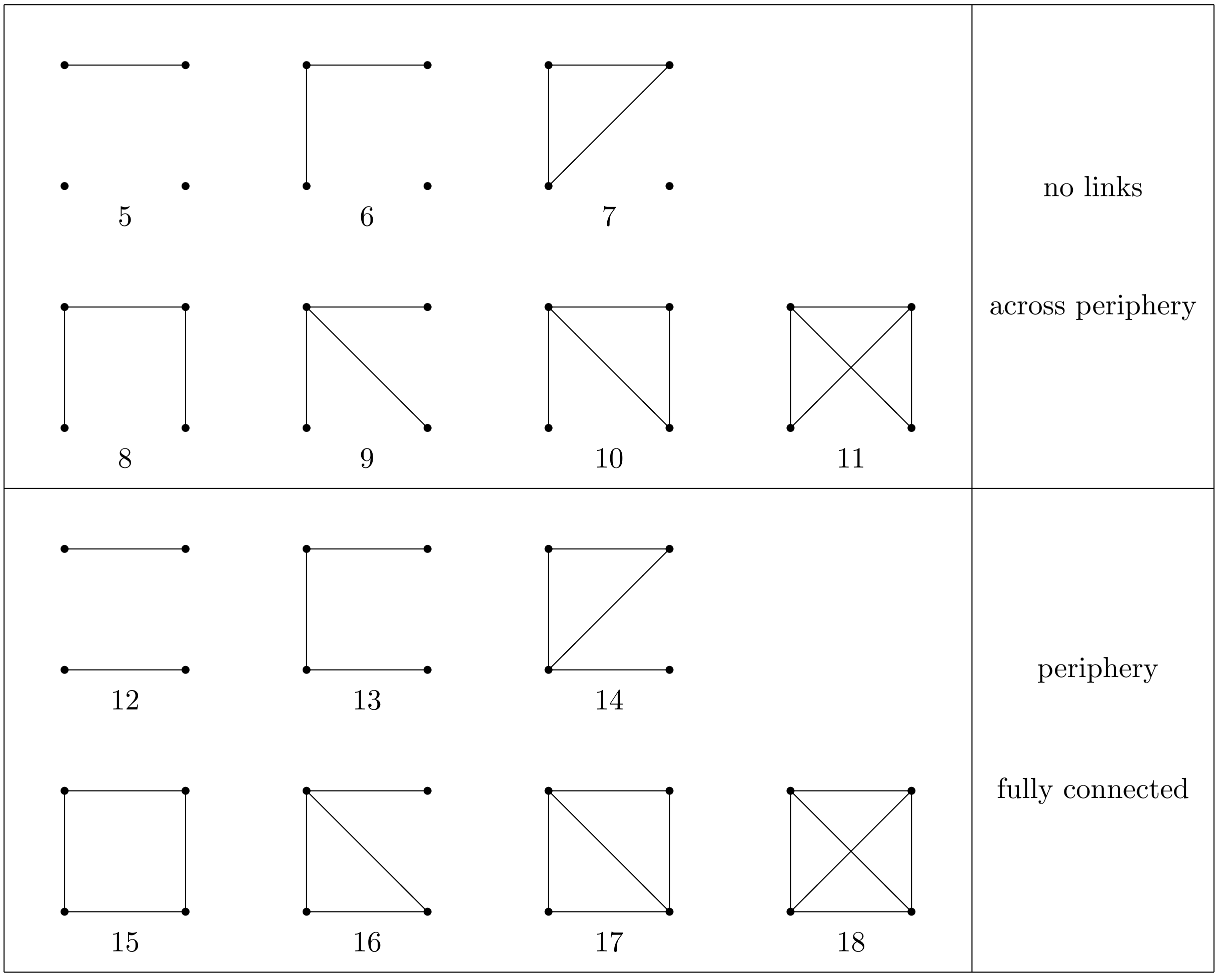

FIGURE 2

Note that structures number 2 is the same as structures 10, 14 and 16. Also structure 3 coincides with structure 17. Structure 4 is equal to structure 18. The difference is in the investment decision of the banks. In structures 2, 3 and 4 there are three banks choosing the safe asset, while in the other structures there are only two banks doing so. That is, in the latter structures the core is smaller.

Again, under the conditions $\eta f(3) > f(2)$ and $E < E_4$, the optimal network structure would be 18 (for the same reason as before). This means that the planner would choose the fully connected structure also in this case. However, structure 18 implies a financial fragility higher than the one in structure 4 since in the former structure there are two banks gambling while in the latter there was only one bank doing so. Again, the same (fully connected) structure can imply different degree of financial fragility.

However, for different parameter values, other structures can arise as the optimal one. For example, let us consider network structure 15, which is the incomplete structure analyzed in Allen and Gale [2]. As already seen, assuming $E < E_1$ and $\eta f\,(3) < f\,(2)$, no structure with a core made of three banks can be an INE (there is not enough capital available.) Recall that the function $f(k)$ is quasiconcave. This means that the ratio $\frac{f(k)}{f(k+1)}$ is increasing in k and $f(k) > \frac{f(k+1)+f(k-1)}{2}$. Furthermore, assume that η satisfies the following condition

$$\frac{f(1)}{f(2)} < \eta < \frac{f(2)}{f(3)}.$$

Note that the condition $\eta f(2) > f(1)$, which implies $\eta f(1) > 1$, guarantees that whenever there is a bank connected to at most one other bank, it is better to connect it to another bank even if the new bank is gambling. This implies that structure 8 yields an higher amount of money (whenever feasible) than structure 6. For the same reason, structure 6 is preferred to structure 5; structure 15 is preferred to structures 8, 12 and 13; and structure 16 is preferred to structure 9.

Moreover, since $f(1) > \frac{f(2)+f(0)}{2}$, structure 8 is preferred to structure 7, and, since $f(2) > \frac{f(1)+f(3)}{2}$, structure 15 is preferred to structure 14. It remains to compare then structures 10, 11, 15, 16, 17 and 18. Note that $\eta f(3) < f(2)$ implies that $I^*(3,\eta,2) > I^*(2,\eta,1)$, and that $f(1) < \eta f(2)$ implies that $I^*(1,\eta,0) > I^*(2,\eta,1)$. Let

$$\tilde{E} \equiv \min\{I^*(2,\eta,1) + I^*(3,\eta,2), I^*(2,\eta,1) + I^*(1,\eta,0)\}.$$

If the total endowment of bank capital E lies in the interval $\left[2I^*(2,\eta,1), \tilde{E}\right)$ then structures 10, 11, 16, 17 and 18 are not feasible anymore. So the CFB allocation is given by structure 15 when

1. $\frac{f(1)}{f(2)} < \eta < \frac{f(2)}{f(3)}$, and
2. $2I^*(2,\eta,1) \leq E < \tilde{E}$ and $E < 3I^*(3,\eta,1)$.

Note that this is not the unique case in which structure 15 could be the CFB allocation. Finally, observe that contingent on the failing of one bank, the incomplete structure 15 is more resilient than the complete structure 18. Indeed, when one of the gambling bank fails, structure 15 implies that one safe bank will survive. Structure 18 instead implies that all the system will collapse. In other words, the probability of failing of the entire system under structure 18 (i.e., η) is higher than the same probability under structure 15 (i.e., η^2).

5 Financial Network Formation

In this section we analyze the decentralized financial network formation. To fulfill such task, we start with the concept of INE. Given an INE, we define a Transfer Nash Equilibrium (TNE) where the network structure is taken as given and agents know that, once transfers are realized, banks play an INE in choosing the investment (i.e., in the last stage). Finally, using the notion of pairwaise stability (Jackson and Wolinsky, [9]), the agents choose the network taking into account the TNE and the INE. In other words, given the sequence of events described in Table 1, we solve the model backwards.

Let the payoff function $m(K,x,s)$ be as defined in (2). Recall that an allocation (K,x,s) is an INE for a given economy if taking the financial network and capital as given there are no unilateral profitable deviations in the investment choice. Recall also that an INE allocation has to satisfy for all $i \in N$ that

$$s_i = \begin{cases} b, & \text{if } x_i \geq I^*(k_i, \eta, g_i), \\ g, & \text{otherwise.} \end{cases}$$

In order to analyze the decentralized network formation, we have to specify the transfers that lead to a particular allocation of bank capital x. So far, we did not specify any rule about transfers. We have just assumed that a bank i can only give a transfer to another bank j that is connected to it in the financial network.

Our equilibrium concept will select the allocations where banks *do not* have any further incentive to transfer money through the financial network.

Definition 2 *Given an INE (K, x, s) and a bank i, another allocation $(K, \tilde{x}_i, s(\tilde{x}_i))$ is called a* short-sighted profitable deviation *for a bank i from (K, x) if there exists a transfer $0 < t_i \leq x_i$ and a set of neighbors $J \subseteq G_i$ such that:*

1. $\tilde{x}_i(K, x) = x_i - t_i$, $\tilde{x}_j(K, x) = x_j + t_{ij} \geq I^*(k_j, \eta, g_j - |J \cap G_j|)$, with $\sum_{j \in J} t_{ij} = t_i$, for all $j \in J$, and $\tilde{x}_r(K, x) = x_r$ for all $r \notin J$;

2. $$s_i(\tilde{x}_i) = \begin{cases} b, & \text{if } x_i - t_i \geq I^*(k_i, \eta, g_i - |J|), \\ g, & \text{otherwise,} \end{cases}$$

   $s_j(\tilde{x}_i) = b$ for all $j \in J$ and $s_r(\tilde{x}_i) = s_r$ for all $r \notin J$;

3. $m_i(K, \tilde{x}_i, s(\tilde{x}_i)) > m_i(K, x, s)$ and $m_j(K, \tilde{x}_i, s(\tilde{x}_i)) \geq m_j(K, x, s)$ for all $j \in J$.

Note that a profitable deviation for a bank i occurs if it can make transfers to a set of neighbors in the network to avoid those neighbors' moral hazard, expecting these neighbors to accept the transfer. If the transfer does not avoid the moral hazard problem bank i cannot gain anything from making such transfers. Therefore that case is not considered in a profitable deviation.

To keep things tractable, we will assume that after making a transfer to a set of neighbors J, bank i can anticipate changes to the investment decisions corresponding only to that set of neighbors J receiving the transfer and itself. Hence, $s(\tilde{x}_i)$ is restricted to $s_r(\tilde{x}_i) = s_r$ for all $r \notin J$. Note that a short-sighted profitable deviation for a bank i given an INE is not necessarily an INE itself, as some neighbors of banks in J could decide to move to the safe asset when the banks in J do so.

An allocation (K, x, s) is a *Transfer Nash Equilibrium* (TNE) if it is an INE and for all $i \in N$ there is no short-sighted profitable deviation from (K, x). Let the set $\mathcal{T}(K)$ denote the set of all possible TNE capital reallocations and investment strategies, once the financial network has been fixed to be equal to K. Formally,

$$\mathcal{T}(K) = \{(x, s) \in \mathcal{X} \times \mathcal{S} \text{ such that } (K, x, s) \text{ is a TNE}\}.$$

Note that in the definition of the TNE there is no specification about the order or dynamics of transfers. If we allow banks to make transfers to one of their neighbors that are choosing the gambling asset at any point, a TNE is a stable point in a short-sighted way. The reader should note that there are no guarantees that a free process of making transfers might end up in a cycle of allocations instead of a single allocation. Nevertheless, we can find conditions under which the transfers are never made. In other words, we can identify the conditions under which for every K there exists a strategy $s(K) \in \mathcal{S}$ such that the allocation $(K, e, s(K))$ is a TNE.

Let $e^{\max}$ the biggest realization of e_i, and let $e^{\max}_{<1}$ the biggest realization of the e_i's that are strictly smaller than 1, if any. We have the following

Proposition 5 *Given the economy (N, e) and η, assume that one of the following conditions is satisfied:*

1. $e_i \geq 1$ *for all* $i \in N$,
2. $e^{\max} + e^{\max}_{<1} < I^* (n-1, \eta, 0)$,
3. $\eta > \tilde{\eta}(e)$, *for a threshold* $\tilde{\eta}(e) \in (0,1)$,

then for any $s \in \mathcal{S}$ such that (K, e, s) is an INE it is true that (K, e, s) is a TNE.

The intuition of the proposition is as follows. Condition 1 means that all the banks on every possible network are sufficiently capitalized, and this implies that they will invest in the safe asset. This implies that there is no need of any bank capital transfer.

Condition 2 implies that $e^{\max} < 1$, and given the definition of $e^{\max}_{<1}$, this implies that

$$e^{\max} = e^{\max}_{<1} < \frac{1}{2} I^* (n-1, \eta, 0) .$$

This means that everybody plays gamble in every possible financial network, and that two banks never have enough bank capital to make at least one of them choosing the safe asset. Again, there are no incentives any bank capital transfer.

Condition 3 states that η has to be sufficiently large, given the initial bank capital endowments, when neither condition 1 nor condition 2 is satisfied. Recall that η sufficiently large means that the risk of bankruptcy of the gambling banks is low. The only reason it might be profitable for a bank to initiate transferring bank capital to its neighbors is to avoid their gambling behavior. However, if the gambling behavior is not too risky, i.e. there is a low probability of bankruptcy, banks will not find it worthwhile to give away capital in exchange of financial stability.

We define an **economy without transfers** an economy (N, e), and the given η, in which either condition 1, or condition 2, or condition 3 in Proposition 5 is satisfied.

As already anticipated, our notion for network formation is an adapted version of *pairwise stability* introduced by Jackson and Wolinsky [9]. Before giving the definition we need to introduce some notation. Given a network K, we can define a new network $K \cup ij$ resulting from adding a link joining banks i and j to the existing network K. Formally, $K \cup ij = \left(\tilde{K}_1, ..., \tilde{K}_n\right)$ such that $\tilde{K}_i = K_i \cup \{j\}$, $\tilde{K}_j = K_j \cup \{i\}$ and $\tilde{K}_r = K_r$ for all $r \neq i, j$. On the contrary, for any two banks i and j connected in K, let $K \backslash ij$ denotes the resulting network from severing the link joining banks i and j from K. Formally, $K \backslash ij = \left(\tilde{K}_1, ..., \tilde{K}_n\right)$ such that $\tilde{K}_i = K_i \backslash \{j\}$, $\tilde{K}_j = K_j \backslash \{i\}$ and $\tilde{K}_r = K_r$ for all $r \neq i, j$.

Definition 3 *An allocation* (K, e, s) *is pairwise stable without transfers (PSWT) if the following holds:*

1. For all i and j directly connected in K: $m_i(K, e, s) \geq m_i(K \backslash ij, e, \tilde{s})$ and $m_j(K, e, s) \geq m_j(K \backslash ij, e, \tilde{s})$ for all allocations $(K \backslash ij, e, \tilde{s})$ that are INE.

2. For all i and j not directly connected in K: if there is an INE $(K \cup ij, e, \tilde{s})$ such that $m_i(K, e, s) < m_i(K \cup ij, e, \tilde{s})$, then $m_j(K, e, s) \geq m_j(K \cup ij, e, \tilde{s})$.

The definition of PSWT captures two ideas that directly derive from the notion of pairwise stability (Jackson and Wolinsky, [9]). The first idea refers to the network internal stability: No pair of banks directly connected in the current financial network individually gain from severing their financial link. The idea implicitly states that any of the two banks could sever the link unilaterally. The second idea establishes the network external stability: If one bank could gain from creating a link with another bank, it has to be that the other bank cannot gain from that link. This idea implicitly assumes that both banks have to agree in order to create a new link. The willingness of one bank in creating a new link is not enough to change the network structure.

Note that we define the equilibrium as pairwise stable without transfers since the banks are assuming that no transfers are going to be made once the network is formed, and the resulting outcome is going to be an INE fixing all bank capitals to be equal to the initial endowments. From Proposition 5, pairwise stability without transfers makes sense in the context of an economy without transfers. If none of the conditions that guarantee that no transfers are taking place, some banks in the financial system could be willing to transfer bank capital to their neighbors. In such a case, it is not guaranteed that the system will rest in a TNE once the dynamic of transfers has started, or it will rest in a cycle of transferring behavior.[5]

Definition 4 *An allocation* (K, e, s) *is a decentralized equilibrium without transfers (DEWT) if it is INE, TNE and PSWT.*

We proceed to describe the set of decentralized equilibria for a given economy (N, e) by means of the following proposition.

Proposition 6 *Assume that* η *and* (N, e) *define an economy without transfers. Then, a DEWT is a core-periphery structure, i.e., if* (K^e, e, s^e) *is a DEWT, then, for every pair of banks* i *and* j *such that* $s_i^e = s_j^e = b$*, we have that* $i \in K_j^e$ *and* $j \in K_i^e$*.*

On the one hand, a bank agrees to be connected to any neighbor that is choosing the safe asset since this decision entails no extra risk of bankruptcy. On the other hand, if a bank plays safe any other bank would like to be connected to it for the same reason. Since links are expected to be beneficial for both participating banks, two banks choosing safe will normally be connected. Therefore a core-periphery structures appears where all banks choosing the safe asset are connected among themselves. The connectivity of banks choosing gambling (the low capitalized banks) depends on how the parameters of the model relate.

[5] One could impose additional conditions on the problem, but they will make the paper technically less tractable and the spirit of the paper would remain the same.

Note that Proposition 6 does not imply that the network structure in a DEWT is the same as in the optimal one. The core in the CFB might have a different size than the core in any DEWT.

6 Financial Network Efficiency

With the presence of moral hazard the (unconstrained) first-best cannot be reached when the total capital available is not large enough. However, for vanishing probability of bankruptcy we show that the structure of the decentralized network is equal to the efficient one. Furthermore, the total payoff delivered by the decentralized network is arbitrarily close to the efficient one. We have the following.

Proposition 7 *If η and (N, e) define an economy without transfers, and $\eta \geq \dfrac{f(n-2)}{f(n-1)}$ then the only network structure for any DEWT is the complete network structure.*

This result states formally the idea that when the risk of financial contagion is sufficiently low, it is always worthwhile to take the risk of being connected to a low capitalized bank in order to obtain the advantages resulting from investment diversification.

Note that Proposition 7, even if it establishes that the decentralized network is equal to the optimal one, does not imply the investment decisions are the same in both networks. In the DEWT the investment decisions might be suboptimal, as the example in Section 3 showed, since some banks can gamble while in the efficient network they would invest in the safe asset. However, when η tends to 1, all investment profiles yield the same total amount of money for each given network structure. In other words, as the moral hazard problem vanishes, the only factor that determines the payoff *is* the network structure.

Proposition 8 *If η and (N, e) define an economy without transfers, and $\eta < \dfrac{1}{f(1)}$ the only network structure for any DEWT is a core-periphery structure where the periphery has no links.*

Proposition 8 states that when the risk of financial contagion is sufficiently high, the network formed by agents has zero probability of contagion. Only agents that are choosing the safe asset have connections in the network. Let us compare Proposition 8 with Proposition 4. The former establishes that $\eta < \dfrac{1}{f(1)}$ is sufficient to observe an empty periphery in the decentralized network, while the latter states that the same condition is not sufficient for the empty periphery to be optimal. Recall that $\eta < \dfrac{1}{f(1)}$ guarantees that no safe bank wants to connect to a gambling bank, but it could be that the gains for the gambling bank outweight the loss for the safe bank if they get connected and the planner links them together. As a consequence, gambling banks are (inefficiently) under-connected in the decentralized network for $\eta \in [\frac{1}{\rho+2(n-1)(\rho-1)}, \frac{1}{f(1)}]$.

Finally, in order to establish fully the efficiency of the network structures, we have to combine the conditions in Propositions 7 and 8 with the conditions stated in Proposition 5.

Condition 1 in Proposition 5 implies that everybody plays safe in every possible network. This means that the fully connected network will be formed and everybody chooses the safe asset. This is indeed the (unconstrained) first-best outcome. This result is reached independently of the value taken by η.

Condition 2 in Proposition 5 implies that everybody plays gamble in every possible network. On the one hand, if the condition in Proposition 7 holds, the complete network will be formed, which is the same structure as in the CFB. Again, this does not mean that the optimal investment profile is to choose the gambling asset for all banks, since the social planner could pool the bank capitals inducing one (or some) bank to choose the safe investment. On the other hand, if the condition in Proposition 8 holds, the resulting structure is the empty network since all banks are periphery banks. This structure could not coincide with the CFB structure if by pooling all the bank capital endowments the planner can get at least one bank choosing the safe asset.

Condition 3 in Proposition 5 implies that η has to be high enough given the bank capital endowments. As before, under the condition in Proposition 7, the resulting structure is the CFB, although the investment profiles need not to be the optimal ones. Under the condition in Proposition 8 it is not guaranteed that the decentralized network structure is equal to the CFB one, as it depends on how the capital endowments are distributed.

Overall, when η is sufficiently high (according to the condition in Proposition 7), we have established that the decentralized financial network is characterized with a structure that coincides either with the (unconstrained) first-best structure or the CFB structure. On the contrary, when η is sufficiently low, the structure of the decentralized network is not the same as the CFB structure.

7 Discussion

7.1 Alternative Motivation for the Network Benefits

We motivated the benefit of the financial network with the assumption that the return to investment increases with the number of banks connected. Alternatively, another rationale for the benefits of establishing financial links among banks can be found also in the banking literature. If banks face idiosyncratic liquidity shocks due to consumers' consumption preferences (Diamond and Dybvig, [6]), and as long as there is no aggregate liquidity shortage, the uncertainty arising from these shocks can be eliminated by establishing financial links (Allen and Gale, [2]). Moreover, banks in autarky would need to invest more resources in short term liquidity to prevent high idiosyncratic liquidity shocks, and consequently less resources can be invested in more profitable long term projects. The same result holds when banks are affected by moral hazard problem (Brusco and Castiglionesi, [5]).

To capture this feature we can assume that the per unit cost of the investment (both safe and gambling) is decreasing in the number of banks linked with the investing bank $i \in N$, where $K_i \subseteq N$ is still the set of banks to whom bank i is directly linked. Then the number of banks connected to bank i is $k \in \{0, 1, ..., n-1\}$. We indicate the cost of investment with the function $C(k)$ with $C'(k) < 0$, and $C''(k) > 0$ for all $k \in [0, n-1]$. We assume that $C(0) = 1$ and $C(n-1) = c$, with $0 < c < 1$, that is $C(k) \in [c, 1]$. Consequently a bank that makes the investment in autarky ($k = 0$) faces the highest cost, while a bank that is connected with all the other banks ($k = n-1$) faces the lowest investment cost. With a little change in notation, this assumption conserves the same mathematical structure of the one used in the text and consequently all our results go through.

8 Conclusion

We present a model where fragile financial networks can be optimal when aggregate bank capital is not large enough to avoid any problem of moral hazard. From a decentralized point of view, banks can rationally join a fragile financial network as far as the risky asset fails with a sufficiently low probability. We characterize the set of optimal financial networks as core-periphery structures that get more and more connected as the probability of failure becomes smaller. The financial networks resulting from a decentralized process of network formation are also core-periphery structures provided there are no bank capital transfers, although may not be exactly equal to the optimal ones. Furthermore, the investment profiles in the decentralized system are in general different from the efficient. However, our main conclusion is that the decentralized system works close enough to the social planner's solution when the probability that the network collapses is negligible. On the contrary, when the same probability is sufficiently high, inefficient structures of the decentralized financial network arise.

Appendix

Proof of Proposition 1. The first statement is proved by contradiction. Assume that (K^*, x^*, s^*) is a CFB for (N, e) such that there is a bank i such that $s_i^* = g$ and $x_i^* > 0$, and there is another bank j with $k_j^* \geq k_i^*$ such that $s_j^* = b$ and $g_j \geq g_i$ (the proof of the case for $s_j^* = g$ and $g_j < g_i$ is equivalent and therefore omitted). Take $(\hat{x}, K^*, s^*)$ a new allocation for the same economy (N, e), where the network and investment strategies are the same. The allocation of capital differs in bank j receiving all the capital endowment of bank i. Formally, $\hat{x}_i = 0$, $\hat{x}_j = x_j^* + x_i^*$, and $\hat{x}_r = x_r^*$ for all $r \neq i, j$. We show that $(\hat{x}, K^*, s^*)$ yields a higher expected total money generated in the economy. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planners problem and statement 1 follows. Note that if (K^*, x^*, s^*) is a CFB then it has to be an INE. Therefore, as j is choosing the safe asset, we have that $x_j^* \geq I^*(k_j^*, \eta, g_j)$. By definition of $\hat{x}$, we have that $\hat{x}_j > x_j^* \geq I^*(k_j^*, \eta, g_j)$ and therefore the allocation $(\hat{x}, K^*, s^*)$ is an INE. Furthermore,

$$\begin{aligned} & \sum_{r\in N} p_r(K^*, s^*) f(k_r^*) R\left(\hat{x}_r + 1\right) - \sum_{r\in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \quad & R x_i^* \left[\eta^{g_j} f(k_j^*) - \eta^{g_i+1} f(k_i^*)\right] \geq 0, \end{aligned} \tag{13}$$

as $k_j^* \geq k_i^*$, the function $f(k)$ is increasing on k and $g_j \leq g_i$. By equation (13), the allocation $(\hat{x}, K^*, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB.

Also the second statement is proved by contradiction. Assume that (K^*, x^*, s^*) is a CFB for (N, e) but there is a bank i such that $s_i^* = b$ and $x_i^* > I^*(k_i^*, \eta, g_i)$, but there is another bank j with $k_j^* \geq k_i^*$ such that $s_j^* = b$ and $g_j < g_i$ (the proof of the case for $s_j^* = g$ and $g_j < g_i - 1$ is equivalent and therefore omitted). Take $(\hat{x}, K^*, s^*)$ a new allocation for the same economy (N, e), where the network and investment strategies are the same. The allocation of capital differs in bank j receiving all *extra* capital endowment of bank i. Formally, $\hat{x}_i = I^*(k_i^*, \eta, g_i)$, $\hat{x}_j = x_j^* + x_i^* - I^*(k_i^*, \eta, g_i)$, and $\hat{x}_r = x_r^*$ for all $r \neq i, j$. We show now that $(\hat{x}, K^*, s^*)$ yields a higher expected total money generated in the economy. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 2 follows.

As before, since (K^*, x^*, s^*) is a CFB then it has to be an INE. This means that, as $\hat{x}_j > x_j^* \geq I^*(k_j^*, \eta, g_j)$, the allocation $(\hat{x}, K^*, s^*)$ is also an INE. Furthermore,

$$\begin{aligned} & \sum_{r\in N} p_r(K^*, s^*) f(k_r^*) R\left(\hat{x}_r + 1\right) - \sum_{r\in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \quad & R[x_i^* - I^*(k_i^*, \eta, g_i)] \left[\eta^{g_j} f(k_j^*) - \eta^{g_i} f(k_i^*)\right] \geq 0, \end{aligned} \tag{14}$$

given that $k_j^* \geq k_i^*$, and the function $f(k)$ is increasing in k and $g_j \leq g_i$. By equation (14), the allocation $(\hat{x}, K^*, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB. ■

Proof of Proposition 2. This proof is done by contradiction. Assume that (K^*, x^*, s^*) is a CFB for (N, e) but there are two unconnected banks i and j such that $s_i^* = s_j^* = b$. Take $(x^*, \hat{K}, s^*)$ a new allocation for the same economy (N, e), where the allocation of capital and strategies are the same but the network structure only adds the link of banks i and j. Formally, $\hat{K}_i = K_i^* \cup \{j\}$, $\hat{K}_i = K_j^* \cup \{i\}$, and $\hat{K}_r = K_r^*$ for all $r \neq i, j$. We show now that this allocation $(x^*, \hat{K}, s^*)$ just defined derives a higher expected total money generated in the economy. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planners problem and the statement of Proposition 2 follows.

Note first that, by definition of $\hat{K}$, $p_r(\hat{K}, s^*) = p_r(K^*, s^*)$, for all $r \in N$, and

$$\hat{k}_r = \begin{cases} k_r^* + 1, & \text{if } r = i \text{ or } r = j \\ k_r^*, & \text{otherwise.} \end{cases}$$

On the other hand, if (K^*, x^*, s^*) is a CFB then it has to be an INE. Therefore, as both i and j are choosing the safe asset, we have that $x_i^* \geq I^*(k_i^*, \eta, g_i)$ and $x_j^* \geq I^*(k_j^*, \eta, g_j)$. By definition of the function $I^*(., \eta, .)$, it is true then that $x_i^* \geq I^*(k_i^* + 1, \eta, g_i)$ and $x_j^* \geq I^*(k_j^* + 1, \eta, g_j)$. Therefore, the allocation $(x^*, \hat{K}, s^*)$ is an INE. Furthermore,

$$\begin{aligned} & \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R\left(x_r^* + 1\right) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \; & \eta^{g_i} R\left(x_i^* + 1\right) \left[f(k_i^* + 1) - f(k_i^*)\right] + \eta^{g_j} R\left(x_j^* + 1\right) \left[f(k_j^* + 1) - f(k_j^*)\right] \geq 0, \end{aligned} \tag{15}$$

as the function $f(k)$ is increasing on k. By equation (15), the allocation $(x^*, \hat{K}, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB. ■

Proof of Proposition 3. We prove statement 1 by contradiction. Assume that (K^*, x^*, s^*) is an CFB allocation where there are at least two banks i and j not directly connected in K^*. Take $(x^*, \hat{K}, s^*)$ a new allocation for the same economy (N, e), where the allocation of capital and strategies are the same but the network structure only adds the link of banks i and j. Formally, $\hat{K}_i = K_i^* \cup \{j\}$, $\hat{K}_j = K_j^* \cup \{i\}$, and $\hat{K}_r = K_r^*$ for all $r \neq i, j$. We show that there is an INE that yields at least the payment of this allocation $(x^*, \hat{K}, s^*)$, which is a higher expected total money generated in the economy for η high enough. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 1 follows.

First note that if (K^*, x^*, s^*) is a CFB then it has to be an INE. If at least one of them, for example i, is choosing safe (the equivalent applies for j), it means that $x_i^* \geq I^*(k_i^*, \eta, g_i)$ and the other one is playing gamble (by Proposition 2 we know that they cannot be both investing safe, otherwise (K^*, x^*, s^*) would not be a CFB allocation). Note that $I^*(k_i^*, \eta, g_i) \geq I^*(k_i^* + 1, \eta, g_i + 1)$ if and only if $\eta f(k_i^* + 1) \geq f(k_i^*)$, which is true for $\eta \geq \dfrac{f(n-2)}{f(n-1)}$. Recall that by quasiconcavity of $f(k_i)$ the ratio $\dfrac{f(k_i)}{f(k_i + 1)}$ is increasing on k_i, so $\dfrac{f(n-2)}{f(n-1)} \geq \dfrac{f(k_i)}{f(k_i + 1)}$ for $k_i \leq n - 2$. Therefore, the allocation $(x^*, \hat{K}, s^*)$ could be an INE, in which case,

$$\begin{aligned} & \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R\left(x_r^* + 1\right) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \; & \eta^{g_i} R\left(x_i^* + 1\right) \left[\eta f(k_i^* + 1) - f(k_i^*)\right] + \eta^{g_j + 1} R\left(x_j^* + 1\right) \left[f(k_j^* + 1) - f(k_j^*)\right], \end{aligned} \tag{16}$$

if i chooses the safe asset, or

$$\begin{aligned}&\sum_{r\in N}p_r(\hat{K},s^*)f(\hat{k}_r)R\left(x_r^*+1\right)-\sum_{r\in N}p_r(K^*,s^*)f(k_r^*)R\left(x_r^*+1\right)\\ =\;&\eta^{g_i+1}R\left(x_i^*+1\right)\left[\eta f(k_i^*+1)-f(k_i^*)\right]+\eta^{g_j+1}R\left(x_j^*+1\right)\left[\eta f(k_j^*+1)-f(k_j^*)\right],\end{aligned}\tag{17}$$

if i chooses the gambling project. Note that both (16) and (17) are greater than zero since $\eta \geq \dfrac{f\,(n-2)}{f\,(n-1)} \geq \dfrac{f\,(k_i)}{f\,(k_i+1)}$ by assumption and the function $f(k)$ is increasing in k. Therefore, the allocation $(x^*, \hat{K}, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB. If $\left(x^*, \hat{K}, s^*\right)$ is not an INE is because j would choose the safe asset as well, once $\hat{K}$ is given, or, in the case when both i and j choose the gambling asset in (K^*, x^*, s^*) an INE would select at least one of them choosing the safe asset. In all these cases, the new INE will yield a higher total amount of expected money higher than the one in $\left(x^*, \hat{K}, s^*\right)$. This proves statement 1.

Once established that the only optimal financial network for η big enough is the complete network structure, it is easy to see that c^* as defined in statement 2 is the highest number of banks that can choose safe given the financial network being equal to the complete network structure. Any allocation with a lower number of banks choosing the safe project yields a lower total expected money in the economy. ■

Proof of Proposition 4. We prove statement 1 by contradiction. Assume that (K^*, x^*, s^*) is a CFB allocation where there is at least one bank j with $s_j^* = g$ and $g_i^* \neq 0$. Let $i \in K_j^*$ with $s_i^* = g$. Take $(x^*, \hat{K}, s^*)$ a new allocation for the same economy (N, e), where the allocation of capital and strategies are the same but the network structure only severs the link of banks i and j. Formally, $\hat{K}_i = K_i^* \backslash \{j\}$, $\hat{K}_j = K_j^* \backslash \{i\}$, and $\hat{K}_r = K_r^*$ for all $r \neq i, j$. Again, we show that this allocation $(x^*, \hat{K}, s^*)$ yields a higher expected total money generated in the economy for η low enough. As before, it is easily seen that if $(x^*, \hat{K}, s^*)$ is not an INE, i or j or both prefer choosing the safe asset in $\left(x^*, \hat{K}\right)$. Such a case yields a higher expected total amount of money than $(x^*, \hat{K}, s^*)$. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 1 follows.

By definition,

$$\begin{aligned}&\sum_{r\in N}p_r(\hat{K},s^*)f(\hat{k}_r)R\left(x_r^*+1\right)-\sum_{r\in N}p_r(K^*,s^*)f(k_r^*)R\left(x_r^*+1\right)\\ =\;&\eta^{g_i}R\left(x_i^*+1\right)\left[f(k_i^*-1)-\eta f(k_i^*)\right]+\eta^{g_j}R\left(x_j^*+1\right)\left[f(k_j^*-1)-\eta f(k_j^*)\right],\end{aligned}\tag{18}$$

which is greater than zero for $0 < \eta < \dfrac{1}{f\,(1)}$ by quasiconcavity of $f\,(k)$. This means that, if (K^*, x^*, s^*) is a CFB allocation, two banks choosing the gambling asset cannot be connected. Therefore, for any allocation (K^*, x^*, s^*) to be CFB: $x_j^* = g$ implies $g_j = 0$ for $0 < \eta < \dfrac{1}{f\,(1)}$. This proves statement 1.

We prove now that no bank choosing the safe asset can be connected to a gambling bank in an optimal allocation, for $0 < \eta < \dfrac{1}{\rho + 2\,(n-1)\,[\rho - 1]} < \dfrac{1}{\rho} < \dfrac{1}{f\,(1)}$. Assume that a bank i is choosing the safe project in the CFB allocation (K^*, x^*, s^*), with $g_i^* > 0$. As we have just shown, $g_j = 0$ for any $j \in G_i$ if $\eta < \dfrac{1}{\rho + 2(n-1)[\rho - 1]}$ since $\dfrac{1}{\rho + 2(n-1)[\rho - 1]} < \dfrac{1}{f(1)}$. Take $(x^*, \hat{K}, s^*)$ a new allocation for the same economy (N, e), where the allocation of capital and strategies are the same but the network structure only severs all the risky links of bank i. Formally, $\hat{K}_i = K_i^* \backslash \{G_i\}$, $\hat{K}_j = K_j^* \backslash \{i\}$, for all $j \in G_i$ and $\hat{K}_r = K_r^*$ for all $r \notin G_i$. Again, we show that this allocation $(x^*, \hat{K}, s^*)$ is an INE and yields a higher expected total money generated in the economy for η low enough. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 2 follows.

First note that if (K^*, x^*, s^*) is a CFB then it has to be an INE. If i and j are choosing the gambling project, the new allocation is also INE as x_i^* and x_j^* are greater or equal to zero. If i is choosing the safe asset, $x_i^* \geq I^*(k_i^*, \eta, g_i)$. Note that $I^*(k_i^* \eta, g_i) \geq I^*(k_i^* - g_i, \eta, 0)$ if and only if $f(k_i^* - g_i) \geq \eta f(k_i^*)$, which is true for $\eta < \dfrac{1}{\rho}$ since, by increasiness of $f(k_i)$, $\dfrac{1}{\rho} \leq \dfrac{f(k_i - g_i)}{f(k_i)}$ for $k_i \geq g_i$. On the other hand, $x_j^* < I^*\left(k_j^*, \eta, g_j\right) < I^*\left(k_j^* - 1, \eta, g_j\right)$ for any $j \in G_i$. Hence, the allocation $(x^*, \hat{K}, s^*)$ is an INE. Then,

$$\begin{aligned}
& \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R(x_r^* + 1) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R(x_r^* + 1) \\
= \;& R(x_i^* + 1)\left[f(k_i^* - g_i) - \eta^{g_i} f(k_i^*)\right] + \eta \sum_{j \in G_i} R\left(x_j^* + 1\right)\left[f(k_j^* - 1) - f(k_j^*)\right] \geq \\
\geq \;& R[1 - \eta\rho + \eta 2(n-1)(1-\rho)].
\end{aligned} \tag{19}$$

Recall that $x_i^* \geq 0$. The inequality is true, since, as $1 \leq g_i \leq k_i^* \leq n-1$, and by increasingness of f, $f(k_i^* - g_i) - \eta^{g_i} f(k_i^*) \geq 1 - \eta\rho$, $0 \geq f(k_j^* - 1) - f(k_j^*) \geq 1 - \rho$, and $x_j^* \leq 1$, for each j. Note that, since $\eta < \frac{1}{\rho + 2(n-1)(\rho-1)}$, $1 - \eta\rho + \eta 2(n-1)(1-\rho) \geq 0$. Therefore, if (K^*, x^*, s^*) is a CFB allocation, it has to be that peripheral agents have no links.

If the optimal financial network for $0 < \eta < \frac{1}{\rho + 2(n-1)(\rho-1)}$ is the complete network structure linking only banks that choose the safe project, it is easy to see that c^* as defined in statement 2 is the highest number of banks that can choose safe under such a network structure. Any allocation with a lower number of banks choosing the safe project yields a lower total expected money in the economy. ■

Proof of Proposition 5. Given an economy (N, e) fix any network K. We prove that for any (K, e, s) that is an INE there are no short-sighted profitable deviations for any bank i as far as either condition 1, or 2 or 3 is satisfied. Fix a bank i such that K_i contains at least one neighbor j choosing the gambling asset. If we cannot find such an i it means that no bank can have a short-sighted profitable deviation. This case is true whenever condition 1 is satisfied. Now assume that condition 1 is not satisfied and such a bank i has at least one neighbor j choosing the gambling asset in the INE (K, e, s). Define $e^{\max}$ as the biggest realization of e_i, and let $e_{<1}^{\max}$ the biggest realization of the e_i's that are strictly smaller than 1, if any. Let $\widetilde{\eta}(e)$ be defined as the minimal η for which all this conditions are satisfied:

1. $\eta^{n-1} R - 1 > 0$

2. $\left(1 - \eta^{n-1}\right) \rho R - 1 < 0$

3. $[1 + (1 - \eta^n) \rho R] e^{\max}_{<1} < 1$

4. $\frac{1 - \left[1 + \left(1 - \eta^{n-1}\right) \rho R\right] e^{\max}_{<1}}{\rho R} \geq \left(1 - \eta^{n-1}\right) e^{\max}$

5. $\frac{1 - \left[1 + \left(1 - \eta^{n-1}\right) \rho R\right] e^{\max}_{<1}}{\rho R} \geq \frac{1 - I^*(n-1, \eta, 0)\left[1 - \left(1 - \eta^{n-1}\right) \rho R\right]}{\eta^{n-1} R - 1}$

6. $\frac{1 - \left[1 + \left(1 - \eta^{n-1}\right) \rho R\right] e^{\max}_{<1}}{\rho R} \geq \frac{\rho R \left(1 - \eta^{n-1}\right) e^{\max}_{<1}}{\eta^{n-1} R - 1}$

It is easy to see that if a given η_1 satisfies the six previously stated conditions, any $\eta > \eta_1$ would satisfy all of them as well.

If there were a short-sighted profitable deviation for bank i there has to be a transfer $0 < t_i \leq e_i$ that is bounded depending on the choices of bank i before and after t_i. Notice that the bank transfer is directed in principle towards all the neighbor gambling banks $j \in J$, however feasibility implies that $t_i = \sum_{j \in J} t_{ij}$. We consider 4 different cases:

1. Bank i chooses the safe asset before and after the transfer, that is $e_i \geq I^*(k_i, \eta, g_i)$, $e_i - t_i \geq I^*(k_i, \eta, g_i - |J|)$ and $e_j + t_{ij} \geq I^*(k_j, \eta, g_j - |J \cap G_j|)$, where $e_j < I^*(k_j, \eta, g_j)$ for all $j \in J$. Therefore

   $$e_i + \sum_{j \in J} e_j \geq I^*(k_i, \eta, g_i - |J|) + \sum_{j \in J} I^*(k_j, \eta, g_j - |J \cap G_j|),$$

   since (K, e, s) is an INE and t_i generates a short-sighted profitable deviation. When condition 2 holds, this implies that

   $$|J| I^*(n-1, \eta, 0) \geq e^{\max} + |J| e^{\max}_{<1},$$

   then the condition

   $$e_i + \sum_{j \in J} e_j \geq I^*(k_i, \eta, g_i - |J|) + \sum_{j \in J} I^*(k_j, \eta, g_j - |J \cap G_j|)$$

   cannot be satisfied since

   $$\begin{aligned} e_i + \sum_{j \in J} e_j &\leq e^{\max} + |J| e^{\max}_{<1} \leq |J| I^*(n-1, \eta, 0) \\ &\leq \sum_{j \in J} I^*(k_j, \eta, g_j - |J \cap G_j|) < I^*(k_i, \eta, g_i - |J|) + \sum_{j \in J} I^*(k_j, \eta, g_j - |J \cap G_j|). \end{aligned}$$

   Assume that Condition 2 does not hold. Bank i expects to gain from that transfer if and only if

   $$\eta^{g_i} f(k_i) R e_i < \eta^{g_i - |J|} f(k_i) R [e_i - t_i], \tag{20}$$

   since it invests safe before and after the transfer, which implies

   $$t_i < (1 - \eta^{|J|}) e_i.$$

   Note that, since $|J| < n-1$, and $e_i \leq e^{\max}$, $t_i < (1-\eta^{|J|}) e_i$ implies that $t_i < (1 - \eta^{n-1}) e^{\max}$. Bank $j \in J$ accepts the transfer if and only if

   $$\eta^{g_j + 1} f(k_j) R e_j + 1 - e_j \leq \eta^{g_j - |J \cap G_j|} f(k_j) R [e_j + t_{ij}] \tag{21}$$

since it switches from investing gamble to investing safe, which implies

$$t_{ij} \geq \frac{1 - e_j[1 + \left(1 - \eta^{|J \cap G_j|+1}\right) \eta^{g_j - |J \cap G_j|} f\left(k_j\right) R]}{\eta^{g_j} f(k_j) R},$$

for any $j \in J$. Equation (21) implies that $t_{ij} \geq \frac{1 - e^{\max}_{<1}\left(1 + \rho R\left(1 - \eta^{n-1}\right)\right)}{\rho R}$, given that $|J \cap G_j| + 1 \leq n - 1$, $e_j \leq e^{\max}_{<1}$ and $\eta^{g_j} f(k_j) R < \rho R$. By definition of $\tilde{\eta}(e)$, if $\eta \geq \tilde{\eta}(e)$ then $\frac{1 - e^{\max}_{<1}\left(1 + \rho R\left(1 - \eta^{n-1}\right)\right)}{\rho R} \geq (1 - \eta^{n-1}) e^{\max}$. This would imply that $t_{ij} > t_i$ for any $j \in J$, a contradiction given that $\sum_{j \in J} t_{ij} = t_i$. Hence, under Condition 3 there is no short-sighted profitable deviation for this case.

2. Bank i chooses the safe asset before the transfer, but chooses the gambling asset after the transfer. This means that $e_i \geq I^*(k_i, \eta, g_i)$, $e_i - t_i < I^*(k_i, \eta, g_i - |J|)$ and $e_j + t_{ij} \geq I^*(k_j, \eta, g_j + 1 - |J \cap G_j|)$, where $e_j < I^*(k_j, \eta, g_j)$ for any $j \in J$ since (K, e, s) is an INE and t_i generates a short-sighted profitable deviation. Again, if Condition 2 is satisfied, the inequality

$$e_j + t_{ij} \geq I^*(k_j, \eta, g_j + 1 - |J \cap G_j|)$$

cannot hold since it implies that

$$\sum_{j \in J} e_j + e_i \geq \sum_{j \in J} I^*(k_j, \eta, g_j + 1 - |J \cap G_j|),$$

a contradiction with the fact that

$$e_i + \sum_{j \in J} e_j \leq e^{\max} + |J| e^{\max}_{<1} < |J| I^*(n-1, \eta, 0) < \sum_{j \in J} I^*(k_j, \eta, g_j + 1 - |J \cap G_j|).$$

Assume that Condition 2 is not satisfied. Bank i expects to gain from that transfer if and only if

$$\eta^{g_i} f(k_i) R e_i < \eta^{g_i - |J| + 1} f(k_i) R[e_i - t_i] + 1 - (e_i - t_i), \tag{22}$$

as it invests safe before and gambling after the transfer, which implies

$$t_i[1 - \eta^{g_i - |J| + 1} f(k_i) R] > \left(1 - \eta^{g_i - |J| + 1} \left(1 - \eta^{|J| - 1}\right) f\left(k_i\right) R\right) e_i - 1.$$

Bank j accepts the transfer if and only if

$$\eta^{g_j + 1} f(k_j) R e_j + 1 - e_j \leq \eta^{g_j + 1 - |J \cap G_j|} f(k_j) R[e_j + t_{ij}] \tag{23}$$

since it switches from investing gamble to investing safe, which implies

$$t_{ij} \geq \frac{1 - e_j\left(1 + \eta^{g_j + 1 - |J \cap G_j|}\left(1 - \eta^{|J \cap G_j|}\right) f\left(k_j\right) R\right)}{\eta^{g_j + 1 - |J \cap G_j|} f(k_j) R} \geq \frac{1 - e^{\max}_{<1}\left(1 + \rho R\left(1 - \eta^{n-1}\right)\right)}{\rho R},$$

given that $\eta^{g_j + 1 - |J \cap G_j|} f(k_j) R \leq \rho R$, $e_j \leq e^{\max}_{<1}$ and $\eta^{g_j + 1 - |J \cap G_j|}\left(1 - \eta^{|J \cap G_j|}\right) \leq 1 - \eta^{n-1}$. Note that $1 - \eta^{g_i - |J| + 1} f(k_i) R$ is negative when Condition 3 is satisfied. Indeed, the condition $\eta > \frac{1}{\sqrt[n-1]{R}}$ implies that

$$1 < \eta^{n-1} R < \eta^{n-1} f(k_i) R \leq \eta^{g_i - |J| + 1} f(k_i) R,$$

given that $g_i \leq n-1$ and $|J| \geq 1$. This means that

$$t_i < \frac{1 - e_i \left(1 - \eta^{g_i - |J| + 1} \left(1 - \eta^{|J|-1}\right) f(k_i) R\right)}{\eta^{g_i - |J| + 1} f(k_i) R - 1} < \frac{1 + e_i \left((1 - \eta^{g_i})\rho R - 1\right)}{\eta^{n-1} R - 1} < \frac{1 + e_i \left((1 - \eta^{n-1})\rho R - 1\right)}{\eta^{n-1} R - 1}$$

since

$$\eta^{g_i - |J| + 1} f(k_i) R - 1 > \eta^{n-1} R - 1.$$

Note that given Condition 3, we have $(1-\eta^{n-1})\rho R - 1 < 0$. Recall that $e_i \geq I^*(n-1, \eta, 0)$. Therefore, equation (22) together with Condition 3 imply that

$$t_i < \frac{1 - I^*(n-1, \eta, 0)\left[1 - (1 - \eta^{n-1})\rho R\right]}{\eta^{n-1} R - 1} < \frac{1 - e^{\max}_{<1} \left(1 + \rho R \left(1 - \eta^{n-1}\right)\right)}{\rho R}$$

that implies $t_{ij} > t_i$, which is again a contradiction.

3. Bank i chooses the gambling asset before the transfer, but chooses the safe asset after the transfer. This means that $e_i < I^*(k_i, \eta, g_i)$, $e_i - t_i \geq I^*(k_i, \eta, g_i - |J|)$ and $e_j + t_{ij} \geq I^*(k_j, \eta, g_j - 1 - |J \cap G_j|)$, where $e_j < I^*(k_j, \eta, g_j)$ since (K, e, s) is an INE and t_i generates a short-sighted profitable deviation. Again, if Condition 2 is satisfied, the inequalities $e_i - t_i \geq I^*(k_i, \eta, g_i - |J|)$ and $e_j + t_{ij} \geq I^*(k_j, \eta, g_j - |J \cap G_j|)$ for all $j \in J$ cannot simultaneously hold since they imply that

$$e_i + \sum_{j \in J} e_j \geq I^*(k_i, \eta, g_i - |J|) + \sum_{j \in J} I^*(k_j, \eta, g_j - 1 - |J \cap G_j|),$$

a contradiction with the fact that

$$e_i + \sum_{j \in J} e_j \leq e^{\max} + |J| e^{\max}_{<1} \leq |J| I^*(n-1, \eta, 0) < I^*(k_i, \eta, g_i - |J|) + \sum_{j \in J} I^*(k_j, \eta, g_j - 1 - |J \cap G_j|).$$

Assume that Condition 2 does not hold. Bank i expects to gain from that transfer if and only if

$$\eta^{g_i+1} f(k_i) R e_i + 1 - e_i < \eta^{g_i - |J|} f(k_i) R [e_i - t_i], \tag{24}$$

since it invests safe before and gambling after the transfer, which implies

$$t_i < e_i - \frac{\eta^{g_i+1} f(k_i) R e_i + 1 - e_i}{\eta^{g_i - |J|} f(k_i) R} < \frac{(1 + (1 - \eta^n)\rho R) e^{\max}_{<1} - 1}{\eta^{g_i - |J|} f(k_i) R},$$

given that $g_i \leq n-1$. Note that, given Condition 3, it is $(1 + (1 - \eta^n)\rho R) e^{\max}_{<1} - 1 < 0$. This implies that t_i has to be a negative number, a contradiction.

4. Finally, assume that bank i chooses the gambling asset before and after the transfer. This implies that $e_i < I^*(k_i, \eta, g_i)$, $e_i - t_i < I^*(k_i, \eta, g_i - |J|)$ and $e_j + t_{ij} \geq I^*(k_j, \eta, g_j - |J \cap G_j|)$, where $e_j < I^*(k_j, \eta, g_j)$ since (K, e, s) is an INE and t_i generates a short-sighted profitable deviation. Again, if Condition 2 is satisfied, the inequality $e_j + t_{ij} \geq I^*(k_j, \eta, g_j - |J \cap G_j|)$ cannot be satisfied since it implies that

$$e_i + \sum_{j \in J} e_j \geq \sum_{j \in J} I^*(k_j, \eta, g_j - |J \cap G_j|),$$

a contradiction with the fact that

$$e_i + \sum_{j \in J} e_j \leq e^{\max} + |J| e^{\max}_{<1} \leq |J| I^*(n-1, \eta, 0) < \sum_{j \in J} I^*(k_j, \eta, g_j - |J \cap G_j|).$$

Assume that Condition 2 does not hold. Bank i expects to gain from that transfer if and only if

$$\eta^{g_i+1} f(k_i) R e_i + 1 - e_i < \eta^{g_i+1-|J|} f(k_i) R [e_i - t_i] + 1 - e_i + t_i, \tag{25}$$

since it invests gambling before and after the transfer, which implies

$$t_i [\eta^{g_i+1-|J|} f(k_i) R - 1] < (1 - \eta^{|J|}) \eta^{g_i+1-|J|} f(k_i) R e_i.$$

By Condition 3, it is $\eta^{n-1} R - 1 > 0$. Recall that $e_i \leq e^{\max}_{<1}$ and $|J| \leq G_i \leq n-1$. Then,

$$t_i < \frac{\left(1 - \eta^{n-1}\right) \rho R e^{\max}_{<1}}{\eta^{n-1} R - 1}.$$

Bank j accepts the transfer if and only if

$$\eta^{g_j+1} f(k_j) R e_j + 1 - e_j \leq \eta^{g_j - |J \cap G_j|} f(k_j) R [e_j + t_{ij}] \tag{26}$$

since it switches from investing gamble to investing safe, which implies

$$t_{ij} \geq \frac{1 - e_j \left[1 + \left(1 - \eta^{|J \cap G_j|+1}\right) \eta^{g_j - |J \cap G_j|} f\left(k_j\right) R\right]}{\eta^{g_j+1} f(k_j) R} > \frac{1 - e^{\max}_{<1} [1 + \left(1 - \eta^{n-1}\right) \rho R]}{\rho R},$$

given that $e_j \leq e^{\max}_{<1}$ and $|J \cap G_j| \leq n-2$. Given Condition 3, we have

$$\frac{1 - e^{\max}_{<1} [1 + \left(1 - \eta^{n-1}\right) \rho R]}{\rho R} \geq \frac{\left(1 - \eta^{n-1}\right) \rho R e^{\max}_{<1}}{\eta^{n-1} R - 1}$$

and therefore $t_{ij} > t_i$, again a contradiction.

Hence, there is no short-sighted profitable deviation for any possible case. ■

Proof of Proposition 6. The assumption on (N, e) and η guarantees that an INE without transfers is also a TNE. Assume by contradiction that (K^e, e, s^e) is a DEWT, but there exist two banks i and j such that $s^e_i = s^e_j = b$ but $i \notin K^e_j$, and therefore $j \notin K^e_i$. We prove that $(K^e \cup ij, e, s^e)$ is an INE and then that both $m_i(K^e \cup ij, e, s^e) > m_i(K^e, e, s^e)$ and $m_j(K^e \cup ij, e, s^e) > m_j(K^e, e, s^e)$, contradicting the fact that (K^e, e, s^e) is a PSWT, and therefore it cannot be a DEWT.

Note first that, since (K^e, e, s^e) is a DEWT, it has to be an INE. This means that both $e_i \geq I^*(k_i, \eta, g_i)$ and $e_j \geq I^*(k_j, \eta, g_j)$. Furthermore, it has to be $I^*(k_i, \eta, g_i) \geq I^*(k_i + 1, \eta, g_i)$ and $I^*(k_j, \eta, g_j) \geq I^*(k_j + 1, \eta, g_j)$ given that $f(k)$ is increasing in k. This implies that $e_i \geq I^*(k_i + 1, \eta, g_i)$ and $e_j \geq I^*(k_j + 1, \eta, g_j)$. Therefore, $(K^e \cup ij, e, s^e)$ is also an INE. Finally, note that

$$m_i(K^e \cup ij, e, s^e) = \eta^{g_i} f(k_i + 1) R e_i > \eta^{g_i} f(k_i) R e_i = m_i(K^e, e, s^e),$$

and

$$m_j(K^e \cup ij, e, s^e) = \eta^{g_j} f(k_j + 1) R e_j > \eta^{g_j} f(k_j) R e_j = m_j(K^e, e, s^e),$$

since the function $f(k)$ is increasing in k. Therefore, (K^e, e, s^e) as defined above cannot be a DEWT. ■

Proof of Proposition 7. Again, the assumption on (N, e) and η guarantees that an INE without transfers is also a TNE. Assume by contradiction that (K^e, e, s^e) is a DEWT, but there exist two banks i and j such that $i \notin K^e_j$, and therefore $j \notin K^e_i$. We prove that there exists an $(K^e \cup ij, e, \tilde{s})$ that is an INE such that $m_i(K^e \cup ij, e, \tilde{s}) > m_i(K^e, e, s^e)$ and $m_j(K^e \cup ij, e, \tilde{s}) > m_j(K^e, e, s^e)$, contradicting the fact that (K^e, e, s^e) is a PSWT, and therefore it cannot be a DEWT. We distinguish two cases: In the first one, a bank chooses the safe asset while the other chooses the gambling asset. In the second, both banks choose the gambling asset. Note that since we know that a DEWT is a core-periphery structure we do not need to check the case of both banks choosing the safe asset.

Consider the first case, where $s^e_i = b$ and $s^e_i = g$. Note first that, since (K^e, e, s^e) is a DEWT, it has to be an INE. This means that $e_i \geq I^*(k_i, \eta, g_i)$ and $e_j < I^*(k_j, \eta, g_j)$. Given that $f(k)$ is increasing in k, $e_i \geq I^*(k_i+1, \eta, g_i)$. Nevertheless, $e_j \geq I^*(k_j+1, \eta, g_j)$ could also be true or not. Therefore, $(K^e \cup ij, e, s^e)$ could also be an INE, in which case

$$m_i(K^e \cup ij, e, s^e) = \eta^{g_i+1} f(k_i+1) Re_i > \eta^{g_i} f(k_i) Re_i = m_i(K^e, e, s^e),$$

for $\eta \geq \dfrac{f\,(n-2)}{f\,(n-1)} \geq \dfrac{f\,(k_i)}{f\,(k_i+1)}$, and

$$m_j(K^e \cup ij, e, s^e) = \eta^{g_j+1} f(k_j+1) Re_j + 1 - e_j > \eta^{g_j+1} f(k_j) Re_j + 1 - e_j = m_j(K^e, e, s^e),$$

since the function $f(k)$ is increasing in k. In the case when $e_j \geq I^*(k_j+1, \eta, g_j)$ there is an INE where at least j switches from choosing the gambling asset in s^e to choosing the safe asset in $s = \tilde{s}$. This case is even more profitable for banks i and j than in $(K^e \cup ij, e, s^e)$, so we have $m_i(K^e \cup ij, e, \tilde{s}) \geq m_i(K^e \cup ij, e, s^e)$ and $m_j(K^e \cup ij, e, \tilde{s}) \geq m_j(K^e \cup ij, e, s^e)$.

Consider now the second case, when both $s^e_i = s^e_i = g$. The least profitable case would be when $(K^e \cup ij, e, s^e)$ is also an INE. Otherwise we can find an INE where at least one of them switches from investing in the gambling asset to investing in the safe asset. Note that

$$m_i(K^e \cup ij, e, s^e) = \eta^{g_i+2} f(k_i+1) Re_i + 1 - e_i > \eta^{g_i+1} f(k_i) Re_i + 1 - e_i = m_i(K^e, e, s^e),$$

and

$$m_j(K^e \cup ij, e, s^e) = \eta^{g_j+2} f(k_j+1) Re_j + 1 - e_j > \eta^{g_j+1} f(k_j) Re_j + 1 - e_j = m_j(K^e, x^e, s^e),$$

again for $\eta \geq \dfrac{f\,(n-2)}{f\,(n-1)}$ and given that $f(k)$ is increasing and quasiconcave in k. ■

Proof of Proposition 8. Again, the assumption on (N, e) and η guarantees that an INE without transfers is also a TNE. Assume by contradiction that (K^e, e, s^e) is a DEWT, but there exist two banks i and j such that $i \in K^e_j$, and therefore $j \in K^e_i$ and such that at least one of them is choosing the gambling asset. We prove that there exists an $(K^e \backslash\, ij, e, \tilde{s})$ that is INE and one $m_i(K^e \backslash\, ij, e, \tilde{s}) > m_i(K^e, e, s^e)$ or $m_j(K^e \backslash\, ij, e, \tilde{s}) > m_j(K^e, e, s^e)$, contradicting the fact that (K^e, e, s^e) is a PSWT, and therefore it cannot be a DEWT. We distinguish two cases: In the first one, a bank chooses the safe asset while the other chooses the gambling asset. In the second, both banks choose the gambling asset. As before, we know that a DEWT is a core-periphery structure and therefore we do not need to check the case of both banks choosing the safe asset.

Consider the first case, where $s_i^e = b$ and $s_i^e = g$. Note first that, since (K^e, e, s^e) is a DEWT, it has to be an INE. This means that $e_i \geq I^*(k_i, \eta, g_i)$ and $e_j < I^*(k_j, \eta, g_j)$. On the one hand, note that $I^*(k_i, \eta, g_i) \geq I^*(k_i, \eta, g_i)$ if and only if $\eta \leq \frac{f(k_i - 1)}{f(k_i)}$. By assumption $\eta < \frac{1}{f(1)} \leq \frac{f(k_i - 1)}{f(k_i)}$ for $k_i \geq 1$ and therefore $e_i \geq I^*(k_i - 1, \eta, g_i - 1)$. On the other hand, given that $f(k)$ is increasing in k, we have $I^*(k_j, \eta, g_j) \leq I^*(k_j - 1, \eta, g_j)$ and therefore $e_j < I^*(k_j - 1, \eta, g_j)$. Therefore, $(K^e \backslash ij, e, s^e)$ is also an INE. Furthermore,

$$m_i(K^e \backslash ij, e, s^e) = \eta^{g_i - 1} f(k_i - 1) Re_i > \eta^{g_i} f(k_i) Re_i = m_i(K^e, e, s^e),$$

for $\eta < \frac{1}{f(1)} \leq \frac{f(k_i - 1)}{f(k_i)}$.

Consider now the second case, when both $s_i^e = s_i^e = g$. Since (K^e, e, s^e) is an INE we know that both $e_i < I^*(k_i, \eta, g_i)$ and $e_j < I^*(k_j, \eta, g_j)$. Like before, the assumption $\eta < \frac{1}{f(1)}$, which by quasiconcavity of $f(k)$ implies both $\eta \leq \frac{f(k_i - 1)}{f(k_i)}$ and $\eta \leq \frac{f(k_j - 1)}{f(k_j)}$, means that $I^*(k_j, \eta, g_i) \geq I^*(k_j - 1, \eta, g_i - 1)$ and $I^*(k_j, \eta, g_j) \geq I^*(k_j - 1, \eta, g_j)$. Therefore, $(K^e \backslash ij, e, s^e)$ could also be an INE, in which case

$$m_i(K^e \backslash ij, e, s^e) = \eta^{g_i} f(k_i - 1) Re_i + 1 - e_i > \eta^{g_i + 1} f(k_i) Re_i + 1 - e_i = m_i(K^e, e, s^e),$$

and

$$m_j(K^e \backslash ij, e, s^e) = \eta^{g_j} f(k_j - 1) Re_j + 1 - e_j > \eta^{g_j + 1} f(k_j) Re_j + 1 - e_j = m_j(K^e, e, s^e),$$

again given that $\eta < \frac{1}{f(1)}$ and $f(k)$ is quasiconcave in k. Otherwise we can find an INE where at least one of them switches from investing in the gambling asset to investing in the safe asset, earning more than $m_i(K^e \backslash ij, e, s^e)$ and $m_j(K^e \backslash ij, e, s^e)$ respectively, and therefore earning more than in the proposed DEWT (K^e, x^e, s^e). ■

References

[1] Aghion, P., P. Bolton and M. Dewatripont (2000), 'Contagion Bank Failures in a Free Banking System', *European Economic Review,* **44**: 713-718.

[2] Allen, F. and D. Gale (2000), 'Financial Contagion', *Journal of Political Economy*, **108**: 1-33.

[3] Allen, F. and D. Gale (2004), 'Financial Intermediaries and Markets', *Econometrica*, **72**: 1023-1061.

[4] Babus, A. (2007), 'The Formation of Financial Networks', Manuscript, Erasmus University of Rotterdam.

[5] Brusco, S. and F. Castiglionesi (2007), 'Liquidity Coinsurance, Moral Hazard, and Financial Contagion', *Journal of Finance*, **62**: 2275 - 2302.

[6] Diamond, D. and P. Dybvig (1983), 'Bunk Runs, Deposit Insurance and Liquidity', *Journal of Political Economy*, **91**: 401-419.

[7] Freixas, X., B. Parigi and J.C. Rochet (2000), 'Systemic Risk, Interbank Relations and Liquidity Provision by the Central Bank', *Journal of Money, Credit and Banking*, **32**: 611-638.

[8] Gale, D. and S. Kariv (2007), 'Financial Networks', mimeo, NYU, http://www.econ.nyu.edu/user/galed/papers.html.

[9] Jackson, M. O. and A. Wolinsky (1996), 'A Strategic Model of Social and Economic Networks', *Journal of Economic Theory*, **71**: 44-74.

[10] Leitner, Y. (2005), 'Financial Networks: Contagion, Commitment, and Private Sector Bailouts', *Journal of Finance*, **6**: 2925-2953.

[11] Nier, E., J. Yang, T. Yorulmazer, and A. Alerton (2007), 'Network Models and Financial Stability', *Journal of Economic Dynamics and Control*, **31**: 2033-2060.

[12] Rochet, J.C. and J. Tirole (1996), 'Interbank Lending and Systemic Risk', *Journal of Money, Credit and Banking*, **28**: 733-762.