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Perfection and Stability of Stationary Points with Applications to Noncooperative Games [1]

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Abstract

It is well known that an upper semi-continuous compact- and convex-valued mapping ϕ from a nonempty compact and convex set X to the Euclidean space of which X is a subset has at least one stationary point, being a point in X at which the image $\phi(x)$ has a nonempty intersection with the normal cone at x. In many circumstances there may be more than one stationary point. In this paper we refine the concept of stationary point by perturbing simultaneously both the set X and the solution concept. In case a stationary point is the limit of a sequence of perturbed solutions on a sequence of sets converging continuously to X we say that the stationary point is stable with respect to this sequence of sets and the mapping which defines the perturbed solution. It is shown that stable stationary points exist for a large class of perturbations. A specific refinement, called robustness, is obtained if a stationary point is the limit of stationary points on a sequence of sets converging to X. It is shown that a robust stationary point always exists for any sequence of sets which starts from an interior point and converges to X in a continuous way.

We also discuss several applications in noncooperative game theory. We first show that two well known refinements of the Nash equilibrium, namely, perfect Nash equilibrium and proper Nash equilibrium, are special cases of our robustness concept. Further, a third special case of robustness refines the concept of properness and a robust Nash equilibrium is shown to exist for every game. In symmetric bimatrix games, our results imply the existence of a symmetric proper equilibrium. Applying our results to the field of evolutionary game theory yields a refinement of the stationary points of the replicator dynamics. We show that the refined solution always exists, contrary to many well known refinement concepts in the field that may fail to exist under the same conditions.

Keywords: stationary point, stability, perfectness, perturbation, equilibrium, games

1 Introduction

Let X be a nonempty subset of the n-dimensional Euclidean space $\mathbb{R}^n$ and let f be a function from X to $\mathbb{R}^n$. Then a *stationary point* or solution to the variational inequality problem with respect to f is a point x^* in X satisfying

$$(x^* - x)^\top f(x^*) \geq 0, \qquad \text{for all } x \in X. \tag{1.1}$$

In case of a point-to-set mapping ϕ from X to the collection of non-empty subsets of $\mathbb{R}^n$, a point x^* in X is called a stationary point of ϕ if there exists an element $y^* \in \phi(x^*)$ satisfying

$$(x^* - x)^\top y^* \geq 0, \qquad \text{for all } x \in X. \tag{1.2}$$

The concept of stationary point has many important applications in various fields. For instance, in noncooperative game theory, economic equilibrium theory, fixed point theory, nonlinear optimization theory and engineering a stationary point gives a solution to the problem under investigation. In many of these applications the multiplicity of stationary points may ask for a more refined solution concept; see for example van Damme (1987), Kehoe (1991), and Yamamoto (1993). Although the conditions to guarantee the existence of a stationary point are quite weak, conditions to guarantee the existence of a unique stationary point are often very demanding and are usually not satisfied. For instance, in game theoretical applications there can be any finite number of equilibria, being stationary points of some specific function or mapping, and there may even exist higher-dimensional sets of equilibria. Then a refinement may reduce the number of stationary points or equilibria considerably by requiring additional properties to be satisfied. Within the field of noncooperative game theory two well-known refinements of Nash equilibria, being stationary points of the marginal payoff funtion on the strategy space of the game, are the so-called *perfect* equilibria introduced by Selten (1975) and the *proper* equilibria by Myerson (1978). In these references it has been shown that the set of perfect equilibria is a non-empty subset of the set of equilibria and that the set of proper equilibria is a non-empty subset of the set of perfect equilibria. In van der Laan, Talman and Yang (1998) the concept of properness has been generalized to the concept of a robust stationary point for arbitrary (continuous) functions on polytopes. Proper and perfect equilibria in noncooperative are known to exist under the same conditions guaranteeing the existence of a (Nash) equilibrium, in sharp contrast to the solution concept of *evolutionary stability* in evolutionary game theory that selects a possibly empty subset of the set of equilibria.

In this paper we provide a general refinement concept for point-to set mappings on arbitrary convex compact subsets, by introducing the concept of *stable stationary point*. A stable stationary point will be shown to exist under the same conditions under which a stationary point is known to exist. The concept of stable stationary points contains the above mentioned concepts of perfect and proper equilibria in noncooperative game theory and robust points for functions on polytopes as special cases.

The main idea of the refinement is to perturb simultaneously both the domain X and the concept of stationary point. The set X will be perturbed by taking a sequence of subsets of X converging to X, while the concept of stationary point is replaced by a more general concept. The refinement depends both on the way the sequence of subsets of X is chosen and the way in which the concept of stationary point on those subsets is generalized. For both choices there are many possibilities. The only restrictions will be that a generalized stationary point exists on each subset of the sequence and that every convergent subsequence of generalized stationary points converges to a stationary point on X of the given mapping. Such a stationary point being the limit of a sequence of generalized stationary points on a sequence of subsets is then called stable with respect to the underlying sequence of subsets and the chosen concept of generalized stationary point. Given the way the sequence of subsets is chosen and the concept of generalized stationary point, an induced stable stationary point has additional properties that other stationary points may not have. Doing this gives one the possibility to select stationary points having certain desirable additional properties, by choosing in an appropriate way the sequence of subsets and the concept of generalized stationary point on the subsets. In case we only perturb the set X and take the standard concept of stationary point on each subset of the sequence, we call a stable stationary point a *robust* stationary point for the chosen sequence of subsets converging to X. On the other hand, a stable stationary point is called *perfect* with respect to the concept of generalized stationary point when the sequence of subsets converging to X linearly expands to X from an arbitrarily chosen point in the interior of X.

As an application we consider the special case that the set X is a polytope. In that case some of the refinements will have some specific appealing and intuitive properties, due to the special structure of a polytope as the intersection of a finite number of half spaces. In particular, we will give explicit conditions for stationary point to be robust, respectively, perfect. When applied to noncooperative games perfectness coincides with the usual concept of perfectness of Nash equilibria, as was introduced by Selten (1975). We show that the concept of robustness, introduced earlier on polytopes in van der Laan, Talman and Yang (1998), follows as a special case from the general concept given in this paper. Furthermore, when applied to noncooperative games, the concept of robustness yields two very interesting special cases. One special case gives the concept of a proper Nash equilibrium, as introduced by Myerson (1978), another special case results in a new solution concept to noncooperative games, which we call robust Nash equilibrium. Such a robust Nash equilibrium is also proper and every noncooperative game has a robust Nash equilibrium. Hence, this concept of robustness yields a further refinement of properness. We also show that every symmetric two-person game has a symmetric proper equilibrium. To the best of our knowledge, this result is unknown within the field of noncooperative game theory.

We then apply the concept of stable stationary point to *replicator dynamics* in the field of evolutionary game theory. It is well known that the set of stationary points of the replicator dynamics contains the set of equilibria; see Weibull (1995). By taking an appropriate generalized stationary point solution concept, we are able to refine the stationary points of the replicator dynamics in such a way that every stable stationary point is an equilibrium. Moreover, it is shown that such a stable stationary point always exists. This result is in sharp contrast to many well known equilibrium refinement concepts in evolutionary game theory that may fail to exist under the same conditions.

The paper is organized as follows. Section 2 introduces the concepts of stability, robustness and perfectness on an arbitrary nonempty compact and convex set. Section 3 discusses the refinements on polytopes. Finally, Section 4 discusses several applications both in the field of noncooperative games and in the field of evolutionary games.

2 Stable stationary points

In this paper we assume that X is a nonempty compact and convex subset of $\mathbb{R}^n$. It is well-known that any continuous function f from X to $\mathbb{R}^n$ has at least one solution to the variational inequality problem (1.1); see for instance Eaves (1971) and Hartman and Stampacchia (1966). In case of a point-to-set mapping ϕ from X to the collection of nonempty subsets of $\mathbb{R}^n$ a solution to the variational inequality problem (1.2) exists if ϕ is upper semi-continuous and, for all $x \in X$, $\phi(x)$ is a convex and compact subset of $\mathbb{R}^n$; see for example Yang (1999).

Without loss of generality we assume that X is full-dimensional. For $x \in X$, let

$$N(X, x) = \{y \in \mathbb{R}^n | y^\top x \geq y^\top x', \text{ for all } x' \in X\}$$

denote the normal cone of X at x. Due to the properties of X it holds that $N(X, \cdot)$ is an upper semi-continuous mapping on X, that for every $x \in X$ the set $N(X, x)$ is a nonempty, closed and convex cone, and that $N(X, x) = \{0^n\}$ when x lies in the interior of X, where 0^n denotes the n-vector of zeros. Clearly, $x^* \in X$ is a stationary point of a point-to-set mapping ϕ on X if and only if $\phi(x^*) \cap N(X, x^*) \neq \emptyset$. For a function f the latter condition reduces to $f(x^*) \in N(X, x^*)$.

As has been discussed in the introduction there can be more than one or even an infinite number of solutions to the variational inequality problem. In this section we introduce a general refinement concept, which may select a subset of the set of stationary points and gives a certain stability property to the stationary points within this subset. The general idea is to perturb both the set X and the concept of stationary point in such a way that every convergent subsequence of generalized stationary points converges to a solution of the variational inequality problem. A solution that is not the limit of any such subsequence is not stable with respect to the chosen perturbations, selecting a subset of stationary points. To guarantee the existence of a stable stationary point it is sufficient to assume that a generalized stationary point exists on any perturbed subset and that there exists a convergent subsequence of generalized stationary points converging to a stationary point.

To describe formally the idea of refinement we introduce two mappings. The first mapping defines the perturbation of the set X and is given by a mapping $\mathcal{X}\colon [0,1] \rightarrow X$ satisfying the following two conditions, where Int denotes the interior of a set.

(X1) $\mathcal{X}$ is continuous and for each $\epsilon \in [0,1]$ the set $\mathcal{X}(\epsilon)$ is a non-empty, convex and compact subset of X.

(X2) $\mathcal{X}(0) = X$ and $\mathcal{X}(\epsilon') \subset \text{ Int } \mathcal{X}(\epsilon)$ for every $0 \leq \epsilon < \epsilon' \leq 1$.

For example, let X be described by the set $\{x \in \mathbb{R}^n | h(x) \leq 0\}$ for some convex function h from $\mathbb{R}^n$ to $\mathbb{R}$. Notice that such a function h always exists, since X is compact and convex. Then we may take $\mathcal{X}(\epsilon) = \{x \in \mathbb{R}^n | h(x) \leq -\epsilon\}$, where we assume that $\mathcal{X}(1) \neq \emptyset$. Another possibility is to take $\mathcal{X}(\epsilon) = \epsilon\{v\} + (1-\epsilon)X$ for some point v in the interior of X.

For a given mapping $\mathcal{X}$ satisfying conditions (X1) and (X2), the second mapping defines the concept of generalized stationary point on each set $\mathcal{X}(\epsilon)$. This mapping is given by a mapping $G\colon X \rightarrow \mathbb{R}^n$ satisfying the following three conditions, where Bnd denotes the boundary of a set.

(G1) G is upper semi-continuous on X and for each $x \in X$ the set $G(x)$ is a non-empty, convex, closed cone in $\mathbb{R}^n$.

(G2) For every $x \in \text{ Bnd } \mathcal{X}(\epsilon)$ and $y \in N(\mathcal{X}(\epsilon), x) \setminus \{0^n\}$, $0 < \epsilon < 1$, there exists $w \in G(x)$ such that $y^\top w > 0$.

(G3) For every $x \in \text{ Bnd } X$ it holds that $G(x) = N(X, x)$.

The first condition means that like in the normal cone the length of a vector in $G(x)$

is not important, only the direction into which the vector points matters. The second condition will guarantee the existence of generalized stationary points in $\mathcal{X}(\epsilon)$ for every ϵ, $0 < \epsilon < 1$. The condition says that when x lies in the boundary of $\mathcal{X}(\epsilon)$ the set $G(x)$ must point in the same direction as the normal cone $N(\mathcal{X}(\epsilon), x)$ in the sense that for every nonzero element of $N(\mathcal{X}(\epsilon), x)$ there is an element in $G(x)$ making a positive angle with it. Notice that due to both conditons (X1) and (X2) it holds that for every $x \in X \setminus \mathcal{X}(1)$ there exists a unique ϵ, $0 \leq \epsilon < 1$, such that $x \in$ Bnd $\mathcal{X}(\epsilon)$. The third condition says that G maps a point x in the boundary of X to the normal cone $N(X, x)$ of X at x and guarantees that a convergent sequence of generalized stationary points in $X(\epsilon)$ converges to a stationary point when ϵ goes to zero. Notice that the conditions on G do not depend on ϕ and that condition (G2) depends on the chosen mapping $\mathcal{X}$.

Definition 2.1 *A pair $(\mathcal{X}, G)$ of mappings is regular when it satisfies the conditions (X1), (X2), (G1), (G2) and (G3).*

Let ϕ be a point-to-set mapping from X to the collection of non-empty subsets of $\mathbb{R}^n$. For any pair $(\mathcal{X}, G)$ and $\epsilon \in [0, 1)$ a generalized stationary point of ϕ on $\mathcal{X}(\epsilon)$ is defined as follows.

Definition 2.2 *For a pair $(\mathcal{X}, G)$ and $0 \leq \epsilon < 1$, a point $x \in \mathcal{X}(\epsilon)$ is a generalized stationary point of ϕ on $\mathcal{X}(\epsilon)$ if $0^n \in \phi(x)$ when $x \in \text{Int } \mathcal{X}(\epsilon)$ and $\phi(x) \cap G(x) \neq \emptyset$ when $x \in \text{Bnd } \mathcal{X}(\epsilon)$.*

In case ϕ is a function f from X to $\mathbb{R}^n$ the vector $f(x)$ should be an element of $G(x)$. Observe that a generalized stationary point of ϕ on $X(\epsilon)$ is just a stationary point of ϕ on $X(\epsilon)$ when for all $x \in$ Bnd $\mathcal{X}(\epsilon)$ it holds that

$$G(x) = N(\mathcal{X}(\epsilon), x),$$

i.e. when for every x in the boundary of $\mathcal{X}(\epsilon)$ the set $G(x)$ is equal to the normal cone of $\mathcal{X}(\epsilon)$ at x. Under condition (G3) this necessarily holds when $\epsilon = 0$, i.e. under (G3) a generalized stationary point of ϕ on $\mathcal{X}(0) = X$ is a stationary point of ϕ on X. Next, we define for $\epsilon \in (0, 1)$ the concept of ϵ-stable stationary point of ϕ on X with respect to $(\mathcal{X}, G)$.

Definition 2.3 *For given $(\mathcal{X}, G)$ and $0 < \epsilon < 1$, a point $x \in X$ is an ϵ-stable stationary point of ϕ with respect to $(\mathcal{X}, G)$ if $x \in \mathcal{X}(\epsilon)$ and x is a generalized stationary point of ϕ on $\mathcal{X}(\epsilon)$.*

Together with Definition 2.2, the definition says that a point x in X is an ϵ-stable stationary point of ϕ with respect to $(\mathcal{X}, G)$ if either x lies in the interior of $X(\epsilon)$ and is a zero point and therefore a stationary point of ϕ or x lies on the boundary of $\mathcal{X}(\epsilon)$ and its image under ϕ has a nonempty intersection with $G(x)$. When a stationary point x^* of ϕ is the limit of a sequence of ϵ-stable stationary points with respect to the pair $(\mathcal{X}, G)$ for ϵ going to zero, we call x^* a stable stationary point of ϕ with respect to the pair $(\mathcal{X}, G)$.

Definition 2.4 *A stationary point x^* of ϕ is stable with respect to the pair $(\mathcal{X}, G)$, shortly $(\mathcal{X}, G)$-stable, if there exists a sequence of positive numbers $(\epsilon_k)_{k \in \mathrm{N}}$ with limit 0 such that x^* is the limit of a sequence of ϵ_k-stable stationary points of ϕ with respect to $(\mathcal{X}, G)$ for k going to infinity.*

Stability of a stationary point x^* with respect to $(\mathcal{X}, G)$ means that either x^* lies in the interior of X and is a zero point of ϕ or x^* lies in the boundary of X and in every small neighborhood of x^* there exists a point x in the interior of X such that $\phi(x)$ has a nonempty intersection with $G(x)$. When $(\mathcal{X}, G)$ is regular and thus G is upper semi-continuous and $G(x^*) = N(X, x^*)$ if x^* lies in the boundary of X, a stable stationary point $x^* \in$ Bnd X satisfies the property that when X is slightly perturbed to $X(\epsilon_k)$, there exists a point x in $X(\epsilon_k)$ that is close to x^* and that is approximately a stationary point of ϕ on $\mathcal{X}(\epsilon_k)$ in the sense that $\phi(x) \cap G(x) \neq \emptyset$. This property gives a stationary point x^* in the boundary of X a certain stability because for any small perturbation of X according to $\mathcal{X}$ an approximate solution exists arbitrarily close to x^*. Observe that a stationary point of ϕ in the interior of X is always stable.

The next theorem states that every mapping ϕ satisfying the standard conditions has an $(\mathcal{X}, G)$-stable stationary point for any regular pair $(\mathcal{X}, G)$.

Theorem 2.5 *Let ϕ be an upper semi-continuous mapping from a full-dimensional convex, compact set X to $\mathbb{R}^n$ such that $\phi(x)$ is convex and compact for all $x \in X$ and let $(\mathcal{X}, G)$ be a regular pair of mappings. Then there exists a $(\mathcal{X}, G)$-stable stationary point of ϕ on X.*

Proof. First we prove that for every ϵ, $0 < \epsilon < 1$, an ϵ-stable stationary point of ϕ with respect to $(\mathcal{X}, G)$ exists. For ϵ, $0 < \epsilon < 1$, let the mapping $G^\epsilon \colon \mathcal{X}(\epsilon) \to \mathbb{R}^n$ be defined by

$$G^\epsilon(x) = \{0^n\}, \ x \in \ \text{Int}\ \mathcal{X}(\epsilon),$$

$$G^\epsilon(x) = G(x) \cap \{y \in \mathbb{R}^n | \max_j |y_j| \leq M\}, \ x \in \ \text{Bnd}\ \mathcal{X}(\epsilon),$$

for some $M > 0$. Due to condition (G1) it holds that for every ϵ, $0 < \epsilon < 1$, and any given $M > 0$, the mapping G^ϵ is upper semi-continuous and $G^\epsilon(x)$, $x \in \mathcal{X}(\epsilon)$, is nonempty, convex and compact. Since for all $x \in X$ the set $G(x)$ is a cone and due to condition (G2), for every ϵ, $0 < \epsilon < 1$, we can choose the number $M > 0$ such that for all $x \in$ Bnd $\mathcal{X}(\epsilon)$ and $y \in N(\mathcal{X}(\epsilon), x)$, there exists $w \in G^\epsilon(x)$ and $f \in \phi(x)$ satisfying $y^\top w \geq y^\top f$. From Fan's coincidence theorem applied to the mappings ϕ and G^ϵ restricted to the non-empty, convex and compact set $\mathcal{X}(\epsilon)$ it follows that for every ϵ, $0 < \epsilon < 1$, there exists x^ϵ in $\mathcal{X}(\epsilon)$ satisfying that $\phi(x^\epsilon) \cap G^\epsilon(x^\epsilon) \neq \emptyset$. Hence, $0^n \in \phi(x^\epsilon)$ if $x^\epsilon \in$ Int $\mathcal{X}(\epsilon)$ and $\phi(x^\epsilon) \cap G(x^\epsilon) \neq \emptyset$ if $x^\epsilon \in$ Bnd $\mathcal{X}(\epsilon)$, i.e. x^ϵ is an ϵ-stable stationary point of ϕ with respect to $(\mathcal{X}, G)$.

Now take any sequence of positive numbers $\epsilon_k, k \in \mathbb{N}$, converging to zero, and for every $k \in \mathbb{N}$ let x^k be an ϵ_k-stable stationary point of ϕ with respect to $(\mathcal{X}, G)$. Since X is compact, without loss of generality we may assume that the sequence $(x^k)_{k \in \mathbb{N}}$ is convergent and converges to some x^* in X. Hence, x^* is the limit of a sequence of ϵ_k-stable stationary points of ϕ with respect to $(\mathcal{X}, G)$ for ϵ_k converging to zero when k goes to infinity. We still have to prove that x^* is a stationary point of ϕ. If x^* lies in the interior of X, then because of the continuity of $\mathcal{X}$ and the properties of the mapping $\mathcal{X}$ given in (X2), the point x^k lies in the interior of $\mathcal{X}(\epsilon_k)$ for k large enough, which implies that x^* is a zero point and therefore a stationary point of ϕ. If x^* lies in the boundary of X we may assume without loss of generality that for every $k \in \mathbb{N}$ the point x^k lies in the boundary of $\mathcal{X}(\epsilon_k)$. Since for every $k \in \mathbb{N}$ the set $\phi(x^k) \cap G(x^k) \neq \emptyset$, let f^k be an element in this intersection. Because all f^k, $k \in \mathbb{N}$, lie in a compact set there exists a convergent subsequence to some f^*. Since ϕ both G are upper semi-continous on X and $G(x^*) = N(X, x^*)$, we obtain that $f^* \in \phi(x^*) \cap N(X, x^*)$, and hence x^* is a stationary point of ϕ. □

Notice that the conditions on $(\mathcal{X}, G)$ are completely independent of the mapping ϕ. However, as can be seen from the end of the proof, it is enough to have the condition that

$$\phi(x) \cap G(x) \subset \phi(x) \cap N(X, x),$$

for all $x \in$ Bnd X. Clearly, this condition is satisfied when condition (G3) holds.

The theorem implies that for every given regular $(\mathcal{X}, G)$ any mapping ϕ satisfying the same conditions under which a stationary point is known to exist, has a stationary point being stable with respect to $(\mathcal{X}, G)$. Of course the reverse does not hold. Not every stationary point needs to be a stable stationary point with respect to a chosen pair $(\mathcal{X}, G)$. Also, the stableness of a stationary point depends on the chosen pair. This means that a stationary point may be stable for some pair, but not for another pair. It may also happen that a stationary point is not stable for any pair. So, the set of stable points depends on the pair $(\mathcal{X}, G)$ and is a (nonempty) subset of the set of stationary points. Notice that a zero point can only be not stable if it lies on the boundary of X.

Let us consider two special cases, the first with respect to the mapping G, the second with respect to the mapping $\mathcal{X}$. Concerning the first case, let $\mathcal{X}$ be any given mapping satisfying the conditions (X1) and (X2). Recall that by condition (X2) it holds that for any $x \in X \setminus \mathcal{X}(1)$, there is a unique ϵ, $0 \le \epsilon < 1$, such that x lies in Bnd $\mathcal{X}(\epsilon)$. A natural choice for the mapping G is to take the mapping $C\colon X \to \mathbb{R}^n$ defined by $C(x) = N(\mathcal{X}(\epsilon), x)$ when x lies in the boundary of $\mathcal{X}(\epsilon)$ for some ϵ, $0 \le \epsilon < 1$, and $C(x) = \mathbb{R}^n$ when $x \in \mathcal{X}(1)$. Then, for ϵ, $0 \le \epsilon < 1$, a generalized stationary point of a mapping ϕ on $X(\epsilon)$ is just a stationary point of ϕ on $\mathcal{X}(\epsilon)$. For this particular choice of the mapping G, for any ϵ, $0 < \epsilon < 1$, an ϵ-stable stationary point on X of a mapping ϕ with respect to $(\mathcal{X}, C)$ is said to be ϵ-robust with respect to $\mathcal{X}$, and an $(\mathcal{X}, C)$-stable stationary point on X of ϕ is said to be robust with respect to $\mathcal{X}$, or shortly $\mathcal{X}$-robust. The next theorem states that every mapping satisfying the standard conditions has an $\mathcal{X}$-robust stationary point for any mapping $\mathcal{X}$ satisfying (X1) and (X2).

Theorem 2.6 *Let ϕ be an upper semi-continuous mapping from a full-dimensional convex, compact set X to $\mathbb{R}^n$ such that $\phi(x)$ is convex and compact for all $x \in X$ and let $\mathcal{X} : [0,1] \to X$ be a mapping satisfying (X1) and (X2). Then ϕ has an $\mathcal{X}$-robust stationary point on X.*

Proof. For $x \in X$ define $G(x) = \mathbb{R}^n$ when $x \in \mathcal{X}(1)$ and $G(x) = N(\mathcal{X}(\epsilon), x)$ otherwise, where ϵ, $0 \le \epsilon < 1$, is uniquely determined by $x \in$ Bnd $\mathcal{X}(\epsilon)$. It is sufficient to show that $(\mathcal{X}, G)$ is regular, i.e. G satisfies the conditions (G1)-(G3). Clearly, G satisfies (G2) and (G3). Moreover, for each $x \in X$, $G(x)$ is a non-empty, convex and closed cone in $\mathbb{R}^n$. So, to prove (G1), we only need to show that G is upper semi-continuous on X. By definition, G is upper semi-continuous on $\mathcal{X}(1)$. Take any $y \in X \setminus \mathcal{X}(1)$. Let $(y^k)_{k\in\mathbb{N}}$ be a sequence of points in X converging to y and let $(f^k)_{k\in\mathbb{N}}$ be a sequence satisfying $f^k \in G(y^k)$ for all $k \in \mathbb{N}$ and converging to f. Since $y \notin \mathcal{X}(1)$, we may assume without loss of generality that for all $k \in \mathbb{N}$ it holds that $y^k \in X \setminus \mathcal{X}(1)$. Let ϵ, $0 \le \epsilon < 1$, be such that $y \in$ Bnd $\mathcal{X}(\epsilon)$. Due to conditions (X1) and (X2) on $\mathcal{X}$ there exists a unique sequence of nonnegative numbers $(\epsilon_k)_{k\in\mathbb{N}}$ converging to ϵ and satisfying that $y^k \in$ Bnd $\mathcal{X}(\epsilon_k)$ for all $k \in \mathbb{N}$. To show that $f \in G(y)$, take any x in $\mathcal{X}(\epsilon)$. Then, again according to conditions (X1) and (X2) there exists a sequence $(x^k)_{k\in\mathbb{N}}$ satisfying $x^k \in \mathcal{X}(\epsilon_k)$ for all $k \in \mathbb{N}$ and converging to x. Since $x^k \in \mathcal{X}(\epsilon_k)$ and $f^k \in G(y^k) = N(\mathcal{X}(\epsilon_k), y^k)$, we have for all $k \in \mathbb{N}$ that

$$x^{k\top} f^k \le y^{k\top} f^k.$$

Taking the limits on both sides for k going to infinity, x being the limit of x^k, y being the limit of y^k, f being the limit of f^k, we obtain that

$$x^\top f \le y^\top f.$$

Since x is an arbitrary point in $\mathcal{X}(\epsilon)$, we obtain that $f \in N(\mathcal{X}(\epsilon), y) = G(y)$, showing that G is upper semi-continuous on X and thus G satisfies (G1). Hence the pair $(\mathcal{X}, G)$ is regular and Theorem 2.5 applies. □

Theorem 2.6 implies that there always exists a stationary point which is the limit point of a sequence of stationary points restricted to $\mathcal{X}(\epsilon_k)$, $k \in \mathbb{N}$, with $\lim_{k\to\infty} \epsilon_k = 0$. Of course, the same remarks made after the proof of Theorem 2.5 for the set of stable points apply to the set of robust points.

Next, we consider the case that the mapping $\mathcal{X}$ is chosen to be in a more specific way. Let v be an arbitrarily chosen point in the interior of X. Then we consider the mapping $\mathcal{E}$ given by

$$\mathcal{E}(\epsilon) = \epsilon\{v\} + (1-\epsilon)X, \quad 0 \le \epsilon \le 1, \tag{2.3}$$

i.e. $\mathcal{E}(\epsilon)$ expands linearly from the single point $\{v\}$ to the full set X when ϵ goes from one to zero. Clearly, this mapping satisfies (X1) and (X2). Taking $\mathcal{X} = \mathcal{E}$, for $0 < \epsilon < 1$ an ϵ-stable stationary point on X of a mapping ϕ with respect to $(\mathcal{E}, G)$ is called ϵ-perfect with respect to G, and an $(\mathcal{E}, G)$-stable stationary point on X of ϕ is called perfect with respect to G, or shortly G-perfect. Moreover, an ϵ-perfect (perfect) stationary point with respect to the mapping C is simply said to be ϵ-perfect (perfect). It follows from the results above that every mapping ϕ satisfying the standard conditions has a G-perfect stationary point for any mapping G satisfying (G1), (G2) and (G3) and therefore also always has a perfect stationary point.

Theorem 2.7 *Let ϕ be an upper semi-continuous mapping from a full-dimensional convex, compact set X to $\mathbb{R}^n$ such that $\phi(x)$ is non-empty, convex and compact for all $x \in X$. Then ϕ has a G-perfect stationary point on X for any mapping G satisfying (G1), (G2) and (G3). In particular ϕ has a perfect stationary point on X.*

Proof. That ϕ has a G-perfect stationary point on X for any mapping G satisfying (G1), (G2) and (G3) follows from Theorem 2.5 and the fact that $\mathcal{E}$ satifies conditions (X1) and (X2). The existence of a perfect stationary point follows from Theorem 2.6. □

Notice that the concept of G-perfectness depends on the chosen point v in X. In applications there is often a natural choice for the point v, for example the origin, the barycenter of a simplex or some other specific point.

Example 1 Let X be the two-dimensional unit ball $B = \{x \in \mathbb{R}^2 \mid \|x\|_2 \le 1\}$ and let the function $f\colon B \to \mathbb{R}^2$ be given by $(f_1(x), f_2(x)) = (x_1 + 1, x_2)$. Clearly, $x^* \in \text{Bnd } B$ is a stationary point of f if and only if $f(x^*) = \lambda x^*$ for some $\lambda \ge 0$. The function F has two stationary points of f and both lie in the boundary of B, $(-1,0)$ with function value $f(-1,0) = (0,0)$ and $(1,0)$ with function value $f(1,0) = (2,0)$. However, only $(1,0)$ is a perfect stationary point of f, i.e. $(1,0)$ is the unique $(\mathcal{B}, G)$-stable stationary point when $\mathcal{B}$ is defined by

$$\mathcal{B}(\epsilon) = \{x \in \mathbb{R}^2 \mid \| x \|_2 \leq 1 - \epsilon\}, \quad 0 \leq \epsilon \leq 1,$$

i.e. $\mathcal{B}(\epsilon)$ expands from the zero point to B when ϵ goes from one to zero, and $G(x)$ is taken to be the normal cone to $B(\epsilon)$ at x when x lies on the boundary of $B(\epsilon)$.

3 Perfect and robust stationary points on polytopes

In this section we consider the special case that the set X is a (full-dimensional) polytope P in $\mathbb{R}^n$ and ϕ is a function f from P to $\mathbb{R}^n$. Since a polytope is compact and convex, Theorems 2.5, 2.6 and 2.7 immediately apply to any function or mapping from P to $\mathbb{R}^n$. Due to the special structure of polytopes, ϵ-robust and ϵ-perfect stationary points possess appealing and interesting properties. In the next section these properties will be shown to have intuitive and natural interpretations in the context of both game theory and equilibrium theory.

We consider the case that the polytope P is simple and full-dimensional and is described as a bounded polyhedron by

$$P = \{x \in \mathbb{R}^n \mid a^{i\top} x \leq b_i, \text{ for all } i \in I_m\},$$

where $I_m = \{1, \cdots, m\}$, $a^i \in \mathbb{R}^n \setminus \{0^n\}$ and $b_i \in \mathbb{R}$, for all $i \in I_m$. We assume that none of the constraints is redundant. For each subset I of I_m, let

$$F(I) = \{x \in P \mid a^{i\top} x = b_i, \text{ for all } i \in I\}.$$

Note that $F(\emptyset) = P$. Further, let $\mathcal{I}$ be the collection of subsets of I defined by

$$\mathcal{I} = \{I \subseteq I_m \mid F(I) \neq \emptyset\},$$

i.e. $I \in \mathcal{I}$ when $F(I)$ is not empty. A non-empty $F(I)$ is called a *face* of P. The polytope P is said to be *simple* if the dimension of every face $F(I)$ of P is equal to $n - |I|$, where $|I|$ denotes the cardinality of I. Finally, for $I \in \mathcal{I}$, define

$$A(I) = \{y \in \mathbb{R}^n \mid y = \sum_{i \in I} \mu_i a^i, \ \mu_i \geq 0, \text{ for all } i \in I\}.$$

Since P is simple and there are no redundant constraints, it holds that for any $y \in \mathbb{R}^n$ there is a unique $I \in \mathcal{I}$ such that $y = \sum_{i \in I} \mu_i a^i$ with $\mu_i > 0$, for all $i \in I$. Notice that $A(\emptyset) = \{0^n\}$. Moreover, if $x \in \text{Int } F(I)$, then $A(I) = N(X, x)$. Hence, we have the following straightforward but important observation; see also Talman and Yamamoto (1989) and Burke and More (1994).

Lemma 3.1 *A point $x^* \in P$ is a stationary point of a function f from P to $\mathbb{R}^n$ if and only if there exists $I^* \in \mathcal{I}$ satisfying $x^* \in F(I^*)$ and $f(x^*) \in A(I^*)$.*

Proof. The result immediately follows from linear optimization. □

Let $\mathcal{P}\colon [0,1] \to P$ be a mapping satisfying the conditions (X1) and (X2) and let $G = C$, i.e. $G(x) = \mathbb{R}^n$ when $x \in \mathcal{P}(1)$ and $G(x) = N(\mathcal{P}(\epsilon), x)$ otherwise, where ϵ, $0 \le \epsilon < 1$, is uniquely determined by $x \in$ Bnd $\mathcal{P}(\epsilon)$. Then, according to Theorem 2.6, any continuous function f from P to $\mathbb{R}^n$ has a $\mathcal{P}$-robust stationary point x^* on P, i.e. f has a stationary point x^* satisfying that there exists a sequence of positive numbers $(\epsilon_k)_{k\in\mathbb{N}}$ with limit 0 such that x^* is the limit of a sequence of ϵ_k-robust stationary points of f with respect to $\mathcal{P}$.

A special mapping $\mathcal{P}$ has been considered in van der Laan, Talman and Yang (1998). To define the sets $\mathcal{P}(\epsilon)$, define for $x \in P$, $\gamma(x) = \min_{i\in I_m}(b_i - a^{i\top}x)$ and $\Gamma = \max_{x\in P}\gamma(x)$. Further, take some $\omega \in (0, \Gamma]$ and define for $I \in \mathcal{I}$ and $\epsilon \in [0,1]$,

$$a^I = \sum_{h\in I} a^h \text{ and } b_I(\epsilon) = \sum_{h\in I} b_h - \omega \sum_{k=n+1-|I|}^{n} \left(\frac{\epsilon}{2}\right)^k .$$

Then the mapping $\mathcal{P}$ with $\mathcal{P}(\epsilon) \subset P$ and $\mathcal{P}(0) = P$ is defined by

$$\mathcal{P}(\epsilon) = \{x \in \mathbb{R}^n \mid a^{I\top}x \le b_I(\epsilon),\ I \in \mathcal{I}\}, \quad \epsilon \in [0,1]. \tag{3.1}$$

In van der Laan, Talman and Yang (1998) the next lemma is shown.

Lemma 3.2 *Let $x \in P$ be a ϵ-robust stationary point with respect to $\mathcal{P}$ of a function f on P for some $0 < \epsilon < 1$ and let $I \in \mathcal{I}$ be such that $f(x) = \sum_{h\in I_m}\mu_h a^h$ with $\mu_h = 0$ if $h \notin I$ and $\mu_h > 0$ if $h \in I$. Then for any pair of indices l and k in I_m it holds that*

$$b_l - a^{l\top}x \le \frac{\epsilon}{2}(b_k - a^{k\top}x) \text{ when } \mu_l > \mu_k.$$

Recall that the set I and the μ_h's are uniquely determined. The lemma states two facts. First, when $\mu_h > 0$, then $0 \le b_h - a^{h\top}x \le \frac{\epsilon}{2}\max_{k\in I_m}(b_k - a^{k\top}x)$, saying that x lies arbitrarily close to the face $F(I)$ for ϵ small enough. Second, for $h \in I$ it holds that the larger the coefficient μ_h is, the closer the point x lies to the facet $F(\{h\})$ of P. For this choice of $\mathcal{P}$, in van der Laan, Talman and Yang (1998), an ϵ-robust stationary point with respect to $\mathcal{P}$ is called shortly ϵ-robust and a $\mathcal{P}$-robust stationary point a robust stationary point. Clearly, when f is a continuous function from P to $\mathbb{R}^n$, an ϵ-robust stationary point of f exists for every $\epsilon \in (0,1)$ and therefore f has a robust stationary point on P. The lemma can be easily generalized for the case of a point-to-mapping instead of a function or when the polytope P is a lower-dimensional case. Applying the above lemma to a noncooperative normal form game in the next section, we will show that a robust stationary point of the marginal payoff function on the strategy space of the game yields what we will call a robust Nash equilibrium.

We now turn to discuss the special case that $\mathcal{P} = \mathcal{E}$ with

$$\mathcal{E}(\epsilon) = \epsilon\{v\} + (1-\epsilon)P, \quad 0 \leq \epsilon \leq 1, \tag{3.2}$$

where v is some arbitrary point in the interior of P, i.e. $a^{i\top}v < b_i$ for all $i \in I_m$. As defined in the previous section, for this mapping of expanding sets an $\mathcal{E}$-robust stationary point is called a perfect stationary point, and an ϵ-robust stationary point with respect to $\mathcal{E}$ is said to be ϵ-perfect. Define $M = \max_{i \in I_m}(b_i - a^{i\top}v)$ and notice that $M > 0$ since $v \in$ Int P. An ϵ-perfect point satisfies the next property.

Lemma 3.3 *Let $x \in P$ be an ϵ-perfect stationary point of f. Then there exists $I \in \mathcal{I}$ such that $f(x) \in A(I)$ and*

$$a^{i\top}x \geq b_i - M\epsilon \text{ for all } i \in I.$$

Proof. Let x be an ϵ-perfect stationary point of f. By definition, x is a stationary point of f on $\mathcal{E}(\epsilon) = \epsilon\{v\} + (1-\epsilon)P$ with v some arbitrary point in the interior of P. Since P is simple, the set $\mathcal{E}(\epsilon)$ is a simple polytope and can be written as

$$\mathcal{E}(\epsilon) = \{x \in \mathbb{R}^n \mid a^{i\top}x \leq b_i(\epsilon),\ i \in I_m\},$$

where

$$b_i(\epsilon) = \epsilon a^{i\top}v + (1-\epsilon)b_i,\ i \in I_m.$$

From applying Lemma 3.1 to $\mathcal{E}(\epsilon)$ it follows that there exists a set of indices $I \in \mathcal{I}$ such that $F(x) \in A(I)$ and $x \in F(I) = \{x \in \mathcal{E}(\epsilon) \mid a^{i\top}x = b_i(\epsilon),\ \text{ for all } i \in I\}$. Hence, $a^{i\top}x = \epsilon a^{i\top}v + (1-\epsilon)b_i = b_i - \epsilon(b_i - a^{i\top}v)$ for all $i \in I$. Since $M = \max_{i \in I_m}(b_i - a^{i\top}v)$ this proves the lemma. □

The lemma says that if x is an ϵ-perfect stationary point of f then there exists $I \in \mathcal{I}$ such that $f(x) \in A(I)$ and $0 \leq b_i - a^{i\top}x \leq M\epsilon$ for every $i \in I$, i.e. x lies arbitrarily close to the face $F(I)$ for ϵ small enough. If f is a continuous function from P to $\mathbb{R}^n$ then an ϵ-perfect stationary point of f exists for ever $\epsilon \in (0,1)$ and therefore any continuous f has a perfect stationary point on P. Notice that every robust stationary point is also perfect, but that the reverse is not true. It is again easy to generalize the lemma in case of a point-to-set mapping ϕ instead of a function or when P is a lower-dimensional polytope. In the next section it will be shown that a perfect stationary point of the marginal payoff function on the strategy space of a noncooperative game yields a perfect Nash equilibrium.

4 Applications

4.1 Noncooperative games in normal form

Two special cases of a polytope are the $(n-1)$-dimensional unit simplex $S^n = \{x \in \mathbb{R}^n \,|\, x_j \geq 0,\ j \in I_N,\ \sum_{j=1}^n x_j = 1\}$ and the simplotope, being the cartesian product of a finite number of unit simplices. It should be noticed that for the special case of the unit simplex the notion of a robust stationary point was introduced in Yang (1996,1999).

The first application we consider concerns noncooperative games in normal form. Let there be N players. Player j, $j \in I_N$, can choose out of n_j different actions in the set A^j. If player j, $j \in I_N$, chooses action a_j, then the payoff to player i, $i \in I_N$, is equal to some number $u_i(a)$, where $a = (a_1, \cdots, a_N)$ is an element of the action space $A = \Pi_{j \in I_N} A^j$. Each player j, $j \in I_N$, can randomize the choice of his actions by taking a strategy $x^j = (x_1^j, \cdots, x_{n_j}^j)$ in the $(n_j - 1)$-dimensional unit simplex S^{n_j}, where x_k^j, $k \in I_{n_j}$, denotes the probability with which player j chooses his kth action. The cartesian product of the strategy set S^{n_j}, $j \in I_N$, is the strategy set of the game and is denoted by the simplotope S with typical element $x = (x^1, \cdots, x^N)$. Clearly, S is a simple polytope with dimension equal to $n - N$ where n is the total number of actions in the game, i.e. $n = \sum_{j=1}^N n_j$.

For $x \in S$, $v_j(x)$ denotes the expected payoff for player j, $j \in I_N$, when strategy x is being played, i.e.

$$v_j(x) = \sum_{a \in A} \Pi_{i \in I_N} x_{a_i}^i u_j(a),$$

and $f_k^j(x)$ denotes the marginal payoff for player j, $j \in I_N$, when player j chooses action k, $k \in A^j$, and the other players play according to strategy x, i.e.

$$f_k^j(x) = \sum_{\{a \in A | a_j = k\}} \Pi_{i \neq j} x_{a_i}^i u_j(a).$$

We now have the following definitions, where (x^j, x^{*-j}) denotes the strategy vector x^* with x^{*j} replaced by x^j.

Definition 4.1 1. *(Nash, 1950) A strategy $x^* \in S$ is a Nash equilibrium if for every $j \in I_N$ it holds that $v_j(x^*) \geq v_j(x^j, x^{*-j})$ for all $x^j \in S^{n_j}$.*

2. *(Selten, 1975) A strategy $x^* \in S$ is a perfect Nash equilibrium if it is the limit of a sequence of ϵ_k-perfect equilibria for a sequence of positive numbers ϵ_k, $k \in \mathbb{N}$, converging to zero, where a strategy x is called an ϵ-perfect equilibrium if $x \in$ Int S and $x_k^j \leq \epsilon$ whenever $f_k^j(x) < \max_h\ f_h^j(x)$.*

3. *(Myerson, 1978) A strategy $x^* \in S$ is a proper Nash equilibrium is the limit of a sequence of ϵ_k-proper equilibria for a sequence of positive numbers ϵ_k, $k \in \mathbb{N}$,*

converging to zero, where a strategy x is called an ϵ-proper equilibrium if $x \in$ Int S and $x^j_k < \epsilon x^j_h$ whenever $f^j_k(x) < f^j_h(x)$.

Clearly, $x^* \in S$ is a Nash equilibrium if and only if $f^j_k(x^*) = \max_h f^j_h(x^*)$ whenever $x^{*j}_k > 0$, i.e. x^* is a stationary point of the marginal payoff function f on S. A Nash equilibrium is perfect when it is the limit of a sequence of ϵ-perfect equilibria, where a strategy x is called ϵ-perfect if each player j plays each non-optimal action k with probability at most equal to ϵ. A proper equilibrium is the limit of a sequence of ϵ-proper equilibria where a strategy x is called an ϵ-proper equilibrium if 'the lower the marginal payoff of an action of a player is, the smaller the probability should be with which this player chooses that action'. Clearly, every proper equilibrium is perfect and any perfect equilibrium is a Nash equilibrium. The existence of a perfect and proper Nash equilibrium follows from our results in Sections 2 and 3. First we consider the existence of a perfect Nash equilibrium.

Proposition 4.2 *Any noncooperative game in normal form has a perfect Nash equilibrium.*

Proof. Take S as the polytope P, the mapping $\mathcal{P} = \mathcal{E}$ given by

$$\mathcal{E}(\epsilon) = \epsilon\{v\} + (1-\epsilon)S, \quad 0 \le \epsilon \le 1,$$

for some v in the (relative) interior of S, and the mapping $G = C$ given by $C(v) = \mathbb{R}^n$ and $C(x) = N(\mathcal{E}(\epsilon), x)$ when $x \in$ Bnd $\mathcal{E}(\epsilon)$ for ϵ, $0 \le \epsilon < 1$. In polyhedral form S can be written as

$$S = \{x \in \mathbb{R}^n \mid -x^j_k \le 0, \text{ for all } j \text{ and } k, \sum_{k=1}^{n_j} x^j_k = 1, \text{ for all } j\}. \tag{4.3}$$

Clearly, there are no redundant constraints. From Theorem 2.7 it follows that the marginal payoff function f has a perfect stationary point x^* on S. Hence, x^* is the limit of a sequence of ϵ-perfect stationary points of f on S. Applying Lemma 3.3 and taking into account the above formulation of S, so that $M \le 1$, learns that x is an ϵ-perfect Nash equilibrium if x is an ϵ-perfect stationary point of f. Hence, the limit point x^* is a perfect Nash equilibrium. □

Next we consider the exisitence of a proper Nash equilibrium.

Proposition 4.3 *Any noncooperative game in normal form has a proper Nash equilibrium.*

Proof. Again take S as the polytope P. The mapping $\mathcal{P} = \mathcal{R}$ is given by

$$\mathcal{R}(\epsilon) = \Pi_{j=1}^N \mathcal{P}^j(\epsilon), \quad 0 \le \epsilon \le 1,$$

where for $j \in I_N$ the set $\mathcal{P}^j(\epsilon)$ is defined as in (3.1) for the $n_j - 1$-dimensional unit simplex S^{n_j} written in polyhedral form as

$$S^{n_j} = \{x \in \mathbb{R}^{n_j} \mid -x^j_k \leq 0, \text{ for all } k, \sum_{k=1}^{n_j} x^j_k = 1\}.$$

The mapping $G = C$ is given by $C(v) = \mathbb{R}^n$ and $C(x) = N(\mathcal{R}(\epsilon), x)$ when $x \in$ Bnd $\mathcal{R}(\epsilon)$ for ϵ, $0 \leq \epsilon < 1$. From Theorem 2.6 it follows that the marginal payoff function f has a $\mathcal{R}$-robust stationary point x^* on S. Hence, x^* is the limit of a sequence of ϵ-perfect stationary points of f on S with respect to $\mathcal{R}$. Applying Lemma 3.2 for $\mathcal{R}$ and taking into account the formulation (4.3) of S, learns that x is an ϵ-proper Nash equilibrium if x is an ϵ-proper stationary point of f. Hence, the limit point x^* is a proper Nash equilibrium. □

In the literature, properness is known to be the most refined concept of a Nash equilibrium that still exists for every noncooperative game in normal form. The concept of robustness as introduced on a polytope in the previous section, suggests that we may refine properness to robustness.

Definition 4.4 *A strategy $x^* \in S$ is a robust Nash equilibrium if it is the limit of a sequence of ϵ_k-robust equilibria for a sequence of positive numbers ϵ_k, $k \in \mathbb{N}$, converging to zero, where a strategy x is called an ϵ-robust equilibrium if $x \in$ Int S and $x^j_k < \epsilon x^i_l$ whenever $\max_h f^j_h(x) - f^j_k(x) > \max_h f^i_h(x) - f^i_l(x)$.*

The definition implies that the worser an action in the game is, the smaller the probability should be with which that action is chosen. So, robustness refines properness in the sense that the condition saying that the probability of an action decreases by at least a factor ϵ if the marginal payoff becomes worser, is taken over all players simultaneously instead of per player seperately.

Proposition 4.5 *Any noncooperative game in normal form has a robust Nash equilibrium and the set of robust Nash equilibria is a subset of proper Nash equilibria.*

Proof. Take $P = S$, the mapping $\mathcal{P}$ as defined in (3.1), and $G = C$. Then Theorem 2.6 says that the marginal payoff function f has a $\mathcal{P}$-robust stationary point x^*. Hence, x^* is the limit of a sequence of ϵ-robust stationary points of f with respect to $\mathcal{P}$ for ϵ going to zero. Applying Lemma 3.2 and taking into account formulation (4.3) of S, learns that if x is an ϵ-robust stationary point of f then x is an ϵ-robust Nash equilibrium. Therefore, the point x^* is a robust Nash equilibrium. □

Notice that a robust Nash equilibrium is always proper and therefore also perfect.

4.2 Symmetric bimatrix games

In this subsection we consider two-player games in normal form. Such a two-player game can be summarized by the $n_1 \times n_2$ payoff matrices $A = (a_{hk})$ and $B = (b_{hk})$, $h = 1, \ldots, n_1$, $k = 1, \ldots, n_2$, where n_j is the number of pure actions for player j, $j = 1, 2$. Given a mixed strategy pair $(x^1, x^2) \in S$, the payoff of the players is then given by $v_1(x^1, x^2) = x^{1\top} A x^2$ for player 1 and $v_2(x^1, x^2) = x^{1\top} B x^2$ for player 2.

The class of symmetric bimatrix games is given by the class of bimatrix games (A, B) such that $B = A^\top$. As a consequence we have that $n_1 = n_2 = n$. Such games have appeared to be very important in evolutionary game theoretic models, in which individuals are repeatedly drawn from a large monomorphic population to play a symmetric two-person game. If $(x, x) \in S^n \times S^n$ is a Nash equilibrium of the symmetric bimatrix game $(A, A^\top)$, then strategy x is called an *equilibrium strategy* of the game. As introduced by Maynard Smith (1982), see also Maynard Smith and Price (1973), an equilibrium strategy $x \in S^n$ is said to be an *evolutionary stable strategy*, shortly ESS, if for any mixed strategy $y \neq x$ in S^n there exists some $\epsilon_y \in (0, 1)$ such that for all $\epsilon \in (0, \epsilon_y)$ it holds that

$$x^\top A w > y^\top A w \text{ where } w = \epsilon y + (1 - \epsilon) x.$$

We now have the following results.

Lemma 4.6 (see e.g. Nash (1951), Van Damme (1987) or Weibull (1995))
Every symmetric bimatrix game has an equilibrium strategy $x \in S^n$, i.e. a symmetric Nash equilibrium $(x, x) \in S^n \times S^n$.

Lemma 4.7 (see e.g. Van Damme (1987) or Weibull (1995))
Let $x \in S^n$ be an ESS, then $(x, x) \in S^n \times S^n$ is a symmetric proper Nash equilibrium.

However, the existence of an ESS is not guaranteed. Indeed there exist many symmetric bimatrix games games not having an ESS. So, although we know from Myerson (1978) that every symmetric bimatrix game has at least one proper Nash equilibrium and Lemma 4.6 states that any such game has at least one symmetric Nash equilibrium, these results do not guarantee the existence of a symmetric proper Nash equilibrium. As we will show now, the existence of a symmetric proper Nash equilibrium in a symmetric bimatrix game follows immediately from the existence of a robust stationary point of a continuous function on the unit simplex.

For a symmetric bimatrix game $(A, A^\top)$ with A an $n \times n$ matrix we define the function $f \colon S^n \to \mathbb{R}^n$ by

$$f(x) = Ax.$$

So, given strategy $x \in S^n$ of player i, $f_k(x)$ is the expected marginal payoff of player $j \neq i$ when the latter player chooses his kth action with probability 1, $k = 1, \ldots, n$. Then we have the following results.

Lemma 4.8
For $\epsilon \in (0, 1)$, let $x \in S^n$ be a completely mixed strategy such that

$$x_k \leq \epsilon x_l \text{ if } f_k(x) < f_l(x) \text{ for all } k,\ l \in \{1, \ldots, n\}.$$

Then the pair $(x, x) \in S^n \times S^n$ is a symmetric ϵ-proper Nash equilibrium.

Proof. Clearly (x, x) satisfies the conditions of an ϵ-proper equilibrium given in Definition 4.1 with $f^1 = f^2 = f$. □

Proposition 4.9 *Any symmetrix bimatrix game has a symmetric proper Nash equilibrium.*

Proof. Take $P = S^n$ and the mapping $\mathcal{P}$ as defined in formula (3.1), where the set S^n is written in polyhedral form as

$$S^n = \{x \in \mathbb{R}^n \mid -x_k \leq 0 \text{ for all } k,\ \sum_{k=1}^{n} x_k = 1\}.$$

Then Theorem 2.6 says that the marginal payoff function f has a robust stationary point x^*. Hence, x^* is the limit of a sequence of ϵ-robust stationary points of f on S^n. Applying Lemma 3.2 and taking into account the formulation of S^n as a polytope, it follows from Lemma 4.8 that if x is an ϵ-robust stationary point of f on S^n then (x, x) is a symmetric ϵ-proper Nash equilibrium. Therefore, the limit point x^* is a symmetric proper Nash equilibrium. □

The proposition shows that the existence of a symmetric proper equilibrium follows as a corollary from the existence of a robust stationary point on the polytope. To the best of our knowledge, this existence result is unknown within the field of noncooperative game theory. It might be worthwile to mention that by using the algorithm given in Yang (1996) to approximate a robust stationary point on the unit simplex, we can also compute a symmetric proper Nash equilibrium.

4.3 Replicator and price dynamics

In this subsection we consider a function $z\colon S^n \to \mathbb{R}^n$ satisfying $x^\top z(x) = 0$ for all $x \in S^n$. The function z could be the excess demand function of a pure exchange economy with

n commodities. Then S^n is the set of nonnegative prices normalized to sum up to one and $z_j(x)$ is the excess demand of commodity j at price vector x. In an evolutionary game theory, $z_j(x)$ could be the excess fitness of or action j in a symmetric bimatrix game $(A, A^\top)$ at mixed strategy x, i.e. for $j = 1, \ldots, n$,

$$z_j(x) = A_j x - x^\top A x,$$

is the marginal payoff of action j minus the average payoff over all actions at strategy x, where A_j denotes the jth column of the matrix A. A stationary point x^* of z gives a vector at which $z(x^*) \leq 0^n$ and $z_j(x^*) < 0$ implies that $x^*_j = 0$. In case of a pure exchange economy, a stationary point of z gives a Walrasian or general equilibrium price system, at which the excess demand of every commodity is nonpositive and can be only negative if its price is equal to zero. A stationary point of an excess fitness function gives a solution satisfying that the fitness of every action is maximal unless it is played with probability zero, i.e. a stationary point is an equilibrium strategy.

In evolutionary game theory the probability x_j is considered to be the fraction of players using action j within a monomorphic population of a large number of players. So, the fitness can be seen as the difference of the (expected) payoff of a player of population j and the expected payoff in the population as a whole. It is further assumed that players with a higher fitness get more offspring, resulting in the so-called replicator dynamics given by

$$dx(t)/dt = f(x(t)), \ t \geq 0,$$

with $f\colon S^n \to \mathbb{R}^n$ given by

$$f_j(x) = x_j z_j(x), \ j = 1, \ldots, n.$$

In game theoretic models the replicator dynamics models the population dynamics, in economic exchange models terms the function f is called the excess value function and the dynamics corresponds to some price adjustment. The function f has the property that $\sum_{j=1}^n f_j(x) = 0$ for any $x \in S^n$, so that the solution path of the replicator dynamics $dx(t)/dt = f(x(t))$ stays in S^n; see for example, Weibull (1995).

Clearly, each stationary point of z (and thus each equilibrium strategy of a symmetric bimatrix game and each equilibrium price system of a pure exchange economy) is a stationary point of the corresponding function f and is even a zero point of f. The reverse is not true. Not every stationary point of f is a stationary point of z. For example, all vertices of S^n are stationary points of f, but not all of them need to be equilibrium points. However, we will show that a so-called "sign-stable" stationary point of the function f is a stationary point of z and therefore an equilibrium and we will also prove that such a point always exists. A point $x \in S^n$ is called a *sign-stable stationary point* of f if it is the limit of a convergent subsequence of ϵ_k-sign-stable stationary points of f for a sequence of positive real numbers $(\epsilon_k)_{k \in \mathrm{N}}$ with $\lim_k \epsilon_k = 0$. For $0 < \epsilon < n^{-1}$, a point $x \in \operatorname{Int} S^n$ is an ϵ-sign-stable stationary point of f if $x_j \leq \frac{\epsilon}{n}$ when $f_j(x) < 0$ and $x_j \geq n^{-1}$ when $f_j(x) > 0$.

In the following, $e(i)$ denotes the i-th unit vector in $\mathbb{R}^n$ and e the n-vector of ones.

Theorem 4.10 *Let $z : S^n \to \mathbb{R}^n$ be a continuous function satisfying $x^\top z(x) = 0$ for all $x \in S^n$ and let $f : S^n \to \mathbb{R}^n$ be defined by $f_j(x) = x_j z_j(x)$ for all $j \in I_n$ and $x \in S^n$. Then a sign-stable stationary point of f exists and every sign-stable stationary point of f is a stationary point of z.*

Proof. For ϵ, $0 \leq \epsilon \leq 1$, let

$$\mathcal{P}(\epsilon) = \{x \in S^n \mid \min_j x_j \geq \frac{\epsilon}{n}\}.$$

Clearly, $\mathcal{P}(\cdot)$ is a continuous mapping, $\mathcal{P}(0) = S^n$, for every ϵ, $0 \leq \epsilon \leq 1$, the set $P(\epsilon)$ is a nonempty, compact and convex set, $\mathcal{P}(1) = \{\frac{e}{n}\}$, and $\mathcal{P}(\epsilon') \subset \operatorname{Int} \mathcal{P}(\epsilon)$ for every $0 \leq \epsilon < \epsilon' \leq 1$. For ϵ, $0 < \epsilon \leq 1$, and I being a proper subset of the set $I_n = \{1, \cdots, n\}$, the face $F^\epsilon(I)$ of $P(\epsilon)$ is given by $F^\epsilon(I) = \{x \in \mathcal{P}(\epsilon) | x_i = \frac{\epsilon}{n},\ i \in I\}$ and the normal cone $N(\mathcal{P}(\epsilon), x)$ at a point $x \in \operatorname{Int} F^\epsilon(I)$ is given by the set

$$A(I) = \{y \in \mathbb{R}^n | y = \mu_0 e - \sum_{i \in I} \mu_i e(i),\ \mu_0 \in \mathbb{R},\ \mu_i \geq 0,\ i \in I\}.$$

For $x \in S^n$, define $G(x) = \mathbb{R}^n$ if $x = \frac{1}{n} e$ and otherwise

$$\begin{aligned} G(x) = \{w \in \mathbb{R}^n \mid\ & w_i \leq 0 \text{ if } x_i = \min\nolimits_h\ x_h, \\ & w_i \geq 0 \text{ if } x_i = \max\nolimits_h\ x_h, \\ & w_i = 0 \text{ otherwise}\}. \end{aligned}$$

Clearly, $G(\cdot)$ satisfies condition (G1). To show that $G(\cdot)$ satisfies condition (G2), take any $x \in F^\epsilon(I)$ and $y \in A(I) \setminus \{0^n\}$, for ϵ, $0 < \epsilon < 1$, and I being a proper subset of I_n, so $y = \mu_0 e - \sum_{i \in I} \mu_i e(i)$ for some $\mu_0 \in \mathbb{R}$ and $\mu_i \geq 0$, $i \in I$, not all equal to zero. If $\mu_0 > 0$ take $w = e(j)$ for some j with $x_j = \max_h x_h$, then $w \in G(x)$ and $w^\top y = \mu_0 > 0$. If $\mu_0 = 0$ take $w = -e(j)$ for some j with $x_j = \min_h x_h$ and $\mu_j > 0$, then $w \in G(x)$ and $w^\top y = \mu_j > 0$. And if $\mu_0 < 0$ take $w = -e(j)$ for some j with $x_j = \max_h x_h$, then $w \in G(x)$ and $w^\top y = \mu_j - \mu_0 > 0$. Hence, $G(\cdot)$ satisfies condition (G2). With respect to (G3), it should be noticed that $G(\cdot)$ satisfies the weaker, but sufficient condition that $\phi(x) \cap G(x) \subset \phi(x) \cap N(X, x)$ for all $x \in \operatorname{Bnd} X$. Modifying the proof of Theorem 2.5 to the lower-dimensional set S^n it follows that there exists an $(\mathcal{P}, G)$-stable stationary point of f on S^n. Hence, for every ϵ, $0 < \epsilon < 1$, there exists $x^\epsilon \in P(\epsilon)$ satisfying $f(x^\epsilon) = \mu_0 e$ for some $\mu_0 \in \mathbb{R}$ if $x^\epsilon \in \operatorname{Int} \mathcal{P}(\epsilon)$ and $f(x^\epsilon) \in G(x^\epsilon)$ if $x^\epsilon \in \operatorname{Bnd} \mathcal{P}(\epsilon)$. Since $\sum_{i=1}^n f_i(x^\epsilon) = 0$

we obtain that $\mu_0 = 0$ and so $f(x^\epsilon) = 0^n$ if $x^\epsilon \in$ Int $P(\epsilon)$. If $x^\epsilon \in$ Bnd $\mathcal{P}(\epsilon)$ then there exists $\delta(\epsilon) \geq n^{-1}$ such that $x_j^\epsilon = \frac{\epsilon}{n}$ if $f_j(x^\epsilon) < 0$, $x_j^\epsilon = \delta(\epsilon)$ if $f_j(x^\epsilon) > 0$, $\epsilon \leq x_j^\epsilon \leq \delta(\epsilon)$ if $f_j(x^\epsilon) = 0$, i.e. x^ϵ is an ϵ-sign-stable stationary point of f. Take any convergent subsequence $(x^{\epsilon_k})_{k\in\mathbb{N}}$ of such points with $\lim_k \epsilon_k = 0$ and let x^* be the limit of this subsequence. Suppose $z_j(x^*) < 0$ for some component j, then for large enough k it holds that $f_j(x^{\epsilon_k}) < 0$ and therefore $x^{\epsilon_k} = \frac{\epsilon_k}{n}$ for k large enough. Hence, after taking limits we obtain that $z_j(x^*) < 0$ implies $x_j^* = 0$. Since $n^{-1} \leq \delta(\epsilon_k) \leq 1$ for all $k \in \mathbb{N}$, we may assume without loss of generality that the sequence $(\delta(\epsilon_k))_{k\in\mathbb{N}}$ converges to some $\delta^* > 0$. This implies that $x_j^* > 0$ if $z_j(x^*) > 0$. Since $\sum_{j=1}^n f_j(x^*) = 0$ we get that $f_j(x^*) = 0$ for all $j \in I_n$, and therefore $z_j(x^*) = 0$ if $x_j^* > 0$. Hence, x^* is a stationary point of z. □

The theorem says that the replicator dynamics function f has always a sign-stable stationary point and that every sign-stable stationary point of f induces an equilibrium for the underlying function z. It remains an open question to consider the conditions on z under which the replicator dynamics will converge to a sign-stable solution.

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