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Abstract

Airport profit games are a generalization of airport cost games as well as of bankruptcy games. In this paper we present a simple algorithm to compute the nucleolus of airport profit games. In addition we prove that there exists an unique consistent allocation rule in airport profit problems, and it coincides with the nucleolus of the associated TU game.

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1 Introduction

Littlechild and Thompson (1977) introduced airport problems when they investigated the cost sharing question arising at the construction of a landing strip for the Birmingham's airport (see also Thompson, 1971).

They proposed a game-theoretic approach to solve these problems. In the corresponding characteristic function only the airport runway cost function was taken into account. Taking advantage of the special structure of these "airport *cost* games", Littlechild and Owen (1973) gave a remarkably simplified formula for the Shapley value of these games (see also Dubey, 1982); later an extremely simple algorithm for the nucleolus was derived by Littlechild (1974). More recently, Potters and Sudhölter (1999) have studied several consistency properties for these problems.

In other work Littlechild and Owen (1977) noticed: "airport *cost* games are only a partial representation of the actual situation, for it takes no account of the revenues or other benefits generated by the aircraft movements". For instance, revenues are important to determine the optimal size of the airport runway. In addition, revenues affect the payoff vectors of the players, since none of the agents will accept a fee schedule higher than his revenue. Thus these authors thought more appropriate to work with what they named after "airport *profit* problems". Accordingly they defined the characteristic function value of a given coalition as the maximum revenue minus construction cost attainable for this coalition. In the same work they proved that Littlechild's (1974) algorithm for the nucleolus remains valid for these "airport *profit* games".

However, in the characteristic function proposed by Littlechild and Owen (1977) some coalitions may obtain a negative worth. And it is reasonable to think that agents in such coalitions would prefer not to build any airport runway at all, obtaining a surplus of zero. Consequently we have adopted in this work a different definition of the surplus of coalition for a public facility provision problem (namely, the zero-monotonic cover of Littlechild and Owen's (1977) game; see also Brânzei *et al.*, 2002). That is, the characteristic function considered in this paper coincides with the one of these authors only for coalitions having a nonnegative worth, and it is zero on the rest. It turns out that bankruptcy games (Aumann and Maschler's, 1985) are a special subclass of these new "airport *profit* games".

In this work we propose a simple algorithm to calculate the nucleolus of these "airport *profit* games". This algorithm coincides with Littlechild's (1974) one on the special class of "airport *cost* games" (and, needless to say, with the Aumann and Maschler's (1985) one for the consistent solution on the class of bankruptcy problems). To obtain the algorithm we will make use of a different approach to the one used by Littlechild (1974). Instead of it, we use a result of Arin and Iñarra (1998) about the structure of the family of proper coalitions with maximal excess at the nucleolus. According to Littlechild's (1974) algorithm, only coalitions formed by agents demanding a runway shorter than a given length are taking into account. However, the algorithm presented here also considers coalitions of agents demanding a runway larger than a given size.

This algorithm consists of a sequence of airport profit problems, starting with the original one. Each problem is a "reduced" problem of the preceding one. That is, in each step we calculate the nucleolus for a subset of agents, and then we consider a reduced problem. This reducing airport profit problems lead us to the necessity of proving a result related to a property of consistency for these problems. Actually this result is an extension of another one due to Potters and Sudhölter (1999) for airport cost problems (these authors call it ν-consistency). Moreover, our result can be also considered as an extension of Aumann and Maschler's (1985) one about the consistent solution of a bankruptcy problem.

The outline of the paper is as follows. In Section 2 we introduce the model and the preliminaries. In Section 3 we obtain the main results referred to the coalitions with maximal excess (some of the proofs are postponed to an appendix). Sections 4 is devoted to consistency, and finally Section 5 contains the full description of the algorithm offered here.

2 Airport profit problems

We say that the tuple $(N, \preceq, C, b)$ is an *airport profit problem* if:

a) N is a finite nonempty set.

b) $\preceq$ is a total order relation on N.

c) $C : N \to \mathbf{R}_+$ is non-decreasing (i.e., $i \preceq j$ implies $C(i) \leq C(j)$).

d) $b \in \mathbf{R}^N_{++}$.

The interpretation is as follows. N represents a set of agents. Every nonempty subset of N is called coalition. Agent i wants to carry out a project that generates a cost $C(i) \geq 0$. Then $i \preceq j$ means that project of agent j is an extension of project of agent i, in the sense that if agent i is served then j is automatically served too. Consequently we assume $C(i) \leq C(j)$ whenever $i \preceq j$. In addition, if the project of agent i is complemented, then he receives a revenue $b_i > 0$.

If there is no confusion we shall simply write (N, C, b).

Remark 1. The ordering $\preceq$ has been included in the description of the model for clarity in the exposition. Of course this ordering could have been anyone induced by the mapping C.

Given $(N, \preceq, C, b)$, and a coalition $S \subseteq N$, denote

$$\ell_S := \max\{i : i \in S\},$$
$$C(S) := \max\{C(i) : i \in S\} = C(\ell_S).$$

Player ℓ_S is the last player in S according to $\preceq$. The real number $C(S)$ represents the cost of serving all the members in S. By convention define $C(\emptyset) = 0$.

If $x \in \mathbf{R}^N$, we write x_S for the restriction of x to $\mathbf{R}^S$, and $x(S) = \sum_{i \in S} x_i$; hence the total benefit of members in S when all of them are attended is $b(S)$ $(b(\emptyset) = 0)$.

A *TU game* (or simply a *game*) is a pair (N, v), where N is a coalition of *players*, and $v : 2^N \to \mathbf{R}$ is a mapping that associates with every $S \subseteq N$ a real number $v(S)$, s. t. $v(\emptyset) = 0$.

A player $i \in N$ is said to be a *null player* in the game (N, v) if $v\big(S \cup \{i\}\big) = v(S)$ for every $S \subseteq N \backslash \{i\}$.

A game (N, v) is said to be *convex* if for every $S, T \subseteq N$ it holds: $v(S) + v(T) \leq v\big(S \cup T\big) + v(S \cap T)$.

Given a game (N, v), the set of *imputations* is defined to be

$$\mathcal{I}(N, v) := \big\{x \in R^N : x(N) = v(N), \text{ and } x_i \geq v\big(\{i\}\big) \text{ for every } i \in N\big\},$$

and the core of this game is the set

$$\mathcal{C}(N, v) := \big\{x \in R^N : x(N) = v(N), \text{ and } v(S) \leq x(S) \text{ for every } S \subseteq N\big\}.$$

The *excess of* $S \subseteq N$ *with respect to* $x \in \mathbf{R}^N$ is

$$e^v(S, x) := v(S) - x(S).$$

We often write $e(S, x)$ instead of $e^v(S, x)$.

For any payoff vector $x \in \mathbf{R}^N$, let $\theta(x)$ be the 2^N-tuple whose components are the excesses $e^v(S, x)$, $S \subseteq N$, arranged in non-increasing order. Denote by $\ll_{\mathcal{L}}$ the lexicographic order. The nucleolus $\nu(v)$ of the game (N, v) is the imputation $\nu(v) \in \mathcal{I}(N, v)$ satisfying

$$\theta\big(\nu(v)\big) \ll_{\mathcal{L}} \theta(y), \qquad \text{for all } y \in \mathcal{I}(N, v).$$

We shall simply write ν if there is no confusion.

Define the *surplus of player* i *against player* j $(i \neq j)$ at payoff vector $x \in \mathbf{R}^N$:

$$s_{ij}(x) := \max\left\{v(S) - x(S) : S \subseteq N,\ i \in S, j \notin S\right\}.$$

The *prekernel* of game (N, v) is the set

$$PK(N, v) := \big\{x \in \mathbf{R}^N : x(N) = v(N),\ s_{ij}(x) = s_{ji}(x),\ \text{for all } i, j \in N\big\}.$$

The *standard solution* of the 2-person TU game $\big(\{i, j\}, v\big)$ is the payoff vector given by

$$x_k = v\big(\{k\}\big) + \frac{v\big(\{i, j\}\big) - v\big(\{i\}\big) - v\big(\{j\}\big)}{2} \qquad k = i, j. \tag{1}$$

The standard solution assigns to each player k the quantity that can be assured by himself, $v\big(\{k\}\big)$, plus the amount obtained when the rest is divided equally. As Aumann and Maschler (1985) point out, the nucleolus, the kernel, the prekernel, the Shapley value of a 2-person game all of them coincide with the standard solution, and so do the better-known bargaining solutions (Nash, 1950; Kalai and Smorodinsky, 1975; Maschler and Perles, 1981). Furthermore, the standard solution is the only symmetric and efficient single-valued solution for 2-person games that is invariant under strategic equivalence (Peleg, 1986).

Let (N, v) be a TU game, $S \subset N$ be a proper coalition, and $x \in \mathbf{R}^N$. Define the *reduced game of* (N, v) *with respect to* S *and* x (Davis and Maschler, 1965), as the game $(S, v^{S,x})$, defined by

$$v^{S,x}(T) := \begin{cases} v(N) - x(N \setminus S) & \text{if } T = S, \\ \max\big\{v(T \cup Q) - x(Q) : Q \subset N \setminus S\big\} & \text{if } T \neq \emptyset, S \\ 0 & \text{if } T = \emptyset, \end{cases}$$

Remark 2. a) If $T \subseteq S \subset N$ are two coalitions, then $\left(v^{S,x}\right)^{T,x} = v^{T,x}$.

b) Notice that $v^{S,x}$ depends only on $x_{N \setminus S}$, so by abusing the notation we will also write $v^{S,x}$ when $x \in \mathbf{R}^{N \setminus S}$.

With every problem (N, C, b), we associate the TU game $\big(N, v^{(N,C,b)}\big)$, where

$$v^{(N,C,b)}(S) := \max\big\{b(R) - C(R) : R \subseteq S\big\} \quad \text{for each } S \subseteq N. \tag{2}$$

The game $\big(N, v^{(N,C,b)}\big)$ is called an *airport profit game*. If there is no confusion we shall simply write v instead of $v^{(N,C,b)}$.

It is assumed that members in a coalition $S \subseteq N$ would carry out the most profitable project that is feasible for this coalition, if there exists such a project. Thus $v(S)$ represents the total earnings of coalition S for building this project. This most profitable project may or may not coincide with the largest project demanded by agents in S (for instance, if the benefit of the last agent is smaller than the increase in the cost for fulfilling his project, it is rational for this coalition to realize a smaller project).

The game defined in expression (2) differs from the one considered by Littlechild and Owen (1977). These authors consider the following characteristic function

$$\tilde{v}^{(N,C,b)}(S) := \max_{k \in N} \left\{ \sum\nolimits_{\substack{i \in S: \\ i \preceq k}} b_i - C(k) \right\}$$

If there is no confusion we shall write simply $\tilde{v}$.

Thus $\tilde{v}(S)$ can be negative for some coalitions (for instance, if $b_i < C(i)$, then $\tilde{v}\big(\{i\}\big) < 0$), but if we assume that agents in N are rational, it is reasonable to think that they will not make any project at all. Consequently we have adopted in this work the definition given by expression (2). Of course, both games v and $\tilde{v}$ coincide if and only if for every agent the benefit of carrying out his project by himself is higher than the generated individual cost.

Notice that the game v is actually the 0-monotonic cover of the game $\tilde{v}$.

Remark 3. Littlechild and Thompson (1977) (see also Littlechild and Owen, 1973; and Thompson, 1971) introduced the so called "airport cost problems", to find the landing fees generated by Birmingham airport runway costs. An *airport cost problem* is a tuple $(N, \preceq, C)$, where N is a finite and totally ordered set of players, and $C : N \to \mathbf{R}_+$ is a cost function satisfying $C(i) \leq C(j)$ whenever $i \preceq j$.

Given an airport cost problem $(N, \preceq, C)$ considere the game $\left(N, c^{(N,C)}\right)$, where $c^{(N,C)}(S) = C\big(\ell_S\big)$ for each $S \subseteq N$. The game $(N, c^{(N,C)})$ is called an *airport cost game.*

Now define $b_i = C(i)$, and consider the airport profit problem $(N, \preceq, C, b)$ and its associated game $\left(N, v^{(N,C,b)}\right)$. Then it holds

$$v^{(N,C,b)}(S) = \sum_{i \in S} C(i) - C\left(\ell_S\right) = \sum_{i \in S} c^{(N,C)}\big(\{i\}\big) - c^{(N,C)}(S).$$

That is, $\left(N, v^{(N,C,b)}\right)$ is the savings TU game associated with the cost game $\left(N, c^{(N,C)}\right)$.

So we could associate with an airport cost problem a special class of airport profit problems. Namely, the class in which for every player his benefit is not lower than his cost, so he can carry out his project by himself.

Remark 4. The well known class of bankruptcy games is a subclass of airport profit games.

A *bankruptcy problem on N* (Aumann and Maschler, 1985) is a pair (E, d), where $d = (d_i)_{i \in N}$, and $E, d_i \in \mathbf{R}_+$ for every $i \in N$. The real number E stands for the estate left to a group of creditors, represented by the set N, and d_i is the claim of creditor i. It is assumed the estate does not cover all the claims, that is $E \leq d(N)$.

The *bankruptcy game* $\left(N, v_{(E,d)}\right)$ associated with the bankruptcy problem (E, d) is defined by

$$v_{(E,d)}(S) = \big(E - d(N \backslash S)\big)_+.$$

(If $a \in \mathbf{R}$, we write $a_+ = \max\{a, 0\}$.)

Let (E, d) be a bankruptcy problem on N, and define $C(i) = d(N) - E$, and $b_i = d_i$ for every $i \in N$. Consider the airport profit problem (N, C, b). It is easy to check that

$$v^{(N,C,b)}(S) = v_{(E,d)}(S), \quad \text{for every } S \subseteq N.$$

Thus every bankruptcy game is actually a surplus airport game.

Proposition 5. *Let (N, C, b) be an airport profit problem, then the associated game (N, v) is convex.*

Proof. Let $S, T \subseteq N$, and let $S_0 \subseteq S$, and $T_0 \subseteq T$ such that $v\left(S\right) = b\left(S_0\right) -$

$C(S_0)$, and $v(T) = b(T_0) - C(T_0)$ respectively. Then

$$\begin{aligned} v(S) + v(T) &= b(S_0) - C(S_0) + b(T_0) - C(T_0) \\ &= b(S_0 \cup T_0) + b(S_0 \cap T_0) - C(S_0) - C(T_0) \\ &\leq b(S_0 \cup T_0) + b(S_0 \cap T_0) - C(S_0 \cup T_0) - C(S_0 \cap T_0) \\ &\leq v(S \cup T) + v(S \cap T). \end{aligned}$$

Consequently the proof is complete. ∎

Remark 6. For an alternative proof of the proposition above see Theorem 3.3 in Brânzei et al. (2002).

3 The maximal excess of the nucleolus of an airport profit game

Littlechild (1974) proposed a simple algorithm to calculate the nucleolus on the class of airport cost games. Later Littlechild and Owen (1977) showed that the same algorithm can be applied to the case of airport profit games when $v = \tilde{v}$. The aim of this section is to obtain some coalitions with maximal excess at the nucleolus of an airport profit game, and then determinate the nucleolus for agents in these coalitions. Later in Section 5 this will be used for design an algorithm to calculate the nucleolus for all the agents.

This algorithm is obtained with a different approach to the one used by Littlechild (1974), who considered a procedure of Kopelowitz (1967) to calculate the nucleolus based on a sequence of linear programs. Instead of it, we use a result of Arin and Iñarra (1998) about the structure of the family of proper coalitions with maximal excess at the nucleolus, based in Kohlberg's (1971) characterization of the nucleolus by means of balanced collections.

Let (N, v) be a TU game, and $x \in \mathbf{R}^N$. Denote

$$\mathcal{D}_1(x) := \{S \subseteq N : e(S, x) \geq e(T, x), \text{ for all } T \subseteq N, \ S \neq N, \emptyset\}.$$

That is, $\mathcal{D}_1(x)$ stands for the set of proper coalitions of N with maximal excess at x.

Let $\{S_1, \ldots, S_k\}$ be a partition of N. The family $\{N \backslash S_1, \ldots, N \backslash S_k\}$ formed by the complements will be called an *antipartition*.

Remark 7. Let (N, v) be a TU game, and $x \in \mathbf{R}^N$, and let $\mathcal{B}$ be a family of coalitions of N. Define $e(\mathcal{B}, x) := \frac{\sum_{S \in \mathcal{B}} e(S,x)}{|\mathcal{B}|}$ (i.e., the average excess at x of coalitions in $\mathcal{B}$).

(a) For every $\mathcal{B} \subseteq \mathcal{D}_1(x)$, and every $S \in \mathcal{D}_1(x)$, it holds:

$$e(\mathcal{B}, x) = e(S, x) = e(\mathcal{D}_1(x), x).$$

(b) If $\mathcal{B}$ is a partition or an antipartition, and $x(N) = v(N)$ then $e(\mathcal{B}, x)$ does not depend explicitly on x. Indeed, it is easy to check that

– if $\mathcal{P}$ is a partition: $e(\mathcal{P}, x) = \dfrac{\sum_{S\in\mathcal{P}} v(S) - v(N)}{|\mathcal{P}|}$

– if $\mathcal{A}$ is an antipartition: $e(\mathcal{A}, x) = \dfrac{\sum_{S\in\mathcal{A}} v(S) - (|\mathcal{A}| - 1)v(N)}{|\mathcal{A}|}$.

Theorem 8. *[Arin and Iñarra (1998)] If (N, v) is convex then $\mathcal{D}_1\big(\nu(v)\big)$ contains a partition or an antipartition of N.*

The next proposition will be proved in Appendix A, and permit us to identify some partitions or antipartitions contained in $\mathcal{D}_1\big(\nu(v)\big)$ for an airport profit game.

Proposition 9. *Let (N, C, b) be an airport profit problem and (N, v) its associated game. Then at least one of the following statements is true.*

(I) $\mathcal{P} = \big\{\{i\} : i \in N\big\} \subseteq \mathcal{D}_1(\nu)$, *and* $v\big(\{i\}\big) = 0$ *for every* $i \in N$.

(II) there exists $i_0 \in N\backslash\{\ell_N\}$, *s. t.* $\mathcal{P} = \big\{\{i_0\}, N\backslash\{i_0\}\big\} \subseteq \mathcal{D}_1(\nu)$, *and* $v\big(\{i_0\}\big) = 0$.

(III) there exists $i_0 \in N\backslash\{\ell_N\}$, *such that*

- *(i)* $\mathcal{P} = \big\{\{k : k \preceq i_0\}\big\} \cup \big\{\{k\} : i_0 \prec k\big\} \subseteq \mathcal{D}_1(\nu)$, *and*
- *(ii)* $v\big(\{k : k \preceq i_0\}\big) \neq 0$, *and* $v\big(\{k\}\big) = 0$ *for every* $k \succ i_0$.

(IV) there exists $i_0 \in N\backslash\{\ell_N\}$, *such that*

$$\mathcal{P} = \big\{\{k \in N : k \preceq i_0\}\big\} \cup \big\{N\backslash\{k\} : k \preceq i_0\big\} \subseteq \mathcal{D}_1(\nu), \text{ and } |\mathcal{P}| \geq 3.$$

Let (N, v) be an airport profit game. A coalition $S \subseteq N$ is said to be *effective* if $v(S) = b(S) - C(S)$, and has not null players.

The next lemmas will be proved in Appendix B. They permit us to calculate very easily the nucleolus for some agents in an airport profit game.

Lemma 10. *Let (N, C, b) be an airport profit problem and (N, v) its associated game. Define $\alpha^v := \frac{b(N)-C(N)}{|N|}$. Let $\mathcal{P} = \big\{\{i\} : i \in N\big\}$; if N is effective then:*

(a) $e(\mathcal{P}, \nu) = -\alpha^v$ *if and only if* $v\big(\{i\}\big) = 0$ *for every* $i \in N$.

(b) $e(\mathcal{P}, \nu) \geq -\alpha^v$.

(c) If $\mathcal{P} \subseteq \mathcal{D}_1(\nu)$, *and* $e(\mathcal{P}, \nu) = -\alpha^v$, *then* $\nu_i = \alpha^v$ *for every* $i \in N$.

Lemma 11. *Let (N,C,b) be an airport profit problem and (N,v) its associated game. Let also $\{i_0\} \in N\backslash\{\ell_N\}$, and define $\beta^v_{i_0} := \frac{b_{i_0}}{2}$. Let $\mathcal{P} = \big\{\{i_0\}, N\backslash\{i_0\}\big\}$; if N is effective then:*

(a) $e(\mathcal{P},\nu) = -\beta^v_{i_0}$ if and only if $v\big(\{i_0\}\big) = 0$.

(b) $e(\mathcal{P},\nu) \geq -\beta^v_{i_0}$.

(c) If $\mathcal{P} \subseteq \mathcal{D}_1(\nu)$, and $e(\mathcal{P},\nu) = -\beta^v_{i_0}$, then $\nu_{i_0} = \beta^v_{i_0}$.

Lemma 12. *Let (N,C,b) be an airport profit problem and (N,v) its associated game. Let also $\{i_0\} \in N\backslash\{\ell_N\}$, and define $\gamma^v_{i_0} := \frac{C(i_0)+b\left(\left\{k\in N: i_0\prec k\right\}\right)-C(N)}{\left|\left\{k\in N: i_0\prec k\right\}\right|+1}$. Let $\mathcal{P} = \big\{\{i\in N : i\preceq i_0\}\big\}\cup\big\{\{k\} : k\in N,\ i_0\prec k\big\}$; if N is effective then:*

a) $e(\mathcal{P},\nu) = -\gamma^v_{i_0}$ if and only if $v\big(\{k\}\big) = 0$ for every $k \succ i_0$.

b) $e(\mathcal{P},\nu) \geq -\gamma^v_{i_0}$.

(c) If $\mathcal{P} \subseteq \mathcal{D}_1(\nu)$, and $e(\mathcal{P},\nu) = -\gamma^v_{i_0}$, then $\nu_k = \gamma^v_{i_0}$ for every $k \succ i_0$.

Lemma 13. *Let (N,C,b) be an airport profit problem and (N,v) its associated game. Let also $\{i_0\} \in N\backslash\{\ell_N\}$, and define $\delta^v_{i_0} := \frac{C(i_0)}{\left|\left\{k\in N: k\preceq i_0\right\}\right|+1}$. Let $\mathcal{P} = \big\{\{i\in N : i\preceq i_0\}\big\}\cup\big\{N\backslash\{i\} : i\preceq i_0\big\}$; if N is effective then:*

(a) $e(\mathcal{P},\nu) = -\delta^v_{i_0}$.

(b) If $\mathcal{P} \subseteq \mathcal{D}_1(\nu)$, and $e(\mathcal{P},\nu) = -\delta^v_{i_0}$, then $\nu_i = b_i - \delta^v_{i_0}$ for every $i \preceq i_0$.

Lemma 14. *Let (N,C,b) an airport profit problem and (N,v) the associated game. Define*

$$\lambda^v := \min\{\alpha^v\}\cup\{\beta^v_i,\gamma^v_i,\delta^v_i : i\neq\ell_N\}.$$

(Since there will be no confusion we will simply write $\lambda,\alpha,\beta_{i_0},\gamma_{i_0}$, and δ_{i_0}.)

(a) If $\lambda = \gamma_{i_0} \leq 0$ for some $i_0 \in N$, and $i_0 \prec i$, then i is a null player.

(b) If $\lambda = \alpha \leq 0$ then for every $S \subseteq N$ it holds $v(S) = 0$.

(c) If $\lambda > 0$, then N is effective.

Proof. (a) Let $i \in N$ such that $i_0 \prec i$. First notice that since $\lambda = \gamma_{i_0} \leq \gamma_i$, and $\gamma_{i_0} < 0$ it holds

$$C(i_0) + b\big(\{k\in N : i_0\prec k\}\big) \leq C(i) + b\big(\{k\in N : i\preceq k\}\big),$$

and hence

$$b\big(\{k \in N : i_0 \prec k \preceq i\}\big) \leq C(i) - C(i_0)\,. \tag{3}$$

Now let us see that i is a null player. Take any coalition $S \subseteq N\backslash\{i\}$, and let $Q \subseteq S \cup \{i\}$ such that $v\big(S \cup \{i\}\big) = b(Q) - C(Q)$. If $Q = \emptyset$, then $v\big(S \cup \{i\}\big) = v(S)$. So assume $Q \neq \emptyset$, then

$$\begin{aligned}
&v\big(S \cup \{i\}\big) = b(Q) - C(Q)\\
&\quad = b\big(\{k \in Q : k \preceq i_0\}\big) - C(i_0) + b\big(\{k \in Q : i_0 \prec k\}\big) - \big(C(\ell_Q) - C(i_0)\big)\\
&\qquad\qquad \leq v(S) + b\big(\{k \in Q : i_0 \prec k\}\big) - \big(C(\ell_Q) - C(i_0)\big)\\
&\qquad\qquad \leq v(S) + b\big(\{k \in N : i_0 \prec k \preceq \ell_Q\}\big) - \big(C(\ell_Q) - C(i_0)\big) \leq v(S),
\end{aligned}$$

where the first inequality follows since $\{k \in Q : k \preceq i_0\} \subseteq S$ because $i_0 \prec i$, and the last one from expression (3) since $i_0 \preceq \ell_Q$. By the monotonicity of v it holds $v\big(S \cup \{i\}\big) = v(S)$, and we can conclude that i is a null player.

(b) First notice that since $\alpha \leq \gamma_i$ for every $i \in N$, and $\alpha < 0$ it holds for every $i \in N\backslash\{\ell_N\}$

$$b(N) \leq C(i) + b\big(\{k \in N : i \preceq k\}\big). \tag{4}$$

Now take $S \subseteq N$, and let $Q \subseteq S$ such that $v(S) = b(Q) - C(Q)$. If $Q \neq \emptyset$ then

$$\begin{aligned}
v(S) = b(Q) - C(Q) &\leq b\big(\{k \in Q : k \preceq \ell_Q\}\big) - C(\ell_Q)\\
&= b(N) - b\big(\{k \in Q : \ell_Q \prec k\}\big) - C(\ell_Q) \leq 0,
\end{aligned}$$

where the last inequality follows from expression (4). Since $v(S) \geq 0$ we conclude the result.

(c) Assume on the contrary that N is not effective and $\lambda > 0$. If $v(S) = 0$ for every $S \subseteq N$, then $0 = \alpha \geq \lambda$, so let S be the maximal proper coalition that is effective. Then $v(N) = v(S)$, and since $b_i > 0$ for each $i \preceq \ell_S$ it holds $i \in S$. Then

$$0 \geq \big(b(N) - C(N)\big) - \big(b(S) - C(S)\big) = b\big(\{k \in N : k \succ \ell_S\}\big) - \big(C(N) - C(\ell_S)\big).$$

Consequently $0 \geq \gamma_{\ell_S} \geq \lambda$, that is a contradiction. ∎

Let $\bar{S}$ be the set of the players $k \in N$ that satisfy at least one of the following conditions:

a) $\lambda = \alpha$ (notice that in this case $S = N$),

b) $\lambda = \beta_k$,

c) there exists $i \in N$ such that $i \prec k$ and $\lambda = \gamma_i$,

d) there exists $i \in N$ such that $k \preceq i$ and $\lambda = \delta_i$.

Proposition 15. *Let (N, C, b) be an airport profit problem and (N, v) the associated game.*

(a) If $S \in \mathcal{D}_1(\nu)$ then $e(S, \nu) = (\lambda)_+$.

(b) For every $k \in \bar{S}$ it holds

$$\nu_k = \begin{cases} (\lambda)_+ & \text{if } \lambda = \alpha, \\ \lambda & \text{if } \lambda = \beta_k, \\ (\lambda)_+ & \text{if } \lambda = \gamma_i \quad \textit{for some } i \prec k, \\ b_k - \lambda & \text{if } \lambda = \delta_i \quad \textit{for some } i \succeq k. \end{cases}$$

Proof. (a) If $\lambda \leq 0$, then either $\lambda = \alpha$, or $\lambda = \gamma_i$ for some $i \prec k$. From Lemma 14 parts (a) and (b) there are null players in v. Thus $e(S, \nu) = 0 = (\lambda)_+$ for every $S \in \mathcal{D}_1(\nu)$.

If $\lambda > 0$, then N is effective by Lemma 14 part (c). Applying Proposition 9, and parts (a) of lemmas 10, 11, 12 and 13 we conclude the result.

(b) Let $k \in \bar{S}$. If $\lambda \leq 0$, then either $\lambda = \alpha$, or $\lambda = \gamma_i$ for some $i \prec k$. From Lemma 14, parts (a) and (b) it follows that k is a null player, hence $\nu_k = 0 = (\lambda)_+$.

So assume that $\lambda > 0$. Then either $\lambda = \alpha$, or $\lambda = \beta_{i_0}$ or γ_{i_0} or δ_{i_0} for some $i_0 \prec k$.

Consider for instance the case $\lambda = \alpha$. By Lemma 14 part (c), N is effective, and by part (a) of the present lemma and part (b) of Lemma 10 it holds $\mathcal{P} = \{\{i\} : i \in N\} \subseteq \mathcal{D}_1(\nu)$. From part (c) of Lemma 10, we conclude $\nu_i = \alpha = (\lambda)_+$ for every $i \in N$.

The remaining cases are similar applying respectively lemmas 11, 12, and 13 instead of Lemma 10, so we conclude the result. ∎

Remark 16. It can be easily checked that if (N, v) is an airport cost game then there exists $i \in N$ such that $\lambda = \delta_i$; that is, according to Proposition 9 we are always in case (IV. On other hand, in case (N, v) is an airport game corresponding to a bankruptcy problem, then $\lambda = \alpha$ or there exists $i \in N$ such that $\lambda = \beta_i$; that is, at least one of the statements (I) or (II) in Proposition 9 is true.

4 Allocation rules and consistency

In an airport problem once the agents have decided to build an *optimal* project, i.e., a project that maximizes the net present value of total benefits, the question is how to share the common costs. Following Littlechild and Owen (1977) we will consider allocation rules for which the associated fee schedules, satisfy

(a) no group of agents is charged more than the cost of building a project for that group alone, and

(b) the total cost of the optimal project is exactly covered.

Accordingly, a payoff vector $x \in \mathbf{R}_+^N$ is said to be a *solution* of the airport profit problem (N, C, b) if it satisfies

$$b\Big(\{j \in N : j \preceq i\}\Big) - x\Big(\{j \in N : j \preceq i\}\Big) \leq C(i) \quad \text{for each } i \in N. \tag{5}$$

$$b(M) - C(M) = \max\big\{b(R) - C(R) : R \subseteq N\big\} \quad \text{implies} \\ x(N) = x(M) = b(M) - C(M). \tag{6}$$

Let us examine the particular case of a 2-person problem (N, C, b) with $N = \{1, 2\}$. If agent, let us say i, decides to achieve his project without the help of his partner, he would obtain $b_i - C(i)$, whenever this amount is non-negative; otherwise it is rational for him to desist of making his project. Therefore the benefits generated by the cooperation are given by

$$\alpha = b\big(\{1,2\}\big) - C\big(\{1,2\}\big) - \big(b_1 - C(1)\big)_+ - \big(b_2 - C(2)\big)_+.$$

This is the amount at issue, and if it is shared equally, agent i will receive

$$x_i = \frac{\alpha}{2} + \big(b_i - C(i)\big)_+ \qquad i = 1, 2. \tag{7}$$

The payoff vector defined above will be called the *standard solution* of the 2-person problem (N, C, b).

Now let us consider a general airport profit problem with more participants. To find allocation rules for such a problem we shall use the property of consistency as follows. Assume all the agents have reached an agreement, and subsequently some of the agents are paid accordingly and leave out. Assume also the remaining agents do renegotiate what is left to them. A *consistent* payoff vector will prescribe for this new reduced situation the same payoffs as in the former one. A formal definition of consistency will clearly depend upon the way we describe the new situation.

Let $(N, \preceq, C, b)$ be an airport profit problem, S a proper coalition of N, and x a solution of $(N, \preceq, C, b)$. The *reduced airport profit problem of* (N, C, b) *with respect to* S *and* x, is the problem $\big(S, \preceq_S, C^{S,x}, b_S\big)$, where

$$C^{S,x}(i) = \min\Big\{C\big(Q \cup \{i\}\big) - \big(b(Q) - x(Q)\big) : Q \subseteq N \backslash S\Big\}, \quad \text{for every } i \in S. \tag{8}$$

Remark 17. Notice that $C^{S,x}$ depends only on $x_{N \backslash S}$, so by abusing the notation we will also write $C^{S,x}$ when $x \in \mathbf{R}^{N \backslash S}$.

If there is no confusion, we will simply write C^x, and (S, C^x, b_S).

To illustrate the meaning of the reduced problem let us suppose that agents in N are constructing a landing runway, and they agree upon agent $i \in N$

receiving x_i, so his payment to finance the runway is $b_i - x_i$. Before leaving the others in the negotiation assume agent i has to specify the precise part of runway that he will pay. It is reasonable to think he will choose a fragment between the origin and the last point of the runway he will use. If this agent defrays the piece of runway closest to this last point, the remaining agents will face a 'reduced' airport profit problem. The cost function of the new problem will be precisely C^x, and their benefits will be the same as in the original situation.

We say that a solution x of the problem (N, C, b) is *consistent with the standard solution*, or simply *consistent*, if for every 2-person coalition S, the standard solution of $\left(S, C^{S,x}, b_S\right)$ is x_S.

Potters and Sudhölter (1999) showed that there is only one allocation rule consistent with the standard solution in "airport *cost* problems" (they called it ν-consistent, to distinguish between other alternative definitions for consistency suggested by them), and it turns out to be the nucleolus of the corresponding games. An extension of this result to "airport *profit* games" is the following theorem.

Theorem 18. *For every airport profit problem there exists a unique consistent solution.*

Remark 19. As it was mentioned in Remark 4 every bankruptcy problem can be considered as an airport profit problem. It can be easily proved that the definition of consistency established above is equivalent to the definition of consistency given by Aumann and Maschler (1985) for solutions on bankruptcy games. Thus Theorem 18 can be viewed as an extension of the corresponding theorem given by these authors.

To prove Theorem 18 we will show that the consistent solution of an airport profit problem is the only payoff in the prekernel of the corresponding game, i.e., the nucleolus of this game.

Lemma 20. *Let $T \subseteq S \subseteq N$ be two proper coalitions of N. If x is a solution of (N, C, b) then*

$$\left(T, \left(C^{S,x}\right)^{T,x}, \left(b_S\right)_T\right) = \left(T, C^{T,x}, b_T\right).$$

Proof. It is straightforward. ∎

Lemma 21. *Let x be a solution of (N, C, b), and let S be a proper coalition of N. Then*

$$\left(v^{(N,C,b)}\right)^{S,x} = v^{(S,C^x,b_S)}.$$

(In words, the reduced surplus airport game is the game corresponding to the reduced airport profit problem.)

Proof. By Remark 2 part (a), and Lemma 20, it is enough to consider the case in which $S = N \setminus \{i\}$ for some $i \in N$. To simplify the notation, write $v = v^{(N,C,b)}$, and $w = v^{\left(N\setminus\{i\},C^x,b_{N\setminus\{i\}}\right)}$.

First notice that $C^x(R) = \min\Big\{C(R), C\big(R \cup \{i\}\big) - \big(b_i - x_i\big)\Big\}$ for every $R \subseteq N \setminus \{i\}$.

Thus we have to show that $w(T) = v^x(T)$, for every $T \subseteq N\setminus\{i\}$. First we prove this identity for $T \neq N\setminus\{i\}$. Indeed,

$$\begin{aligned}
v^x(T) &= \max\Bigg\{v(T), v\big(T \cup \{i\}\big) - x_i\Bigg\} \\
&= \max_{R\subseteq T}\Bigg\{\max\Big\{b(R) - C(R), b\big(R\cup\{i\}\big) - C\big(R\cup\{i\}\big) - x_i\Big\}\Bigg\} \\
&= \max_{R\subseteq T}\Bigg\{b(R) - \min\Big\{C(R), C\big(R\cup\{i\}\big) - \big(b_i - x_i\big)\Big\}\Bigg\} \\
&= \max\Big\{b(R) - C^x(R) : R \subseteq T\Big\} = w(T).
\end{aligned}$$

Now consider the case $T = N\setminus\{i\}$. Let $M \subseteq N$ be the maximal coalition such that $v(N) = b(M) - C(M)$. We consider two cases

1) $i \notin M$. Then $C(M) < C(i) - b_i = C\big(M \cup \{i\}\big) - b_i$, and by condition (6) it is also $x_i = 0$. Hence, $C^x(M) = C(M)$, and it holds

$$w\big(N\setminus\{i\}\big) \geq b(M) - C^x(M) = b(M) - C(M) = v(N) = v(N) - x_i.$$

On other hand, for every $R \subseteq N\setminus\{i\}$ it holds $C^x\,(R) \leq C\,(R)$, and since $x_i = 0$ it follows

$$\begin{aligned}
w\big(N\setminus\{i\}\big) &= \max\Big\{b(R) - C^x\,(R) : R \subseteq N\setminus\{i\}\Big\} \\
&\leq \max\Big\{b(R) - C\,(R) : R \subseteq N\setminus\{i\}\Big\} \leq v(N) = v(N) - x_i.
\end{aligned}$$

2) $i \in M$. Then $C^x\big(M\setminus\{i\}\big) = C(M) - b_i + x_i$ implying that

$$\begin{aligned}
w\big(N\setminus\{i\}\big) &\geq b\big(M\setminus\{i\}\big) - C^x\big(M\setminus\{i\}\big) = b\big(M\setminus\{i\}\big) - C(M) + b_i - x_i \\
&= b\,(M) - C\,(M) - x_i = v(N) - x_i.
\end{aligned}$$

Again it will be enough to show that for every $R \subseteq N\setminus\{i\}$ it holds $b(R) - C^x\,(R) \leq v(N) - x_i$. We have two cases

2a) $C^x\,(R) = C\big(R \cup \{i\}\big) - (b_i - x_i)$. In this case

$$\begin{aligned}
b(R) - C^x\,(R) &= b(R) - C\big(R\cup\{i\}\big) + (b_i - x_i) \\
&= b\big(R\cup\{i\}\big) - C\big(R\cup\{i\}\big) - x_i \leq v(N) - x_i.
\end{aligned}$$

2b) $C^x(R) = C(R)$. Let $M' \subseteq N\backslash\{i\}$ be the maximal coalition such that $v\big(N\backslash\{i\}\big) = b(M') - C(M')$. From expressions (5) and (6):

$$C(M) - C(M') \leq \big(b(M) - x(M)\big) - \big(b(M') - x(M')\big) \leq b(M\backslash M') - x_i,$$

Hence

$$\begin{aligned} b(R) - C^x(R) &= b(R) - C(R) \leq b(M') - C(M') \\ &= b(M) - x_i - C(M) - b(M\backslash M') + x_i + C(M) - C(M') \\ &\leq b(M) - x_i - C(M) = v(N) - x_i. \end{aligned}$$

Thus the proof is complete. ∎

Proposition 22. *The payoff vector x is in the prekernel of the TU game (N,v) if and only if for every $S \subset N$ such that $|S| = 2$, the payoff vector x_S is the standard solution of $\big(S, v^{S,x}\big)$.*

Proof. See Aumann and Maschler (1985) or Peleg (1986). ∎

Proposition 23. *The standard solution of a 2-person airport profit problem (N,C,b) is the standard solution of the associated game.*

Proof. It follows from (7), (2) and (1). ∎

Proposition 24. *A solution of (N,C,b) is consistent if and only if it is the nucleolus of the corresponding game.*

Proof. Let v be the TU game associated to (N,C,b), and ν the nucleolus of this game.

First let us see that ν is a solution. Indeed, since ν is in the core of v it holds

$$\begin{aligned} \nu\Big(\big\{j \in N : C(j) \leq C(i)\big\}\Big) &\geq v\Big(\big\{j \in N : C(j) \leq C(i)\big\}\Big) \\ &\geq b\Big(\big\{j \in N : C(j) \leq C(i)\big\}\Big) - C(i). \end{aligned}$$

Hence, ν satisfies (5).

Furthermore, if $b(M) - C(M) = \max\big\{b(R) - C(R) : R \subseteq N\big\}$, then

$$\nu(N) = v(N) = \max\big\{b(R) - C(R) : R \subseteq N\big\} = v(M) = \nu(M).$$

So ν also satisfies (6).

Now let us show that ν is consistent. Let $S \subseteq N$ such that $|S| = 2$. Since $\nu \in PK(v)$, by Proposition 22, ν_S is the standard solution of $v^{S,\nu}$, and hence by Lemma 21, it is also the standard solution of the TU game corresponding to the problem (S, C^ν, b_S). By Proposition 23, ν is the standard solution of (S, C^ν, b_S). That is ν is consistent.

Now assume that x is a consistent solution of (N, C, b). Then for every two-person coalition $S \subseteq N$, the payoff vector x_S is the standard solution of (S, C^{ν}, b_S). By Proposition 23 and Lemma 21, x_S is the standard solution of the 2-person TU game corresponding to this problem. And by Proposition 22, $x \in PK(v)$. But, by Proposition 5, v is convex and hence $PK(v)$ consists of only an unique payoff vector, namely its nucleolus. So $x = \nu$, and the proof is complete. ∎

Proof of Theorem 18. It is direct consequence of Proposition 24.

5 The algorithm

Taking into account propositions 15 and 24, and Lemma 21, we can construct a simple algorithm to determinate the nucleolus of an airport profit game.

Let (N, C, b) be an airport profit problem, with $|N| \geq 3$.

The algorithm can be described as follows: We construct a finite sequence of airport profit problems (N_m, C_m, b_{N_m}), with $m = 1, \ldots, M$, where $(N_1, C_1, b_{N_1})) = (N, C, b)$. In each step we calculate the consistent solution x for a subgroup of agents $S^m \subseteq N_m$. If $S^m = N_m$ the algorithm stops. Otherwise we reduce the problem with respect to the complement of S^m at x, and consider a new problem $\left(N_{m+1}, C_{m+1}, b_{N_{m+1}}\right)$. Thus the i-th problem is a reduced problem of the $(i-1)$-th one for $i = 2, \ldots, M$.

In each step m calculate the following numbers

$$\alpha^m = \frac{b\left(N_m\right) - C_m\left(N_m\right)}{\left|N_m\right|},$$

and for every $i \in N_m \backslash \left\{\ell_{N_m}\right\}$,

$$\begin{aligned}
\beta_i^m &= \frac{b_i}{2}, \\
\gamma_i^m &= \frac{C_m(i) + b\Big(\left\{k \in N_m : C_m(k) > C_m(i)\right\}\Big) - C_m\left(N_m\right)}{\Big|\left\{k \in N_m : C_m(k) > C_m(i)\right\}\Big| + 1}, \qquad \text{and} \\
\delta_i^m &= \frac{C_m(i)}{\Big|\left\{k \in N_m : C_m(k) \leq C_m(i)\right\}\Big| + 1}.
\end{aligned}$$

Next consider the minimum of all these numbers, that is

$$\lambda^m = \min\left\{\alpha^m\right\} \cup \left\{\beta_i^m, \gamma_i^m, \delta_i^m, : i \in N_m \backslash \left\{\ell_{N_m}\right\}\right\}.$$

Now define the set S^m of the players $k \in N_m$ that satisfy at least one of the following conditions:

a) $\lambda^m = \alpha^m$ (notice that in this case $S^m = N_m$).

b) $\lambda^m = \beta_k^m$, or

c) There exists $i \in N_m \setminus \{\ell_{N_m}\}$ such that $C_m(k) > C_m(i)$ and $\lambda^m = \gamma_i^m$, or

d) There exists $i \in N_m \setminus \{\ell_{N_m}\}$ such that $C_m(k) \leq C_m(i)$ and $\lambda^m = \delta_i^m$.

and for every $k \in S^m$ define

$$x_k^m = \begin{cases} (\lambda)_+ & \text{if } \lambda = \alpha, \\ \lambda & \text{if } \lambda = \beta_k, \\ (\lambda)_+ & \text{if } \lambda = \gamma_i \quad \text{for some } i \prec k, \\ b_k - \lambda & \text{if } \lambda = \delta_i \quad \text{for some } i \succeq k. \end{cases}$$

Finally, let

$$N_{m+1} = N_m \setminus S^m, \quad C_{m+1} = (C_m)^x, \quad \text{and} \quad M = \max\{m : S^m = N_m\}.$$

Theorem 25. *If (N, C, b) is an airport profit problem, its consistent solution is the payoff vector $x \in \mathbf{R}^N$ such that*

$$x_i = x_i^m, \qquad \textit{for every } i \in S^m.$$

Proof. Follows from propositions 15 and 24, and Lemma 21. ∎

Example 26. To illustrate the algorithm consider the following example with $N = \{1, 2, 3, 4, 5, 6\}$ and the following data

	Agents					
	1	2	3	4	5	6
$C(i)$	6	15	20	30	36	38
b_i	12	12	4	23	6	5

For this problem we have: $N_1 = N$, $C_1 = C$. In this case notice that

$$\alpha^1 = \frac{b(N) - C(N)}{n},$$
$$\beta_i^1 = \frac{b_i}{2},$$
$$\gamma_i^1 = \frac{C(i) + b\big(\{i+1, \ldots, n\}\big) - C(N)}{n - i + 1}, \text{ and}$$
$$\delta_i^1 = \frac{C(i)}{i+1}.$$

Consequently we have the following table summarizing the data:

Stage 1	Agents					
	1	2	3	4	5	
α^1	–	–	–	–	–	4
β_i^1	6	6	2	11.5	3	–
γ_i^1	3	3	4	1	1.5	–
δ_i^1	3	5	5	6	6	–

Thus the minimum in this table is attained in $\lambda^1 = \gamma_4^1 = 1$. Hence $S^1 = \{5, 6\}$, and agents 5 and 6 get 1 (and accordingly they pay 5 and 4 respectively). The rest of the agents face the reduced problem

	Agents			
N_2	1	2	3	4
$C_2(i)$	6	15	20	29
b_i	12	12	4	23

For this reduced problem similar calculations give the following table

Stage 2	Agents			
	1	2	3	
α^2	–	–	–	11/2
β_i^2	6	6	2	–
γ_i^2	4	13/3	7	–
δ_i^2	3	5	5	–

Now the minimum is attained in $\beta_3^2 = 2$. This means that $S^2 = \{3\}$, and agent 3 gets 2 (consequently he pays 2). The rest of the agents face the reduced problem

	Agents		
N_3	1	2	4
$C_3(i)$	6	15	27
b_i	12	12	23

For the new reduced problem the calculations give the following table

Stage 3	Agents		
	1	2	
α^3	–	–	20/3
β_i^3	6	6	–
γ_i^3	14/3	11/2	–
δ_i^3	3	5	–

The minimum is attained in $\delta_1^3 = 3$. So $S^3 = \{1\}$, and agent 1 pays 3 (so he gets 9). The remaining agents 2 and 3 face the reduced two person problem:

	Agents	
N_4	2	4
$C_4(i)$	12	24
b_i	12	23

This is a 2-person problem, whose standard solution is $x_2 = x_4 = 5.5$.

Finally the consistent solution of this problem is: $(9, 5.5, 2, 5.5, 1, 1)$.

Appendix A

Before proving Proposition 9, we need several results.

The next lemma refers to general convex games.

Lemma 27. *Let (N, v) be a convex game.*

(a) If $S, T \in \mathcal{D}_1(\nu)$, and $S \cup T \neq N$, then $S \cup T \in \mathcal{D}_1(\nu)$.

(b) If $S_1, \ldots, S_k \in \mathcal{D}_1(\nu)$, and $\bigcup_{i=1}^k S_k \neq N$, then $\bigcup_{i=1}^k S_k \in \mathcal{D}_1(\nu)$.

(c) $i \in N$ is a null player in (N, v) if and only if $\nu_i = 0$.

Proof. (a) See Maschler *et al.* (1972).

(b) By induction.

(c) It is straightforward to show that if i is a null player then $\nu_i = 0$.

For the converse assume that $\nu_i = 0$. Since i is a null player of (N, v) it follows $v(\{i\}) = 0$, and hence $e(\{i\}, \nu) = 0$. Since the core of (N, v) is not empty, $e(S, \nu) \leq 0$ for every $S \subseteq N$, and consequently $\{i\} \in \mathcal{D}_1(\nu)$. Hence $s_{ij}(\nu) = 0$ for all $j \in N \backslash \{i\}$.

Since $\nu \in PK(v)$, for every $j \in N \backslash \{i\}$ it holds $s_{ji}(\nu) = 0$, and hence there exists a coalition $S(j) \subseteq N \backslash \{i\}$ such that $j \in S(j)$, and $e(S(j), \nu) = 0$.

From part (b) of this lemma it follows $\bigcup_{j \neq i} S(j) = N \backslash \{i\} \in \mathcal{D}_1(\nu)$. But then we have:

$$0 = e(N \backslash \{i\}, \nu) = v(N \backslash \{i\}) - \nu(N \backslash \{i\}) = v(N \backslash \{i\}) - \nu(N),$$

where the last equality follows since $\nu_i = 0$. Since ν is efficient we can conclude $v(N) - v\big(N\backslash\{i\}\big) = 0$. By the convexity of (N, v) it follows $v(S) - v\big(S\backslash\{i\}\big) = 0$ for every $S \subseteq N$ such that $i \in S$; and this means that i is a null player. ∎

Throughout the rest of this appendix (N, C, b) will be an airport profit problem. By (N, v) we denote the corresponding TU game, and ν its nucleolus.

Lemma 28. *If i is a null player, and $j \succeq i$, then j is also a null player in (N, v).*

Proof. By contradiction, assume that j is not a null player, then there exists a coalition $S \subseteq N\backslash\{j\}$ such that $v\big(S \cup \{j\}\big) > v(S) \geq 0$. Let $Q \subseteq S \cup \{j\}$ be the maximal coalition such that $v\big(S \cup \{j\}\big) = b(Q) - C\,(Q)$. Consider also the coalition $P = \{k \in N : k \preceq \ell_Q\}$. Since $j \in Q$ (otherwise $Q \subseteq S$ and $v\big(S \cup \{j\}\big) = v(S)$), it follows $j \in P$, and hence $i \in P$. Moreover, since $C(P) = C(Q)$ it holds

$$
\begin{aligned}
v(P) = \max\big\{b(R) - C(R) : R \subseteq P\big\} \\
\leq \max\big\{b(R \cap Q) + b(R\backslash Q) - C(Q) : R \subseteq P\big\} \\
\leq b(Q) - C(Q) + \max\big\{b(R\backslash Q) : R \subseteq P\big\} \leq b(P) - C(Q) = b(P) - C(Q).
\end{aligned}
$$

Thus $v(P) = b(P) - C(P)$.

In addition $v(P) - v\big(P\backslash\{i\}\big) \geq b_i > 0$. Thus i would not be a null player. ∎

Lemma 29. *Let $S \in \mathcal{D}_1(\nu)$. If $v(S) = 0$, then $\{i\} \in \mathcal{D}_1(\nu)$, for every $i \in S$.*

Proof. Let $S \in \mathcal{D}_1(\nu)$ such that $v(S) = 0$, and $i \in S$. Since v is monotonic it follows $v\big(\{i\}\big) = 0$, and hence

$$e\big(\{i\}, \nu\big) = -\nu_i \geq -\nu(S) = v(S) - \nu(S) = e(S, \nu).$$

This is only possible if $\{i\} \in \mathcal{D}_1(\nu)$. ∎

Lemma 30. *Let (N, C, b) be an airport profit problem and (N, v) its associated game. If there are not null players in (N, v), and $S \in \mathcal{D}_1(\nu)$ is not effective, then $|S| = 1$.*

Proof. By contradiction. Assume that S is not effective and $|S| > 1$. Take $R \subseteq S$, $R \neq \emptyset, S$, such that $v(S) = v(R)$. By Lemma 27 part (c), it follows $\nu(S\backslash R) > 0$, and hence

$$e(S, \nu) = v(S) - \nu(S) = v(R) - \nu(R) - \nu(S\backslash R) = e(R, \nu) - \nu(S\backslash R) < e(R, \nu).$$

But this contradicts $S \in \mathcal{D}_1(\nu)$. ∎

Lemma 31. *Let $R \in \mathcal{D}_1(\nu)$ be effective. If $C(S) \leq C(R)$, and $S \neq N\backslash R$, then $R \cup S \in \mathcal{D}_1(\nu)$.*

Proof. If R is effective and $C(S) \leq C(R)$, then $R \cup S$ is also effective and moreover $C(R \cup S) = C(R)$. Then we have

$$\begin{aligned} e(R \cup S, \nu) - e(R, \nu) &= v(R \cup S) - v(R) - \nu(S) \\ &= b(R \cup S) - C(R \cup S) - b(R) + C(R) - \nu(S) \\ &= b(S) - \nu(S) \geq 0. \end{aligned}$$

Since $R \in \mathcal{D}_1(\nu)$, this is only possible if $R \cup S \in \mathcal{D}_1(\nu)$. ∎

Lemma 32. *Let $R, S \in \mathcal{D}_1(\nu)$ be effective, and $T \subseteq S$, $T \neq \emptyset, S$ such that $T \cap R = \emptyset$. If $C(S \backslash T) \leq C(R)$, then $T \in \mathcal{D}_1(\nu)$.*

Proof. Since $C(S \backslash T) \leq C(R)$, and R is effective, then $R \cup (S \backslash T)$ is also effective by Lemma 31. Let us compare the excesses of coalitions $R \cup (S \backslash T)$, R, T, and S. Since $C\big(R \cup (S \backslash T)\big) = C(R)$ it holds

$$\begin{aligned} &e\big(R \cup (S \backslash T), \nu\big) - e(R, \nu) + e(T, \nu) - e(S, \nu) \\ &\quad= v\big(R \cup (S \backslash T)\big) - v(R) + v(T) - v(S) \\ &\quad= v(T) - v(S) + b(S \backslash T) \\ &\quad= v(T) - \big(b(S) - c(S)\big) + b(S \backslash T) \\ &\quad= v(T) - \big(b(T) - c(T)\big) + c(S) - c(T) \\ &\quad\geq c(S) - c(T) \geq 0. \end{aligned}$$

Since $S, R \in \mathcal{D}_1(\nu)$, this can only be possible if $T, R \cup (S \backslash T) \in \mathcal{D}_1(\nu)$. ∎

The next lemma identifies some partitions in $\mathcal{D}_1(\nu)$.

Lemma 33. *If $\mathcal{D}_1(\nu)$ contains a partition of N, then at least one of the following statements is true.*

(I) $\{i\} \in \mathcal{D}_1(\nu)$, and $v\big(\{i\}\big) = 0$ for every $i \in N$.

(II) there exists $i_0 \in N$ such that $i_0 \neq \ell_N$, and it holds

- *i) $\{i \in N : i \preceq i_0\} \in \mathcal{D}_1(\nu)$, and $v\big(\{i \in N : i \preceq i_0\}\big) \neq 0$.*
- *ii) $\{k\} \in \mathcal{D}_1(\nu)$, and $v\big(\{k\}\big) = 0$ for every $k \in N$ s. t. $i_0 \prec k$.*

(III) there exists $i_0 \in N$ such that $i_0 \neq \ell_N$, and $\{i_0\}, N \backslash \{i_0\} \in \mathcal{D}_1(\nu)$.

(Notice that (I) is actually a special case of (II).)

Proof. Firstly assume that there are no null players in (N, v), and let $\mathcal{P}$ be a partition included in $\mathcal{D}_1(\nu)$. If $v(S) = 0$ for every $S \in \mathcal{P}$, by the monotonicity of v it holds $v\big(\{i\}\big) = 0$, and also from Lemma 29 it follows $\{i\} \in \mathcal{D}_1(\nu)$ for every $i \in N$, so we are in case (I).

So assume that there exists $S \in \mathcal{P}$, such that $v(S) \neq 0$. Let $R \in \mathcal{P}$ such that $\ell_R = \max_{\preceq} \bigcup_{\substack{S \in \mathcal{P}: \\ v(S) \neq 0}} S$. Then $v(R) \neq 0$, and by Lemma 30, R is effective, Let us consider first the case in which $\ell_N \neq \ell_R$. From Lemma 31 it follows $\{i \in N : i \preceq \ell_R\} = R \cup \{i \in N : i \preceq \ell_R\} \in \mathcal{D}_1(\nu)$. Moreover, if $\ell_R \prec i$, then there exists $S \in \mathcal{P}$ such that $i \in S$ and $v(S) = 0$, and by Lemma 29, $\{i\} \in \mathcal{D}_1(\nu)$. Then we are in case (II) by choosing $i_0 = \ell_R$.

Assume now that $\ell_N = \ell_R$. Let $S \in \mathcal{P}$, $S \neq R$ and take $k \in S$. By Lemma 31, since R is effective it holds $N\backslash\{k\} = R \cup \big(N\backslash\{k\}\big) \in \mathcal{D}_1(\nu)$. If $S = \{k\}$ then $\{k\} \in \mathcal{D}_1(\nu)$, and we are in case (III) by taking $i_0 = k$. If S is not a singleton, then S is effective and $v(S) \neq 0$. Thus $C(S) \leq C(R)$, and applying Lemma 32 we obtain $\{k\} \in \mathcal{D}_1(\nu)$. So we are again in case (III) by taking $i_0 = k$.

Now assume there are null players in (N, v). If all the players are null then for every $i \in N$ it holds $\nu_i = 0$, and consequently we are in case (I). Otherwise let $k = \max_{\preceq}\{i \in N : i \text{ is not a null player}\}$. By Lemma 28, if $k \prec i$ then i is a null player, hence $\nu_i = 0 = v\big(\{i\}\big)$ and $\{i\} \in \mathcal{D}_1(\nu)$. Moreover, by the efficiency of ν, it holds $v\big(\{i \in N : i \preceq k\}\big) = \nu\Big(\{i \in N : i \preceq k\}\Big)$, and consequently $\{i \in N : i \preceq k\} \in \mathcal{D}_1(\nu)$. So it will be enough to take $i_0 = k$, and we will be in case (II). ∎

In the next lemma we identify some antipartitions in $\mathcal{D}_1(\nu)$.

Lemma 34. *If $\mathcal{D}_1(\nu)$ contains an antipartition $\mathcal{A}$, with $|\mathcal{A}| \geq 3$, then there exists $i_0 \in N$, $i_0 \neq \ell_N$, such that $\{i_0\}, N\backslash\{i_0\} \in \mathcal{D}_1(\nu)$.*

Proof. Assume that $\mathcal{A} = \{N\backslash Q_1, \ldots, N\backslash Q_r\}$ ($r \geq 3$, and w. l. o. g. also assume that $\ell_N \in Q_1$. Let us consider $k \in Q_2$. Since $|\mathcal{A}| \geq 3$ necessarily $|N\backslash Q_1|, |N\backslash Q_2| \geq 2$, and hence $N\backslash Q_1$ and $N\backslash Q_2$ are effective.

On one hand, by Lemma 31 it holds $N\backslash\{k\} = \big(N\backslash Q_2\big) \cup \big(N\backslash\{k\}\big) \in \mathcal{D}_1(\nu)$. On the other hand by Lemma 32 it holds $\{k\} \in \mathcal{D}_1(\nu)$ (with $N\backslash Q_2$ and $N\backslash Q_1$ in the role of R and S respectively). Taking $i_0 = k$ we conclude the result. ∎

Proof of Proposition 9:

Proof. By Theorem 8, and Lemmas 33, and 34 we are in case (I) or (III) or there exist $i_0 \in N\backslash\{\ell_N\}$ such that $\{i_0\}, N\backslash\{i_0\} \subseteq \mathcal{D}_1(\nu)$. So assume we are in this case.

If $v\big(\{i_0\}\big) = 0$ we are in case (II). So we can assume that $v\big(\{i_0\}\big) \neq 0$. Then $\{i_0\}$ is effective and from Lemma 31 it follows $\{i \in N : i \preceq i_0\} \in \mathcal{D}_1(\nu)$.

If for every $i \in N$ it holds $i \succ i_0$, then we are in case (III)

So we can assume that there exists $i \preceq i_0$. From Lemma 32 it holds $N\backslash\{i, i_0\} \in \mathcal{D}_1(\nu)$ (with $\{1, \ldots, i_0\}$, $N\backslash\{i_0\}$, and $N\backslash\{i, i_0\}$, in the role of R, S and T respectively). And by Lemma 31 (with $N\backslash\{i, i_0\}$ in the role of R) it holds $N\backslash\{i\} \in \mathcal{D}_1(\nu)$. Consequently we are in case (IV). ∎

Appendix B

Proof of Lemma 10.

Proof. (a) and (b): By Remark 7 part b) it holds:

$$\begin{aligned} e(\mathcal{P},x) &= \frac{\sum_{i\in N} v\big(\{i\}\big) - v(N)}{|\mathcal{P}|} \\ &= \frac{\sum_{i\in N} v\big(\{i\}\big) - \big(b(N) - C(N)\big)}{|\mathcal{P}|} = -\alpha^v + \frac{\sum_{i\in N} v\big(\{i\}\big)}{|\mathcal{P}|}. \end{aligned}$$

And the result follows.

(c) Since $v\big(\{i\}\big) = 0$, and $\{i\} \in \mathcal{D}_1(\nu)$, it follows $\nu_i = -e(\mathcal{P},\nu)$. ∎

Proof of Lemma 11.

Proof. (a) and (b): By Remark 7 part b), and taking into account that $N\backslash\{i_0\}$ is effective it holds:

$$\begin{aligned} e(\mathcal{P},\nu) &= \frac{v\big(N\backslash\{i_0\}\big) + v\big(\{i_0\}\big) - v(N)}{2} \\ &= \frac{b\big(N\backslash\{i_0\}\big) - C(N) + v\big(\{i_0\}\big) - b(N) + C(N)}{2} = -\beta^v_{i_0} + \frac{v\big(\{i_0\}\big)}{2}. \end{aligned}$$

And we conclude easily the result.

(c) Since $v\big(\{i_0\}\big) = 0$, and $\{i_0\} \in \mathcal{D}_1(\nu)$, it follows $\nu_{i_0} = -e(\mathcal{P},\nu)$. ∎

Proof of Lemma 12.

Proof. (a) and (b): By Remark 7 part b), and taking into account that coalition $\{i \in N : i \preceq i_0\}$ is effective it holds:

$$\begin{aligned} e(\mathcal{P},\nu) &= \frac{v\big(\{i\in N : i \preceq i_0\}\big) + \sum_{k\succ i_0} v\big(\{k\}\big) - v(N)}{n - i_0 + 1} \\ &= \frac{b\big(\{i\in N : i \preceq i_0\}\big) - C\,(i_0) + \sum_{k\succ i_0} v\big(\{k\}\big) - b(N) + C(N)}{n - i_0 + 1} \\ &= \frac{\sum_{k\succ i_0} v\big(\{k\}\big) - C(i_0) - b\big(\{k\in N : k \succ i_0\}\big) + C\,(N)}{n - i_0 + 1} \\ &= -\gamma^v_{i_0} + \frac{\sum_{k\succ i_0} v\big(\{k\}\big)}{n - i_0 + 1}. \end{aligned}$$

And the result follows.

(c): Since for every $k \succ i_0$, it holds $\{k\} \in \mathcal{D}_1(\nu)$, and $v\big(\{k\}\big) = 0$, it follows $\nu_k = -e(\mathcal{P},\nu)$. ∎

Proof of Lemma 13.

Proof. (a) By Remark 7 part b), and taking into account that all the coalitions in $\mathcal{P}$ are effective it holds:

$$e(\mathcal{P},\nu) = \frac{v\big(\{k \in N : i_0 \prec k\}\big) + \sum_{k \preceq i_0} v\big(N\backslash\{k\}\big) - |\mathcal{P}-1| \cdot v(N)}{|\mathcal{P}|}$$

$$= \frac{b\big(\{k \in N : i_0 \prec k\}\big) - C\big(\{k \in N : i_0 \prec k\}\big) - |\mathcal{P}-1| \cdot \big(b(N) - C(N)\big)}{|\mathcal{P}|}$$

$$+ \frac{\sum_{k \preceq i_0} \Big(b\big(N\backslash\{i\}\big) - C(N)\Big)}{|\mathcal{P}|} = \frac{C(i_0)}{|\mathcal{P}|} = -\delta^v_{i_0}.$$

(b) Now let $i \preceq i_0$. Then

$$e\big(N\backslash\{i\},\nu\big) = v\big(N\backslash\{i\}\big) - \nu\big(N\backslash\{i\}\big) = v\big(N\backslash\{i\}\big) - \nu(N) + \nu_i$$

$$b\big(N\backslash\{i\}\big) - C(N) - v(N) + C(N) + \nu_i = -b_i + \nu_i,$$

Hence $\nu_i = b_i - \delta^v_{i_0}$. ∎

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