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The Optimal Defense of Networks of Targets

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The Optimal Defense of Networks of Targets[1]

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Abstract

This paper examines a game-theoretic model of attack and defense of multiple networks of targets in which there exist intra-network strategic complementarities among targets. The defender's objective is to successfully defend all of the networks and the attacker's objective is to successfully attack at least one network of targets. In this context, our results highlight the importance of modeling asymmetric attack and defense as a conflict between "fully" strategic actors with endogenous entry and force expenditure decisions as well as allowing for general correlation structures for force expenditures within and across the networks of targets.

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1 Introduction

In the literature on the optimal defense against intentional attack there has been growing interest in not only the attack and defense of isolated targets[1] but also networks of targets[2] and even complex supra-networks of targets.[3] This move towards increasing network complexity emphasizes the role that strategic complementarities among targets play in creating structural asymmetries between attack and defense. For example in complex infrastructure supra-networks — such as communication systems, electrical power grids, water and sewage systems, oil pipeline systems, transportation systems, and cyber security systems — there often exist particular targets or combinations of targets which if destroyed would be sufficient to either: (a) disable the entire supra-network or (b) create a terrorist "spectacular."

In order to highlight the importance of modeling the attack and defense of complex supra-networks as a conflict between "fully" strategic actors with endogenous entry and force expenditure decisions, we examine a contest-theoretic model that allows for the use of general correlation structures for force expenditures within and across the networks of targets. The supra-network of targets is made up of an arbitrary combination of two simple types of networks which capture the two extreme endpoints of an exposure-redundancy spectrum of network types. The maximal exposure network, which we label a *weakest-link network*, is successfully defended if and only if the defender successfully defends all targets within the network.[4] The maximal redundancy network, which we label a *best-shot network*, is successfully defended if the defender successfully defends at least one target within the network. At each target the conflict is modeled as a deterministic contest in which the player who allocates the higher level of force wins the target with probability one. Given

[1]See for example Bier et al. (2007), Powell (2007a, b), and Rosendorff and Sandler (2004).

[2]See for example Bier and Abhichandani (2003), Bier et al. (2005), and Clark and Konrad (2007).

[3]See for example Azaiez and Bier (2007), Hausken (2008, 2009), and Levitin and Ben-Haim (2008).

[4]See Hirshleifer (1983) who coins the terms best-shot and weakest-link in the context of voluntary provision of public goods.

that the loss of a single network may be sufficient to either disable the entire supra-network or create a terrorist "spectacular," we focus on the case in which the attacker's objective is to successfully attack a single network and the defender's objective is to successfully defend all of the networks.

A distinctive feature of this environment is that a mixed strategy is a joint distribution function in which the randomization in the force allocation to each target is represented as a separate dimension. A pair of equilibrium joint distribution functions specifies not only each player's randomization in force expenditures to each target but also the correlation structure of the force expenditures within and across the networks of targets. For all parameter configurations, we completely characterize the unique set of Nash equilibrium univariate marginal distributions for each player as well as the unique equilibrium payoff for each player. Furthermore, in any equilibrium we find that the attacker launches an attack on at most one network of targets, and there exist parameter configurations for which the attacker optimally launches no attack with positive probability. Although at most one network is attacked, the attacker randomizes over which network is attacked, and each of the networks is attacked with positive probability. In the event that a weakest-link network is attacked, the attacker optimally launches an attack on only a single target. When a best-shot network is attacked, the attacker optimally attacks every target in that network with a strictly positive force level.

As emphasized in the *National Strategy for Homeland Security*, "terrorists are strategic actors." However, much of the existing literature [e.g. Azaiez and Bier (2007), Bier and Abhichandani (2003), Bier et al. (2005), Bier et al. (2007), Levitin and Ben-Haim (2008), Powell (2007a, b), and Rosendorff and Sandler (2004).] assumes that terrorists (henceforth attackers) are not 'fully' strategic in the sense that the number of attacks (which is usually set to one) is exogenously specified. By endogenizing the attacker's entry and force expenditure decisions, we examine not only the conditions under which the assumption of one attack is likely to hold, but also related issues such as how the defender's actions can decrease the number of terrorist attacks. Furthermore, the few previous models which allow for the attacker to endogenously choose the number of targets to attack [e.g. Clark and Konrad (2007) and Hausken (2008)][5] obtain the result that even when the attacker's objective is to disable a single network — and the attacker derives no additional benefit from successfully disabling more than one network — the attacker optimally chooses to attack every target in every network with certainty. By demonstrating that in all equilibria of our model the attacker optimally engages in a form of stochastic guerilla warfare in which at most one network of targets is attacked, but with positive probability each network is chosen as the one to be attacked, our results also provide a sharp contrast with existing models of "fully" strategic attackers.

Section 2 presents the model of attack and defense with networks of targets. Section 3 characterizes a Nash equilibrium and explores properties of the equilibrium distributions of force. Section 4 concludes.

2 The Model

Players

The model is formally described as follows. Two players, an attacker, A, and a defender, D, simultaneously allocate their forces across a finite number, $n \geq 2$, of heterogeneous targets. The players' payoffs depend on the composition of each of the networks of targets in the supra-network. We examine a supra-network consisting of any arbitrary combination of two types of simple networks.

[5]Utilizing probabilistic contest success functions [Clark and Konrad (2007) utilize the Tullock contest success function, Hausken (2008) utilizes both the Tullock and difference-form contest success functions], Clark and Konrad (2007) and Hausken (2008) examine a single weakest-link network and a supra-network consisting of any arbitrary combination of weakest-link and best-shot networks [as in this paper, a successful attack on any one network is sufficient to disable the entire supra-network], respectively.

The targets are partitioned into a finite number $k \geq 1$ of disjoint networks, where network $j \in \{1, \ldots, k\}$ consists of a finite number $n_j \geq 1$ of targets with $\sum_{j=1}^{k} n_j = n$. Let N_j denote the set of targets in network j. Let $\mathcal{W}$ denote the set of weakest-link networks and $\mathcal{B}$ denote the set of best-shot networks.

In a *best-shot network* the network is successfully defended if the defender allocates at least as high a level of force to at least one target within the network. Conversely, an attack on a best-shot network is successful if the attacker allocates a higher level of force to each target in the network. Let x_A^i (x_D^i) denote the level of force allocated by the attacker (defender) to target i. Define

$$\iota_j^B = \begin{cases} 1 & \text{if } \forall\ i \in N_j \mid x_A^i > x_D^i \\ 0 & \text{otherwise} \end{cases}.$$

Observe that for each target, the player that allocates the higher level of force wins that target, but in order to win the network the attacker must win all of the targets. In a best-shot network, a tie arises when player A allocates a level of force to each target in the network that is at least as great as player D's allocation, and there exists at least one target in the network to which the players allocate the same level of force. In this case, the defender wins the network.

In the second type of network, which we label a *weakest-link network*, the network is successfully defended if the defender allocates at least as high a level of force to all targets within the network. Conversely, an attack on a weakest-link network is successful if the attacker allocates a higher level of force to any target in the network. Define

$$\iota_j^W = \begin{cases} 1 & \text{if } \exists\ i \in N_j \mid x_A^i > x_D^i \\ 0 & \text{otherwise} \end{cases}.$$

Again, in the case of a tie, the defender is assumed to win the network.

The players are risk neutral and have asymmetric objectives. The attacker's objective is to successfully attack at least one network, and the attacker's payoff for the successful attack of at least one network is $v_A > 0$. The attacker's payoff function is given by

$$\pi_A(\mathbf{x}_A, \mathbf{x}_D) = v_A \max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) - \sum_{i=1}^{n} x_A^i$$

The defender's objective is to preserve the entire supra-network, and the defender's payoff for successfully defending the supra-network is $v_D > 0$. The defender's payoff function is given by

$$\pi_D(\mathbf{x}_A, \mathbf{x}_D) = v_D\left(1 - \max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right)\right) - \sum_{i=1}^{n} x_D^i.$$

The force allocated to each target must be nonnegative.

It is important to note that our formulation utilizes an auction contest success function.[6] It is well known that, because behavior is invariant with respect to positive affine transformations of utility, all-pay auctions in which players have different constant unit costs of resources may be transformed into behaviorally equivalent all-pay auctions with identical unit costs of resources, but suitably modified valuations. This result extends directly to the environment examined here, and thus, our focus on asymmetric valuations also covers the case in which the players have different constant unit costs of resources.

Also observe that in the formulation described above the supra-network is a weakest-link supra-network. That is if the defender loses a single network then the entire supra-network is inoperable. By interchanging the identities of player A and player D, our results on weakest-link supra-networks apply directly to the case of best-shot supra-networks (where a best-shot supra-network is a supra-network which is successfully defended if the defender successfully defends at least one network).

[6]See Baye, Kovenock, and de Vries (1996).

Figure 1 provides a representative supra-network consisting of 5 networks (A, B, C, D, and E). Networks A, C, and E are weakest-link (series) networks with two targets each. Networks B and D are best-shot (parallel) networks with five targets each. In order to preserve the entire supra-network player D's objective is to preserve a path across the entire network. If a single target in networks A, C, or E is destroyed then the supra-network is inoperable. Conversely, in networks B and D all of the targets must be destroyed in order to render the supra-network inoperable.

[Insert Figure 1 here]

Strategies

It is clear that there is no pure strategy equilibrium for this class of games. A mixed strategy, which we term a *distribution of force*, for player i is an n-variate distribution function $P_i : \mathbb{R}^n_+ \to [0, 1]$. The n-tuple of player i's allocation of force across the n targets is a random n-tuple drawn from the n-variate distribution function P_i.

Model of Attack and Defense with Networks of Targets

The model of attack and defense with networks of targets, which we label

$$ADN \left\{ \{N_j\}_{j \in \mathcal{B}}, \{N_j\}_{j \in \mathcal{W}}, v_A, v_D \right\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of force, each target is won by the player that provides the higher allocation of force for that target, ties are resolved as described above, and players' payoffs, π_A and π_D, are specified above.

3 Optimal Distributions of Force

It is useful to introduce a simple summary statistic that captures both the asymmetry in the players' valuations and the structural asymmetries arising in the supra-network.

Definition 1. Let $\alpha = v_D/(v_A[\sum_{j\in\mathcal{W}} n_j + \sum_{j\in\mathcal{B}} \frac{1}{n_j}])$ denote the *normalized relative strength of the defender.*

Several properties of this summary statistic should be noted. First, the normalized relative strength of the defender is increasing in the relative valuation of the defender to the attacker (v_D/v_A), and is decreasing in the level of exposure arising in the supra-network ($\sum_{j\in\mathcal{W}} n_j + \sum_{j\in\mathcal{B}} \frac{1}{n_j}$). In particular, the defender's exposure is increasing in the number of weakest-link targets ($\sum_{j\in\mathcal{W}} n_j$), and is decreasing in the number of targets within each best-shot network ($\sum_{j\in\mathcal{B}} \frac{1}{n_j}$).

For all parameter ranges, Theorem 1 establishes the uniqueness of: (i) the players' equilibrium expected payoffs and (ii) the players' sets of univariate marginal distributions. Theorem 1 also provides a pair of equilibrium distributions of force for all parameters ranges. Case (1) of Theorem 1 examines the parameter configurations for which the defender has a normalized relative strength advantage, i.e. $\alpha \geq 1$. Case (2) of Theorem 1 addresses the parameter configurations for which the defender has a normalized relative strength disadvantage, i.e. $\alpha < 1$. It is important to note that the stated equilibrium distributions of force (n-variate distributions) are not unique. However, in Propositions 1-3 we characterize properties of optimal attack and defense that hold in all equilibria.

Theorem 1. *For any feasible parameter figuration of the game $ADN\{\{N_j\}_{j\in\mathcal{B}}, \{N_j\}_{j\in\mathcal{W}}, v_A, v_D\}$ there exists a unique set of Nash equilibrium univariate marginal distributions and a unique equilibrium payoff for each player. One such equilibrium is for each player to allocate his forces according to the following n-variate distribution functions:*

(1) *If* $\alpha \geq 1$*, then for player* A *and* $\boldsymbol{x} \in \prod_{j \in \mathcal{W}} [0, v_A]^{n_j} \times \prod_{j \in \mathcal{B}} [0, \frac{v_A}{n_j}]^{n_j}$

$$P_A(\boldsymbol{x}) = 1 - \frac{1}{\alpha} + \frac{\sum_{j \in \mathcal{W}} \sum_{i \in N_j} x^i + \sum_{j \in \mathcal{B}} \min_{i \in N_j} \{x^i\}}{v_D}$$

Similarly for player D *and* $\boldsymbol{x} \in \prod_{j \in \mathcal{W}} [0, v_A]^{n_j} \times \prod_{j \in \mathcal{B}} [0, \frac{v_A}{n_j}]^{n_j}$

$$P_D(\boldsymbol{x}) = \min \left(\left\{ \frac{\min_{i \in N_j} \{x^i\}}{v_A} \right\}_{j \in \mathcal{W}}, \left\{ \frac{\sum_{i \in N_j} x^i}{v_A} \right\}_{j \in \mathcal{B}} \right)$$

The expected payoff for player A *is* 0*, and the expected payoff for player* D *is* $v_D(1 - \frac{1}{\alpha})$*.*

(2) *If* $\alpha < 1$*, then for player* A *and* $\boldsymbol{x} \in \prod_{j \in \mathcal{W}} [0, \alpha v_A]^{n_j} \times \prod_{j \in \mathcal{B}} [0, \frac{\alpha v_A}{n_j}]^{n_j}$

$$P_A(\boldsymbol{x}) = \frac{\sum_{j \in \mathcal{W}} \sum_{i \in N_j} x^i + \sum_{j \in \mathcal{B}} \min_{i \in N_j} \{x^i\}}{v_D}$$

Similarly for player D *and* $\boldsymbol{x} \in \prod_{j \in \mathcal{W}} [0, \alpha v_A]^{n_j} \times \prod_{j \in \mathcal{B}} [0, \frac{\alpha v_A}{n_j}]^{n_j}$

$$P_D(\boldsymbol{x}) = 1 - \alpha + \min \left(\left\{ \frac{\min \{x^i\}_{i \in N_j}}{v_A} \right\}_{j \in \mathcal{W}}, \left\{ \frac{\sum_{i \in N_j} x^i}{v_A} \right\}_{j \in \mathcal{B}} \right)$$

The expected payoff for player D *is* 0*, and the expected payoff for player* A *is* $v_A(1 - \alpha)$*.*

Proof. The proof of the uniqueness of the players' equilibrium expected payoffs and sets of univariate marginal distributions is given in the Appendix. We now establish that the pair of n-variate distribution functions given in case (1) constitute an equilibrium for $\alpha \geq 1$. The proof of case (2) is analogous. The Appendix (see Lemma 5) establishes that in any n-tuple drawn from any equilibrium n-variate distribution P_A player A allocates a strictly positive level of force to at most one network of targets. If the network which receives the strictly positive level of force is a weakest-link network, then exactly one target in that network receives a strictly positive level of force. Although not a necessary condition for equilibrium, the P_A described in Theorem 1 also displays the property that when the network which receives the strictly positive level of force is a best-shot network the force allocated to each target in that network is an almost surely increasing function of the force allocated to any of the other targets in that network. The Appendix (see Lemma 5) also establishes that in any n-tuple drawn from any equilibrium n-variate distribution P_D player D allocates a strictly positive level of force to at most one target in each best-shot network of targets.

We will now show that for each player each point in the support of their equilibrium n-variate distribution function, P_A or P_D, given in case (1) of Theorem 1 results in the same expected payoff, and then show that there are no profitable deviations from this support.

We begin with the case in which player A attacks a single target in a single weakest-link network. The probability that player A wins target i in network $j \in \mathcal{W}$ is given by the univariate marginal distribution $P_D(x_A^i, \{\{v_A\}_{i' \in N_{j'} | x_A^{i'}=0}\}_{j' \in \mathcal{W}}, \{\{\frac{v_A}{n_{j'}}\}_{i' \in N_{j'}}\}_{j' \in \mathcal{B}})$, which we denote as $P_D^i(x_A^i)$. Given that player D is using the equilibrium strategy P_D described above, the payoff to player A for any allocation of force $\mathbf{x}_A \in \mathbb{R}_+^n$ which allocates a strictly positive level of force to a single target i in a weakest-link network $j \in \mathcal{W}$ is

$$\pi_A(\mathbf{x}_A, P_D) = v_A P_D^i(x_A^i) - x_A^i.$$

Simplifying,

$$\pi_A(\mathbf{x}_A, P_D) = v_A \left(\frac{x_A^i}{v_A}\right) - x_A^i = 0.$$

Thus the expected payoff to player A from allocating a strictly positive level of force to only one target in any weakest-link network is 0 regardless of which target is attacked.

Next, we examine the case in which player A attacks a single best-shot network. The probability that player A wins every target in network $j \in \mathcal{B}$ is given by the n_j-variate marginal distribution $P_D(\{x_A^i\}_{i \in N_j}, \{\{v_A\}_{i' \in N_{j'}}\}_{j' \in \mathcal{W}}, \{\{\frac{v_A}{n_{j'}}\}_{i' \in N_{j'}}\}_{j' \in \mathcal{B} | j' \neq j})$, which we denote as $P_D^{N_j}(\{x_A^i\}_{i \in N_j})$. Given that player D is using the equilibrium strategy P_D described above, the payoff to player A for any allocation of force $\mathbf{x}_A \in \mathbb{R}^n_+$ which allocates a strictly positive level of force only to the targets in a best-shot network $j \in \mathcal{B}$, and allocates zero forces to every other network is

$$\pi_A\left(\mathbf{x}_A, P_D\right) = v_A P_D^{N_j}\left(\{x_A^i\}_{i\in N_j}\right) - \sum_{i\in N_j} x_A^i.$$

Simplifying,

$$\pi_A\left(\mathbf{x}_A, P_D\right) = v_A\left(\frac{\sum_{i\in N_j} x_A^i}{v_A}\right) - \sum_{i\in N_j} x_A^i = 0.$$

Thus, the expected payoff to player A from allocating a strictly positive level of force to only one best-shot network is 0 regardless of which best-shot network is attacked.

For player A, possible deviations from the support include allocating a strictly positive level of force to: (a) two or more targets in the same weakest-link network, (b) two or more targets in different weakest-link networks, (c) two or more best-shot networks, and (d) any combination of both weakest-link and best-shot networks.

Beginning with (a), the probability that player A wins both targets i and i' in network $j \in \mathcal{W}$ is given by the bivariate marginal distribution $P_D(x_A^i, x_A^{i'}, \{\{v_A\}_{i''\in N_{j'}|i''\neq i,i'}\}_{j'\in\mathcal{W}}, \{\{\frac{v_A}{n_{j'}}\}_{i''\in N_{j'}}\}_{j'\in\mathcal{B}})$, which we denote as $P_D^{i,i'}(x_A^i, x_A^{i'})$. The payoff to player A for any allocation of force $\mathbf{x}_A \in \mathbb{R}^n_+$ which allocates a strictly positive level of force to two targets i, i' in a weakest-link network $j \in \mathcal{W}$ is

$$\pi_A\left(\mathbf{x}_A, P_D\right) = v_A P_D^i\left(x_A^i\right) + v_A P_D^{i'}\left(x_A^{i'}\right) - v_A P_D^{i,i'}\left(x_A^i, x_A^{i'}\right) - x_A^i - x_A^{i'}.$$

Simplifying,

$$\pi_A\left(\mathbf{x}_A, P_D\right) = v_A\left(\frac{x_A^i}{v_A} + \frac{x_A^{i'}}{v_A} - \frac{\min\left\{x_A^i, x_A^{i'}\right\}}{v_A}\right) - x_A^i - x_A^{i'} < 0.$$

The case of player A allocating a strictly positive level of force to more than two targets in a weakest-link network follows directly. Clearly, in any optimal strategy player A never allocates a strictly positive level of force to more than one target within a weakest-link network.

The proof for type (b) deviations follows along similar lines. Thus, in any optimal strategy player A never allocates a strictly positive level of force to more than one target within a weakest-link network of targets or in different weakest-link networks.

For type (c) deviations, the probability that player A wins all of the targets in both best-shot networks $j,\ j' \in \mathcal{B}$ is given by the $(n_j + n_{j'})$-variate marginal distribution $P_D(\{x_A^i\}_{i\in N_j\cup N_{j'}}, \{\{v_A\}_{i''\in N_{j''}}\}_{j''\in\mathcal{W}}, \{\{\frac{v_A}{n_{j''}}\}_{i''\in N_{j''}}\}_{j''\in\mathcal{B}|j''\neq j,j'})$, which we denote as $P_D^{N_j,N_{j'}}\left(\{x_A^i\}_{i\in N_j\cup N_{j'}}\right)$. The payoff to player A for any allocation of force $\mathbf{x}_A \in \mathbb{R}_+^n$ which allocates a strictly positive level of force to exactly two best-shot networks $j,\ j' \in \mathcal{B}$ is

$$\pi_A\left(\mathbf{x}_A, P_D\right) = $$
$$v_A P_D^{N_j}\left(\{x_A^i\}_{i\in N_j}\right) + v_A P_D^{N_{j'}}\left(\{x_A^i\}_{i\in N_{j'}}\right) - v_A P_D^{N_j,N_{j'}}\left(\{x_A^i\}_{i\in N_j\cup N_{j'}}\right) - \sum_{i\in N_j\cup N_{j'}} x_A^i.$$

Simplifying,

$$\pi_A\left(\mathbf{x}_A, P_D\right) = -v_A \min\left\{\frac{\sum_{i\in N_j} x_A^i}{v_A}, \frac{\sum_{i\in N_{j'}} x_A^i}{v_A}\right\}$$

The case of player A allocating a strictly positive level of force to more than two best-shot networks follows directly. Clearly, in any optimal strategy player A never allocates a strictly positive level of force to more than one best-shot network.

The case of type (d), follows along similar lines. Thus, the expected payoff from each point in the support of the n-variate distribution P_A results in the same expected payoff, 0, and there exist no allocations of force which have a higher expected payoff.

The case for player D follows along similar lines. $\square$

Although the equilibrium distributions of force stated in Theorem 1 are not unique,[7] it is useful to provide some intuition regarding the existence of this particular equilibrium before moving on to the characterization of properties of optimal attack and defense that hold in all equilibria (Propositions 1-3). The supports of the equilibrium distributions of force stated in Theorem 1 are given in Figure 2 for two different parameter configurations. Panels (i) and (ii) of Figure 2 provide the supports for the attacker and defender, respectively, in the case that there is one weakest-link network with two targets ($i = 1, 2$). Panels (iii) and (iv) of Figure 2 provide the supports for the attacker and defender, respectively, in the case that there is one best-shot network with two targets ($i = 1, 2$) and one weakest-link network with one target ($i = 3$).

[Insert Figure 2]

Across all of the Panels (i)-(iv), if $\alpha = 1$ then each player randomizes continuously over their respective shaded line segments. In the event that the defender has a normalized relative strength advantage ($\alpha > 1$), the defender's strategy stays the same, but the attacker now places a mass point of size $1 - (1/\alpha)$ at the origin and randomizes continuously over the respective line segments with the remaining probability. Conversely, if the defender has a normalized relative strength disadvantage ($\alpha < 1$) then it is the defender who places a mass point (of size $1 - \alpha$) at the origin.

Beginning with Panels (i) and (ii), recall that if the attacker successfully attacks a single target in a weakest-link network the entire network is disabled. As shown in Panel (i) the

[7] For example, in the case (1) parameter range of Theorem 1 another equilibrium strategy for player D is to use the distribution of force

$$P_D(\mathbf{x}) = \min\left(\left\{\frac{\prod_{i\in N_j} x^i}{v_A}\right\}_{j\in\mathcal{W}}, \left\{\frac{\sum_{i\in N_j} x^i}{v_A}\right\}_{j\in\mathcal{B}}\right).$$

attacker launches an attack on at most one target. To successfully defend a weakest-link network, the defender must win every target within the network. As shown in Panel (ii) the defender's allocation of force to target i is an almost surely strictly increasing function of the force allocated to target $-i$. Note that if the attacker launches an attack on at most one target, then the probability that any single attack is successful depends only on the univariate marginal distributions of the defender's (n-variate joint) distribution of force. In addition, the defender's expected force expenditure depends only on his set of univariate marginal distributions, and, for a given set of univariate marginal distributions, is invariant to the correlation structure.[8] Finally, note that given the defender's choice of correlation structure [Panel (ii)], the attacker's probability of at least one successful attack depends only on the maximum of his force allocations across the two targets. That is, given the defender's distribution of force, if the set of points such that $x^i_A > x^{-i}_A > 0$ for some $i \in \{1, 2\}$ has positive probability, then the attacker can strictly increase his expected payoff by reducing x^{-i}_A to $x^{-i}_A = 0$ for all such points. In such a deviation, the probability of at least one successful attack is unaffected, but the attacker's expected force expenditure decreases. Thus, at each point in the support of an optimal distribution of force the attacker launches at most one attack.

Panels (iii) and (iv) examine a simple supra-network with one best-shot network and one weakest-link network . In Panel (iii), note that the attacker launches an attack on at most one network. In the event that the best-shot network is attacked, the attacker's allocation of force to target i in the best-shot network is an almost surely strictly increasing function of the force allocated to target $-i$ in the network. In Panel (iv), note that the defender allocates a strictly positive level of force to at most one of the targets $i \in \{1, 2\}$ in the best-shot network, and that the level of force allocated to the sole target in the weakest-link

[8]More formally, for a given set of univariate marginal distribution functions, the expected force expenditure is invariant to the mapping into a joint distribution function, i.e. the n-copula. For further details see Nelsen (1999).

network is an almost surely increasing function of the level of force allocated to the best-shot network. Given these correlation structures, the intuition for why the attacker launches an attack on at most one network in the supra-network follows along the lines given above for the weakest-link network in which at most one target was attacked.

We now characterize the qualitative features arising in all equilibrium distributions of force. Proposition 1 examines the number of networks that are simultaneously attacked as well as the number of targets within each network that are simultaneously attacked and defended. Propositions 2 and 3 examine the likelihood that the attacker optimally chooses to launch an attack on any given network, and the likelihood that the attacker launches no attack or the defender leaves the supra-network undefended.

Proposition 1. *In any equilibrium* $\{P_A, P_D\}$*:*

1. *Player A allocates a strictly positive level of force to at most one network.*

2. *If player A allocates a strictly positive level of force to a weakest-link network, then one target in that network receives a strictly positive level of force*

3. *In each best-shot network player D allocates a strictly positive level of force to at most one target in the network.*

The formal proof of Proposition 1 is given in the appendix (see Lemma 5). The intuition for Proposition 1 follows from the fact that the likelihood that player D successfully defends all of the networks (and therefore player D's expected payoff) is weakly decreasing in the number of networks that player A chooses to simultaneously attack. However, player D has the ability to vary the correlation structure of his force allocations while leaving invariant: (i) his network specific multivariate marginal distributions of force, (ii) his univariate marginal distributions of force, and (iii) his expected expenditure. Furthermore, there exist correlation structures for which the likelihood that player D successfully defends all of the networks depends only on player A's force allocation to the one network which receives the highest level of force from player A. Given that player D is using such a correlation structure, player A optimally attacks at most one network at a time.

Proposition 2. *If $\alpha \geq 1$, then in any equilibrium $\{P_A, P_D\}$:*

1. *The probability that any weakest-link network j is attacked (i.e., the probability that the attacker allocates a strictly positive level of force to weakest-link network j) is $(n_j v_A/v_D)$, which is increasing in the number of targets in network j and the attacker's valuation of success and decreasing in the defender's valuation of successfully defending the entire supra-network.*

2. *The probability that any best-shot network j is attacked is $(v_A)/(n_j v_D)$, which is increasing in the attacker's valuation of success and is decreasing in both the defender's valuation and the number of targets in network j.*

3. *The attacker optimally attacks no network in the supra-network with probability $1 - (1/\alpha)$.*

In the Appendix, we provide the univariate marginal distributions that arise in any equilibrium joint distribution of the attacker. Moreover, we show that if $\alpha > 1$, then in any equilibrium the attacker paces a mass point at the origin. Proposition 2 follows directly. The probability that a network j is attacked is equal to one minus the attacker's mass point at zero in the n_j-variate marginal distribution for network j, $P_A^{N_j}(\{x_i\}_{i \in N_j})$. The likelihood that the attacker optimally chooses to launch no attack is increasing in the defender's valuation of success and decreasing in the attacker's valuation of success.

For $\alpha \geq 1$, the attacker's valuation is low enough relative to the defender's valuation that the optimal strategy includes not launching an attack with positive probability. For $\alpha < 1$, the attacker optimally launches an attack with certainty. In this case the probability that any given network of targets is attacked depends only on the number of targets in the network and the type of network. The proof of Proposition 3 also follows from the characterization of the properties of equilibrium joint distribution given in the Appendix.

Proposition 3. *If $\alpha < 1$, then in any equilibrium $\{P_A, P_D\}$:*

1. *The probability that any weakest-link network j is attacked (i.e., the probability that the attacker allocates a strictly positive level of force to weakest-link network j) is $n_j/([\sum_{j'\in\mathcal{W}} n_{j'} + \sum_{j'\in\mathcal{B}} \frac{1}{n_{j'}}])$, which is increasing in the number of targets in network j.*

2. *The probability that any best-shot network j is attacked is $1/(n_j[\sum_{j'\in\mathcal{W}} n_{j'} + \sum_{j'\in\mathcal{B}} \frac{1}{n_{j'}}])$, which is decreasing in the number of targets in network j.*

3. *The defender optimally leaves the entire supra-network undefended with probability $1 - \alpha$.*

If $\alpha \geq 1$, the defender optimally chooses, with certainty, to allocate a strictly positive level of defensive force. However, if $\alpha < 1$, the defender optimally chooses to leave the entire supra-network undefended with positive probability. Furthermore, the likelihood that the defender chooses to leave the entire supra-network undefended is increasing in the attacker's valuation of success and decreasing in the defender's valuation of successfully defending the entire supra-network.

To summarize, the following conditions hold in all equilibria. If $\alpha > 1$ the attacker optimally chooses not to launch an attack with positive probability. Regardless of the value of α, the attacker optimally launches an attack on at most one network. In the event that a weakest-link network is attacked, only one target within the network is attacked. The likelihood that any individual network is attacked depends on the number of targets within the network. In each weakest-link network the likelihood of attack is increasing in the number of targets. In each best-shot network the likelihood of attack is decreasing in the number of targets. If $\alpha < 1$, the defender optimally leaves the entire supra-network undefended with positive probability. Lastly, regardless of the value of α, when the defender chooses to defend the supra-network, within each best-shot network, the defender randomly chooses at most one target to defend.

4 Conclusion

This paper examines a game theoretic model of attack and defense of a supra-network, made up of a combination of weakest-link and best-shot networks of targets. The model features asymmetric objectives: the defender wishes to successfully defend all networks and the attackers objective is to successfully attack at least one network. Although the model allows for general correlation structures for force expenditures within and across the networks of targets, for any such configuration of networks, we derive the unique equilibrium expected payoffs of the attacker and defender and demonstrate that there exists a unique equilibrium univariate marginal distribution of forces to each target. An equilibrium pair of strategies for the attacker and defender, each of which is a joint distribution governing the allocation of forces to all targets, is also constructed, although these are generally non-unique.

Our approach leads to a wealth of interesting extensions and applications. Because the game examined here is a set of complete information all-pay auctions linked by payoff complementarities, almost any extension of the standard one-dimensional strategic allocation problem represented by the standard all-pay auction with complete information has a corresponding extension in this game. Examples include, incomplete information, about values or unit costs of forces, affine handicapping of players within target contests, and nonlinear costs of forces.[9] In addition, as in other models of strategic multidimensional resource allocation,

[9]Examples of these extensions for the one-dimensional strategic allocation problem include Amann and Leininger (1996), Krishna and Morgan (1997), Moldovanu and Sela (2001, 2006), Gale and Stegeman (1994),

such as Colonel Blotto games, interesting extensions arise by introducing more heterogeneity across targets, such as allowing for differential target values for attacker and defender within the weakest-link and best-shot structure, or other linkages across targets, such as budget constraints or "infrastructure technologies" that allow lumpy force expenditure across sets of multiple targets or networks. Furthermore, because our model succeeds in pinning down unique equilibrium payoffs for arbitrary network configurations and player valuations, it readily serves as a component model for multistage models of network investment, where uniqueness of subgame equilibrium payoffs avoids a multiplicity of equilibria supported by finite horizon trigger strategies. Hence, theories of strategic network investment and systems redundancies may be simply addressed in the framework, in which each best shot network j employed may be viewed as a network with $n_j - 1$ redundant components.

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Appendix

This appendix characterizes the supports of the equilibrium joint distributions, the unique equilibrium payoffs, and the unique sets of equilibrium univariate marginal distributions. Before proceeding, observe the following notational conventions which will be used throughout the appendix. For points in $\mathbb{R}^n$, we will use the vector notation $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. For $a_k \leq b_k$ for all $k = 1, 2, \ldots, n$, let $[\mathbf{a}, \mathbf{b}]$ denote the n-box $B = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$, the Cartesian product of n closed intervals. The vertices of the n-box B are the points $(c_1, c_2, \ldots, c_n)$ where c_k is equal to a_k or b_k. Lastly, let $\bar{s}_i^j$ and $\underline{s}_i^j$ denote the upper and lower bounds, respectively, for player i's distribution of force for target j.

Given that the defender is using the distribution of force P_D, let

$$Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) = 1\middle|P_D, \mathbf{x}_A\right) \tag{1}$$

denote the probability that with a force allocation of $\mathbf{x}_A$ the attacker wins at least one network. Thus, the attacker's expected payoff from any pure strategy $\mathbf{x}_A$ is

$$v_A Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) = 1\middle|P_D, \mathbf{x}_A\right) - \sum_i x_A^i. \tag{2}$$

It will also be useful to note that the attacker's expected payoff from any distribution of force P_A is

$$v_A E_{P_A}\left[Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) = 1\middle|P_D, \mathbf{x}_A\right)\right] - \sum_i E_{P_A^i}\left[x_A^i\right] \tag{3}$$

where E_{P_A} denotes the expectation with respect to the joint distribution of force P_A and $E_{P_A^i}$ denotes the expectation with respect to the univariate marginal distribution for target i, henceforth P_A^i, of the joint distribution of force P_A.

Similarly, given that the attacker is using the distribution of force P_A, let

$$Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}},\left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right)=0\middle|P_A,\mathbf{x}_D\right) \tag{4}$$

denote the probability that with a force allocation of $\mathbf{x}_D$ the defender wins all of the networks in the supra-network. Thus, the defender's expected payoff from any pure strategy $\mathbf{x}_D$ is

$$v_D Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}},\left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right)=0\middle|P_A,\mathbf{x}_D\right)-\sum_i x_D^i. \tag{5}$$

Lastly, the defender's expected payoff from any distribution of force P_D is

$$v_D E_{P_D}\left[Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}},\left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right)=0\middle|P_A,\mathbf{x}_D\right)\right]-\sum_i E_{P_D^i}\left[x_D^i\right] \tag{6}$$

where E_{P_D} and $E_{P_D^i}$ denote the expectation with respect to the joint distribution of force P_D and the expectation with respect to the univariate marginal distribution for target i, P_D^i, respectively.

We begin by showing that for each target i within weakest-link (best-shot) network j, both players' distributions of force have the same upper bound, denoted $\bar{s}_W^j$ ($\bar{s}_B^j$), and a lower bound of 0 (0).

Lemma 1. *In any equilibrium: (i) for each $j\in\mathcal{W}$, $\bar{s}_A^i=\bar{s}_D^i=\bar{s}_W^j>0$ and $\underline{s}_A^i=\underline{s}_D^i=0$ for all $i\in N_j$, and (ii) for each $j\in\mathcal{B}$, $\bar{s}_A^i=\bar{s}_D^i=\bar{s}_B^j>0$ and $\underline{s}_A^i=\underline{s}_D^i=0$ for all $i\in N_j$.*

Proof. We begin with the proof that $\underline{s}_A^i=\underline{s}_D^i=0$ for all i. By way of contradiction, suppose $\underline{s}_A^i\neq\underline{s}_D^i$. Let $\hat{\underline{s}}^i\equiv\max\{\underline{s}_A^i,\underline{s}_D^i\}$, and let l be the identity of the player attaining $\hat{\underline{s}}^i$ (that is $\hat{\underline{s}}^i=\underline{s}_l^i$ and $\hat{\underline{s}}^i>\underline{s}_{-l}^i$).

If $\underline{s}_{-l}^i>0$, when player $-l$ allocates $\underline{s}_{-l}^i$ to target i player $-l$ loses target i with certainty and can strictly increase his payoff by setting $\underline{s}_{-l}^i=0$. It follows directly, that player $-l$ does not randomize over the open interval $(0, \underline{\hat{s}}^i)$, and thus player $-l$ must have a mass point at 0.

In the case that $\underline{s}^i_{-l} = 0$ (where player $-l$ does not randomize over the open interval $(0, \underline{\hat{s}}^i)$ and has a mass point at 0), we know that (i) both players cannot have a mass point at $\underline{s}^i_l$, (ii) player $-l$ cannot place mass at $\underline{s}^i_l$, and (iii) player l can strictly increase his payoff by lowering $\underline{s}^i_l$ to a neighborhood above 0. Thus, we conclude that $\underline{s}^i_A = \underline{s}^i_D = 0$ for all i.

Lastly, for the proof that for each $j \in \mathcal{W}$, $\bar{s}^i_A = \bar{s}^i_D = \bar{s}^j_W > 0$ for all $i \in N_j$, note that if there exists a target i such that $\bar{s}^i_A = \bar{s}^i_D = 0$, then player A can strictly increase his payoff by allocating an arbitrarily small, but strictly positive, level of force to weakest-link target i. Similarly, for any pair $i', i'' \in N_j$ it follows that if $\bar{s}^{i'}_A = \bar{s}^{i'}_D < \bar{s}^{i''}_A = \bar{s}^{i''}_D$ then player A would do better by moving mass from $\bar{s}^{i''}_A$ to $\bar{s}^{i'}_A$. The proof that for each $j \in \mathcal{B}$, $\bar{s}^i_A = \bar{s}^i_D = \bar{s}^j_B > 0$ for all $i \in N_j$ follows from a similar argument. $\square$

Lemma 2. *In any equilibrium $\{P_A, P_D\}$ with the set of univariate marginal distributions $\{P^i_A, P^i_D\}_{i=1}^n$, for each target i neither player's univariate marginal distribution places positive mass on any point except possibly at zero.*

Proof. If for target i, $x^i_l > 0$ is such a point for player l, then player $-l$ would either benefit from moving mass from an ϵ-neighborhood below x^i_l to zero or to a δ-neighborhood above x^i_l. $\square$

Lemma 3. *In any equilibrium, each player's expected payoff (equations (2) and (5) for the attacker and defender respectively) is constant over the support of his joint distribution except possibly at points of discontinuity of his expected payoff function.*

Proof. Except for possibly at points of discontinuity of his expected payoff function, each player l must make his equilibrium expected payoff at each point in the support of his equilibrium strategy, P_l. Otherwise, player l would benefit by moving mass to the n-tuple(s) in his support with the highest expected payoff. $\square$

Lemma 4. *In any equilibrium $\{P_A, P_D\}$ with the set of univariate marginal distributions $\{P_A^i, P_D^i\}_{i=1}^n$, for each target i each player l's univariate marginal distribution P_l^i randomizes continuously over the interval $(0, \bar{s}^i]$.*

Proof. Lemma 2 rules out mass points of P_l^i in the interval $(0, \bar{s}^i]$. To rule out gaps, by way of contradiction, suppose that there exists an equilibrium in which for some target i, player l's univariate marginal distribution for target i, P_l^i, is constant over the interval $[\alpha, \beta) \subset (0, \bar{s}^i]$ and strictly increasing above β in its support. For this to be an equilibrium, it must be the case that P_{-l}^i is also constant over the interval $[\alpha, \beta)$. Otherwise, player $-l$ could increase his payoff.

If $P_{-l}^i(\alpha) = P_{-l}^i(\beta)$, then for sufficiently small $\epsilon > 0$ spending $\beta + \epsilon$ in target i cannot be optimal for player l. Indeed, by discretely reducing his expenditure from $\beta + \epsilon$ to $\alpha + \epsilon$ player l's payoff would strictly increase. Consequently, if P_l^i is constant over $[\alpha, \beta)$ it must also be constant over $[\alpha, \bar{s}^i]$, a contradiction to the definition of $\bar{s}^i$. □

Lemma 5. *In any equilibrium $\{P_A, P_D\}$:*

(a) If $\mathbf{x}_A$ is an n-tuple contained in the support of P_A, then $\mathbf{x}_A$ allocates a strictly positive level of force to at most one network.

(b) If the n-tuple $\mathbf{x}_A$ (contained in the support of P_A) allocates a strictly positive level of force to a weakest-link network, then one target in that weakest-link network receives a strictly positive level of force.

(c) If $\mathbf{x}_D$ is an n-tuple contained in the support of P_D, then within each best-shot network $\mathbf{x}_D$ allocates a strictly positive level of force to at most one target in the network.

Proof. We begin with the proof of part (a). By way of contradiction suppose that there exists an equilibrium $\{P_A, P_D\}$ such that at one or more points in the support of P_A at least two networks simultaneously receive strictly positive levels of force (henceforth, simultaneously attacked). Let $\mathbf{x}_A^j$ denote the restriction of the vector $\mathbf{x}_A$ to the set of targets contained in network j (i.e., $\{x_A^i\}_{i\in N_j}$). Denote the set of points in the support of P_A that simultaneously attack at least two networks as

$$\Omega_A \equiv \{\mathbf{x}_A \in \text{Supp}\{P_A\} \big| \exists \text{ at least two } j \in \mathcal{B} \cup \mathcal{W} \text{ s.t. } \mathbf{x}_A^j \neq 0\}.$$

For each point $\mathbf{x}_A \in \Omega_A$ let $\mathcal{P}(j \in \mathcal{B} \cup \mathcal{W} | \mathbf{x}_A^j \neq \mathbf{0})$ denote the power set of the indices of networks that player A simultaneously attacks at the point $\mathbf{x}_A$. Let ψ denote an arbitrary element of this power set, let $|\psi|$ denote the cardinality of the set ψ, and let $\mathbf{x}_A^\psi$ denote the restriction of the vector $\mathbf{x}_A$ to the set of targets contained in the networks in ψ (i.e., $\{x_A^i\}_{i\in\cup_{j\in\psi}N_j}$).

Before stating the probability that at an arbitrary point $\mathbf{x}_A \in \Omega_A$ player A wins at least one network, consider the probability that player A wins at least one network in the special case that at $\mathbf{x}_A \in \Omega_A$ player A simultaneously attacks two networks j' and j''. In this special case,

$$\begin{aligned} &Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) = 1 \Big| P_D, \mathbf{x}_A\right) = \\ &\qquad Pr\left(\iota_{j'} = 1 \Big| P_D, \mathbf{x}_A^{j'}\right) + Pr\left(\iota_{j''} = 1 \Big| P_D, \mathbf{x}_A^{j''}\right) - Pr\left(\iota_{j'}, \iota_{j''} = 1 \Big| P_D, \mathbf{x}_A^{j',j''}\right) \end{aligned} \tag{7}$$

where the third term in the second line of (7) corrects for the first two terms' multiple countings of player A winning at least one network. Similarly, if at $\mathbf{x}_A \in \Omega_A$ player A simultaneously attacks three networks j', j'', and j''', then

$$
\begin{aligned}
Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) = \\
Pr\left(\iota_{j'} = 1 \middle| P_D, \mathbf{x}_A^{j'}\right) + Pr\left(\iota_{j''} = 1 \middle| P_D, \mathbf{x}_A^{j''}\right) + Pr\left(\iota_{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j'''}\right) \\
- Pr\left(\iota_{j'}, \iota_{j''} = 1 \middle| P_D, \mathbf{x}_A^{j',j''}\right) - Pr\left(\iota_{j'}, \iota_{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j',j'''}\right) - Pr\left(\iota_{j''}, \iota_{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j'',j'''}\right) \\
+ Pr\left(\iota_{j''}, \iota_{j''}, \iota_{j'''} = 1 \middle| P_D, \mathbf{x}_A^{j',j'',j'''}\right).
\end{aligned}
\tag{8}
$$

where, again, the third and fourth lines of (8) correct for the second line's multiple countings of player A winning at least one network. A straightforward proof by induction shows that for any arbitrary point $\mathbf{x}_A \in \Omega_A$ the probability that player A wins at least one network is given by

$$
\begin{aligned}
Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) = 1 \middle| P_D, \mathbf{x}_A\right) = \\
\sum_{\psi\in\mathcal{P}(j\in\mathcal{B}\cup\mathcal{W}|\mathbf{x}_A^j\neq\mathbf{0})-\emptyset} (-1)^{(|\psi|-1)}\, Pr\left(\iota_j = 1\ \forall\ j\in\psi \middle| P_D, \mathbf{x}_A^{\psi}\right).
\end{aligned}
\tag{9}
$$

We begin by examining the case in which for all $\mathbf{x}_A \in \Omega_A$ only best-shot networks are simultaneously attacked and refer to this as case (i). We then move on to case (ii) in which for all $\mathbf{x}_A \in \Omega_A$ only weakest-link networks are simultaneously attacked. Case (iii) includes all remaining configurations of simultaneous attacks (i.e., there exists at least one point $\mathbf{x}_A \in \Omega_A$ such that player A simultaneously attacks an arbitrary combination of both weakest-link and best-shot networks and/or there exist points in Ω_A at which only best-shot networks are simultaneously attacked and points at which only weakest-link networks are simultaneously attacked).

In case (i), simultaneous attacks occur on only best-shot networks, and the probability that player A wins every target in a best-shot network j, and hence wins network j is

$$Pr(\iota_j = 1 | P_D^{N_j}, \mathbf{x}_A^j) = P_D^{N_j}(\mathbf{x}_A^j), \tag{10}$$

where $P_D^{N_j}$ is the n_j-variate marginal distribution for network j. Similarly, the probability that player A wins every target in each best-shot network $j \in \psi$ is

$$Pr\left(\iota_j = 1 \ \forall \ j \in \psi \middle| P_D, \mathbf{x}_A\right) = P_D^{\psi}(\mathbf{x}_A^{\psi}). \tag{11}$$

where P_D^{ψ} is the $(\sum_{j\in\psi} n_j)$-variate marginal distribution over all of the networks $j \in \psi$. Note that if player D uses the strategy $\hat{P}_D(\mathbf{x}_A) = \min_{j\in\mathcal{B}\cup\mathcal{W}}\{P_D^{N_j}(\mathbf{x}_A^j)\}$ then for each $j \in \mathcal{B}\cup\mathcal{W}$ the n_j-variate marginal distribution $P_D^{N_j}(\mathbf{x}_A^j)$ is preserved, for each $i \in \cup_{j\in\mathcal{B}\cup\mathcal{W}} N_j$ the univariate marginal distribution $P_D^i(x_A^i)$ is preserved, and for each $\psi \in \mathcal{P}(j \in \mathcal{B} | \mathbf{x}_A^j \neq \mathbf{0}) - \emptyset$,

$$\hat{P}_D^{\psi}(\{\mathbf{x}_A^j\}_{j\in\psi}) = \min_{j\in\psi}\{P_D^{N_j}(\mathbf{x}_A^j)\}. \tag{12}$$

Because the expected cost of the strategy P_D — given in the second term in (6) — depends only on the set of univariate marginal distributions $\{P_D^i\}_{i\in\cup_{j\in\mathcal{B}\cup\mathcal{W}}N_j}$, the strategy $\hat{P}_D(\mathbf{x}_A)$ has the same expected cost as $P_D(\mathbf{x}_A)$. However, inserting (10), (11), and (12) into (9) a straightforward proof by induction (beginning with (7) and (8)) yields

$$Pr\left(\max\left(\left\{\iota_j^B\right\}_{j\in\mathcal{B}}, \left\{\iota_j^W\right\}_{j\in\mathcal{W}}\right) = 1 \middle| \hat{P}_D, \mathbf{x}_A\right) = \max_{j\in\mathcal{B}|\mathbf{x}_A^j\neq\mathbf{0}}\{P_D^{N_j}(\mathbf{x}_A^j)\}. \tag{13}$$

That is, if player D uses the strategy $\hat{P}_D(\mathbf{x}_A)$, then, in the event that player A simultaneously attacks two or more best-shot networks, the probability that player A successfully attacks at least one of the best-shot networks is equal to the probability that player A successfully attacks best-shot network $\bar{j} = \arg\max_{j\in\mathcal{B}|\mathbf{x}_A^j\neq\mathbf{0}}\{P_D^{N_j}(\mathbf{x}_A^j)\}$.

Therefore, if $P_D = \hat{P}_D$ then from (2) player A could increase his payoff by attacking only network $\bar{j}$. A contradiction to the assumption that $\{P_A, P_D\}$ is an equilibrium. Conversely, if $P_D \neq \hat{P}_D$, then the deviation to the strategy $\hat{P}_D$ leaves player D's expected costs invariant and at each $\mathbf{x}_A \in \Omega_A$ essentially nullifies player A's attacks on all but network $\bar{j}$. Thus, the expected cost-invariant deviation from P_D to $\hat{P}_D$ increases player D's probability of successful defense for each $\mathbf{x}_A \in \Omega_A$. Because each n_j-variate marginal distribution $P_D^{N_j}(\mathbf{x}_A^j)$ is preserved, the deviation from P_D to $\hat{P}_D$ also maintains player D's probability of successful defense at each $\mathbf{x}_A \notin \Omega_A$. From (4) this is a profitable deviation, and also a contradiction to the assumption that $\{P_A, P_D\}$ is an equilibrium. This completes the proof of part (i) of the proof of (a).

Before moving on to cases (ii) and (iii) in the proof of part (a), note that the argument given above can be used to establish part (b) of Lemma 5 (i.e., within each weakest-link network player A attacks at most one target). In particular, at target i in weakest-link network j let $\iota_{j,i} = 1$ if $x_A^i > x_D^i$ and $\iota_{j,i} = 0$ otherwise. The probability that player A wins weakest-link network j is given by

$$Pr\left(\iota_j = 1 \middle| P_D^{N_j}, \mathbf{x}_A\right) = Pr\left(\max_{i \in N_j}\{\iota_{j,i}\} = 1 \middle| P_D^{N_j}, \mathbf{x}_A\right) \tag{14}$$

Then by choosing $P_D^{N_j}(\mathbf{x}_A^j) = \min_{i \in N_j}\{P_D^i(x_A^i)\}$, player D's univariate marginals and hence the expected cost remain the same, the correlation of player D's allocation of force among the networks is unaffected, and the correlation of player D's allocation of force among the targets in weakest-link network j renders all simultaneous attacks among the targets in weakest-link network j equivalent to an attack on only $\bar{i} = \arg\max_{i \in N_j}\{P_D^i(x_A^i)\}$. Thus, in equilibrium the attacker allocates a strictly positive level of force to at most one target in each weakest-link network. The proof for part (c) of Lemma 5 follows from a symmetric argument.

Returning to the proof of case (ii) of part (a) of Lemma 5, from part (b) of Lemma 5, player A attacks at most one target in any weakest-link network, and the probability that player A wins weakest-link network j with an allocation of $x_A^{i'} > 0$ and $x_A^i = 0 \; \forall i \in N_j - i'$, is

$$Pr(\iota_j = 1 | P_D^{N_j}, \mathbf{x}_A^j) = P_D^{i'}(x_A^{i'}). \tag{15}$$

If player D uses the strategy $\hat{P}_D$, then for each $\psi \in \mathcal{P}(j \in \mathcal{W} | \mathbf{x}_A^j \neq \mathbf{0}) - \emptyset$ it follows from (15) that the probability that player A wins every weakest-link network $j \in \psi$ is

$$Pr(\iota_j = 1 \; \forall \; j \in \psi | \hat{P}_D, \mathbf{x}_A) = \min_{i \in \cup_{j \in \psi} N_j | x_A^i > 0} \{P_D^i(x_A^i)\}. \tag{16}$$

Inserting (15) and (16) into (9) a straightforward proof by induction shows that for each point $\mathbf{x}_A \in \Omega_A$ satisfying the conditions of case (ii)

$$Pr\left(\max\left(\left\{\iota_j^B\right\}_{j \in \mathcal{B}}, \left\{\iota_j^W\right\}_{j \in \mathcal{W}}\right) = 1 \middle| \hat{P}_D, \mathbf{x}_A\right) = \max_{i \in \cup_{j \in \mathcal{W}} N_j | x_A^i > 0} \{P_D^i(x_A^i)\}. \tag{17}$$

From (17) it is clear that an argument similar to that used to establish case (i) applies. This completes the proof of case (ii). The proof of case (iii) follows along similar lines. □

Lemma 6. *In any equilibrium,* $\bar{s}_W^j = \bar{s}_W^{j'} \equiv \bar{s}_W$, $\forall \; j', j'' \in \mathcal{W}$.

Proof. Following from Lemmas 1, 2 and 5, in the support of any equilibrium strategy, when player A allocates $\bar{s}_W^{j'}$ to a single target in network j' the force allocated to each of the remaining targets is 0, player A wins network j' with certainty, and player A's expected payoff is $v_A - \bar{s}_W^{j'}$.

From Lemma 3, player A's expected payoff is constant across all points in the support of P_A except for points of discontinuity of the expected payoff function. Thus, from Lemma 4 $\forall \; j', j'' \in \mathcal{W}$, $v_A - \bar{s}_W^{j'} = v_A - \bar{s}_W^{j''}$, or equivalently $\bar{s}_W^{j'} = \bar{s}_W^{j''} \equiv \bar{s}_W$. □

Lemma 7. *In any equilibrium $\{P_A, P_D\}$, there exists a $k_A \geq 0$ such that for any best-shot network j and every n_j-tuple $\mathbf{x}_A^j \in [0, \bar{s}_B^j]^{n_j}$, $P_D^{N_j}(\mathbf{x}_A^j) = \frac{k_A}{v_A} + \frac{\sum_{i \in N_j} x_A^i}{v_A}$.*

Proof. From Lemma 5 part (c) in the support of any optimal strategy player D allocates a strictly positive level of force to at most one target in network j, and thus the support of player D's n_j-variate marginal distribution for network j, $P_D^{N_j}$, is located on the axes in $\mathbb{R}_+^{n_j}$. Because from Lemma 4 each of player D's univariate marginals randomizes continuously over the interval $(0, \bar{s}_B^j]$, there are no mass points in the support of player D's n_j-variate marginal distribution for network j, $P_D^{N_j}$, except for possibly at the origin in $\mathbb{R}_+^{n_j}$.

From Lemma 5 part (a) in the support of any equilibrium strategy player A attacks at most one network. In the event that player A attacks a best-shot network j, Lemmas 3 and 4 show that there exists a $k_A \geq 0$ such that for each $\mathbf{x}_A$ in the support of P_A in which $\mathbf{x}_A^j \in (0, \bar{s}_B^j]^{n_j}$

$$Pr\left(\iota_j^B = 1 \middle| P_D^{N_j}, \mathbf{x}_A\right) = P_D^{N_j}(\mathbf{x}_A^j) = \frac{k_A}{v_A} + \frac{\sum_{i \in N_j} x_A^i}{v_A}. \tag{18}$$

Moreover, from the definition of ι_j^B it is clear that for each $\mathbf{x}_A$ in the support of any equilibrium strategy P_A such that $\mathbf{x}_A^j \neq \mathbf{0}$, it must be that $\mathbf{x}_A^j \in (0, \bar{s}_B^j]^{n_j}$. Otherwise, player A could increase his payoff by setting $\mathbf{x}_A^j = \mathbf{0}$.

The proof that follows shows that the second inequality in equation (18) holds not only for each $\mathbf{x}_A$ in the support of P_A such that $\mathbf{x}_A^j \in (0, \bar{s}_B^j]^{n_j}$, but for all n_j-tuples $\mathbf{x}^\mathbf{j} \in [0, \bar{s}_B^j]^{n_j}$.

Consider an arbitrary point $\mathbf{x}_A \in \text{Supp}(P_A)$ in which $x_A^{i'} \in (0, \bar{s}_B^j)$ for $i' \in N_j$. Because $\mathbf{x}_A \in \text{Supp}(P_A)$ and $\mathbf{x}_A^j \neq \mathbf{0}$, it must be that $\mathbf{x}_A^j \in (0, \bar{s}_B^j]^{n_j}$. Thus, equation (18) holds. From Lemma 4, there exists an $\epsilon^{i'} > 0$ such that $(x_A^{i'} + \epsilon^{i'}) \in (0, \bar{s}_B^j]$. Furthermore, there exists a point $\tilde{\mathbf{x}}_A \in \text{Supp}(P_A)$ such that $\tilde{x}_A^{i'} = (x_A^{i'} + \epsilon^{i'})$. Similarly, for each $i \in N_j$ such that $i \neq i'$ define ϵ^i as $\epsilon^i = \tilde{x}_A^i - x_A^i$.

Because from Lemma 5 part (a) player A attacks at most one network and in both $\mathbf{x}_A$ and $\tilde{\mathbf{x}}_A$ player A attacks network j, we know that for each $i \notin N_j$, $\tilde{x}_A^i = x_A^i = 0$, and we can restrict our focus to player D's n_j-variate marginal distribution for best-shot network j, $P_D^{N_j}$. Recall that for any $\mathbf{x}^j \in \mathbb{R}_+^{n_j}$, $P_D^{N_j}(\mathbf{x}^j)$ is equal to the $P_D^{N_j}$-volume of the n_j-box $[\mathbf{0}, \mathbf{x}^j]$.

Let $\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}^j)$ denote the first-order differences of the function $P_D^{N_j}$ as follows:

$$\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}^j) = P_D^{N_j}(x^1, \ldots, x^{i-1}, \tilde{x}_A^i, x^{i+1}, \ldots, x^{n_j}) - P_D^{N_j}(x^1, \ldots, x^{i-1}, x_A^i, x^{i+1}, \ldots, x^{n_j}). \tag{19}$$

Because the support of $P_D^{N_j}$ is located on the axes in $\mathbb{R}_+^{n_j}$, the expression $\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j)$ is the measure of the support of $P_D^{N_j}$ over the interval $(x_A^i, \tilde{x}_A^i)$ on the ith axis.[10] Note that the difference in (19) involves one point in the support of P_A, $(x^1, \ldots, x^{i-1}, x_A^i, x^{i+1}, \ldots, x^{n_j})$, and one point, $(x^1, \ldots, x^{i-1}, \tilde{x}_A^i, x^{i+1}, \ldots, x^{n_j}) \in (0, \bar{s}_B^j]^{n_j}$, that may or may not be in the support of P_A. Because the expected payoff from the n_j-tuple $(x^1, \ldots, x^{i-1}, \tilde{x}_A^i, x^{i+1}, \ldots, x^{n_j})$ must be less than or equal to the equilibrium expected payoff and from Lemma 4 the first equality in equation 18 holds at this point we know that

$$\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j) \leq \frac{\epsilon^i}{v_A}. \tag{20}$$

Because the support of $P_D^{N_j}$ is located on the axes in $\mathbb{R}_+^{n_j}$, we also know that

$$P_D^{N_j}(\tilde{\mathbf{x}}_A^j) = P_D^{N_j}(\mathbf{x}_A^j) + \sum_{i \in N_j} \Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j). \tag{21}$$

That is, the $P_D^{N_j}$-volume of the n_j-box $[\mathbf{0}, \tilde{\mathbf{x}}_A^j]$ is equal to the $P_D^{N_j}$-volume of the n_j-box $[\mathbf{0}, \mathbf{x}_A^j]$ plus the measure of the support of $P_D^{N_j}$ over the interval $(x_A^i, \tilde{x}_A^i)$ on each of the $i \in N_j$ axes, where the caveat in footnote 10 applies.

Because both $\mathbf{x}_A$ and $\tilde{\mathbf{x}}_A$ are contained in the support of P_A and $\mathbf{x}_A, \tilde{\mathbf{x}}_A \in (0, \bar{s}_B^j]^{n_j}$ it

[10] This interval is for the case that $x_A^i \leq \tilde{x}_A^i$, or equivalently $\epsilon^i \geq 0$, for all $i \in N_j$. If $x_A^i > \tilde{x}_A^i$ for one or more $i \in N_j$, then $\Delta_{x_A^i}^{\tilde{x}_A^i} P_D^{N_j}(\mathbf{x}_A^j)$ should be replaced with $\Delta_{\min\{x_A^i, \tilde{x}_A^i\}}^{\max\{x_A^i, \tilde{x}_A^i\}} P_D^{N_j}(\mathbf{x}_A^j)$ and the relevant interval is $(\min\{x_A^i, \tilde{x}_A^i\}, \max\{x_A^i, \tilde{x}_A^i\})$.

follows from equation (18), Lemma 1, and Lemma 2 that

$$P_D^{N_j}(\tilde{\mathbf{x}}_A^j) - P_D^{N_j}(\mathbf{x}_A^j) = \sum_{i \in N_j} \frac{\epsilon^i}{v_A} \tag{22}$$

Combining equations (21) and (22) it follows that for each $i \in N_j$ equation (20) holds with equality. That is the measure of the support of $P_D^{N_j}$ over the interval $(x_A^i, \tilde{x}_A^i)$ on the ith axis is equal to ϵ^i/v_A.

Given that the points $\mathbf{x}_A$ and $\tilde{\mathbf{x}}_A$ were arbitrarily chosen from the support of P_A and that there are no mass points in the support of player D's n_j-variate marginal distribution for network j, $P_D^{N_j}$, except for possibly at the origin, it follows directly that the measure of the support of $P_D^{N_j}$ over any interval $[a, b] \subset (0, \bar{s}_B^j]$ on the ith axis is equal to $(b - a)/v_A$. Furthermore, player D must place a mass point of size k_A/v_A at the point $\mathbf{x}^j = \mathbf{0}$, and from (18), Lemma 1, and Lemma 2, $k_A = v_A - n_j\bar{s}_B^j \geq 0$. This concludes the proof of Lemma 7. □

Lemma 8. *In any equilibrium,* $\bar{s}_W = n_j\bar{s}_B^j$, $\forall\ j \in \mathcal{B}$.

Proof. From the combination of Lemma 3, Lemma 4, Lemma 5 parts (a) and (b) and Lemma 6, for an attack of $x_A^i \in (0, \bar{s}_W]$ on any weakest-link target i player A's expected payoff is $v_A - \bar{s}_W$. Conversely, from Lemma 7 it follows that within any best-shot network j player A's expected payoff is constant not only for those points in the support of P_A which attack network j, but for all n_j-tuples $\mathbf{x}_A^j \in (0, \bar{s}_B^j]^{n_j}$. If we consider the n_j-tuple consisting of $\bar{s}_B^j$ for each of the n_j elements, then we see that player A's expected payoff from any attack on a best-shot network j is $v_A - n_j\bar{s}_B^j$.

From Lemma 3, player A's expected payoff is constant across all points in the support of P_A, except possibly at points of discontinuity of the expected payoff function. Thus, $\forall\ j \in \mathcal{B}$, $v_A - \bar{s}_W = v_A - n_j\bar{s}_B^j$ or equivalently $\bar{s}_W = n_j\bar{s}_B^j$. □

Lemma 9. *In any equilibrium,* $\bar{s}_W = \min\{v_A, v_D/[\sum_{j\in\mathcal{W}} n_j + \sum_{j\in\mathcal{B}}(1/n_j)]\}$.

Proof. If player D allocates: (i) $\bar{s}_W$ to each target in each weakest-link network, (ii) $\bar{s}_B^j$ to exactly one target in each best-shot network j, and (iii) 0 to each of the remaining targets in the best-shot networks, then from Lemmas 4, 6, and 8 player D wins all networks with certainty and has an expected payoff of $v_D - \sum_{j \in \mathcal{W}} n_j \bar{s}_W + \sum_{j \in \mathcal{B}} (\bar{s}_W / n_j)$. Similarly, if player A allocates $\bar{s}_W$ to a single weakest-link target, then from Lemmas 4 and 6, player A wins the weakest-link network containing that target with certainty, and player A's expected payoff is $v_A - \bar{s}_W$.

If $v_D - \bar{s}_W [\sum_{j \in \mathcal{W}} n_j + \sum_{j \in \mathcal{B}} (1/n_j)] > 0$, then in any equilibrium $\{P_A, P_D\}$ player D must necessarily have a strictly positive expected payoff. As a result, for each $\mathbf{x}_D \in \text{Supp}\{P_D\}$, except for possibly at points of discontinuity of his expected payoff function, player D must simultaneously win all of the networks with a probability that is bounded away from zero. This, combined with part (a) of Lemma 5, Lemma 7, and the fact that in equilibrium at most one player abstains from allocating strictly positive forces to a network with positive probability, implies that in each best-shot network j player D's mixed strategy does not place an atom on the n_j-tuple $\mathbf{x}_D^j = \mathbf{0}$. Recalling from the proof of Lemma 7 that in each best-shot network j player D places an atom of size $(v_A - n_j \bar{s}_B^j)/v_A$ on the n_j-tuple $\mathbf{x}_D^j = \mathbf{0}$, it follows from Lemma 8 that $v_A - \bar{s}_W = 0$.

Next, note that if $v_A - \bar{s}_W > 0$, then in any equilibrium $\{P_A, P_D\}$ player A must necessarily have a strictly positive expected payoff, and a similar argument establishes that in each best-shot network j player D's mixed strategy *does* place an atom on the n_j-tuple $\mathbf{x}_D^j = \mathbf{0}$. But, if with strictly positive probability, player D abstains from allocating a strictly positive level of force to best-shot network j, then player D's expected payoff is necessarily 0 and $v_D - \bar{s}_W [\sum_{j \in \mathcal{W}} n_j + \sum_{j \in \mathcal{B}} (1/n_j)] \leq 0$. To conclude the proof, since player D would never choose to set $\bar{s}_W$ such that $v_D - \bar{s}_W [\sum_{j \in \mathcal{W}} n_j + \sum_{j \in \mathcal{B}} (1/n_j)] < 0$, player A has no incentive to choose a strategy with such a $\bar{s}_W$. It follows that, $\bar{s}_W = \min\{v_A, v_D / [\sum_{j \in \mathcal{W}} n_j + \sum_{j \in \mathcal{B}} (1/n_j)]\}$. $\square$

Lemma 10. *There exists a unique set of equilibrium univariate marginal distributions*

$\{P_A^i, P_D^i\}_{i=1}^n$.

Proof. This proof is for the uniqueness of player D's set of univariate marginal distributions. The proof for player A is analogous. For each best-shot network $j \in \mathcal{B}$, Lemmas 7 and 8 show that for any $\mathbf{x_A^j} \in [0, \bar{s}_B^j]^{n_j}$, $P_D^{N_j}(\mathbf{x}_A^j) = \frac{v_A - \bar{s}_W}{v_A} + \frac{\sum_{i \in N_j} x_A^i}{v_A}$, where from Lemma 9 $\bar{s}_W = \min\{v_A, v_D/[\sum_{j \in \mathcal{W}} n_j + \sum_{j \in \mathcal{B}}(1/n_j)]\}$ and from Lemma 8 $\bar{s}_B^j = \frac{\bar{s}_W}{n_j}$. Thus, in each best-shot network j player D's unique univariate marginal distributions follow from player D's unique n_j-variate marginal distribution for network j.

From Lemma 5 parts (a) and (b), player A attacks at most one target in one weakest-link network. From Lemmas 2, 3, and 4 it follows that for each target i in each weakest-link network $j \in \mathcal{W}$,

$$v_A P_D^i\left(x_A^i\right) - x_A^i = v_A - \bar{s}_W$$

for $x_A^i \in (0, \bar{s}_W]$. Thus, player D's univariate marginal distributions are uniquely determined in each weakest-link network. □

Next, note that because success for player D involves simultaneously defending all networks from attack and that for each network at most one player abstains from allocating a positive level of force to the network, it follows that if with positive probability player D abstains from allocating strictly positive forces to any network then with positive probability player D optimally abstains from allocating strictly positive forces to all networks. Otherwise, player D could increase his expected payoff at such points by allocating zero forces to all networks. Combining this fact with Lemma 10, the next two lemmas follow directly. Recall that $\alpha = v_D/(v_A[\sum_{j \in \mathcal{W}} n_j + \sum_{j \in \mathcal{B}} \frac{1}{n_j}])$.

Lemma 11. *If $\alpha \geq 1$, then in any equilibrium: (i) player A places mass $1 - (1/\alpha)$ at the origin, (ii) player A's expected payoff is 0, (iii) player D does not place positive mass at the origin, and (iv) player D's expected payoff is $v_D - (v_D/\alpha)$.*

Lemma 12. *If $\alpha < 1$, then in any equilibrium: (i) player D places mass $1 - \alpha$ at the origin, (ii) player D's expected payoff is 0, (iii) player A does not place positive mass at the origin, and (iv) player A's expected payoff is $v_A - v_A\alpha$.*

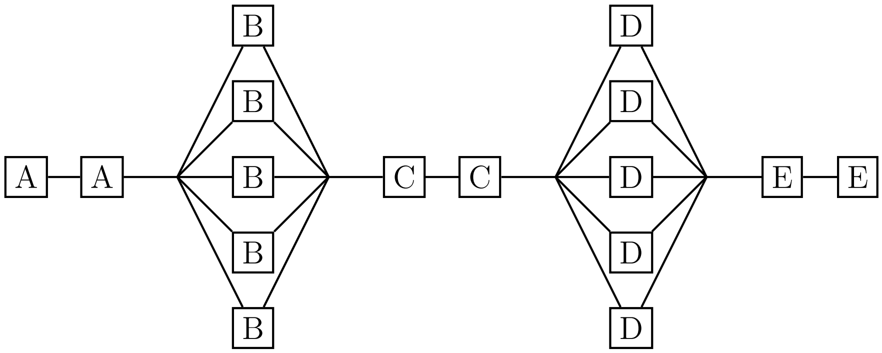

Figure 1: Example Supra-Network with Five Networks (A, B, C, D, and E)

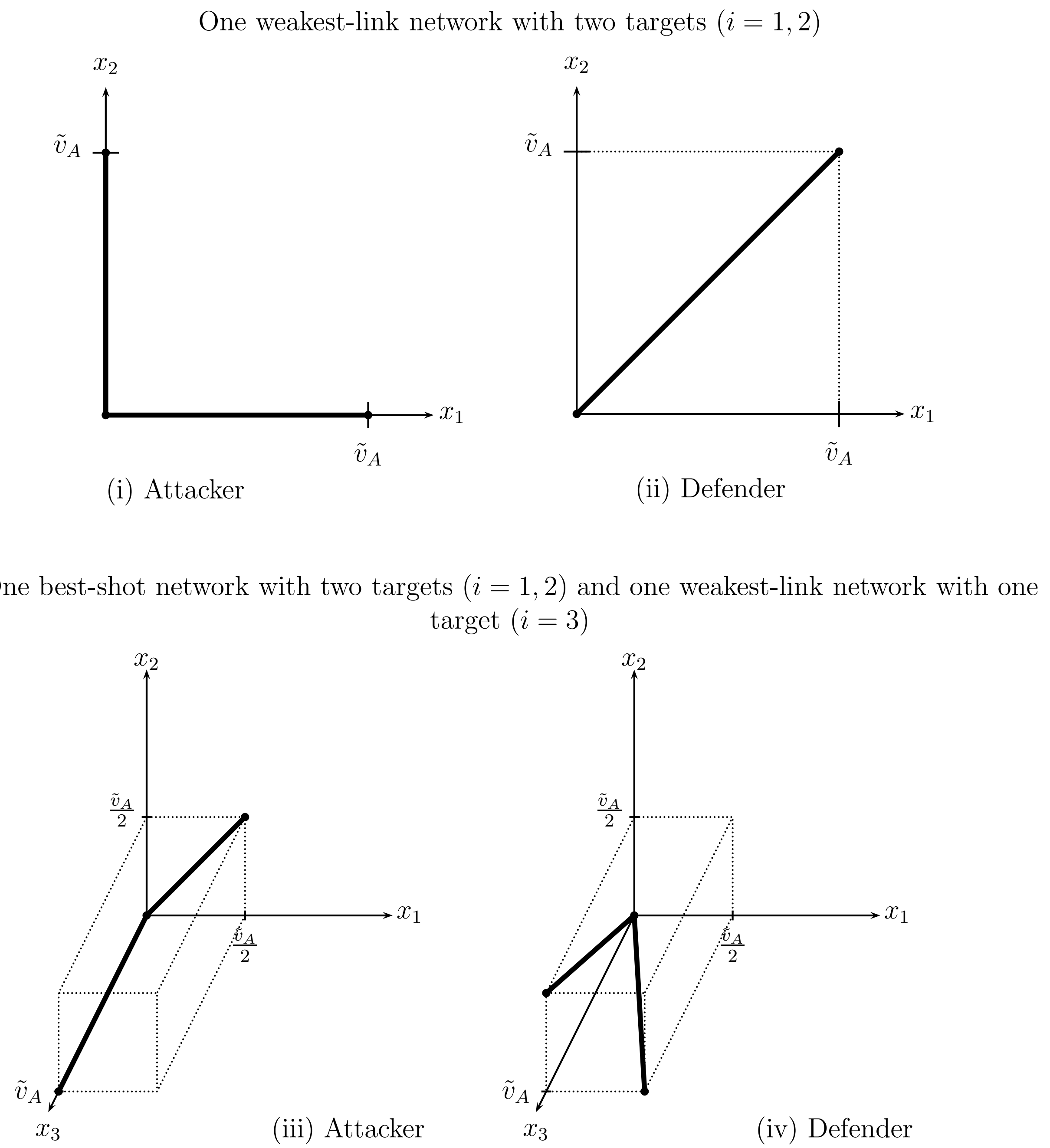

Figure 2: Supports of the equilibrium joint distributions stated in Theorem 1 ($\tilde{v}_A = \min\{\alpha v_A, v_A\}$).