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Spatial Equilibrium in a State Space Approach to Demand Uncertainty

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Abstract

Firms are likely to be uncertain about consumer preferences when launching products. The existing literature models preference uncertainty as an additive shock to the consumer distribution in a characteristic space model. The additive shock only shifts the mean of the consumers' ideal points. We generalize this approach to a state space model in which a vector of parameters can give rise to different distributions of consumer tastes in different states, allowing other moments of the consumer density to be uncertain. An equilibrium existence result is given. In the case of symmetric distributions, the unique subgame-perfect equilibrium can be described by a simple closed-form solution.

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JEL Classification Numbers: C72, D43, D81, L10, L13, R30, R39.

1 Introduction

The Hotelling model of spatial competition can be thought of as one of the earliest problems in information economics: firms make strategic choices (locations/characteristics and prices) without knowing an individual consumer's type (location). However, in typical formulations, the distribution of consumer types is common knowledge giving rise to the familiar mill price equilibrium. Since individual consumer types are not observed in the standard model the assumption that the distribution of types is observable seems internally inconsistent. Furthermore casual empiricism indicates that firms are often less than perfectly informed about consumer tastes and frequently exert considerable effort to generate market research data.[1] The recent demand location uncertainty

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[1]Data may be generated externally by market research survey, stated choice experiments or internally from customer behavior databases or from staff interaction with customers.

literature recasts the Hotelling model in a more general and realistic setting in which firms are uncertain not only about the type of an individual but are also uncertain about the consumer distribution itself.

The existing approaches to preference/demand location uncertainty focus on an additive shock to consumer types. Early approaches to characteristic preference uncertainty, such as De Palma et al. (1985), used the law of large numbers to generate certain demand functions from individual preferences with idiosyncratic shocks. More recently the demand location uncertainty literature (Jovanovic (1981), Harter (1996), Casado-Izaga (2000) and Meagher and Zauner (2004, 2005)) has analyzed situations with perfectly correlated shocks which give rise to residual aggregate uncertainty.

Thus the existing approaches generate no uncertainty at the distributional level or uncertainty only about the mean of preferences. The contributions of this paper are twofold: first we show by way of an example how even simple situations fall beyond the scope of the existing 'additive-shock' approach. Secondly we show how a judicious state space formulation gives an existence theorem for very general forms of uncertainty. For example the distribution of tastes could change shape across states and need not be from a fixed distributional type such as the uniform or Normal. In the case of symmetric distributions this approach gives simple closed form equations for equilibrium prices and locations.

By way of a specific example, in Section 3 we consider a Hotelling style linear city but with the population distributed according to a linear distribution. In addition to the standard characteristic space interpretation this representation also has a geographic interpretation as a coastal city such as New York or Sydney. Two firms, when deciding where to locate, are certain where the coastal boundary is but are unsure how spread out the city will become by the time they build their facilities. In this coastal city example both the mean and the variance of the consumer distribution are uncertain. This issue of uncertainty beyond the mean has not previously been considered in the literature. We briefly lay out a model below to show how this class of problem can be solved by extending location theory. We solve the more general case in Section 4 following our specific uncertainty example in Section 3.

2 Common Features of the Model

We present a model of aggregate taste uncertainty in the spirit of Meagher and Zauner (2004). Both firms and consumers are located at points on the real line $\mathbf{R}$. Each consumer demands either one or zero units of the good and has sufficient income to buy one unit of the good. For a consumer with "ideal" point x, the indirect utility function for consuming firm i's product (located at x_i) at price p_i is given by

$$V(x, x_i, p_i) = A - p_i - \tau(x_i - x)^2, \ \tau > 0. \tag{1}$$

There are two firms, $i = 1, 2$. The marginal cost of production of each firm is constant and normalized to 0. Firms are uncertain about the distribution of consumers but make decisions on the basis of a common prior, which we describe below. Firms choose locations $x_i \in \mathbf{R}$ ($i = 1, 2$ and, without loss of generality, $x_1 \leq x_2$) simultaneously, observe the location of their competitor, then choose prices simultaneously and finally the uncertainty is resolved.

This timing implies that firms are slow in learning about their demand conditions, relative to the product's life cycle, due perhaps to a very short life cycle or to the difficulty or expense of conducting experimentation. Whatever the reason, the limit case of slow learning is when a firm chooses a price under uncertainty and is stuck with that price for the life of the product.

Consumers buy from the firm that gives them the highest (net) utility, hence there exists a unique point ξ, satisfying $V(\xi, x_1, p_1) = V(\xi, x_2, p_2)$, where consumers are indifferent between buying from firm 1 or firm 2. States of the world are indexed by S, which might be a vector, with density $f(S)$. In each state of the world the distribution of consumer locations, x, is given by $g_S(x)$.

3 Ice Cream Sellers on an Asymmetric Beach

Consider a variant of the ice cream sellers story so popularly used to motivate the Hotelling model. Two ice cream sellers must choose locations on a beach of length 1, represented by the unit interval, and post prices before their customers arrive for the day. Furthermore assume this is a beach with a car park at the left end so all consumers enter from the same end and then walk a random distance to the right, determined jointly by their dislike of walking and crowds.

This scenario could plausibly yield an asymmetric density of consumers on the interval $[0, \alpha]$, $0 < \alpha \leq 1$. For simplicity we assume consumers are distributed according to the linear density:

$$g_\alpha(x) = \begin{cases} \frac{2}{\alpha} - \frac{2x}{\alpha^2} & \text{if } 0 \leq x < \alpha \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, since the ice cream sellers are uncertain who will come to the beach they are therefore uncertain of which (linear) density of consumers will occur, which is represented here by a common prior over α, denoted $f(\alpha)$ on $[0, 1]$. For simplicity and to yield explicit solutions we choose f to be the power density of order 2, i.e. $f(\alpha) = 3\alpha^2$.

Assuming firms are risk neutral the expected profits for firm 1 are

$$\begin{aligned} E[\Pi_1(p_1, p_2, x_1, x_2)] &= \int_0^1 \int_0^\xi p_1 g_\alpha(x) f(\alpha) dx d\alpha & (2) \\ &= \int_0^\xi \left(p_1 \int_0^x 0 d\alpha + p_1 \int_x^1 \left(\frac{2}{\alpha} - \frac{2x}{\alpha^2} \right) 3\alpha^2 d\alpha \right) dx & (3) \\ &= p_1 \left[x^3 - 3x^2 + 3x \right]_0^\xi & (4) \\ &= p_1(\xi^3 - 3\xi^2 + 3\xi) & (5) \end{aligned}$$

and similarly for firm 2

$$
\begin{aligned}
E[\Pi_2(p_2, p_2, x_1, x_2)] &= p_2 \left[x^3 - 3x^2 + 3x\right]_\xi^1 && (6) \\
&= p_2(1 - (\xi^3 - 3\xi^2 + 3\xi)) && (7)
\end{aligned}
$$

This problem is quite different from those considered previously in the literature in that clearly more than the mean of the consumer distribution varies with the uncertainty. As the following proposition shows the judicious choice of functional forms made in specifying this problem does in fact give rise to a unique equilibrium.

Proposition 1 *The ice cream sellers on an asymmetric beach problem has a unique subgame perfect location-then-price equilibrium.*

Proof: See appendix.

The obvious way to establish this proposition is by working directly with the specified payoff functions and their derivatives. Naturally one wonders if there are more general economic forces at work and if a more general result is possible. As the next section shows the appropriate formulation of the problem yields a very general existence result.

4 General State Space Results

Consider the more general setting in which the distribution of consumers over locations, x, conditional on the value of a vector of parameters M is given by $g(x|M)$. The marginal density of M, that is, the uncertainty distribution, is given by $f(M)$ with support $\mathcal{S}$. Without loss of generality assume $E[M] = 0$.

We analyze the pure-strategy sub-game-perfect Nash equilibria of this game. Given the above assumptions the distribution of consumers, $h(x)$, is given by:

$$
h(x) = \int_{\mathcal{S}} g(x|M) f(M) dM. \tag{8}
$$

For the following proposition it is useful to define

$$
J(x) \equiv \frac{H(x)(1 - H(x))}{h(x)}, \tag{9}
$$

where $H(\cdot)$ is the distribution function associated with the density function $h(\cdot)$.

Proposition 2 *Assume that the distribution of consumers $h(\cdot)$ is log concave with mean 0 and support $[a, b]$. If $J(x)$ is strictly pseudo-concave and $\lim_{x\to a} J(x) = \lim_{x\to b} J(x)$ then there exists a unique subgame perfect location-then-price equilibrium.*

Proof: The proof establishes the equivalence of our game to another spatial game with a known solution.

Prices are state independent thus for a fixed M firm i's profit, π_i, is $\pi_i = p_i Q_i,\ i = 1, 2$, where Q_i is the demand for firm i. Since firms are risk neutral their payoffs are given by the expectation over M of the state contingent profits. Locations are also state independent implying ξ is independent of M. Hence, we have

$$\begin{aligned} E_M[Q_1(p_1, p_2, x_1, x_2, M)] &= \int_{\mathcal{S}} \int_{-\infty}^{\xi(p_1,p_2,x_1,x_2)} g(z|M) f(M) dz dM \quad (10) \\ &= \int_{-\infty}^{\xi(p_1,p_2,x_1,x_2)} \int_{\mathcal{S}} g(z|M) f(M) dM dz \quad (11) \\ &= \int_{-\infty}^{\xi(p_1,p_2,x_1,x_2)} h(z) dz, \quad (12) \end{aligned}$$

and a similar expression for firm 2. Thus $E_M[\pi_1] = p_1 H(\xi)$ and $E_M[\pi_2] = p_2(1 - H(\xi))$.

Thus the expected payoffs and strategies for each firm are the same as a standard certainty location game with a consumer distribution given by h. Since the two games are equivalent they will have the same equilibrium which is seen to be unique by a direct application of Anderson et al. (1997, Proposition 2). □

Corollary 1 *If, in addition to the conditions of Proposition 2, $h(\cdot)$ is symmetric then the unique sub-game perfect location-then-price equilibrium is:*

$$-x_1^* = x_2^* = \frac{3}{4h(0)}, \quad (13)$$

$$p_1^* = p_2^* = \frac{3\tau}{2h(0)^2}. \quad (14)$$

Proof: Proposition 2 and Anderson et al. (1997, Corollary 1). □

If the density of consumers $h(x)$ is symmetric and neither too concave nor too convex there is a unique equilibrium in which the prices and the locations depend only upon the density of the distribution at its mean. The standard spatial competition model uses a continuum of consumers to aggregate out individual level uncertainty about consumer locations from firm profit functions. Under our state space approach firms know even less: firms must still make their strategic decisions without knowing the locations of individual consumers but in addition they are uncertain about the distribution of consumers as well. However, because all decisions are made prior to the resolution of either source of uncertainty all that matters from the perspective of a firm is the combined effect of both forms of uncertainty on the density of consumers (at the marginal location).

Appendix

Proof of Proposition 1

From Proposition 2 it suffices to that h is log concave and that J is strictly pseudo concave with symmetric limits for its tails. First calculating h:

$$\begin{aligned} h(x) &= \int_x^1 p_1 \left(\frac{2}{\alpha} - \frac{2x}{\alpha^2} \right) 3\alpha^2 d\alpha & (15) \\ &= 3x^2 - 6x + 3 & (16) \end{aligned}$$

and

$$H(x) = x^3 + 3x^2 + 3x \tag{17}$$

Hence

$$\frac{\partial^2 ln(3x^2 - 6x + 3)}{\partial x^2} = -\frac{2}{(x-1)^2}. \tag{18}$$

Which establishes log concavity.

Now for the ice cream sellers problem it can be shown that

$$\begin{aligned} J(x) &= \frac{H(x)(1 - H(x))}{h(x)} & (19) \\ &= \frac{(x^3 + 3x^2 + 3x)(1 - (x^3 + 3x^2 + 3x))}{3x^2 - 6x + 3} & (20) \\ &= -x(x-1)(x^2 - 3x + 3)/3 & (21) \end{aligned}$$

Since this function is continuos the limits can be found by simple substitution

$$J(0) = 0 = J(1). \tag{22}$$

Finally strict pseudo concavity is easily established since the J function in this case is in fact concave:

$$\frac{\partial^2(-x(x-1)(x^2 - 3x + 3)/3)}{\partial x^2} = -4(x-1)^2 < 0 \tag{23}$$

QED

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