DOCTORAL DISSERTATION

TWO CONSTRUCTIONS OF CONTINUA: INVERSE LIMITS AND COMPACTIFICATIONS

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Tina Sovič
Advisor: Assoc. Prof. Dr. Iztok Banič
Co-advisor: Assoc. Prof. Dr. Christopher Mouron
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DVE KONSTRUKCIJI KONTINUUMOV: INVERZNE LIMITE IN KOMP AKTIFIKACIJE

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Tina Sovič
Mentor: izr. prof. dr. Iztok Banič
Somentor: izr. prof. dr. Christopher Mouron
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In the thesis we talk about two different constructions of continua. First we present the generalized inverse limits, with help of which we construct Ważewski’s universal dendrite. What follows is a description of the compactifications of a ray and the presentation of results about their span.

The first chapter will be an introduction to the continuum theory through interesting examples, as \( \sin \frac{1}{x} \)-continuum, Hilbert cube, Brouwer-Janiszewski-Knaster continuum and pseudoarc. We will present some of their properties, among which irreducibility, smoothness and span zero are the most important ones for us.

In the continuation we intend to present some various constructions of continua. The main focus will be on the generalized inverse limits and compactifications of rays, which will also be a central part of the thesis. In this chapter, we also study inverse limits in the category \( \mathcal{CHU} \) of compact Hausdorff spaces with upper semi-continuous functions. We show that the inverse limits with upper semi-continuous bonding functions, together with the projections are weak inverse limits in this category.

The following two are the most important chapters in the thesis. The first is a detailed description of a construction of the family of upper semi-continuous functions \( f : [0, 1] \rightarrow 2^{[0,1]} \), such that the inverse limit of the inverse sequence of unit intervals \( [0, 1] \) and \( f \), as the only bonding function, is homeomorphic to Ważewski’s universal dendrite for each of it. Among other results we will also give a complete characterization of comb-functions, for which the inverse limits of the type described above are dendrites.

The next important chapter will be about compactifications of rays. In the first
part of this chapter we will use compactifications to prove that for each continuum $Y$ there is an irreducible smooth continuum that contains a topological copy of $Y$. The second part presents the main results of this chapter; i.e. the span of a compactification of a ray with a remainder that has a span zero is also zero. In the proofs of this chapter we will help ourselves with a discretization of span.


**Key words:** continua, inverse limit, inverse sequence, upper semi-continuous function, set-valued functions, bonding function, hyperspace, dendrite, universal dendrite, category, compactification, compactification of a ray, smooth continua, irreducible continua, span, span zero
Povzetek

V disertaciji bomo podrobneje opisali dve različni konstrukciji kontinuumov. Najprej bomo predstavili posplošne inverzne limite s pomočjo katerih bomo konstruirali t.i. univerzalni dendrit Ważewskega. Temu bo sledil opis kompaktifikacij žarkov in predstavitev rezultatov o njihovem razponu.

V uvodnem poglavju bomo predstavili kuntinuum z najzanimivejšimi primeri, kot so sin 1/2 kontinuum, Hilbertova kocka, Knasterjev kontinuum in psevdolok. Predstavili bomo hiperprostore in nekatere izmed lastnosti kontinuumov, kot so: ireducibilnost, gladkost in ničelni razpon. Najpomembnejše definicije in izreki prvega poglavja so:

Definicija 1 Kontinuum je neprazen, kompakten in povezan metrični prostor.

Definicija 2 Naj bo $X$ kompakten metrični prostor in naj bo $d$ metrika na $X$.

- $2^X = \{ A \mid A$ je neprazna zaprta podmnožica od $X}\$
- $C(X) = \{ A \in 2^X \mid A$ je povezana} \$
- $N_d(\varepsilon, A) = \{ x \in X \mid d(x, a) < \varepsilon$ za nek $a \in A\}$

Definicija 3 Za poljubni množici $A, B \in 2^X$ definiramo

$$d_H(A, B) = \inf \{ \varepsilon > 0 \mid A \subseteq N_d(\varepsilon, B) \text{ in } B \subseteq N_d(\varepsilon, A) \}.$$  

Preslikavo $d_H : 2^X \times 2^X \rightarrow \mathbb{R}$ imenujemo Hausdorffova metrika. Prostor $2^X$ opremljen z metriko $d_H$ imenujemo hiperprostor prostora $X$. 
Definicija 4 Kontinuum $X$ je ireducibilen, če obstajata točki $x, y \in X$, tako da za vsak pravi podkontinuum $Y$ od $X$ velja, da $\{x, y\}$ ni prava podmnožica od $Y$. V tem primeru rečemo, da je $X$ ireducibilen med $x$ in $y$ ter ga označimo z $xy$.

Definicija 5 Kontinuum $X$ je gladek v točki $p \in X$, če velja:

1. za poljubno točko $x \in X$ obstaja enolično določen podkontinuum $px$, ki je ireducibilen med $p$ in $x$.

2. za poljubno zaporedje točk $x_n \in X$, ki konvergira k točki $x \in X$, zaporedje ireducibilnih kontinuumov $px_n$ konvergira h kontinuumu $px$.

Kontinuum $X$ je gladek, če obstaja točka $p \in X$, v kateri je gladek.

Definicija 6 Naj bo $X$ kontinuum in $Z$ podkontinuum od $X \times X$. Število

$$\sigma(X) = \sup_{Z} \left\{ \inf_{(x_1, x_2) \in Z} d(x_1, x_2) \mid \pi_1(Z) = \pi_2(Z) \right\}$$

imenujemo razpon kontinuuma $X$.

V nadaljevanju bomo predstavili nekaj različnih konstrukcij kontinuumov. Poudarek bo na posplošenih inverznih limitah in na kompaktifikacijah žarkov, ki predstavljajo osrednjo vlogo disertacije. V tem poglavju bomo obravnavali tudi inverzne limite v kategoriji $\mathcal{CHU}$ kompaktnih Hausdorffovih prostorov skupaj z navzgor polzveznimi preslikavami. Pokazali bomo, da so posplošene inverzne limite z navzgor polzveznimi veznimi preslikavami, skupaj s projekcijami šibke inverzne limite v tej kategoriji. Pomembne definicije in izreki tega poglavja so:

Definicija 7 Funkcija $f : X \to 2^Y$ je navzgor polzvezna preslikava, če za vsak $x \in X$ in za vsako odprto množico $U \subseteq Y$, kjer je $f(x) \subseteq U$, obstaja odprta množica $V \in X$, za katero velja

1. $x \in V$,

2. za vse $v \in V$ velja $f(v) \subseteq U$. 
Graf $\Gamma(f)$ funkcije $f : X \to 2^Y$ je množica vseh točk $(x, y) \in X \times Y$, za katere je $y \in f(x)$.

Izrek 1 Naj bosta $X$ in $Y$ kompaktna Hausdorffova prostora in $f : X \to 2^Y$. Teda je $f$ navzgor polzvezna natanko tedaj, ko je njen graf $\Gamma(f)$ zaprt v $X \times Y$.

Definicija 8 Inverzno zaporedje kompaktnih metričnih prostorov $X_k$ z navzgor polzveznimi veznimi preslikavami $f_k$ je zaporedje $\{X_k, f_k\}_{k=1}^{\infty}$, kjer je $f_k : X_{k+1} \to 2^{X_k}$ za vsak $k$.

Definicija 9 Posplošena inverzna limita inverznega zaporedja $\{X_k, f_k\}_{k=1}^{\infty}$ je podprostor produktnega prostora $\prod_{k=1}^{\infty} X_k$, definiran kot

$$\lim \downarrow \{X_k, f_k\}_{k=1}^{\infty} = \left\{(x_1, x_2, \ldots, x_k, \ldots) \in \prod_{k=1}^{\infty} X_k \mid x_k \in f_k(x_{k+1}) \text{ for each } k\right\}.$$

Definicija 10 Za direktno množico $A$, družino objektov $\{X_\alpha \mid \alpha \in A\}$ iz $K$ in družino morfizmov $\{f_\alpha\beta : X_\beta \to X_\alpha \mid \alpha, \beta \in A, \alpha \leq \beta\}$ iz $K$, takšnih da velja

1. za vsak $\alpha \in A$, $f_\alpha\alpha = 1_{X_\alpha}$,
2. za vse $\alpha, \beta, \gamma \in A$ iz $\alpha \leq \beta \leq \gamma$ sledi $f_\alpha\beta \circ f_\beta\gamma = f_\alpha\gamma$,

imenujemo inverzni sistem v $K$ in ga označimo ga z

$$(A, \{X_\alpha\}_{\alpha \in A}, \{f_\alpha\beta\}_{\alpha, \beta \in A}).$$

Definicija 11 Obekt $X \in Ob(K)$ skupaj z morfizmi $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$ je inverzna limita inverznega sistema $(A, \{X_\alpha\}_{\alpha \in A}, \{f_\alpha\beta\}_{\alpha, \beta \in A})$ v kategoriji $K$, če

1. za vse $\alpha, \beta \in A$ iz $\alpha \leq \beta$ sledi, da $p_\alpha(x) = f_\alpha\beta(p_\beta(x))$ velja za vsak $x \in X$,
2. za vsak objekt $Y \in K$ in za vsako družino morfizmov $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ velja: če je $\varphi_\alpha(y) = f_\alpha\beta(\varphi_\beta(y))$ za vsak $y \in Y$, tedaj obstaja enolično določen morfizem $\varphi : Y \to X$, tako da za vsak $\alpha \in A$ velja $\varphi_\alpha(y) = p_\alpha(\varphi(y))$ za vsak $y \in Y$. 
Definicija 12  Kategorija $\mathcal{CHU}$ kompaktnih Hausdorffovih prostorov in navzgor polzveznih preslikav je kategorija sestavljena iz naslednjih razredov objektov, morfizmov in kompozituma morfizmov:

- $\text{Ob}(K)$: kompaktni Hausdorffovi prostori;
- $\text{Mor}(K)$: navzgor polzvezne funkcije (množico vseh morfizmov iz $X \to Y$ (t.j. množico vseh navzgor polzveznih funkcij iz $X \to Y$) označimo z $\text{Mor}(\mathcal{CHU})(X, Y)$;
- $\circ$: Za vsak $f \in \text{Mor}(\mathcal{CHU})(X, Y)$ in $g \in \text{Mor}(\mathcal{CHU})(Y, Z)$ definiramo $g \circ f \in \text{Mor}(\mathcal{CHU})(X, Z)$ s predpisom:

$$\left( g \circ f \right)(x) = g(f(x)) = \bigcup_{y \in f(x)} g(y)$$

za vsak $x \in X$.

Definicija 13  Naj bo $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha \beta}\}_{\alpha, \beta \in A})$ poljuben inverzni sistem v $\mathcal{CHU}$. Objekt

$$\varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha \beta}\}_{\alpha, \beta \in A}) = \left\{ x \in \prod_{\alpha \in A} X_\alpha \mid \text{za vse } \alpha < \beta, x_\alpha \in f_{\alpha \beta}(x_\beta) \right\}$$

imenujeno inverzna limita z navzgor polzveznimi preslikavami.

Definicija 14  Objekt $X \in \text{Ob}(\mathcal{CHU})$ skupaj z morfizmi $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$ je šibka inverzna limita inverznega sistema $(A, \{X_\alpha\}, \{f_{\alpha \beta}\}_{\alpha, \beta \in A})$ v kategoriji $\mathcal{CHU}$, če

1. za vse $\alpha, \beta \in A$ iz $\alpha \leq \beta$ sledi $p_\alpha(x) \subseteq f_{\alpha \beta}(p_\beta(x))$ za vsak $x \in X$,
2. za vsak objekt $Y$ in za vsako družino morfizmov $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ velja:

če je $\varphi_\alpha(y) = \varphi_{\alpha \beta}(\varphi_\beta(y))$ za vsak $y \in Y$, tedaj za poljuben morfizem $\psi : Y \to X$, tak da za vsak $\alpha \in A$ in za vsak $y \in Y$, $p_\alpha(\psi(y)) = \varphi_\alpha(y)$, velja $\psi(y) \subseteq (\prod_{\gamma \in A} \varphi_\gamma(y)) \cap X$ za vse $y \in Y$. 
Izrek 2 Naj bo \((A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A})\) inverzni sistem v \(\text{CHU}\). Potem je inverzna limita z navzgor polzveznimi preslikavami
\[
\lim_{\leftarrow}(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A}),
\]
skupaj s projekcijami
\[
p_\gamma : \lim_{\leftarrow}(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha,\beta \in A}) \to X_\gamma,
\]
šibka inverzna limita inverznega sistema v \(\text{CHU}\).

Definicija 15 Kompaktifikacija prostora \(R\) je kompakten prostor \(X\), ki vsebuje topološko kopijo \(R'\) od \(R\) kot gosto podmnožico. Pri tem \(Y = X \setminus R'\) imenujemo ostanek kompaktifikacije. Pravimo, da je \(X\) kompaktifikacija prostora \(R\) z \(Y\).

Sledita najpomembnejši poglavji disertacije. Prvo podrobneje opisuje konstrukcijo družine navzgor polzveznih veznih preslikav \(f : [0, 1] \to 2^{[0,1]}\), takšnih, da je za vsako izmed njih inverzna limita inverznega zaporedja enotskih intervalov \([0, 1]\) in \(f\), kot edine vezne preslikave, homeomorfnja univerzalnemu dendritu Ważewskega. Poleg drugih rezultatov bomo predstavili tudi popolno karakterizacijo t.i. funkcij glavnik, katerih inverzne limite zgoraj opisanega tipa, so dendriti. Najpomembnejše definicije in izreki tega poglavja so:

Definicija 16 Dendrit je univerzalen, če vsebuje topološko kopijo poljubnega dendrita.

Izrek 3 Poljuben dendrit \(D\), katerega množica razvejiščnih točk je gosta v \(D\) in katerega vsaka razvejiščna točka je neskončnega reda v \(D\), je homeomorfen univerzalnemu dendritu Ważewskega.

Definicija 17 Naj bo \(n\) naravno število in \(\{(a_i, b_i)\}_{i=1}^n\) podmnožica od \([0, 1] \times [0, 1]\), takšna da \(0 < a_i < b_i\) za vsak \(i = 1, 2, 3, \ldots, n\) in \(a_i \neq a_j, \text{če je } i \neq j\). Teda \(f : [0, 1] \to 2^{[0,1]}\) imenujemo funkcija n-glavnik glede na \(\{(a_i, b_i)\}_{i=1}^n\), če je \(f = f_{(a_i, b_i)}\).
Rečemo tudi, da je \(f : [0, 1] \to 2^{[0,1]}\) funkcija n-glavnik, če je \(f\) funkcija n-glavnik glede na neko zaporedje \(\{(a_i, b_i)\}_{i=1}^n\).
Izrek 4  Naj bo \( n \) naravno število in naj bo \( f : [0, 1] \rightarrow 2^{[0,1]} \) poljubna funkcija \( n \)-glavnik. Teda je \( \lim_{k \rightarrow \infty}\{[0, 1], f\}_{k=1}^{\infty} \) dendrit.

Definicija 18  Naj bo \(\{(a_n, b_n)\}_{n=1}^{\infty} \) zaporedje v \([0, 1] \times [0, 1]\), takšno, da je

1. \( a_n < b_n \) za vsak \( n \in \mathbb{N} \),
2. \( a_i \neq a_j \) če je \( i \neq j \),
3. \( \lim_{n \rightarrow \infty} (b_n - a_n) = 0 \).

Teda \( f = f_{\{(a_n, b_n)\}_{n=1}^{\infty}} \) imenujemo funkcija glavnik glede na \(\{(a_n, b_n)\}_{n=1}^{\infty} \).

Rečemo tudi, da je \( f \) funkcija glavnik, če je \( f \) funkcija glavnik glede na neko zaporedje \(\{(a_n, b_n)\}_{n=1}^{\infty} \) v \([0, 1] \times [0, 1]\).

Definicija 19  Zaporedje \(\{(a_n, b_n)\}_{n=1}^{\infty} \) v \([0, 1] \times [0, 1]\) je dopustno, če za poljubno naravno število \( n \) obstaja takšno naravno število \( \mu(n) \geq n \), da za vsak \( m \geq \mu(n) \) velja, da iz \( a_m < a_n \) sledi \( b_m < a_n \).

Izrek 5  Naj bo \( f : [0, 1] \rightarrow 2^{[0,1]} \) funkcija glavnik glede na dopustno zaporedje \(\{(a_n, b_n)\}_{n=1}^{\infty} \). Teda je \( \lim_{k \rightarrow \infty}\{[0, 1], f\}_{k=1}^{\infty} \) dendrit.

Izrek 6  Naj bo \( f : [0, 1] \rightarrow 2^{[0,1]} \) poljubna funkcija glavnik glede na dopustno zaporedje \(\{(a_n, b_n)\}_{n=1}^{\infty} \). Teda je \( \lim_{k \rightarrow \infty}\{[0, 1], f\}_{k=1}^{\infty} \) homeomorfna univerzalnemu dendritu Ważewskega natanko teda, ko je množica \( \{a_n \mid n = 1, 2, 3, \ldots\} \) gosta v \([0, 1]\).

Naslednje pomembno poglavje govori o kompaktifikacijah žarka. V prvem delu poglavja bomo s pomočjo takih kompaktifikacij pokazali, da za vsak kontinuum \( Y \) obstaja ireducibilen gladek kontinuum, ki vsebuje homeomorfno kopijo \( Y \). Drugi del vsebuje osnovno tezo tega poglavja, ki pravi, da imajo vse kompaktifikacije žarka, katerih ostanek ima ničelni rapon, prav tako ničelni razpon. Dokaze tega dela poglavja bomo predstavili s pomočjo diskretizacije razpona. Osrednja izreka tega poglavja sta:

Izrek 7  Za poljuben kontinuum \( Y \) obstaja ireducibilen gladek kontinuum, ki vsebuje homeomorfno kopijo \( Y \).
Izrek 8 Naj bo $X$ kompaktifikacija žarka $[0, \infty)$ z ostankom $Y$, kjer je $\sigma(Y) = 0$. Tedaj je $\sigma(X) = 0$.

V zadnjem poglavju disertacije so predstavljeni še odprti problemi in možnosti za njihovo reševanje.


Ključne besede: kontinuum, inverzna limita, inverzno zaporedje, navzgor polzvena preslikava, večlična preslikava, vezna funkcija, hiperprostor, dendrit, univerzalni dendrit, kategorija, kompaktifikacija, kompaktifikacija žarka, gladki kontinuum, ireducibilni kontinuum, razpon, ničelni razpon

UDK: 515.126(043.3)
Prijazne besede so lahko kratke in preproste, njihov odmev pa je neskončen.
(Mati Terezija)

1.1 Examples

Continuum theory is a well studied branch of topology, while continua have two very important topological properties: compactness and connectedness. Concept of connectness appeared in 1883 when G. Cantor described the classic middle-third Cantor set to show how far perfect alone is from capturing the notion of a curve, [14]. The modern definition of connectness was given independently by N. Lennes and F. Riesz. For more details see [58] and [28].

**Definition 1.1** A topological space $X$ is said to be disconnected if it is the union of two disjoint nonempty open sets. If $X$ is not disconnected, we say that $X$ is connected.

**Remark 1.2** Let $U$ and $V$ be the sets from definition, i.e. let $U$ and $V$ be nonempty, disjoint, open in $X$ and let $U \cup V = X$. Then we say that $U$ and $V$ separate $X$.

**Definition 1.3** A topological space $X$ is totally disconnected provided that $X \neq \emptyset$ and no connected subset of $X$ contains more than one point.

The term of a compact space was introduced by M. Fréchet in 1904 to describe the property that every sequence has a convergent subsequence. The notion of a compact space in the modern sense was introduced by P. Alexandroff and P. Urysohn in 1923. See [18].
Definition 1.4 Topological space $X$ is compact, if for any open cover $\mathcal{U}$ of $X$, there exists a finite subcover.

We summarized some basic properties considering compactness and connectness in the following theorem.

**Theorem 1.5**

1. The continuous image of a connected space is again a connected space.
2. The continuous image of a compact space is again a compact space.
3. A closed subset of a compact space is a compact space.
4. A compact subspace of a Hausdorff space is closed.
5. The product of connected spaces is again a connected space.
6. The product of compact spaces is again a compact space.
7. The product of countable many metrizable spaces is again a metrizable space.
8. Let $X_n$ be totally disconnected metric space for each positive integer $n$. Then the product space $\prod_{n=1}^{\infty} X_n$ is totally disconnected metric space.

**Proof.** Proofs of all propositions above can be found in [44, p. 150, 165, 166,], [34, p. 17, 137] and [33, p. 212].

**Lemma 1.6 (The tube lemma)** Consider the product space $X \times Y$, where $Y$ is compact. If $N$ is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then $N$ contains some tube $W \times Y$, containing $x_0 \times Y$, where $W$ is a neighborhood of $x_0$ in $X$.

**Proof.** See [44, p. 169].

Most continuum theorists define a continuum as a subset of a metric space though some take a somewhat broader view and study $T_1$-spaces or Hausdorff spaces.
1.1 Examples

However, we emphasize that the term continuum always means a metric continuum. We will also restrict most of our attention to one-dimensional continua.

Finally we give a definition of continuum and state a theorem that will help us in the proof that given space is a continuum.

Definition 1.7 A continuum is a nonempty, compact and connected metric space. A subcontinuum is a continuum that is subspace of a continuum.

Theorem 1.8 A metric space that is homeomorphic to some continuum is also a continuum.

Proof. Let $K$ be a continuum, $X$ be a metric space and $f : K \to X$ be a homeomorphism. Since $f$ is continuous and $K$ is compact and connected, it follows from Theorem 1.5 that $X$ is compact and connected too. $X$ is nonempty since $f$ is a bijection and $K$ is nonempty.

Remark 1.9 We say that a topological space is degenerate if it consists of one point. The term nondegenerate will be used for a space that consists of more than one point.

Example 1.10 We can easily see that $\{x\}$ is an example of degenerate continuum for each $x \in \mathbb{R}$.

Example 1.11 The closed unit interval $[0, 1] \subseteq \mathbb{R}$ is obviously a nondegenerate continuum.

We begin our study of continua by giving some well known nondegenerate examples.

Definition 1.12 An arc is a topological space that is homeomorphic to the closed unit interval $[0, 1]$.

Theorem 1.13 An arc is a continuum.
Proof. Since $[0, 1]$ is a continuum, it follows directly from the Theorem 1.8, that an arc is a continuum too.

Let $X$ be an arc and let $h : [0, 1] \to X$ be a homeomorphism with $h(0) = x$ and $h(1) = y$. One can easily see that for any homeomorphism $h' : [0, 1] \to X$ it holds that \{h'(0), h'(1)\} = \{x, y\}.

Remark 1.14 As we can see, $x$ and $y$ are special points of $X$. They are called the end points of $X$. In that case we say that $X$ is an arc from $x$ to $y$.

Example 1.15 Let $X$ be a subset of the Euclidean plane, defined by

$$X = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, \ y \geq 0 \}.$$ 

Then $X$ is an arc from the point $(-1, 0)$ to the point $(1, 0)$, while $f : [0, 1] \to X$, $f(t) = e^{i\pi t}$ is a homeomorphism.

![Figure 1.1: The arc defined in Example 1.15](image)

Definition 1.16 An $n$-cell is a space, homeomorphic to the $B^n = \{ x \in \mathbb{R}^n \mid \| x \| \leq 1 \}$.

Theorem 1.17 An $n$-cell is a continuum for each positive integer $n$.

Proof. Since $B^n$ is a continuum for each $n$, it follows directly from the Theorem 1.8, that also an $n$-cell is a continuum.

Definition 1.18 An $n$-sphere is a space that is homeomorphic to the $S^n = \{ x \in \mathbb{R}^{n+1} \mid \| x \| = 1 \}$. An 1-sphere is called a simple closed curve.
Theorem 1.19  An $n$-sphere is a continuum for each positive integer $n$.

Proof. Since $S^n$ is a continuum for each $n$, it follows directly from the Theorem 1.8, that also an $n$-sphere is a continuum. □

![Figure 1.2: A simple closed curve](image)

Definition 1.20  A Hilbert cube $Q$ is a space that is homeomorphic to the countable Cartesian product $\prod_{n=1}^{\infty} I_n$, where $I_n = [0, 1]$ for each positive integer $n$, with the product topology.

Theorem 1.21  The Hilbert cube is a continuum.

Proof. First, the product of compact and connected spaces is compact and connected by Theorem 1.5. Also, the product of countable many metrizable spaces is also metrizable by Theorem 1.5. Since $(0, 0, 0, ...)$ is an element of the Hilbert cube, it is nonempty. □

Theorem 1.22  Every continuum can be topologically embedded into a Hilbert cube.

Proof. See [33, p. 241]. □

Definition 1.23  We say that continuum is arcwise connected provided that any two points can be joined by an arc.

Before giving new examples of continua we need to introduce some notation.
**Remark 1.24** Let $X$ be a topological space and $A \subseteq X$. By $\text{Cl}(A)$ we denote the closure of $A$ in $X$, while $\text{Int}(A)$ is used for the interior of $A$ in $X$ and $\text{Bd}(A)$ for the boundary of $A$ in $X$.

All nondegenerate continua that we already mentioned are arcwise connected. The following continuum is not.

**Definition 1.25** Let $W = \{(x, \sin \frac{1}{x}) \in \mathbb{R}^2 | 0 < x \leq 1\}$. Space $\text{Cl}(W)$ is called the $\sin \frac{1}{x}$-continuum. The space is also known as Warsaw arc.

![Figure 1.3: sin \(1/x\)-continuum](image)

**Definition 1.26** The Warsaw circle is a continuum which is homeomorphic to $Y \cup Z$ where $Y$ is the $\sin \frac{1}{x}$-continuum and $Z$ is the union of three convex arcs in $\mathbb{R}^2$, one from $(0, -1)$ to $(0, -2)$, another one from $(0, -2)$ to $(1, -2)$, and the last one from $(1, -2)$ to $(1, \sin 1)$.

The Warsaw circle is arcwise connected, but it has a point that does not have small connected neighborhoods. In the context of continuum theory, the following continua are, in a sense, the most fundamental special class of continua to study. They are defined merely by making connectness a local property.
1.1 Examples

Definition 1.27 A metric space $X$ is a Peano space, or locally connected space if for each $p \in X$ and each neighborhood $N$ of $p$, there exists a connected open subset $U$ of $X$ such that $p \in U \subseteq N$. A Peano continuum, or locally connected continuum is a Peano space which is a continuum.

The following examples of continua are very important for us.

Definition 1.28 A dendrite is a Peano continuum which contains no simple closed curve.

Example 1.29 A simple triod is a well known dendrite. It is defined as a space that is homeomorphic to the continuum $X = [-1, 1] \times \{0\} \cup \{0\} \times [0, 1]$ in the plane $\mathbb{R}^2$. In fact, a simple triod $T$ is an union of three arcs $A_1, A_2, A_3$, such that $A_1 \cap A_2 \cap A_3 = \{c\}$. In that case we say that $c$ is the center of triod $T$.

![Figure 1.4: A simple triod](image)

Remark 1.30 A triod is a continuum $T$ which has a subcontinuum $Z$ such that $T \setminus Z$ is the union of three non-empty, mutually separated sets.

The following examples of continua are introduced by some of their properties. Thus we need to state them first.

Definition 1.31 A continuum is said to be unicoherent provided that whenever $A$ and $B$ are subcontinua of $S$, such that $S = A \cup B$, then $A \cap B$ is also a continuum. A continuum is hereditarily unicoherent if each of its subcontinuum is unicoherent.

Example 1.32 An arc is hereditarily unicoherent, while a simple closed curve is not even unicoherent.
Definition 1.33 A dendroid is arcwise connected, hereditarily unicoherent continuum.

Note that a dendrite is, in fact, a locally connected dendroid.

Remark 1.34 By the ramification point of a dendroid $X$ we mean a point that is the center of a simple triod contained in $X$.

Definition 1.35 A dendroid having at most one ramification point is called a fan. If $t$ is the ramification point of a fan, then we say that $t$ is the top of the fan.

Example 1.36 Let $x, y \in \mathbb{R}^2$ and let $I(x, y)$ denote the straight line segment from $x$ to $y$ in the plane. Let $X = ([0, 1] \times \{0\}) \cup \bigcup_{n=1}^{\infty} I((0, 0), (1, \frac{1}{n}))$. Obviously $X$ is a fan. We call it the harmonic fan.

![Figure 1.5: Sketch of the harmonic fan](image)

Before we give another interesting example of a fan, we introduce the famous (middle-third) Cantor set.

Definition 1.37 The middle-third Cantor set is the subspace $C = \bigcap_{n=1}^{\infty} C_n$ of $[0, 1]$, where each $C_n$ is defined inductively as follows. Let $C_1 = [0, 1] \setminus \left( \frac{1}{3}, \frac{2}{3} \right)$ and assume that we already defined $C_n$. Then $C_{n+1}$ is defined by deleting from $C_n$ the middle-third open interval of each component of $C_n$. A Cantor set is any space that is homeomorphic to $C$. 
1.1 Examples

Note that the middle-third Cantor set $C$ may be defined as the space consisting of all numbers in $[0, 1]$ that can be written in the ternary system using only the digits 0 and 2. After the following definition we give a characterization of the Cantor set.

\[ \text{Figure 1.6: The construction of the middle-third Cantor set} \]

**Definition 1.38** Let $X$ be a topological space, $A$ be a subset of $X$ and $x \in A$. We say that $x$ is an isolated point of $A$ if there exists an open set $U$ of $X$, such that $A \cap U = \{x\}$.

**Theorem 1.39** Let $X$ be a nonempty topological space. Then $X$ is a Cantor set if and only if $X$ is totally disconnected compact metric space with no isolated points.

**Proof.** See [59, p. 217].

**Example 1.40** With connecting all points of the middle-third Cantor set $C \subseteq [0, 1] \times \{0\}$ with the point $(\frac{1}{2}, 1)$ we can construct a fan in the plane $\mathbb{R}^2$ that is called the Cantor fan.

\[ \text{Figure 1.7: Sketch of the Cantor fan} \]
The following example of continua can also be described with the middle-third Cantor set. Because of its properties it is one of the most popular continua in the history of continuum theory.

**Definition 1.41** The Knaster continuum, also known as B-J-K continuum (Brouwer, Janiszewski, Knaster) and sometimes as the Buckethandle continuum, has a simple construction using the middle-third Cantor set $C$ in the plane. Note that if $x$ is in $C$ then so is $1 - x$. We link every pair of these points ($x$ and $1 - x$) by a semicircle in the positive $y$ direction. It is easy to see that if $x$ is in $C$, and if it lies between $\frac{2}{3^n}$ and $\frac{1}{3^n}$ for a certain $n$, then the point ($\frac{5}{3^n} - x$) is also in $C$. We link these two points by a semicircle in the negative $y$ direction.

![Figure 1.8: Sketch of the Knaster continuum](image)

1.2 Hyperspaces

In this section we define a metric that measures how far from each other two subsets of a metric space are and then we study convergence of sequences of sets with respect to this metric.
1.2 Hyperspaces

Definition 1.42 Let \((X, d)\) be a compact metric space. Then

- \(2^X = \{A \mid A \text{ is nonempty closed subsets of } X\}\),
- \(C(X) = \{A \in 2^X \mid A \text{ is connected}\}\),
- \(N_d(\varepsilon, A) = \{x \in X \mid d(x, a) < \varepsilon \text{ for some } a \in A\}\) where \(A \in 2^X\) and \(\varepsilon > 0\).

Definition 1.43 For \(A, B \in 2^X\) let

\[
d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N_d(\varepsilon, B), B \subseteq N_d(\varepsilon, A)\}.\]

\(d_H : 2^X \times 2^X \rightarrow \mathbb{R}\) is called the Hausdorff distance or sometimes the Hausdorff metric.

Proposition 1.44 Hausdorff distance is in fact a metric.

Proof. See [45, p. 53]. \(\square\)

Definition 1.45 Let \((X, d)\) be a compact metric space. The space \(2^X\) equipped with the Hausdorff metric is called the hyperspace of the space \((X, d)\).

Informally, two sets are close with respect to the Hausdorff distance if every point of either set is close to some point of the other set. The Hausdorff metric was introduced by F. Hausdorff in 1914. For more details see [27].

Definition 1.46 Let \((X, T)\) be a topological space and \(\{A_n\}_{n=1}^\infty\) a sequence of subsets of \(X\). We define

- \(\liminf A_n = \{x \in X \mid \text{ for each open set } U \text{ of } X, \text{ with } x \in U, \text{ it holds that } U \cap A_n \neq \emptyset \text{ for all but finitely many } n\}\),
- \(\limsup A_n = \{x \in X \mid \text{ for each open set } U \text{ of } X, \text{ with } x \in U, \text{ it holds that } U \cap A_n \neq \emptyset \text{ for infinitely many } n\}\).

Remark 1.47 It follows from definition that \(\liminf A_n \subseteq \limsup A_n\).
**Definition 1.48** Let \((X, \mathcal{T})\) be a topological space and \(\{A_n\}_{n=1}^{\infty}\) a sequence of subsets of \(X\) and let \(A \subseteq X\). Then \(A = \lim A_n\) if \(\lim \inf A_n = A = \lim \sup A_n\).

**Example 1.49** Let \(\{A_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^2\) be the sequence defined as follows:

\[
A_n = \begin{cases} 
\left\{ \frac{1}{n} \right\} \times \left[ \frac{1}{3}, 1 \right] & \text{for odd } n \\
\left\{ \frac{1}{n} \right\} \times \left[ 0, \frac{2}{3} \right] & \text{for even } n.
\end{cases}
\]

The limit \(\lim A_n\) of the sequence does not exist, since \(\lim \inf A_n = \left\{ 0 \right\} \times \left[ \frac{1}{3}, \frac{2}{3} \right]\) and \(\lim \sup A_n = \left\{ 0 \right\} \times [0, 1]\).

![Figure 1.9: Some elements of the sequence \(A_n\), defined in the Example 1.49](image)

There are many results on hyperspaces. The most important ones for us are listed in the next theorem.

**Theorem 1.50** Let \(X\) be a compact metric space, \(2^X\) its hyperspace and let \(\{A_n\}_{n=1}^{\infty}\) be a sequence of nonempty, compact subsets of \(X\). Then

1. \(2^X\) is compact.
2. If \(X\) is a continuum then \(2^X\) is also a continuum.
3. \(A = \lim A_n\) if and only if \(A_n\) converges to \(A\) with respect to the Hausdorff distance.
When we will talk about convergence of a sequence of sets it will be always about the convergence with respect to the Hausdorff distance.

**Definition 1.51** Let \((X,d)\) be a metric space and \(\varepsilon > 0\). A \((d,\varepsilon)\)-chain from \(x_1\) to \(x_n\) in \(X\) is a nonempty finite subset \(\{x_1, x_2, ..., x_n\}\) of \(X\), such that \(d(x_i, x_{i+1}) < \varepsilon\) for all \(i = 1, ..., n - 1\).

A subset \(Z\) of \(X\) is said to be \((d,\varepsilon)\)-chained provided that for any two points of \(Z\) there exists a \((d,\varepsilon)\)-chain from \(x\) to \(y\) in \(Z\).

A subset of \(X\) that is \((d,\varepsilon)\)-chained for each \(\varepsilon > 0\) is said to be \(d\)-well chained.

**Theorem 1.52** Let \((X,d)\) be a compact metric space. Then \(X\) is \(d\)-well chained if and only if it is connected.

**Proof.** See [45, p. 60-64].

---

### 1.3 Properties

Some of the basic properties of continua, like arcwise connectness (see Definition 1.23) and unicoherence (see Definition 1.31) we already mentioned in the first section. More of them follow.

**Definition 1.53** Let \((X,d)\) and \((Y,d')\) be metric spaces and let \(f : (X,d) \rightarrow (Y,d')\). Then \(f\) is called an \(\varepsilon\)-map provided that \(f\) is continuous and \(\text{diam}(f^{-1}(f(x))) < \varepsilon\) for all \(x \in X\).

**Example 1.54** A homeomorphism \(f : X \rightarrow Y\) is an \(\varepsilon\)-map for each \(\varepsilon\).

**Definition 1.55** Let \(X\) be a continuum and let \(\mathcal{P}\) be a given collection of continua. Then \(X\) is said to be \(\mathcal{P}\)-like provided that for each \(\varepsilon > 0\) there is an \(\varepsilon\)-map \(f\) from \(X\) onto some member \(Y\) of \(\mathcal{P}\).
Example 1.56 If \( P = \{ [0, 1] \} \) we say that \( X \) is arc-like.

Example 1.57 Every arc is an arc-like continuum.

Definition 1.58

- A continuum is decomposable if it can be written as the union of two proper subcontinua.
- A continuum which is not decomposable is said to be indecomposable.
- A continuum is hereditarily indecomposable if each of its subcontinua is indecomposable.
- A continuum \( X \) is homogeneous if for every two points \( x, y \in X \), there exists a homeomorphism \( h : X \to X \) such that \( h(x) = y \).

Example 1.59 A simple closed curve is a decomposable homogeneous continuum. An arc is obviously also an example of decomposable continua but it is not homogeneous. The set \( \{ 0 \} \) is an example of indecomposable continuum.

One can easily see that there is no problem with finding some examples of decomposable continua. On the other hand it seems to be a hard task to find a nondegenerate indecomposable continuum. An interesting thing is that there are in fact many examples of indecomposable continua. One of them is the famous Knaster continuum which is actually the first known nondegenerate indecomposable continuum. Knaster continuum is not homogeneous and, since it contains an arc as a subcontinuum, it is not hereditarily indecomposable. Another famous example of indecomposable continuum is given.

Definition 1.60 The pseudo-arc is a nondegenerate hereditarily indecomposable continuum which is arc-like.

Note that the continuum defined above is up to a homeomorphism uniquely determined. Although the fact that something like this exist is surprising, it holds even more. R. H. Bing proved that pseudo-arc is actually typical among the continua in \( \mathbb{R}^n, n \geq 2 \) and Hilbert space, [13].
Definition 1.61 Continuum $X$ is irreducible if there exist $x, y \in X$ such that for each proper subcontinuum $Y$ of $X$ it holds that $\{x, y\}$ is not a subset of $Y$. In that case we say that $X$ is irreducible between $x$ and $y$ and denote it by $X = xy$.

Example 1.62 An arc is irreducible between its end points.

Example 1.63 A simple triod is not irreducible.

Many properties of continua can be defined and studied by sequences of compact sets and their convergence in an appropriate hyperspaces. One of them is the following property, that will be important for us.

Definition 1.64 A continuum $X$ is said to be smooth at a point $p \in X$, if the following both hold:

1. Given any point $x \in X$, there is a unique subcontinuum $px$ which is irreducible between $p$ and $x$.

2. For each sequence of points $x_n \in X$ which is convergent with the limit point $x$, the sequence of irreducible continua $px_n$ is convergent with the limit continuum $px$ with respect to the Hausdorff distance.

A continuum $X$ is said to be smooth if there is a point $p \in X$ such that $X$ is smooth at $p$.

Example 1.65 The closed unit interval $[0, 1]$ is a smooth continuum, while a simple closed curve is obviously not smooth.

Example 1.66 The harmonic fan is a smooth continuum. Actually it is smooth at all of its points except those that lies on the interval $(0, 1) \times \{0\}$.

But not all fans are smooth.
Example 1.67 For any two points \( x, y \in \mathbb{R}^2 \) let \( I(x, y) \) denote the straight line segment from \( x \) to \( y \) in the plane.

We define \( A_n = I((0, 0), (1, \frac{1}{n})) \), \( B_n = \{ (x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 = (\frac{1}{n})^2, x \geq 1 \} \), and \( C_n = I((\frac{1}{2}, -\frac{1}{n}), (1, -\frac{1}{n})) \). Continuum \( X = ([0, 1] \times \{0\}) \cup \bigcup_{n=1}^{\infty} (A_n \cup B_n \cup C_n) \) is a fan that is not smooth, since for each point \( p \in X \) we can easily found a sequence of points \( x_n \) with a limit point \( x \), such that the sequence of continua \( px_n \) is not convergent with the limit \( px \).

![Figure 1.10: The example of a fan that is not smooth](image)

More about smooth continua can be found in [17, 21, 38, 39].

Many interesting and complicated continua can be constructed by using intersections of chains. When a continuum can be covered by chains with arbitrarily small links, the continuum is said to be chainable. The chains give a continuum a sort of linear flavor which facilitates the study of its properties.

**Definition 1.68** A chain \( C = \{U_1, U_2, ..., U_n\} \) in a metric space \( X \) is a nonempty, finite collection of open subsets \( U_i \) of \( X \) such that

\[
U_i \cap U_j \neq \emptyset \text{ if and only if } |i - j| \leq 1.
\]

Sets \( U_i \) are called links of the chain \( C \), with \( U_1 \) and \( U_n \) being the end-links.
Let \( x, y, z \in X \). We say that \( C \) is a chain from \( x \) to \( y \) through \( z \) if \( x \in U_1, y \in U_n \) and \( z \in U_i \) for some \( i \in \{2, \ldots, n-1\} \).

The mesh of a chain \( C \) is defined by \( \text{mesh}(C) = \max \{ \text{diam}(U_i) | i = 1, \ldots, n \} \). If \( C \) is a chain in \( X \) and \( \text{mesh}(C) < \varepsilon \), we say that \( C \) is an \( \varepsilon \)-chain in \( X \).

Continuum \( X \) is chainable provided that for each \( \varepsilon > 0 \) there exists an \( \varepsilon \)-chain in \( X \) covering \( X \).

**Theorem 1.69** A nondegenerate continuum \( X \) is chainable if and only if it is arc-like.

**Proof.** See [45, p. 235].

**Remark 1.70** Chainable continua are sometimes called linearly chainable or snake-like continua.

**Example 1.71** An arc is an obvious example of a chainable continuum. Also Knaster continuum and pseudo-arc are chainable (see the constructions in [45, p. 7, 13]), while simple closed curve and a simple triod are not chainable.

The class of chainable continua is well-studied in continuum theory. For this reason, much information on chainable continua is readily available. Some of them are listed below.

**Theorem 1.72** Let \( X \) be a chainable continuum. Then the following hold true.

1. \( X \) can be embedded in the plane.
2. Every subcontinuum of \( X \) is chainable.
3. \( X \) is atriodic, i.e. it contains no triods.
4. \( X \) is unicoherent.
5. \( X \) is irreducible between two of its points.
6. \( X \) is a continuous image of the pseudo-arc.

**Proof.** See [24].
Chainable continua are long and thin. In an attempt to capture this idea in metric terms A. Lelek introduced the notion of span in [36]. Intuitive, the span of a continuum is the least upper bound of numbers \( \alpha \) such that two points can move over the same portion of the continuum keeping a distance at least \( \alpha \) from each other. The surjective span is obtained if it is required that, in addition, the whole continuum can be covered by each of the moving points. Since the notion of span was introduced, a number of variants of the original definition have been investigated. The most important one for us are collected in the definition below.

**Definition 1.73** By \( \pi_1 \) and \( \pi_2 \) we denote the standard projections of the product \( X \times X \) onto \( X \). That is \( \pi_1(x_1, x_2) = x_1 \) and \( \pi_2(x_1, x_2) = x_2 \) for each \( (x_1, x_2) \in X \times X \).

**Definition 1.74** Let \( X \) be a continuum and let \( P(Z) \) be a property of a subcontinuum \( Z \subseteq X \times X \). The number

\[
\sup_Z \left\{ \inf_{(x_1, x_2) \in Z} d(x_1, x_2) \mid Z \text{ is a subcontinuum of } X \times X \text{ such that } P(Z) \text{ holds} \right\}
\]

is called the

- span of \( X \), denoted by \( \sigma(X) \), if \( P(Z) \) is \( \pi_1(Z) = \pi_2(Z) \);
- semispan of \( X \), denoted by \( \sigma_0(X) \), if \( P(Z) \) is \( \pi_1(Z) \subseteq \pi_2(Z) \);
- surjective span of \( X \), denoted by \( \sigma^*(X) \), if \( P(Z) \) is \( \pi_1(Z) = \pi_2(Z) = X \).

No two versions of span are always equal, but one can easily see that the inequalities \( 0 \leq \sigma(X)^* \leq \sigma(X) \leq \sigma_0(X) \leq \text{diam}(X) \) hold true. While the particular value of the span of a continuum depends on the chosen metric, the distinction between span zero and span nonzero is a topological one. For more details see [24, 37, 35].

**Example 1.75** Span of an arc is obviously zero, while span of a simple closed curve or a simple triod is nonzero.

In [36] Lelek proved that chainable continua have span zero and in 1971 he asked whether the converse also holds. This was known as Lelek’s problem and it has
been one of the most widely investigated problems in continuum theory over the past 40 years. Many positive partial results for Lelek’s problem have been obtained. See [43, 51, 49, 50]. A number of properties of chainable continua have been established for span zero continua. We give a list of some of these.

**Theorem 1.76** Let $X$ be a continuum with span zero. Then the following hold true.

1. $X$ is tree-like.
2. Every subcontinuum of $X$ has span zero.
3. $X$ is atriodic.
4. $X$ is unicoherent.
5. $X$ is irreducible between two of its points.
6. $X$ is a continuous image of the pseudo-arc.

**Proof.** See [24].

However, there has been also a lot of work toward finding a counterexample for Lelek’s problem. For instance, see [53]. Finally, in 2010 L. C. Hoehn gave an example showing that, in general, span zero does not imply chainable, even among continua in the plane; see [25].
In the first chapter we already constructed some examples of continua. In the following one we shall present other methods for constructing new interesting continua. Two of them are well-known techniques, while the other one is brand new.

2.1 Nested intersections

One of the most important and fundamental techniques for obtaining interesting examples of continua is the use of nested intersections. It is not only used to construct examples, but also, is a key idea for the proofs of many theorems, one can say that the nested intersection technique is central to continuum theory.

**Definition 2.1** A sequence of compact subspaces \( \{X_n\}^\infty_{n=1} \) of a metric space \( X \) is said to be a descending sequence of subsets of \( X \) if \( X_n \supseteq X_{n+1} \) for each positive integer \( n \).

**Proposition 2.2** Let \( \{X_n\}^\infty_{n=1} \) be a descending sequence of compact metric spaces and let \( X = \bigcap_{n=1}^\infty X_n \). If \( U \) is an open subset of \( X_1 \) such that \( U \supseteq X \), then there exists a positive integer \( m \) such that \( U \supseteq X_n \) for all \( n \geq m \).

**Proof.** See [45, p. 6].
Theorem 2.3 Let \( \{X_n\}_{n=1}^{\infty} \) be a descending sequence of continua and let \( X = \bigcap_{n=1}^{\infty} X_n \). Then \( X \) is a continuum.

Proof. See [45, p. 6, 7].

We give two examples of continua which are constructed using nested intersections. The first one is a nondegenerate indecomposable continuum in the plane. We construct it as follows.

Example 2.4 Let \( a, b, c \in \mathbb{R}^2 \) with \( a \neq b \neq c \). For each positive integer \( n \) let \( C_n \) be a \( \frac{1}{2^n} \)-chain in the plane \( \mathbb{R}^2 \), such that the following hold true.

1. For each \( n = 0, 1, 2, ... \)
   - \( C_{3n+1} \) goes from \( a \) to \( c \) through \( b \);
   - \( C_{3n+2} \) goes from \( b \) to \( c \) through \( a \);
   - \( C_{3n+3} \) goes from \( a \) to \( b \) through \( c \).

2. For each \( n = 1, 2, 3, ..., \bigcup C_n \supseteq \text{Cl}(\bigcup C_{n+1}) \).

Denote \( X_n = \text{Cl}(\bigcup C_n) \) for each \( n \). Then \( X_1 \supseteq X_2 \supseteq X_3 \supseteq ... \) is a descending sequence of continua and \( X = \bigcap_{n=1}^{\infty} X_n \) is a continuum by Theorem 2.3.

By using the fact that \( X = \bigcap_{n=1}^{\infty} \text{Cl}(\bigcup C_{3n+1}) \), it can be verified that no proper subcontinuum of \( X \) contains \( \{a, c\} \), see [45, p. 7, 8, 9]. Similarly, no proper subcontinuum of \( X \) contains any two of the three points \( a, b, c \). By using the pigeonhole principle it follows easily that \( X \) is indecomposable.

With a similar technique of using chains and nested intersections, as in Example 2.4, the famous hereditarily indecomposable pseudo-arc can be constructed, [45, p. 13]. As another example we construct the Sierpinski Universal Curve which is also a very interesting continuum. It is a generalization of the Cantor set to two dimensions and it was first described by W. Sierpiński in 1916, The term universal refers to the fact that this one-dimensional continuum in the plane contains a topological copy of every one-dimensional continuum in the plane, [54].
Example 2.5 Divide the square $S = [0, 1] \times [0, 1]$ into nine congruent squares $S_1 = [0, 1/3] \times [0, 1/3]$, $S_2 = [1/3, 2/3] \times [0, 1/3]$, $S_3 = [2/3, 1] \times [0, 1/3]$, $S_4 = [0, 1/3] \times [1/3, 2/3]$, $S_5 = [1/3, 2/3] \times [1/3, 2/3]$, $S_6 = [2/3, 1] \times [1/3, 2/3]$, $S_7 = [0, 1/3] \times [2/3, 1]$, $S_8 = [1/3, 2/3] \times [2/3, 1]$, and $S_9 = [2/3, 1] \times [2/3, 1]$. Let $X_1 = S$ and $X_2 = S \setminus \text{Int}(S_5)$. In the next step divide each of the squares $S_i$ into nine congruent squares $S_{ij}$, as before. Let $X_3 = X_2 \setminus (\text{Int}(\bigcup_{i=1}^9 S_i))$; it is the continuum obtained by removing the interiors of each of the resulting central squares.

Continuing in the same fashion, we build a descending sequence of continua $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$. Then $X = \bigcap_{n=1}^\infty X_n$ is a continuum by Theorem 2.3 and is called the Sierpinski Universal Curve.

Any one-dimensional continua can be embedded into the Menger sponge, which can also be constructed using nested intersections, [20].

2.2 Inverse limits

Since searching new examples of continua with particular properties can be very difficult, there were always desires for new, easier constructions of continua. Inverse limits prove to be one of them.

Theorem 2.6 A Cartesian product of countable many continua is a continuum.
Proof. See [33, p. 17, 137]. □

Definition 2.7 An inverse sequence is a double sequence \( \{X_k, f_k\}_{k=1}^{\infty} \) of topological spaces \( X_k \) and continuous functions \( f_k : X_{k+1} \to X_k \). If \( \{X_k, f_k\}_{k=1}^{\infty} \) is an inverse sequence, we say that \( X_k \) are coordinate spaces and \( f_k \) are bonding maps. Sometimes \( \{X_k, f_k\}_{k=1}^{\infty} \) is written as

\[
X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} X_4 \xleftarrow{f_4} \ldots
\]

Definition 2.8 The inverse limit of an inverse sequence \( \{X_k, f_k\}_{k=1}^{\infty} \), denoted by \( \varprojlim \{X_k, f_k\}_{k=1}^{\infty} \), is the subspace of the Cartesian product space \( \prod_{k=1}^{\infty} X_k \) defined by

\[
\varprojlim \{X_k, f_k\}_{k=1}^{\infty} = \left\{ (x_1, x_2, \ldots, x_k, \ldots) \in \prod_{k=1}^{\infty} X_k \mid x_k = f_k(x_{k+1}) \text{ for each } k \right\}.
\]

Example 2.9 Let \( \{X_k, f_k\}_{k=1}^{\infty} \) be the inverse sequence where \( X_k = [0, 1] \) and \( f_k(x) = 0 \) for each positive integer \( k \). It is easy to see that the corresponding inverse limit \( \varprojlim \{X_k, f_k\}_{k=1}^{\infty} \) consists only of the point \((0, 0, 0, \ldots)\).

Example 2.10 Let \( \{X_k, f_k\}_{k=1}^{\infty} \) be the inverse sequence where \( X_k = [0, 1] \) and \( f_k(x) = x \) for each positive integer \( k \). It is easy to see that the corresponding inverse limit \( \varprojlim \{X_k, f_k\}_{k=1}^{\infty} = \{(t, t, t, \ldots) \mid t \in [0, 1]\} \) is an arc.

Figure 2.2: The graph of a bonding function \( f_k \) defined in Example 2.10 and the corresponding inverse limit

The following proposition shows how to think of inverse limits as nested intersections.
2.2 Inverse limits

Proposition 2.11 Let \( \{X_k, f_k\}_{k=1}^{\infty} \) be an inverse sequence. For each positive integer \( n \) let

\[
Q_n(X_k, f_k) = \left\{ x \in \prod_{k=1}^{\infty} X_k \mid f_k(x_{k+1}) = x_k \text{ for each } k \leq n \right\}.
\]

Then the following hold true:

1. \( Q_n(X_k, f_k) \supseteq Q_{n+1}(X_k, f_k) \) for each positive integer \( n \).
2. \( Q_n(X_k, f_k) \) is homeomorphic to \( \prod_{k=n+1}^{\infty} X_k \).
3. \( \varprojlim_{k=1}^{\infty} \{X_k, f_k\} = \bigcap_{n=1}^{\infty} Q_n(X_k, f_k) \).

Proof. See [45, p. 19]. □

Theorem 2.12 Any inverse limit of continua is a continuum.

Proof. Let \( \{X_k, f_k\}_{k=1}^{\infty} \) be an inverse sequence of continua. For each positive integer \( n \) it holds that \( \varprojlim_{k=1}^{\infty} \{X_k, f_k\} = \bigcap_{n=1}^{\infty} Q_n(X_k, f_k) \). Since by Proposition 2.11 also \( Q_n(X_k, f_k) \) is homeomorphic to \( \prod_{k=n+1}^{\infty} X_k \) for each \( n \), it follows that \( Q_n(X_k, f_k) \) is a continuum for each \( n \), by Theorem 2.6. Since the sequence \( Q_n(X_k, f_k) \) is descending sequence of continua by Proposition 2.11, it holds that inverse limit is a continuum by Theorem 2.3. □

By using the fact that an inverse limit of continua is again a continuum, we can get many interesting continua with simple inverse sequences. One of them is already mentioned Knaster continuum. In the following example we show the construction of it as inverse limit of an inverse sequence where all coordinate spaces and all bonding functions are the same.

Example 2.13 Let \( \{X_k, f_k\}_{k=1}^{\infty} \) be the inverse sequence where for each positive integer \( k \), \( X_k = [0,1] \) and \( f_k \) is defined by

\[
f_k = \begin{cases} 
2t & \text{if } 0 \leq t \leq \frac{1}{2} \\
-2t + 2 & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
Then the space $X = \lim_{\leftarrow} \{X_k, f_k\}_{k=1}^{\infty}$ is homeomorphic to the Knaster continuum, see [45, p. 22].

The pseudo-arc can also be written as an inverse limit of an inverse sequence of closed unit intervals and one bonding function, [23]. In fact, when all coordinate spaces are closed unit intervals $[0, 1]$ and all bonding functions are surjective, a continuum $\lim_{\leftarrow} \{X_k, f_k\}_{k=1}^{\infty}$ is always arc-like.

**Theorem 2.14** Let $X$ be a continuum. Then $X$ is arc-like if and only if it can be written as an inverse limit of closed unit intervals $[0, 1]$ and surjective bonding functions.

**Proof.** See [45, p. 246]. □

There is also a generalization of this theorem.

**Theorem 2.15** Let $X$ be a continuum and $\mathcal{P}$ a family of compact and connected polyhedra. Then $X$ is $\mathcal{P}$-like if and only if it can be written as an inverse limit of members of $\mathcal{P}$ and surjective bonding functions.

**Proof.** See [41, p. 148] and [45, p. 24, 247]. □
2.3 Generalized inverse limits

Since the inverse limits have proven to be very useful, there are many researchers who started to work on the generalization of this term. In the year 2004 W. T. Ingram and W. S. Mahavier introduced the concept of generalized inverse sequences with $X_k$ being the compact spaces and $f_k$ being the upper semi-continuous set-valued bonding functions, [30].

**Definition 2.16** A function $f : X \rightarrow 2^Y$, where $X$ and $Y$ are compact metric spaces, is upper semi-continuous (abbreviated u.s.c.) function if for each $x \in X$ and for each open set $U \subseteq Y$ such that $f(x) \subseteq U$ there is an open set $V$ in $X$ such that

1. $x \in V$;
2. for all $v \in V$ it holds that $f(v) \subseteq U$.

A function $f : X \rightarrow 2^Y$ is surjective, if for each $y \in Y$ there is an $x \in X$, such that $y \in f(x)$.

**Definition 2.17** The graph $\Gamma(f)$ of a function $f : X \rightarrow 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

The following is a well-known characterization of u.s.c. functions:

**Theorem 2.18** Let $X$ and $Y$ be compact metric spaces and $f : X \rightarrow 2^Y$ a set-valued function. Then $f$ is u.s.c. if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.

**Proof.** See [30].

**Definition 2.19** An inverse sequence of compact metric spaces $X_k$ with u.s.c. bonding functions $f_k$ is a sequence $\{X_k, f_k\}_{k=1}^{\infty}$, where $f_k : X_{k+1} \rightarrow 2^{X_k}$ for each $k$. 
Since in this chapter we deal only with such inverse sequences, sometimes we call them simply inverse sequences. Note that the inverse sequence defined in Definition 2.7 are really just a special case of those defined in Definition 2.19. See the remark below.

**Remark 2.20** If $f : [0, 1] \to [0, 1]$ is continuous, then $F : [0, 1] \to 2^{[0,1]}$, $F(x) = \{ f(x) \}$ is u.s.c. function. Conversely, if $F : [0, 1] \to 2^{[0,1]}$ is u.s.c. function and for each $x \in [0, 1]$, $F(x) = \{ y_x \}$ then $f : [0, 1] \to [0, 1]$, $f(x) = y_x$ is continuous.

**Definition 2.21** The generalized inverse limit of an inverse sequence $\{ X_k, f_k \}_{k=1}^\infty$, denoted by $\lim\left< X_k, f_k \right>_{k=1}^\infty$, is the subspace of the Cartesian product space $\prod_{k=1}^\infty X_k$ defined by

$$\lim\left< X_k, f_k \right>_{k=1}^\infty = \left\{ (x_1, x_2, \ldots, x_k, \ldots) \in \prod_{k=1}^\infty X_k \mid x_k \in f_k(x_{k+1}) \text{ for each } k \right\}.$$ 

If $x \in X_{k+1}$ and $f_k(x)$ is a set with only one element for each positive integer $k$ then all $f_k$ are single-valued and Definition 2.8 is equivalent to the definition above.

**Theorem 2.22** Let $\left< X_k, f_k \right>_{k=1}^\infty$ be an inverse sequence of nonempty compact metric spaces and u.s.c. bonding functions. Then the inverse limit $\lim\left< X_k, f_k \right>_{k=1}^\infty$ is a nonempty compact metric space.

**Proof.** See [30].

There are many important continua that can be constructed as a generalized inverse limit with a single set-valued bonding function and the coordinate space $[0, 1]$. We give a few examples of them.
Example 2.23 Let $f : [0, 1] \to 2^{[0,1]}$ be an u.s.c. function defined by $f(t) = \{0, t\}$ for each $t \in [0, 1]$. Then the corresponding inverse limit $\varprojlim \{[0, 1], f\}_{k=1}^\infty$ is a fan with the top $(0, 0, 0, \ldots)$. See the illustration below and [29] for more details.

Example 2.24 Let $f : [0, 1] \to 2^{[0,1]}$ be an u.s.c. function defined by

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ [0, 1] & \text{if } t = 1. \end{cases}$$

Then $\varprojlim \{[0, 1], f\}_{k=1}^\infty$ is so called harmonic fan with the top $(1, 1, 1, \ldots)$. See the illustration below and [29] for more details.
Example 2.25 Let $f : [0, 1] \to 2^{[0, 1]}$ be an u.s.c. function defined by

$$f(t) = \begin{cases} 
1 & \text{if } t \neq \frac{1}{2} \\
[0, 1] & \text{if } t = \frac{1}{2}.
\end{cases}$$

Then $\lim_{\leftarrow} \{ [0, 1], f \}_{k=1}^{\infty}$ is the union of a sequence of arcs $(A_n)_{n=1}^{\infty}$ and the point $(1, 1, 1, \ldots)$ such that for each positive integer $i$, $A_i \cap A_{i+1}$ is a single point that is an endpoint of $A_i$ and an interior point of $A_{i+1}$. It is called the harmonic comb. See the illustration below and [29] for more details.

![Figure 2.6: The graph of $f$ and the model of the corresponding inverse limit from Example 2.25](image)

Example 2.26 Let $f : [0, 1] \to 2^{[0, 1]}$ be an u.s.c. function with graph being the set $[0, 1] \times \{0, 1\}$. One can easily see that $\lim_{\leftarrow} \{ X_k, f_k \}_{k=1}^{\infty} = \{0, 1\}^N$, where $\mathbb{N}$ denotes the set of all positive integers, is nonempty, compact, totally disconnected metric space (by Theorem 1.5) with no isolated points and therefore it is the Cantor set (by Theorem 1.39) which is not connected.

![Figure 2.7: The graph of a bonding function $f$, defined in Example 2.26](image)
2.3 Generalized inverse limits

Remember that an inverse limit of an inverse sequence of continua, where all bonding functions are continuous single-valued, is again a continuum. In the following example we show that such a theorem does not hold for generalized inverse limits.

In Example 2.26 we gave an example of u.s.c. set-valued bonding function with a non-connected graph $[0, 1] \times [0, 1]$. In the following example the graph of a given function is connected but the corresponding inverse limit is still not connected.

**Example 2.27** Let $I(x, y)$ denote the straight line segment from $x$ to $y$ in the plane and let $f : [0, 1] \to 2^{[0, 1]}$ be u.s.c. function with graph being the union $I((\frac{1}{4}, \frac{1}{4}), (0, 0)) \cup I((0, 0), (1, 0)) \cup I((1, 0), (1, 1)) \cup I((1, 1), (\frac{3}{4}, \frac{1}{4}))$. Then the corresponding inverse limit $\lim \{[0, 1], f\}_{k=1}^{\infty}$ is not connected. See [30].

![Figure 2.8: The graph of a bonding function $f$, defined in Example 2.27](image)

But there is still a big family of generalized inverse limits that are continua.

**Theorem 2.28** Let for each positive integer $k$, $X_k$ be a continuum and $f_k : X_{k+1} \to 2^{X_k}$ be an u.s.c. function. If for each positive integer $k$ and for each $x \in X_{k+1}$, the space $f_k(x)$ is connected, then the inverse limit $\lim \{X_k, f_k\}_{k=1}^{\infty}$ is a continuum.

**Proof.** See [30].

Since the introduction of generalized inverse limits, there has been much interest in the subject and many papers have appeared [3, 4, 5, 6, 7, 8, 16, 26, 31, 32, 22, 46, 47, 48, 52, 56].
2.4 Inverse limits in the category $\text{CHU}$

The place of inverse limits in category theory also prove to be very useful. We investigate inverse limits in the category $\text{CHU}$ of compact Hausdorff spaces with u.s.c. functions. We introduce the notion of weak inverse limits in this category and show that the inverse limits with u.s.c. set-valued bonding functions together with the projections are not necessarily inverse limits in $\text{CHU}$ but they are always weak inverse limits in this category. W. T. Ingram in his book [29] states the following problem:

"What can be said about inverse limits with set-valued functions if the underlying directed set is not a sequence of integers?"

We present a categorical approach to solving the problem and give some basic definitions first. All results given in this section are from [11].

**Definition 2.29** A category $\mathcal{K}$ consists of the following:

- A class of objects of $\mathcal{K}$, denoted by $\text{Ob}(\mathcal{K})$.

- For each pair of objects $X, Y$, a set $\text{Mor}_\mathcal{K}(X, Y)$ of morphisms. We denote a morphism $f \in \text{Mor}_\mathcal{K}(X, Y)$ as $f : X \to Y$.

The morphisms satisfy the following conditions:

1. For each pair of morphisms $f : X \to Y$, $g : Y \to Z$ (where $X, Y, Z$ are arbitrary objects of $\mathcal{K}$) there is a uniquely determined morphism $g \circ f : X \to Z$.

2. The partial binary operation (composition $\circ$) defined in 1. is associative.

3. For every object $X \in \text{Ob}(\mathcal{K})$, there exists a morphism $1_X : X \to X$ called the identity morphism on $X$, such that for every morphism $f : X \to Y$ we have $1_Y \circ f = f$ and $f \circ 1_X = f$.

**Remark 2.30** From the definition it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.
2.4 Inverse limits in the category $\mathcal{CHU}$

Example 2.31 The category of topological spaces, often denoted by $\text{Top}$ is the category whose objects are topological spaces and whose morphisms are continuous functions. Obviously this is a category since the composition of two continuous functions is again continuous.

Definition 2.32 A directed set is a nonempty set $A$ equipped with a reflexive and transitive binary relation $\leq$ with the property that every pair of elements has an upper bound (for any $a, b \in A$ there exists $c \in A$ such that $a \leq c$ and $b \leq c$).

Definition 2.33 For a directed set $A$, a family of objects $\{X_\alpha \mid \alpha \in A\}$ of $\mathcal{K}$, and a family of morphisms $\{f_{\alpha\beta} : X_\beta \to X_\alpha \mid \alpha, \beta \in A, \alpha \leq \beta\}$ of $\mathcal{K}$, such that

1. for each $\alpha \in A$, $f_{\alpha\alpha} = 1_{X_\alpha}$;
2. for each $\alpha, \beta, \gamma \in A$, from $\alpha \leq \beta \leq \gamma$ it follows that $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$

we call an inverse system (in $\mathcal{K}$) and denote it by

$$(A, \{X_\alpha \}_{\alpha \in A}, \{f_{\alpha\beta} \}_{\alpha, \beta \in A}).$$

We assume throughout the section that $A$ is cofinite, i.e. every $\alpha \in A$ has at most finitely many predecessors.

Definition 2.34 An object $X \in \text{Ob}(\mathcal{K})$, together with morphisms $\{p_\alpha : X \to X_\alpha \mid \alpha \in A\}$ is an inverse limit of an inverse system $(A, \{X_\alpha \}_{\alpha \in A}, \{f_{\alpha\beta} \}_{\alpha, \beta \in A})$ in the category $\mathcal{K}$, if

1. for all $\alpha, \beta \in A$, from $\alpha \leq \beta$ it follows that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p_\alpha} & X_\alpha \\
\downarrow{p_\beta} & & \downarrow{f_{\alpha\beta}} \\
X_\alpha & \xleftarrow{f_{\alpha\beta}} & X_\beta
\end{array}
\]

commutes;
2. for any object $Y \in \mathcal{K}$ and any family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ it follows that if the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi_\alpha} & X_
\beta \\
\varphi_\beta & & \downarrow f_{\alpha\beta} \\
X_\alpha & \xleftarrow{f_{\alpha\beta}} & X_\beta
\end{array}
\]

(2.2)

commutes, then there is a unique morphism $\varphi : Y \to X$ such that for each $\alpha \in A$ the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & X \\
\varphi_\alpha & & \downarrow p_\alpha \\
X_\alpha & \xleftarrow{p_\alpha} & X_\alpha
\end{array}
\]

(2.3)

commutes.

Consider an inverse system $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ of compact Hausdorff spaces and continuous bonding functions. It is a well-known fact, that the space

\[
\lim_{\leftarrow}(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) = \\
\{(x_\gamma)_{\gamma \in A} \in \prod_{\alpha \in A} X_\alpha \mid \text{for all } \alpha, \beta \in A, \alpha < \beta, x_\alpha = f_{\alpha\beta}(x_\beta)\}
\]

together with the projection mappings

\[
p_\gamma : \lim_{\leftarrow}(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) \to X_\gamma,
\]
2.4 Inverse limits in the category $\mathcal{CHU}$

$p_\gamma((x_\alpha)_{\alpha \in A}) = x_\gamma$, is in fact an inverse limit in the category $\mathcal{CHC}$ of compact Hausdorff spaces with continuous functions.

We extend the category $\mathcal{CHC}$ to the category $\mathcal{CHU}$ of compact Hausdorff spaces with u.s.c. functions in such a way that $\mathcal{CHC}$ is interpreted as a proper subcategory of $\mathcal{CHU}$. This can be done since every continuous function between compact Hausdorff spaces can be interpreted as an u.s.c. function, see Remark 2.20.

**Definition 2.35** The category $\mathcal{CHU}$ of compact Hausdorff spaces and u.s.c. functions consists of the following objects and morphisms:

1. $\text{Ob}(\mathcal{CHU})$: compact Hausdorff spaces
2. $\text{Mor}(\mathcal{CHU})$: u.s.c. functions (the u.s.c. functions from $X$ to $Y$ is the set of morphisms from $X$ to $Y$, denoted by $\text{Mor}(\mathcal{CHU})(X,Y)$).

We also define the partial binary operation $\circ$ (the composition) as follows. For each $f \in \text{Mor}(\mathcal{CHU})(X,Y)$ and each $g \in \text{Mor}(\mathcal{CHU})(Y,Z)$ we define $g \circ f \in \text{Mor}(\mathcal{CHU})(X,Z)$ by

$$(g \circ f)(x) = g(f(x)) = \bigcup_{y \in f(x)} g(y)$$

for each $x \in X$.

**Theorem 2.36** $\mathcal{CHU}$ is a category.

**Proof.** First we show that $\circ$ is well-defined. Let $f : X \to Y$ and $g : Y \to Z$ be any morphisms. Let also $x \in X$ be arbitrary and let $U$ be an open set in $Z$ such that $(g \circ f)(x) \subseteq U$. Since $g$ is u.s.c. and $f(x) \subseteq Y$, it holds that for each $y \in f(x)$ there is an open set $W_y$ in $Y$ such that

1. $y \in W_y$,
2. for all $w \in W_y$ it holds that $g(w) \subseteq U$.

Let $W = \bigcup_{y \in f(x)} W_y$. Since $W$ is open in $Y$, $f(x) \subseteq W$, and since $f$ is u.s.c., there is an open set $V$ in $X$ such that
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1. \( x \in V \);

2. for all \( v \in V \) it holds that \( f(v) \subseteq W \).

Let \( v \in V \) be arbitrary. Then

\[
(g \circ f)(v) = g(f(v)) = \bigcup_{z \in f(v)} g(z) \subseteq U
\]

since for each \( z \in f(v) \), it holds that \( g(z) \subseteq U \). Therefore \( \circ \) is well-defined.

It is obvious that the composition \( \circ \) of u.s.c. functions is an associative operation.

All that is left to show is that for each \( X \in Ob(CHU) \) there is a morphism \( 1_X : X \rightarrow X \) such that \( 1_X \circ f = f \) and \( g \circ 1_X = g \) for any morphisms \( f : Y \rightarrow X \) and \( g : X \rightarrow Z \).

We easily see that the identity map \( 1_X : X \rightarrow X \), defined by \( 1_X(x) = \{x\} \) for each \( x \in X \), is the u.s.c. function satisfying the above conditions. \( \square \)

Next we show that if \( (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) \) is an inverse system of compact Hausdorff spaces and u.s.c. set-valued bonding functions, then

\[
\lim \left( A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A} \right)
\]

(see Definition 2.37) together with the projections is not necessarily an inverse limit in the category \( CHU \).

Motivated by \([30, 40]\), we define objects in \( CHU \), that are called inverse limits with u.s.c. set-valued bonding functions.

**Definition 2.37** Let \( (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) \) be any inverse system in \( CHU \). We call the object

\[
\lim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) = \{ x \in \prod_{\alpha \in A} X_\alpha \mid \text{for all } \alpha < \beta, \ x_\alpha \in f_{\alpha\beta}(x_\beta) \}
\]

an inverse limit with u.s.c. set-valued bonding functions.

In the following theorem we prove that \( \lim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) \) is really an object of \( CHU \).
2.4 Inverse limits in the category $\text{CHU}$

**Theorem 2.38** Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ be any inverse system in $\text{CHU}$. Then the inverse limit with u.s.c. set-valued bonding functions

\[
\lim_{\leftarrow} (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})
\]

is a compact Hausdorff space.

**Proof.** For each $\gamma \in A$, $X_\gamma$ is a compact Hausdorff space, therefore the product $\prod_{\gamma \in A} X_\gamma$ is a compact Hausdorff space. Since $\lim_{\leftarrow} (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is a subspace of the Hausdorff space, it is also a Hausdorff space.

We show that $\lim_{\leftarrow} (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is a closed subset of the compact space $\prod_{\gamma \in A} X_\gamma$ to show that it is compact.

Let for all $\alpha, \beta \in A$, $\alpha < \beta$,

\[
G_{\alpha\beta} = \Gamma(f_{\alpha\beta}) \times \prod_{\gamma \in A \setminus \{\alpha, \beta\}} X_\gamma = \{ x \in \prod_{\gamma \in A} X_\gamma | x_\alpha \in f_{\alpha\beta}(x_\beta) \}.
\]

Since the graph $\Gamma(f_{\alpha\beta})$ of $f_{\alpha\beta}$ is by Theorem 2.18 a closed subset of $X_\beta \times X_\alpha$, $G_{\alpha\beta}$ is also a closed subset of $\prod_{\gamma \in A} X_\gamma$. It is obvious that

\[
\lim_{\leftarrow} (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A}) = \bigcap_{\alpha, \beta \in A, \alpha < \beta} G_{\alpha\beta}
\]

and hence $\lim_{\leftarrow} (A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ is a closed subset of $\prod_{\gamma \in A} X_\gamma$. $\square$

In the following example we construct an inverse limit with u.s.c. set-valued bonding functions that is not an inverse limit in $\text{CHU}$, regardless of the choice of morphisms $\{p_\alpha : X \to X_\alpha | \alpha \in A\}$.

**Example 2.39** Let $\mathbb{N}$ denote the set of all positive integers and let $A = \mathbb{N}$, $X_k = [0, 1]$, and $f_{k(k+1)} = f$ for each $k \in \mathbb{N}$, where $f : [0, 1] \to 2^{[0,1]}$ is the function on $[0, 1]$ defined by its graph

\[
\Gamma(f) = \{(t, t) \in [0, 1] \times [0, 1] | t \in [0, 1]\} \cup (\{1\} \times [0, 1]).
\]

Also let $X = \lim_{\leftarrow} (\mathbb{N}, \{[0, 1]\}_{k \in \mathbb{N}}, \{f_{kt}\}_{k, t \in \mathbb{N}})$ and let $\{p_i : X \to X_i | i \in \mathbb{N}\}$ be any set of morphisms in $\text{CHU}$, such that the diagrams (2.1) always commute. We show that $X$ with
\{p_i : X \to X_i \mid i \in \mathbb{N}\} is not an inverse limit of \((\mathbb{N}, \{[0, 1]\}_{k \in \mathbb{N}}, \{f_{k\ell}\}_{k, \ell \in \mathbb{N}})\) in \(\text{CHU}\). Let \(Y = [0, 1]\) be an object in \(\text{CHU}\) and let \(\{\varphi_k : Y \to X_k \mid k \in \mathbb{N}\}\) be the family of morphisms where \(\varphi_k(t) = [0, 1]\) for each \(k\) and each \(t \in Y\). The diagram (2.2) always commutes. We distinguish the following two cases.

1. If there is a positive integer \(i_0\), such that \(1 \notin p_{i_0}(x)\) for each \(x \in X\), then suppose that \(\Phi\) is any morphism \(Y \to X\). Then \(\varphi_{i_0}(t) = [0, 1]\) but \(1 \notin p_{i_0}(\Phi(t))\) for any \(t \in Y\). Therefore the diagram (2.3) does not commute for \(\alpha = i_0\).

2. If for each positive integer \(i\) there is \(x^i \in X\) such that \(1 \in p_i(x^i)\), then let \(s \in X\) be an accumulation point of the sequence \(\{x^i\}_{i=1}^{\infty}\). We show first that \(p_i(s) = [0, 1]\) for each \(i\). Let \(k\) be any positive integer. Then for each \(\ell > k\), it follows from

\[ [0, 1] \supseteq p_k(x^\ell) = f_{k\ell}(p_k(x^\ell)) \supseteq f_{k\ell}(1) \supseteq [0, 1]\]

that \(p_k(x^\ell) = [0, 1]\). Let \(\{n_i\}_{i=1}^{\infty}\) be any increasing sequence of positive integers such that

- \(n_i > k\) for each \(i\);
- \(\lim_{i \to \infty} x^{n_i} = s\).

It follows from \(p_k(x^{n_i}) = [0, 1]\) that \(\{x^{n_i}\} \times [0, 1] \subseteq \Gamma(p_k)\) for each \(i\). This means that for each \(t \in [0, 1]\), the point \((x^{n_i}, t) \in \Gamma(p_k)\). Therefore \(\lim_{i \to \infty} (x^{n_i}, t) = (s, t) \in \Gamma(p_k)\) for each \(t\), since \(\Gamma(p_k)\) is a closed subset of \(X \times [0, 1]\). It follows that \(\{s\} \times [0, 1] \subseteq \Gamma(p_k)\) and hence \(p_k(s) = [0, 1]\).

Next, let \(\Phi, \Psi : Y \to X\) be the morphisms in \(\text{CHU}\), defined by

\[ \Phi(t) = X, \]

\[ \Psi(t) = \{s\} \]

for each \(t \in Y\). It follows from

\[ p_k(\Phi(t)) = p_k(X) = [0, 1] = \varphi_k(t) \]

and

\[ p_k(\Psi(t)) = p_k(\{s\}) = [0, 1] = \varphi_k(t) \]
Lemma 2.40 Let \((A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})\) be any inverse system in \(\text{CHU}\) and let \(X = \varprojlim(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})\). Suppose that for an object \(Y\) of \(\text{CHU}\) and a family of morphisms \(\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}\) the diagram (2.2) commutes for any \(\alpha, \beta, \alpha < \beta\). Then \(\varphi : Y \to X\), defined by \(\varphi(y) = (\prod_{\gamma \in A} \varphi_\gamma(y)) \cap X\) for each \(y \in Y\), is a morphism in \(\text{CHU}\) such that for each \(\alpha \in A\) the diagram (2.3) commutes. Even more, for any morphism \(\Psi : Y \to X\) such that \(p_\alpha(\Psi(y)) = \varphi_\alpha(y)\) for each \(\alpha \in A\) and for each \(y \in Y\), \(\Psi(y) \subseteq \varphi(y)\) holds true for all \(y \in Y\).

**Proof.** We show that \(\varphi\) satisfies all the conditions in the following steps.

1. The set \(\prod_{\gamma \in A} \varphi_\gamma(y)\) is a closed subset of \(\prod_{\alpha \in A} X_\alpha\), therefore \(\varphi(y)\) is a closed subset of \(X\) for any \(y \in Y\).

2. Next we show that for any \(y \in Y\), the set \(\varphi(y)\) is nonempty. Let \(y \in Y\) be arbitrarily chosen. Next, let for each positive integer \(n\), \(A_n \subseteq A\) be the set of all elements \(\alpha \in A\) that have exactly \(n - 1\) predecessors and let \(\{i_n\}_{n=1}^\infty\) be the increasing sequence of all positive integers, such that \(A_{i_n} \neq \emptyset\) for each \(n\). For any \(\alpha \in A_{i_1}\) we arbitrarily choose \(t_\alpha \in \varphi_\alpha(y)\). For any \(\beta \in A_{i_2}\) there is an \(\alpha \in A_{i_1}\) such that \(\alpha < \beta\). For any such \(\alpha\) and \(\beta\) it follows from \(t_\alpha \in \varphi_\alpha(y) \subseteq f_{\alpha\beta}(\varphi_\beta(y))\) that there is \(t_\beta \in \varphi_\beta(y)\) such that \(t_\alpha \in f_{\alpha\beta}(t_\beta)\). We choose and fix such \(t_\beta\) for each \(\beta \in A_{i_2}\). Suppose that we have already constructed \(t_\alpha \in \varphi_\alpha(y)\) for all \(\alpha \in A_{i_n}\). Then for any \(\beta \in A_{i_{n+1}}\) there is an \(\alpha \in A_{i_n}\) such that \(\alpha < \beta\). For any such \(\alpha\) and \(\beta\) it follows from \(t_\alpha \in \varphi_\alpha(y) \subseteq f_{\alpha\beta}(\varphi_\beta(y))\) that there is \(t_\beta \in \varphi_\beta(y)\) such that \(t_\alpha \in f_{\alpha\beta}(t_\beta)\). We choose and fix such \(t_\beta\) for each \(\beta \in A_{i_{n+1}}\).

Then obviously \(x = (t_\alpha)_{\alpha \in A} \in \varphi(y)\) and therefore \(\varphi(y)\) is nonempty.
3. We show that $\varphi$ is an u.s.c. function. Let $y \in Y$ be arbitrary point and let

$$U = (U_{\gamma_1} \times U_{\gamma_2} \times U_{\gamma_3} \times \cdots \times U_{\gamma_n}) \times \prod_{\gamma \in A \setminus \{\gamma_1, \gamma_2, \ldots, \gamma_n\}} X_{\gamma}$$

be an open set in $X$ such that $\varphi(y) \subseteq U$, where for each $i = 1, 2, 3, \ldots, n$, $U_{\gamma_i}$ is an open set in $X_{\gamma_i}$. It follows from the definitions of $\varphi$ and $U$ that $\varphi_{\gamma_i}(y) \subseteq U_{\gamma_i}$ for each $i = 1, 2, 3, \ldots, n$. Since each $\varphi_{\gamma_i}$ is u.s.c., there is an open set $V_i$ in $Y$ such that

(a) $y \in V_i$;

(b) for each $x \in V_i$, it holds that $\varphi_{\gamma_i}(x) \subseteq U_{\gamma_i}$

for each $i$. We define $V = \bigcap_{i=1}^n V_i$. Then $V$ is an open set in $Y$, such that

(a) $y \in V$;

(b) for each $x \in V$, it holds that $\varphi(x) = \prod_{\gamma \in A} \varphi_{\gamma}(x) \subseteq U$.

Therefore $\varphi$ is an u.s.c. function and so it is a morphism from $Y$ to $X$.

4. Next we show that the diagram (2.3) commutes, i.e. for any $\alpha \in A$ and any $y \in Y$, $\varphi_{\alpha}(y) = (p_{\alpha} \circ \varphi)(y)$ holds true. Choose any $\alpha \in A$ and any $y \in Y$. Obviously

$$p_{\alpha}(\varphi(y)) = p_{\alpha}((\prod_{\gamma \in A} \varphi_{\gamma}(y)) \cap X) \subseteq p_{\alpha}(\prod_{\gamma \in A} \varphi_{\gamma}(y)) = \varphi_{\alpha}(y).$$

Next we show that $\varphi_{\alpha}(y) \subseteq p_{\alpha}(\varphi(y))$. Let $z \in \varphi_{\alpha}(y)$ be arbitrarily chosen. We show that $z \in p_{\alpha}(\varphi(y))$ by showing that there is a point $x \in \varphi(y)$ such that $z \in p_{\alpha}(x)$. As before, let $\{i_n\}_{n=1}^\infty$ be the increasing sequence of all positive integers, such that $A_{i_n} \neq \emptyset$ for each $n$. Next let $i_k$ be the positive integer such that $\alpha \in A_{i_k}$. For each $\gamma \in A_{i_k} \setminus \{\alpha\}$ let $t_\gamma \in \varphi_{\gamma}(y)$ be arbitrary and let $t_\alpha = z$. For each $\gamma \in A_{i_{k-1}}$ we choose $t_\gamma \in \varphi_{\gamma}(y)$ such that if $\alpha \in A_{i_{k-1}}$, $\beta \in A_{i_k}$, and $\alpha < \beta$, then $t_\alpha \in f_{\alpha,\beta}(t_\beta)$. This can be done since $f_{\alpha,\beta}(\varphi_{\beta}(y)) = \varphi_{\alpha}(y)$ and therefore $f_{\alpha,\beta}(t_\beta) \subseteq \varphi_{\alpha}(y)$.

Continuing in the same fashion we choose for each $j = 1, 2, 3, \ldots, k - 1$ and each $\gamma \in A_{i_j}$ an element $t_\gamma \in \varphi_{\gamma}(y)$ such that $t_\alpha \in f_{\alpha,\beta}(t_\beta)$ for each $\alpha \in A_{i_j}$,
2.4 Inverse limits in the category $\mathcal{CHU}$

$\beta \in A_{i+1}, \alpha < \beta$.

Next, for each $\beta \in A_{i+1}$ and for each $\alpha \in A_i$ such that $\beta > \alpha$, since $t_{\alpha} \in \varphi_{\alpha}(y) = f_{\alpha\beta}(\varphi_{\beta}(y))$, there is $t_{\beta} \in \varphi_{\beta}(y)$, such that $t_{\alpha} \in f_{\alpha\beta}(t_{\beta})$.

We continue inductively in the same fashion and choose for each $j = k+1, k+2, k+3, \ldots$ and each $\beta \in A_{i+1}$ an element $t_{\beta} \in \varphi_{\alpha}(y)$ such that $t_{\alpha} \in f_{\alpha\beta}(t_{\beta})$ for each $\alpha \in A_i$, such that $\alpha < \beta$.

Let $x \in \prod_{\gamma \in A} X_{\gamma}$ be such an element that $p_{\gamma}(x) = \{t_{\gamma}\}$ for each $\gamma \in A$. It follows from the construction of $x$ that $x \in \varphi(y)$ and $z \in p_{\alpha}(x)$.

5. Suppose that $\psi : Y \to X$ is a morphism in $\mathcal{CHU}$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_{\alpha}(\psi(y)) = \varphi_{\alpha}(y)$. Let $y \in Y$ be arbitrary and let $z \in \psi(y)$. Obviously $z \in X$ since $\psi$ is a morphism from $Y$ to $X$. It follows from $p_{\alpha}(z) \subseteq p_{\alpha}(\psi(y)) = \varphi_{\alpha}(y)$ (for each $\alpha$) that $z \in \prod_{\gamma \in A} \varphi_{\gamma}(y)$. Therefore $z \in \varphi(y)$ and hence $\psi(y) \subseteq \varphi(y)$.

□

Next we introduce the notion of weak inverse limits in $\mathcal{CHU}$ and show that $\lim_{\leftarrow} (A, \{X_{\alpha}\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ (together with the projections) is always a weak inverse limit in $\mathcal{CHU}$. First we define a weak commutation of a diagram in the category $\mathcal{CHU}$.

**Definition 2.41** Let $X, Y, Z \in \text{Ob}(\mathcal{CHU})$ and let $f : X \to Y$, $g : X \to Z$ and $h : Z \to Y$ be any morphisms in $\mathcal{CHU}$. The diagram

\[
\begin{array}{c}
X \\
\downarrow f \\
Y \\
\downarrow h \\
Z \\
\end{array}
\]

\[
\begin{array}{c}
g \\
\downarrow \\
Z
\end{array}
\]

weakly commutes, if for any $x \in X$, $f(x) \subseteq (h \circ g)(x)$.

Example 2.42 Let $f : [0, 1] \to 2^{[0, 1]}$, $g : [0, 1] \to 2^{[0, 1]}$ be identity functions on $[0, 1]$ and let $h : [0, 1] \to 2^{[0, 1]}$ be defined by

$$h(x) = [0, 1]$$

for all $x \in [0, 1]$. Then the diagram

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow{g} & & \downarrow{h} \\
[0, 1] & \xrightarrow{h} & [0, 1]
\end{array}
\]

weakly commutes but does not commute.

In the following definition we generalize the notion of inverse limits in $CHU$.

Definition 2.43 An object $X \in Ob(CHU)$, together with morphisms $\{p_\alpha : X \to X_\alpha | \alpha \in A\}$, is a weak inverse limit of an inverse system

$$\left( A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A} \right)$$

in $CHU$, if

1. for all $\alpha, \beta \in A$, from $\alpha \leq \beta$ it follows that the diagram (2.1) weakly commutes;

2. for any object $Y \in Ob(CHU)$ and any family of morphisms $\{\varphi_\alpha : Y \to X_\alpha | \alpha \in A\}$ it follows that if the diagram (2.2) commutes, then for any morphism $\Psi : Y \to X$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq (\prod_{\gamma \in A} \varphi_\gamma(y)) \cap X$ holds true for all $y \in Y$.

Note that each inverse limit in $CHU$ is always a weak inverse limit in $CHU$.

Example 2.44 Let $X = \varprojlim(\mathbb{N}, \{[0, 1]\}_{k \in \mathbb{N}}, \{f_{k\ell}\}_{k, \ell \in \mathbb{N}})$ be the inverse limit with u.s.c. set-valued bonding functions that we defined in Example 2.39. Then $X$, together with the projection mappings, is obviously not an inverse limit but it is a weak inverse limit in $CHU$. 

In the following theorem we show that the inverse limits with upper semicontinuous set-valued bonding functions together with projections are always weak inverse limits in $\text{CHU}$.

**Theorem 2.45** Let $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ be any inverse system in $\text{CHU}$. Then the inverse limit with u.s.c. set-valued bonding functions

$$\lim\left\langle A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A} \right\rangle,$$

together with projections

$$p_\gamma : \lim\left\langle A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A} \right\rangle \to X_\gamma,$$

$$p_\gamma((x_\alpha)_{\alpha \in A}) = \{x_\gamma\},$$

is a weak inverse limit of the inverse system $(A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A})$ in $\text{CHU}$.

**Proof.** Let $X = \lim\left\langle A, \{X_\alpha\}_{\alpha \in A}, \{f_{\alpha\beta}\}_{\alpha, \beta \in A} \right\rangle$. First we prove that the diagram (2.1) weakly commutes. Choose any $x \in X$ and let $\alpha < \beta$. Then $p_\alpha(x) = \{x_\alpha\} \subseteq f_{\alpha\beta}(\{x_\beta\}) = (f_{\alpha\beta} \circ p_\beta)(x)$.

Next, suppose that for an object $Y \in \text{CHU}$ and a family of morphisms $\{\varphi_\alpha : Y \to X_\alpha \mid \alpha \in A\}$ the diagram (2.2) commutes. By Lemma 2.40, for any morphism $\Psi : Y \to X$ such that for each $\alpha \in A$ and for each $y \in Y$, $p_\alpha(\Psi(y)) = \varphi_\alpha(y)$, $\Psi(y) \subseteq \left(\prod_{\gamma \in A} \varphi_\gamma(y)\right) \cap X$ holds true for all $y \in Y$. $\square$
2.5 Compactifications

In general topology, compactification is the result of making a topological space a compact one. The fact that large and interesting classes of non-compact spaces do in fact have compactifications of particular sorts, makes it a common technique in topology. For instance, see [19] and [55]. For us, compactification is another important apparatus for constructing new examples of continua.

**Definition 2.46** $X$ is a compactification of a space $R$ with a space $S$ if a topological copy $R'$ of $R$ is a dense subset of $X$, where $S = X \setminus R'$ is called the remainder of the compactification.

The following are simple examples of compactifications.

**Example 2.47** $X = [0, 1]$ is obviously the compactification of $(0, 1]$ with $\{0\}$. $X$ can be also represented as a compactification of $(0, 1)$ with the set $\{0, 1\}$.

**Example 2.48** If $R = \mathbb{R}$, then $X = \mathbb{R} \cup \{\infty\}$ is the compactification of $R$ with $\{\infty\}$ and it is a simple closed curve. Informally speaking, the compactification is created by tying together the two ends of $\mathbb{R}$ by adding one new point at infinity.

The compactifications, that will be important for us, are compactifications of a ray with a continuum.

**Definition 2.49** A ray is a space that is homeomorphic to the closed half-line $[0, \infty) \subseteq \mathbb{R}$. Sometimes we will just denote it by $[0, \infty)$.

**Theorem 2.50** Compactification of a ray with a continuum is again a continuum.

**Proof.** Let $S$ be a continuum an let $X = S \cup R$ be a compactification of a ray $R$ with $S$. Obviously $X$ is nonempty and compact metric space. Since $R$ is connected, also $X = \text{Cl}(R)$ is connected. Therefore $X$ is a continuum. □

According to remainder, we know compactifications of rays with a point, an arc, a simple closed curve and so on. See the following examples.
2.5 Compactifications

**Example 2.51** We give an illustration of a compactification of ray with a point and another one, a compactification of a ray with a simple closed curve.

Figure 2.9: An example of compactification of a ray with a point and with a simple closed curve

Also $\sin \frac{1}{x}$-continuum is an example of a compactification of a ray. It is a compactification of a ray with an arc. But not all such compactifications are homeomorphic to the $\sin \frac{1}{x}$-continuum. In fact there exist uncountable many nonhomeomorphic compactifications of a ray with remainder being an arc. See [1] and the following example.

**Example 2.52** We give an illustration of two nonhomeomorphic examples of compactification of a ray with an arc. For more detail see [1].

Figure 2.10: Two nonhomeomorphic examples of compactification of a ray with an arc
There are many researchers that use the above mentioned apparatus as a main tool in their research. For instance, in [12] D. P. Bellamy with techniques involving compactifications demonstrated the existence of an uncountable collection of chainable continua, no member of which can be mapped onto another member, while V. Martínez-de-la-Vega constructed an uncountable family of metric compactifications of the ray with the remainder being the pseudo-arc in [42].
WAŻEWSKI’S UNIVERSAL DENDRITE AS AN INVERSE LIMIT WITH ONE SET-VALUED BONDING FUNCTION

We already mentioned that using inverse sequences, where each of the factor space is a closed unit interval $[0, 1]$ and all of the bonding functions are the same, we can get very complicated continua. The continuum that will be constructed in this chapter is the Ważewski’s universal dendrite. It is an example of a dendrite, which contains a topological copy of any dendrite whatsoever. It was described by T. Ważewski in 1923, [57]. In [45] one can find a construction of Ważewski’s universal dendrite using inverse limits. In particular, it is constructed as the inverse limit of an inverse sequence of planar dendrites $D_n$ and monotone bonding mappings $f_n : D_{n+1} \to D_n$, where dendrites $D_n$ and bonding maps $f_n$ are getting more and more complicated as $n$ increases. This construction gave two results at the same time. First, that there is a universal dendrite and second, every dendrite is embeddable in the plane. In this chapter we construct Ważewski’s universal dendrite as the inverse limit of an inverse sequence of closed unit intervals and a single upper semi-continuous set-valued bonding function. This new presentation of Ważewski’s universal dendrite shows the strength of the theory of the inverse limits with upper semi-continuous set-valued bonding functions. All results in this chapter are collected in [9].

First we introduce some basic terms that will be used.
Ważewski’s universal dendrite as an inverse limit with one set-valued bonding function

**Definition 3.1** Let $D$ be a dendrite, $b \in D$, and $\beta$ a cardinal number. We say that $b$ is of order less than or equal to $\beta$ in $D$, written $\text{ord}(b, D) \leq \beta$, provided that for each open neighborhood $U$ of $b$ in $D$, there is an open neighborhood $V$ of $b$ in $D$, such that $b \in V \subseteq U$ and $|\text{Bd}(V)| \leq \beta$. We say that $b$ is of order $\beta$, $\text{ord}(b, D) = \beta$, provided that $\text{ord}(b, D) \leq \beta$ and $\text{ord}(b, D) \neq \alpha$ for any cardinal number $\alpha < \beta$.

Points of order 1 in a dendrite $D$ are called end points of $D$, the set of all end points of $D$ is denoted by $E(D)$. Points of order $n > 2$ are called ramification points and the set of all ramification points of $D$ is denoted by $R(D)$.

**Definition 3.2** A free arc in a dendrite $D$ is an arc such that all its points except the end points are of order 2 in $D$. In particular, a maximal free arc in a dendrite $D$ is an arc $A$ with end points $x$ and $y$ in $D$ such that $A \cap (E(D) \cup R(D)) = \{x, y\}$.

**Definition 3.3** A star $S$ (with center $c$ and beams $B_n$) is any countable union $\bigcup_{n=1}^{\infty} B_n$ of arcs $B_n$, each having $c$ as an end point, such that $B_n \cap B_m = \{c\}$ when $m \neq n$ and $\lim_{n \to \infty} \text{diam}(B_n) = 0$.

![Figure 3.1: A star](image)

The term universal can be defined for any continuum. We will use this term only for dendrites, see the definition below.

**Definition 3.4** A dendrite is said to be universal if it contains a homeomorphic copy of any other dendrite.

The well known example of universal dendrite is the Ważewski’s universal dendrite, which is constructed as follows.
Let $D_1$ be a star in a compact metric space $X$. Let $c_B$ denote a point in the interior of each beam $B$ of $D_1$. Let $C_1 = \{x_1, x_2, x_3, \ldots\}$ be any countable subset of the set $\{c_B \mid B$ is a beam in $D_1\}$. For each positive integer $i$, form a star $S_i$ in $X$ with center $x_i$ and otherwise disjoint from $D_1$, making sure that $S_i \cap S_j \neq \emptyset$ only when $i = j$ and $\lim_{i \to \infty} \text{diam}(S_i) = 0$. Let $D_2 = D_1 \cup (\bigcup_{i=1}^{\infty} S_i)$. Next define $D_3$ in a similar manner. Let $c_A$ denote a point in the interior of each maximal free arc $A$ in $D_2$. Let $C_2 = \{x_1, x_2, x_3, \ldots\}$ be any countable subset of the set $\{c_A \mid A$ is a free arc in $D_2\}$. For each positive integer $i$, form a star $S_i$ in $X$ with center $x_i$ and otherwise disjoint from $D_2$, making sure that $S_i \cap S_j \neq \emptyset$ only when $i = j$ and $\lim_{i \to \infty} \text{diam}(S_i) = 0$. Let $D_3 = D_2 \cup (\bigcup_{i=1}^{\infty} S_i)$. Continuing in this fashion, we obtain a continuum $D_n$ for each positive integer $n$. The following theorem is a well-known fact.

**Theorem 3.5** For each positive integer $n$, $D_n$ is a dendrite.

**Proof.** See [45].

The construction of the continuum, homeomorphic to Ważewski’s universal dendrite in [45, p. 181] uses the above mentioned construction of a chain of dendrites $D_1 \subseteq D_2 \subseteq D_3 \subseteq \ldots$, then defines certain bonding maps $f_n : D_{n+1} \to D_n$, and then obtains Ważewski’s universal dendrite as $\lim_{k \to \infty} \{D_k, f_k\}_{k=1}^{\infty}$.

Finally we state a result that is characterizing Ważewski’s universal dendrite that will be needed below.

**Theorem 3.6** For any dendrite $D$, $D$ is homeomorphic to Ważewski’s universal dendrite if and only if its set of ramification points is dense in $D$ and each of its ramification points is of infinite order in $D$.

**Proof.** See [15, p. 169] and [57, p. 123].

Next we construct a family of upper semi-continuous set-valued functions $f : [0, 1] \to 2^{[0,1]}$, such that for each the inverse limit of the inverse sequence of intervals $[0, 1]$ with $f$ as the only bonding function is a dendrite. We introduce some notation first.
Let $A \subseteq [0, 1] \times [0, 1]$ be defined by

$$A = \{(t, t) \in [0, 1] \times [0, 1] \mid t \in [0, 1]\}.$$

For any positive integer $n$, let $\{(a_i, b_i)\}_{i=1}^n$ be a finite sequence in $[0, 1] \times [0, 1]$, such that $a_i < b_i$ for each $i = 1, 2, 3, \ldots, n$ and $a_i \neq a_j$ whenever $i \neq j$. Next denote by $A(a_i, b_i)_{i=1}^n$ the union

$$A(a_i, b_i)_{i=1}^n = \bigcup_{i=1}^n ([a_i, b_i] \times \{a_i\}) \subseteq [0, 1] \times [0, 1].$$

Then

$$G(a_i, b_i)_{i=1}^n = A \cup A(a_i, b_i)_{i=1}^n$$

is closed in $[0, 1] \times [0, 1]$, since it is a union of finitely many closed arcs. Furthermore $\pi_1(G(a_i, b_i)_{i=1}^n) = \pi_2(G(a_i, b_i)_{i=1}^n) = [0, 1]$. Therefore by Theorem 2.18 there is a surjective u.s.c. function $f(a_i, b_i)_{i=1}^n : [0, 1] \rightarrow 2^{[0,1]}$ such that its graph $\Gamma(f(a_i, b_i)_{i=1}^n)$ equals to $G(a_i, b_i)_{i=1}^n$.

**Definition 3.7** Let $n$ be a positive integer and $\{(a_i, b_i)\}_{i=1}^n$ be a subset of $[0, 1] \times [0, 1]$, such that $0 < a_i < b_i$ for each $i = 1, 2, 3, \ldots, n$ and $a_i \neq a_j$ whenever $i \neq j$. Then $f : [0, 1] \rightarrow 2^{[0,1]}$ is called an $n$-comb function with respect to $\{(a_i, b_i)\}_{i=1}^n$, if $f = f(a_i, b_i)_{i=1}^n$.

We also say that $f : [0, 1] \rightarrow 2^{[0,1]}$ is an $n$-comb function, if $f$ is an $n$-comb function with respect to some $\{(a_i, b_i)\}_{i=1}^n$. 

![Figure 3.2: The graph of an 8-comb function](image)
It is not necessary to eliminate the possibility \( a_i = 0 \) for some \( i \) (all the proofs in this chapter would go through also in such case), but we have chosen to do so in order to reduce the number of cases that must be examined in the proofs and since the main result can be obtained with this restriction in place.

**Definition 3.8** Let for each \( j, i_j \) be a nonnegative integer. We use

\[
(a_{i_1}^{i_1}, a_{i_2}^{i_2}, a_{i_3}^{i_3}, \ldots)
\]

to denote the point \( (\underbrace{a_1, a_1, \ldots, a_1}_{i_1}, \underbrace{a_2, a_2, \ldots, a_2}_{i_2}, \ldots) \) and

\[
(a_{i_1}^{i_1}, a_{i_2}^{i_2}, a_{i_3}^{i_3}, \ldots, a_{i_j}^{i_j}, t^\infty)
\]

to denote the point \( (\underbrace{a_1, a_1, \ldots, a_1}_{i_1}, \underbrace{a_2, a_2, \ldots, a_2}_{i_2}, \ldots, a_j, a_j, \ldots, a_j, t, t, t, \ldots) \).

**Example 3.9** Let \( f \) be an 1-comb function with respect to \( \{(a_i, b_i)\}_{i=1}^1 \).

Then \( x \in \varprojlim \{[0,1], f\}_{k=1}^\infty \) if and only if

1. either \( x = (t^\infty) \) for some \( t \in [0,1] \) or
2. there is a positive integer \( n \) such that \( x = (a_i^n, t^\infty) \) for some \( t \in (a_1, b_1] \).

Therefore \( \varprojlim \{[0,1], f\}_{k=1}^\infty \) is the star with the center \( (a_1^\infty) \) and beams \( B_0 = \{(t^\infty) \mid t \in [0, a_1]\} \), \( B_0' = \{(t^\infty) \mid t \in [a_1, 1]\} \) and \( B_n = \{(a_1^n, t^\infty) \mid t \in [a_1, b_1]\}, n = 1, 2, 3, \ldots \)

![Figure 3.3: The graph of an 1-comb function and its inverse limit](image)
Example 3.10 Let \( f \) be a 2-comb function with respect to \( \{(a_i, b_i)\}_{i=1}^2 \), where \( a_1 < a_2 \).

We distinguish the following two cases:

1. \( b_1 < a_2 \) Then \( x \in \lim_{k \to \infty} \{[0,1], f\}_{k=1}^\infty \) if and only if

   (a) either \( x = (t^\infty) \) for some \( t \in [0,1] \) or

   (b) there is a positive integer \( n \) such that \( x = (a_1^n, t^\infty) \) for some \( t \in (a_1, b_1] \) or

   (c) there is a positive integer \( n \) such that \( x = (a_2^n, t^\infty) \) for some \( t \in (a_2, b_2] \).

Therefore \( \lim_{k \to \infty} \{[0,1], f\}_{k=1}^\infty \) is the union of two stars. The star \( S \) with the center \( (a_1^\infty) \) and beams \( B_0 = \{(t^\infty) \mid t \in [0, a_1]\} \), \( B'_0 = \{(t^\infty) \mid t \in (a_1, 1]\} \) and \( B_n = \{(a_1^n, t^\infty) \mid t \in [a_1, b_1]\} \), \( n = 1, 2, 3, \ldots \), and the star \( S_0 \) with the center \( (a_2^\infty) \) and beams \( C_n = \{(a_2^n, t^\infty) \mid t \in [a_2, b_2]\} \), \( n = 1, 2, 3, \ldots \).

![Figure 3.4: The graph of a 2-comb function and its inverse limit, \( b_1 < a_2 \)](image)

2. \( b_1 \geq a_2 \)

Then \( x \in \lim_{k \to \infty} \{[0,1], f\}_{k=1}^\infty \) if and only if

(a) either \( x = (t^\infty) \) for some \( t \in [0,1] \) or

(b) there is a positive integer \( n \) such that \( x = (a_1^n, t^\infty) \) for some \( t \in (a_1, b_1] \) or

(c) there is a positive integer \( n \) such that \( x = (a_2^n, t^\infty) \) for some \( t \in (a_2, b_2] \) or

(d) there are positive integers \( n \) and \( m \) such that \( x = (a_1^n, a_2^m, t^\infty) \) for some \( t \in (a_2, b_2] \).

Therefore \( \lim_{k \to \infty} \{[0,1], f\}_{k=1}^\infty \) is the union of countable many stars. The star \( S \) with the center \( (a_1^\infty) \) and beams \( B_0 = \{(t^\infty) \mid t \in [0, a_1]\} \), \( B'_0 = \{(t^\infty) \mid t \in [a_1, 1]\} \) and...
$B_n = \{(a_1^n, t^\infty) \mid t \in [a_1, b_1]\}, \ n = 1, 2, 3, \ldots$, the star $S_0$ with the center $(a_2^\infty)$ and beams

$$C_n = \{(a_2^n, t^\infty) \mid t \in [a_2, b_2]\},$$

$n = 1, 2, 3, \ldots$, and for each positive integer $k$ the star $S_k$ with the center $(a_1^k, a_2^\infty)$ and beams

$$C_n^k = \{(a_1^k, a_2^n, t^\infty) \mid t \in [a_2, b_2]\},$$

$n = 1, 2, 3, \ldots$

Figure 3.5: The graph of a 2-comb function and its inverse limit, $b_1 > a_2$

Figure 3.6: The graph of a 2-comb function and its inverse limit, $b_1 = a_2$

Note that if $b_1 = a_2$ the stars $S_k, \ k = 1, 2, 3, \ldots$, are attached at the end points $(a_1^k, b_1^\infty)$ of $S$, and if $b_1 > a_2$ the stars $S_k, \ k = 1, 2, 3, \ldots$, are attached at the interior points of the maximal free arcs $\{(a_1^k, t^\infty) \mid t \in [a_1, b_1]\}$ of $S, \ k = 1, 2, 3, \ldots$

In the following theorem we show that any inverse limit of intervals $[0, 1]$ and a single $n$-comb function is a dendrite.
Theorem 3.11 Let $n$ be any positive integer and let $f : [0, 1] \to 2^{[0,1]}$ be any $n$-comb function. Then $\lim_{k=1}^{\infty}\{[0, 1], f\}$ is a dendrite.

Proof. We prove Theorem 3.11 by induction on $n$ by proving the more precise claim that includes also information about maximal free arcs and ramification points in the dendrite. For each positive integer $\ell$, let us introduce the following notation for certain statements that will be used in the inductive proof of the theorem:

(a) $\ell$ The inverse limit $D_\ell = \lim_{k=1}^{\infty}\{[0, 1], f_{(a_i,b_i)_{i=1}^{\ell}}\}$ is a dendrite.

(b) $\ell$ The points of the form $(x_1, x_2, x_3, \ldots, x_m, a^{\infty}_j) \in D_\ell$, $j \leq \ell$, are exactly the ramification points of $D_\ell$.

(c) $\ell$ The points of the form $(x_1, x_2, x_3, \ldots, x_m, b^{\infty}_i) \in D_\ell$, $i \leq \ell$, where $m \geq 1$, $a_i = x_m \neq b_i$, and $b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_\ell\}$, are endpoints of $D_\ell$.

(d) $\ell$ All endpoints of $D_\ell$ are of such form, except endpoints $(0^{\infty})$ and $(1^{\infty})$.

(e) $\ell$ The maximal free arc in $D_\ell$ having the point

$$x = (x_1, x_2, x_3, \ldots, x_m, b^{\infty}_\ell)$$

described in (c) as one endpoint, has

$$(x_1, x_2, x_3, \ldots, x_m, a^{\infty}_\ell)$$

as the other endpoint if $a_\ell < b_i$; if $a_\ell > b_i$ then the maximal free arc in $D_\ell$ ending at $x$ equals to the maximal free arc in $D_{\ell-1}$ ending at $x$.

(f) $\ell$ The arc with endpoints $(a^{\infty}_\ell)$ and $(1^{\infty})$ is a maximal free arc in $D_\ell$.

1. Let $n = 1$. There are $a_1, b_1 \in [0, 1]$ such that $a_1 < b_1$ and $f = f_{(a_i,b_i)_{i=1}^1}$. In Example 3.9 it was shown that the inverse limit $D_1 = \lim_{k=1}^{\infty}\{[0, 1], f\}$ is a star, and is therefore a dendrite. We see that $(a^{\infty}_1)$ is the only ramification point of $D_1$, and that maximal free arcs of $D_1$ are exactly the beams $B_0 = \{(t^{\infty}) \mid t \in [0, a_1]\}$, $B'_0 = \{(t^{\infty}) \mid t \in [a_1, 1]\}$ and $B_k = \{(a^k_1, t^{\infty}) \mid t \in [a_1, b_1]\}$, $k = 1, 2, 3, \ldots$ of the star $D_1$. Note that (a)–(f) hold true.
2. Let \( f \) be any \( n \)-comb function, \( n \geq 2 \). Without loss of generality we may assume that \( f = f_{(a_i, b_i)}^{n-1} \), where \( a_1 < a_2 < a_3 < \ldots < a_n \).

Let, as the inductive assumption, \((a)_{n-1}-(f)_{n-1}\) hold true for the function \( f_{(a_i, b_i)}^{n-1} \).

We show that the inverse limit 
\[
\varprojlim \{ [0, 1], f \}_{k=1}^\infty = D_n = \varprojlim \{ [0, 1], f_{(a_i, b_i)}^n \}_{k=1}^\infty
\]
satisfies all the above mentioned properties for \( \ell = n \).

By the inductive assumption \( D_{n-1} = \varprojlim \{ [0, 1], f_{(a_i, b_i)}^{n-1} \}_{k=1}^\infty \) is a dendrite.

Case 1. \( a_n > b_i \) for each \( i = 1, 2, 3, \ldots, n-1 \)

In this case any \( x \in D_n \setminus D_{n-1} \) is of the form \( x = (a^n_k, t^\infty) \), where \( k \) is a positive integer and \( t \in (a_n, b_n) \). Therefore

\[
D_n = D_{n-1} \cup S,
\]

where \( S = \{(a^n_k, t^\infty) \mid k \in \mathbb{N}, t \in (a_n, b_n)\} \), and we see that \( S \) is a star with the center \((a^n_\infty) \in D_n \setminus R(D_{n-1}) \). Obviously \((a)_{n}-(f)_{n}\) hold true.

Case 2. \( a_n \leq b_i \) for some \( i = 1, 2, 3, \ldots, n-1 \)

In this case we show that

\[
D_n = D_{n-1} \cup \left( \bigcup S \right),
\]

where

(a) \( S = \{S_1, S_2, S_3, \ldots\} \) is a countable family of stars with centers \( c_1, c_2, c_3, \ldots \) respectively, where \( c_1, c_2, c_3, \ldots \in D_n \setminus R(D_{n-1}) \), and each of the maximal free arcs in \( D_{n-1} \) contains at most one of these centers,

(b) for each positive integer \( i \), \( S_i \cap D_{n-1} = \{c_i\} \),

(c) \( S_i \cap S_j = \emptyset \) if \( i \neq j \), and

(d) \( \lim_{i \to \infty} \text{diam}(S_i) = 0 \),
and therefore it will follow that $D_n$ is a dendrite by Theorem 3.5, using $(a)_{n-1}$. That will prove $(a)_n$.

Any point of $D_n \setminus D_{n-1}$ is of the form $(x_1, x_2, x_3, \ldots, x_m, a^k_n, t^\infty)$, where $k$ is a positive integer, $m$ is a nonnegative integer, $t \in (a_n, b_n)$, and $x_m \neq a_n$, and vice versa.

The set

$$\{(x_1, x_2, x_3, \ldots, x_m, a^k_n, t^\infty) \mid k \geq 1, t \in [a_n, b_n]\}$$

is a star centered in $(x_1, x_2, x_3, \ldots, x_m, a^\infty_n)$ having the beams

$$\{(x_1, x_2, x_3, \ldots, x_m, a^k_n, t^\infty) \mid t \in [a_n, b_n]\},$$

for each $k \geq 1$. Note that $S$ is infinite since for each $i$ such that $a_n \leq b_i$ the family $S$ contains stars centered at $(a^k_i, a^\infty_n)$ for each positive integer $k$.

From $f^{-1}_{(a_i, b_i)}(a_n) = \{a_n\}$ it follows that if for $x \in D_{n-1}$ and for some positive integer $m$, $p_m(x) = a_n$, then $p_{m+1}(x) = a_n$. Therefore such $x$ ends with the block $a^\infty_n$. Let $X_1 = \{(a^\infty_n)\}$, and let for each positive integer $m \geq 2$,

$$X_m = \{x \in D_{n-1} \mid p_m(x) = a_n, p_{m-1}(x) \neq a_n\}.$$ 

Then $X_m$ is a finite set for each $m$. Therefore $X = \bigcup_{m=1}^{\infty} X_m$ is a finite or countable infinite subset of $D_{n-1} \setminus R(D_{n-1})$ ($\langle b \rangle_{n-1}$ is also used). Also, each maximal free arc of $D_{n-1}$ contains at most one $x \in X$. To prove this, we shall, for each $x \in X$, find the uniquely determined maximal free arc of $D_{n-1}$ containing $x$. Let

$$x = (x_1, x_2, x_3, \ldots, x_m, a^\infty_n) \in X,$$

where $x_m \neq a_n$. Then $x_m = a_i$ for some $i < n$. Note that since $a_i \in f^{-1}_{(a_i, b_i)}(a_n)$, it follows that $a_n \in [a_i, b_i]$ and therefore $b_i \geq a_n$. Now we distinguish two cases, $b_i > a_n$ and $b_i = a_n$.

If $b_i > a_n$, then $b_i \not\in \{a_{i+1}, a_{i+2}, \ldots, a_n\}$, hence $(x_1, x_2, \ldots, x_m, b^\infty_i)$ is an endpoint of $D_{n-1}$ by $(c)_{n-1}$ and the arc

$$\{(x_1, x_2, x_3, \ldots, x_m, t^\infty) \mid t \in [a_{n-1}, b_i]\}$$
is a maximal free arc of $D_{n-1}$ by $(e)_{n-1}$. Obviously $x$ belongs to the arc, since $a_n \in [a_{n-1}, b]$. If $b_i = a_n$, then $x$ is an endpoint of $D_{n-1}$ by $(c)_{n-1}$, and clearly it belongs to the maximal free arc \{(x_1, x_2, x_3, \ldots, x_m, t^\infty) \mid t \in [a_{n-1}, b_i]\} of D_{n-1}, which is a maximal free arc in $D_{n-1}$ by $(e)_{n-1}$.

Now, when we have the explicit description of all maximal free arcs in $D_{n-1}$ containing elements of $X$, we see that each such maximal free arc contains exactly one point from $X$.

Take any $x = (x_1, x_2, x_3, \ldots, x_m, a_n^\infty) \in X$, where $x_m \neq a_n$. Then $x_m = a_i$ for some $i < n$. For each positive integer $k$, let

$$B_k = \{(x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty) \mid t \in [a_n, b_n]\}.$$

Obviously, $B_k$ is an arc in $D_n$ and $S(x) = \bigcup_{k=1}^\infty B_k$ is a star centered at $x$. The diameter of $S(x)$ satisfies

$$\text{diam}(S(x)) \leq D((x_1, x_2, \ldots, x_m, 0^\infty), (x_1, x_2, \ldots, x_m, 1^\infty)) \leq \frac{1}{2^{m+1}}.$$

Since for each $m$ there are only finitely many such points $x \in X (X_m$ is finite), it follows that the set $S = \{S(x) \mid x \in X\}$ is finite or it can be presented as $S = \{S_1, S_2, S_3, \ldots\}$. From the above upper bound for the diameters of the stars in the infinite case it follows that $\lim_{i \to \infty} \text{diam}(S_i) = 0$.

Take any point $x \in D_n \setminus D_{n-1}$. As already noticed, it is of the form $x = (x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty)$, where $k$ is a positive integer, $m$ is a nonnegative integer, $t \in (a_n, b_n]$, and $x_m \neq a_n$. Therefore $x \in S(y)$, where $y = (x_1, x_2, \ldots, x_m, a_n^\infty) \in X$. Therefore

$$D_n \setminus D_{n-1} = \left( \bigcup_{x \in X} S(x) \right) \setminus X = \left( \bigcup S \right) \setminus X,$$

and finally

$$D_n = D_{n-1} \cup \left( \bigcup_{x \in X} S(x) \right) = D_{n-1} \cup \left( \bigcup S \right),$$

proving $(a)_n$. 

To prove that the points of the form \( x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \in D_n \), where \( i \leq n, x_m = a_i \), and \( b_i \notin \{a_{i+1}, a_{i+2}, \ldots, a_n\} \), are endpoints of \( D_n \), we distinguish two cases. If \( i \leq n - 1 \) then \( x \in D_{n-1} \), and then \( x \) is an endpoint of \( D_{n-1} \) by \((c)_{n-1}\), since \( b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{n-1}\} \). Since \( b_i \neq a_n \), the only star attached to the maximal free arc in \( D_{n-1} \) ending at \( x \) is centered at a point that differs from \( x \), or no star is attached to that arc at all, it follows that \( x \in E(D_n) \). If \( i = n \), then \( b_i = b_n \), and therefore \( x \) is an endpoint of a star from \( S \). That proves \((c)_n\).

Also each endpoint of \( D_n \) which belongs to \( D_{n-1} \), is also an endpoint in \( D_{n-1} \), therefore it is of the form \( x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \in D_n \), where \( x_m = b_i \) and \( b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{n-1}\} \), by \((d)_{n-1}\). Points of such form with \( b_i = a_n \) are centers of the newly attached stars and therefore are not endpoints of \( D_n \). It follows that \( b_i \neq a_n \), and therefore \( b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_n\} \). Each endpoint of \( D_n \), which belongs to \( D_n \) \( D_{n-1} \), is necessarily an endpoint of a newly attached star and therefore is of the form \( x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \), \( a_n = x_m \neq b_i \). Additional condition \( b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_n\} \) is satisfied vacuously for \( i = n \). Obviously \((0^\infty)\) and \((1^\infty)\) are endpoints of \( D_n \), too. That proves \((d)_n\).

Let \( x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \in D_n \) be any endpoint of \( D_n \) mentioned in \((c)_n\), where \( a_i = x_m \neq b_i \) and \( b_i \notin \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_n\} \). If \( i < n \) then by \((c)_{n-1}\), \( x \) is an endpoint of \( D_{n-1} \). If \( a_n = b_i \) then we have already proved that a new star centered at \( (x_1, x_2, x_3, \ldots, x_m, a_n^\infty) \) is attached to the maximal free arc of \( D_{n-1} \) ending at \( x \) and since no other star was attached to this arc it follows that \( (x_1, x_2, x_3, \ldots, x_m, a_n^\infty) \) is the other endpoint of the maximal free arc of \( D_n \) ending at \( x \). If \( a_n > b_i \) then no star was attached to the maximal free arc of \( D_{n-1} \) ending at \( x = (x_1, x_2, x_3, \ldots, x_m, b_i^\infty) \), and therefore it remained a maximal free arc of \( D_n \) as well. This proves \((e)_n\).

By \((f)_{n-1}\) the maximal free arc of \( D_{n-1} \) having \((1^\infty)\) as one endpoint has \((a_n^\infty)\) as the other endpoint. Since a star centered at \((a_n^\infty)\) was attached to \( D_{n-1} \), and since no other star was attached to the above mentioned arc, \((f)_n\) follows.
Finally (b)\textsubscript{n} follows from (b)\textsubscript{n-1} and from the fact that at each point of the form \((x_1, x_2, x_3, \ldots, x_m, a_\infty) \in D_n\) a new star was attached to \(D_{n-1}\).

\[
\square
\]

In the following remark we extract certain parts of the above proof for later use.

**Remark 3.12** Let \(n\) be a positive integer.

1. For each positive integer \(n\) and for each \(y \in D_n\), \(y\) is either of the form \(y = (t_\infty), t \in [0, 1]\), or of the form \(y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_m}^{k_m}, t_\infty),\) where \(m\) is a positive integer and for each \(\ell \leq m\) it holds that \(i_\ell \leq n, k_\ell > 0, a_{i_\ell} < a_{i_{\ell+1}} \leq b_\ell,\) and \(a_m \leq t \leq b_m\).

2. Any point of \(D_{n+1} \setminus D_n\) is of the form

\[
(x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, t_\infty),
\]

where \(k\) is a positive integer, \(m\) is a nonnegative integer, \(t \in (a_{n+1}, b_{n+1}]\), and \(x_m \neq a_{n+1}\).

3. \(x \in D_n\) is a ramification point in \(D_n\) if and only if there are positive integers \(m\) and \(j \leq n\), such that \(p_k(x) = a_j\) for each positive integer \(k \geq m\).

**Definition 3.13** We will use \(D_n\) to denote the dendrite

\[
D_n = \lim_{k \to \infty} \{[0, 1], f(a_i, b_i)^n\}_{k=1}^\infty.
\]

Next we define functions that we shall use later in proof of the main result of this chapter.

**Definition 3.14** We define the function \(f_n : D_{n+1} \to D_n\) by

\[
f_n(x) = \begin{cases} 
g_n(x) & ; \quad x \in \text{Cl}(D_{n+1} \setminus D_n), \\ 
x & ; \quad x \in D_n, 
\end{cases}
\]
Ważewski’s universal dendrite as an inverse limit with one set-valued bonding function

where \( g_n : \text{Cl}(D_{n+1} \setminus D_n) \to D_n \) is defined as follows. Any point of \( \text{Cl}(D_{n+1} \setminus D_n) \) is of the form

\[
x = (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, t) \in D_n,
\]

where \( k \) is a positive integer, \( m \) is a nonnegative integer, \( t \in [a_{n+1}, b_{n+1}] \), and \( x_m \neq a_{n+1} \) (see Remark 3.12), and we define

\[
g_n(x) = (x_1, x_2, x_3, \ldots, x_m, a^\infty_{n+1}).
\]

Note that \( f_n \) is continuous for each \( n \) by the pasting lemma (see [44, p. 108]).

**Lemma 3.15** Let \( x \in D_n \).

1. If

\[
x = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_j}^{k_j}, t) \in D_n,
\]

where \( j > 0, i_1, i_2, i_3, \ldots, i_j \leq n, a_{i_1} < a_{i_2} < \cdots < a_{i_j}, k_1, k_2, \ldots, k_j > 0 \), and \( t \in [a_{i_j}, b_{i_j}] \), then for each

\[
y \in f_n^{-1}(x)
\]

and for each \( i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1 \) it holds that \( p_i(x) = p_i(y) \).

2. If \( x = (t) \), \( t \in [0, 1] \), then for each

\[
y \in f_n^{-1}(x)
\]

it holds that \( p_1(x) = p_1(y) = t \).

**Proof.** If \( y \in D_n \), then \( y = x \) and the claim is obviously true. Note that in 1. from \( t = a_{i_j} \), it follows that \( y \in D_n \).

If \( y \in D_{n+1} \setminus D_n \), then by 2. from Remark 3.12, \( y \) is of the form

\[
(x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, s^\infty),
\]

where \( k \) is a positive integer, \( m \) is a nonnegative integer, \( s \in (a_{n+1}, b_{n+1}] \), and \( x_m \neq a_{n+1} \). Then \( x = f_n(y) = g_n(y) = (x_1, x_2, x_3, \ldots, x_m, a^\infty_{n+1}) \).

In 1. in the remaining case \( t \neq a_{i_j} \), it follows that \( m = k_1 + k_2 + k_3 + \ldots + k_j \) and
\[ t = a_{n+1}. \text{ Therefore } (x_1, x_2, x_3, \ldots, x_m, a_{n+1}) = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_j}^{k_j}, t). \]

In 2. it follows that \( m = 0 \) and \( t = a_{n+1}. \)

\[ \square \]

**Lemma 3.16** Let \( x = (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^\infty) \in D_n \), where \( n \) is a positive integer, \( m \) is a nonnegative integer, and \( x_m \neq a_{n+1}. \) Then \( f_n^{-1}(x) \) is a star centered in \( x. \)

**Proof.** From what we have seen in the proof of Lemma 3.15 it follows that

\[ f_n^{-1}(x) = \bigcup_{k=1}^\infty \{ (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, s^\infty) \mid s \in [a_{n+1}, b_{n+1}] \}, \]

and for each \( k \) the set

\[ B_k = \{ (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, s^\infty) \mid s \in [a_{n+1}, b_{n+1}] \} \]

is an arc with endpoints \( x \) and \( (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^k, b_{n+1}^\infty) \), \( B_i \cap B_j = \{ x \} \) for any \( i \neq j \), and \( \lim_{k \to \infty} \text{diam}(B_k) = 0. \)

Let \( \{(a_n, b_n)\}_{n=1}^\infty \) be any sequence in \([0, 1] \times [0, 1]\), such that \( a_n < b_n \) for each positive integer \( n \), and \( a_i \neq a_j \) whenever \( i \neq j \). Next denote by \( A(a_n, b_n)^\infty_{n=1} \) the union

\[ A(a_n, b_n)^\infty_{n=1} = \bigcup_{n=1}^\infty ([a_n, b_n] \times \{ a_n \}) \subseteq [0, 1] \times [0, 1]. \]

and by \( G(a_n, b_n)^\infty_{n=1} \) the subset of \([0, 1] \times [0, 1]\), defined by

\[ G(a_n, b_n)^\infty_{n=1} = A \cup A(a_n, b_n)^\infty_{n=1}, \]

where \( A = \{(t, t) \mid t \in [0, 1]\} \) as above.

It is easy to see that \( \pi_1(G(a_i, b_i)^n_{i=1}) = \pi_2(G(a_i, b_i)^n_{i=1}) = [0, 1]. \)

Obviously \( G(a_n, b_n)^\infty_{n=1} \) is not necessarily closed in \([0, 1] \times [0, 1]\). The following theorem gives a whole family of sets \( G(a_n, b_n)^\infty_{n=1} \) that are closed in \([0, 1] \times [0, 1]\).
Theorem 3.17 Let \( \{(a_n, b_n)\}_{n=1}^\infty \) be any sequence in \([0, 1] \times [0, 1]\), such that

1. \( a_n < b_n \) for each positive integer \( n \),
2. \( a_i \neq a_j \) whenever \( i \neq j \),
3. \( \lim_{n \to \infty} (b_n - a_n) = 0 \).

Then \( G(a_n, b_n)_{n=1}^\infty \) is a closed subset of \([0, 1] \times [0, 1]\).

Proof. Let \( \{x_n\}_{n=1}^\infty \) be any sequence in \( G(a_n, b_n)_{n=1}^\infty \), which is convergent in \([0, 1] \times [0, 1]\) with the limit \( x_0 \in [0, 1] \times [0, 1] \). We show that \( x_0 \in G(a_n, b_n)_{n=1}^\infty \).

If there are positive integers \( k \) and \( n_0 \), such that \( x_n \in [a_k, b_k] \times \{a_k\} \) for each \( n \geq n_0 \), then, since \([a_k, b_k] \times \{a_k\}\) is compact, \( x_0 \in [a_k, b_k] \times \{a_k\} \) and therefore \( x_0 \in G(a_n, b_n)_{n=1}^\infty \). Otherwise there are strictly increasing sequences \( \{i_n\}_{n=1}^\infty \) and \( \{j_n\}_{n=1}^\infty \) of integers such that \( x_{i_n} \in ([a_{j_n}, b_{j_n}] \times \{a_{j_n}\}) \cup A \), where \( A = \{(t, t) \in [0, 1] \times [0, 1] \mid t \in [0, 1]\} \), for each positive integer \( n \). Since \( \lim_{n \to \infty} (b_n - a_n) = 0 \), it follows that \( x_0 \in A \) and therefore \( x_0 \in G(a_n, b_n)_{n=1}^\infty \).

Therefore by Theorem 2.18 it follows that for any sequence \( \{(a_n, b_n)\}_{n=1}^\infty \) satisfying

1. \( a_n < b_n \) for each positive integer \( n \),
2. \( a_i \neq a_j \) whenever \( i \neq j \),
3. \( \lim_{n \to \infty} (b_n - a_n) = 0 \),

there is a surjective u.s.c. function \( f_{(a_n, b_n)}_{n=1}^\infty : [0, 1] \to 2^{[0,1]} \) such that its graph \( \Gamma(f_{(a_n, b_n)}_{n=1}^\infty) \) equals to \( G(a_n, b_n)_{n=1}^\infty \), since \( G(a_n, b_n)_{n=1}^\infty \) is a closed subset of \([0, 1] \times [0, 1]\) by Theorem 3.17, and since

\[
\pi_1(G(a_i, b_i)_{i=1}^n) = \pi_2(G(a_i, b_i)_{i=1}^n) = [0, 1].
\]
Definition 3.18 Let \( \{(a_n, b_n)\}_{n=1}^{\infty} \) be any sequence in \([0, 1] \times [0, 1]\), such that

1. \( a_n < b_n \) for each positive integer \( n \),
2. \( a_i \neq a_j \) whenever \( i \neq j \),
3. \( \lim_{n \to \infty} (b_n - a_n) = 0 \).

Then \( f_{(a_n, b_n)}^{\infty} \) is called the comb function with respect to \( \{(a_n, b_n)\}_{n=1}^{\infty} \).

We also say that \( f : [0, 1] \to 2^{[0,1]} \) is a comb function, if \( f \) is the comb function with respect to some sequence \( \{(a_n, b_n)\}_{n=1}^{\infty} \) in \([0, 1] \times [0, 1]\) satisfying 1., 2. and 3.

Theorem 3.19 Let \( f : [0, 1] \to 2^{[0,1]} \) be the comb function with respect to the sequence \( \{(a_n, b_n)\}_{n=1}^{\infty} \). Then

\[
\lim \{[0, 1], f\}_{k=1}^{\infty} = \text{Cl} \left( \bigcup_{n=1}^{\infty} D_n \right).
\]

Proof. Obviously, since \( \lim \{[0, 1], f\}_{k=1}^{\infty} \) is closed in \( \prod_{n=1}^{\infty} [0, 1] \),

\[
\lim \{[0, 1], f\}_{k=1}^{\infty} \supseteq \text{Cl} \left( \bigcup_{n=1}^{\infty} D_n \right).
\]

Next we show that

\[
\lim \{[0, 1], f\}_{k=1}^{\infty} \subseteq \text{Cl} \left( \bigcup_{n=1}^{\infty} D_n \right).
\]

Let \( x \in \lim \{[0, 1], f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n \). Obviously \( x \) is of the form

\[
x = (a_{i_1}, a_{i_2}, a_{i_3}, \ldots),
\]

where \( \{a_n \mid n = 1, 2, 3, \ldots\} \) is an infinite subset of \( \{a_n \mid n = 1, 2, 3, \ldots\} \).

Take any open ball \( U = B(x, \varepsilon) \) in \( \prod_{n=1}^{\infty} [0, 1] \) with respect to the metric \( D \). Let \( m \) be a positive integer such that \( \frac{1}{2^m} < \varepsilon \). Then

\[
(a_{i_1}, a_{i_2}, \ldots, a_{i_{m-1}}, a_{i_m}^\infty) \in U \cap D_{i_m}.
\]
In the above proof we noticed that any \( x \in \lim \{[0, 1], f\}^\infty_{k=1} \setminus \bigcup_{n=1}^\infty D_n \) is of the form \( x = (a_{i_1}, a_{i_2}, a_{i_3}, \ldots) \), where \( \{a_{i_n} \mid n = 1, 2, 3, \ldots\} \) is an infinite subset of \( \{a_n \mid n = 1, 2, 3, \ldots\} \). We can make this statement more precise taking into account the definitions of inverse limits and comb functions as follows:

**Remark 3.20** Let \( f : [0, 1] \to 2^{[0,1]} \) be the comb function with respect to the sequence \( \{(a_n, b_n)\}_{n=1}^\infty \). Any point \( x \in \lim \{[0, 1], f\}^\infty_{k=1} \setminus \bigcup_{n=1}^\infty D_n \) is of the form

\[
(a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots),
\]

where for each \( \ell \) it holds that \( k_\ell > 0 \) and \( a_{i_\ell} < a_{i_{\ell+1}} \leq b_{i_\ell} \).

In Examples 3.21 and 3.22 we show that there are comb functions \( f \), such that the inverse limits \( \lim \{[0, 1], f\}^\infty_{k=1} \) are not dendrites.

**Example 3.21** Let \( (a_1, b_1) = \left( \frac{1}{2}, 1 \right) \), and let for each positive integer \( n \geq 2 \), \( (a_n, b_n) = \left( \frac{1}{2} - \frac{1}{n+1}, \frac{1}{2} + \frac{1}{n+1} \right) \). We show that \( \lim \{[0, 1], f(a_n, b_n)\}^\infty_{n=1} \) is not locally connected, and therefore it is not a dendrite. Let

\[
x_0 = \left( \frac{1}{2}, \frac{1}{2}, 1^\infty \right) \in \lim \{[0, 1], f(a_n, b_n)\}^\infty_{n=1}
\]

and \( \varepsilon = \min\{d(x_0, K), \frac{1}{2^{2n}}\} > 0 \), where \( K = \{t^\infty \mid t \in [0, 1]\} \). Let \( r \leq \varepsilon \) and \( y = \ldots

Figure 3.7: The graph of the comb function from Example 3.21
\((y_1, y_2, y_3, \ldots) \in B(x_0, r) \cap \lim_{k=1}^{\infty} \{[0, 1], f(a_n, b_n)\}_{n=1}^{\infty}\) be arbitrarily chosen. Then, since 
\(r > D(x_0, y) \geq \frac{1 - y_1}{2^2}\), it follows that \(y_3 > 1 - 2^3 r\). Therefore \(y_3 > 1 - 2^3 r \geq 1 - \frac{2^3}{6^2} = \frac{5}{6}\),
and hence \(y_i = y_3\) for each \(i \geq 3\). Furthermore, \(y_2 \in f(y_3) = \{\frac{1}{2}, y_3\}\). If \(y_2 = y_3\), then
\[
D(x_0, y) \geq \frac{y_2 - \frac{1}{2}}{2^2} > \frac{5}{6} - \frac{1}{2^2} = \frac{1}{12} > r,
\]
a contradiction.

Therefore \(y_2 = \frac{1}{2}\), and hence \(y_1 \in f(\frac{1}{2})\). Clearly there is a positive integer \(n\) such that
\(y_1 = a_n\) and \(\frac{\frac{1}{2} - a_n}{2} = \frac{x_1 - y_1}{2} < r\).

Therefore for each \(r \leq \varepsilon\), \(y \in B(x_0, r)\) if and only if there is a positive integer \(n\) such that
\(y = (a_n, \frac{1}{2}, t^\infty)\), where \(\frac{\frac{1}{2} - a_n}{2} < r\) and \(t > 1 - 2^3 r\).

Therefore for each \(r \leq \varepsilon\) the intersection \(B(x_0, r) \cap \lim_{k=1}^{\infty} \{[0, 1], f(a_n, b_n)\}_{n=1}^{\infty}\) is the union of countably many mutually disjoint intervals
\[
\{(a_n, \frac{1}{2}, t^\infty) \mid t \in (1 - 2^3 r, 1]\},
\]
where \(\frac{\frac{1}{2} - a_n}{2} < r\). See Figure 3.8.

Figure 3.8: The continuum from Example 3.21
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Example 3.22 Let \((a_1, b_1) = (\frac{1}{2}, 1)\), and let for each positive integer \(n \geq 2\), \((a_n, b_n) = (\frac{1}{2} - \frac{1}{n}, \frac{1}{2})\). A similar argument as in Example 3.21 shows that the inverse limit

\[ \lim \{ [0, 1], f_{(a_n, b_n)} \}_{n=1}^{\infty} \]  

is not locally connected, and therefore it is not a dendrite.

In Theorem 3.26 we prove that under rather weak additional assumptions the inverse limit of a comb function is a dendrite. Essentially, the conditions say that the only comb functions for which the inverse limits are not dendrites are similar to those from Examples 3.21 and 3.22. Before stating and proving the theorem we introduce the necessary notion of admissible sequences and prove a few lemmas.

Definition 3.23 The sequence \(\{(a_n, b_n)\}_{n=1}^{\infty}\) in \([0, 1] \times [0, 1]\) is admissible if for each positive integer \(n\) there is a positive integer \(\mu(n) \geq n\), such that for each \(m \geq \mu(n)\) it holds that if \(a_m < a_n\), then \(b_m < b_n\).

Lemma 3.24 Let \(f : [0, 1] \to 2^{[0,1]}\) be any comb function with respect to a sequence \(\{(a_n, b_n)\}_{n=1}^{\infty}\), and let

\[ x = (a_1^{k_1}, a_2^{k_2}, a_3^{k_3}, \ldots, a_j^{k_j}, t^\infty) \in D_n, \]

\(j \geq 0, i_1, i_2, i_3, \ldots, i_j \leq n, a_{i_1} < a_{i_2} < \cdots < a_{i_j}, k_1, k_2, \ldots, k_j > 0,\) and \(t \in [a_{i_j}, b_{i_j}]\). Let \(f_\ell\) be the functions defined in Definition 3.14. Then for each

\[ y \in \text{Cl}(\bigcup_{k \geq n} (f_n \circ f_{n+1} \circ \cdots \circ f_k)^{-1}(x)) \]
and for each \(i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1\) it holds that \(p_i(x) = p_i(y)\) (where \(x\) and \(y\) are interpreted as elements of \(\Pi_{n=1}^{\infty}[0,1]\)).

**Proof.** By induction on \(k - n\) we prove the following claim:

For each \(y \in (f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x)\) and for each \(i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1\) it holds that \(p_i(x) = p_i(y)\).

For \(k - n = 0\) the statement holds true by Lemma 3.15 (part 1. for \(j > 0\) and part 2. for \(j = 0\)).

Let \(k - n = \ell\) and assume that the claim is true for \(\ell - 1\). Since

\[
(f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x) = \bigcup_{z \in (f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x)} f_k^{-1}(z)
\]

for any \(y \in (f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x)\) we choose \(z \in (f_n \circ f_{n+1} \circ \ldots \circ f_k)^{-1}(x)\) such that \(y \in f_k^{-1}(z)\). By the induction assumption \(p_i(x) = p_i(z)\) for each \(i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1\), and by Lemma 3.15 \(p_i(y) = p_i(z)\) again for each \(i \leq k_1 + k_2 + k_3 + \ldots + k_j + 1\). This completes the proof since the limits of sequences of points with the required property have the property. \(\square\)

We will also need the following lemma about point preimages.

**Lemma 3.25** Let \(f : [0,1] \to 2^{[0,1]}\) be the comb function with respect to any admissible sequence \(\{(a_n, b_n)\}_{n=1}^{\infty}\). For each \(\varepsilon > 0\) there is a positive integer \(k\) such that

\[
\text{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p)) < \varepsilon
\]

for each \(p \in D_k\), where maps \(f_n\) are defined as in Definition 3.14.

**Proof.** Let \(\varepsilon > 0\) and \(m\) be a positive integer such that \(\frac{1}{2^{m-1}} < \varepsilon\). Also let \(n_0 > m\) be any positive integer such that for each \(n \geq n_0\), it holds that \(b_n - a_n < \frac{\varepsilon}{m}\). For each positive integer \(\ell\), let \(\mu(\ell)\) be a positive integer such that for each \(n \geq \mu(\ell)\) it holds that if \(a_n < a_\ell\), then \(b_n < a_\ell\) (here we use the admissibility of the sequence.
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\[(a_n, b_n)_{n=1}^\infty\).

Let

\[k_0 = \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(n_0)\},\]
\[k_1 = \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_0)\},\]
\[k_2 = \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_1)\},\]
\[\vdots\]
\[k_m = \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_{m-1})\}.

Then we show that

\[k = \max\{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_m)\}\]

is a positive integer, such that

\[\text{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p)) < \varepsilon\]

for each \(p \in D_k\).

Take any \(p \in D_k\). Then by Remark 3.12 (1) \(p\) is either of the form \(p = (t^\infty), t \in [0, 1]\), or of the form \(p = (p_1, p_2, p_3, \ldots, p_j, t^\infty)\), where \(p_j = a_s\) for some \(s \leq k\) and \(t \in (a_s, b_s]\).

Clearly, it holds that

\[\text{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p)) \leq \text{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty))\]

since

\[(f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}((p_1, p_2, p_3, \ldots, p_j, t^\infty)) =
\[\{(p_1, p_2, p_3, \ldots, p_j, x_1, x_2, x_3, \ldots) | (x_1, x_2, x_3, \ldots) \in (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty)\}.

If \(t \neq a_i\) for all \(i > k\), then \(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty) = \{(t^\infty)\},\) and therefore \(\text{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty)) = 0.\)

If \(t = a_i\) for some \(i > k\), then we shall prove that

\[\text{diam}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty)) < \varepsilon\]
by proving that

\[ D(y, (t^\infty)) = D(y, (a_i^\infty)) < \varepsilon / 2 \]

for arbitrary \( y = (y_1, y_2, y_3, \ldots, y_m, \ldots) \in \bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(t^\infty) \).

Since \( y_1 = t = a_i \) by Lemma 3.24 (and therefore \( y_1 - a_i = 0 \)), and since \( y_1 \leq y_2 \leq y_3 \leq \ldots \), it follows that

\[ D(y, (a_i^\infty)) \leq \sup \{ \frac{y_2 - a_i}{2^2}, \frac{y_3 - a_i}{2^3}, \ldots, \frac{y_m - a_i}{2^m}, \frac{1}{2^{m+1}} \}. \]

Let \( j \in \{2, 3, 4, \ldots, m\} \) be arbitrary. We show that

\[ \frac{y_j - a_i}{2^j} < \varepsilon / 2. \]

First we show that for each \( s \in \{2, 3, 4, \ldots, j\} \) there is a positive integer \( \ell > n_0 \) such that \( y_s, y_{s-1} \in [a_i, b_i] \).

For \( s = 2 \), the claim is true since \( y_2 \in [a_i, b_i], y_1 = a_i, \) and \( i > k \geq n_0 \).

If \( y_2 \notin \{a_n \mid n = 1, 2, 3, \ldots\} \), then \( y_2 = y_3 = y_4 = \cdots \) and therefore for each \( s \in \{3, 4, 5, \ldots, j\} \), \( y_s = y_{s-1} = y_2 \in [a_i, b_i] \).

In the rest of the proof we consider the case \( y_2 = a_{i_0} \) for some positive integer \( i_0 \). If \( i_0 \leq k_m \), then \( \mu(i_0) \in \{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_m)\} \), and therefore \( \mu(i_0) \leq k \). Since \( k < i \), it follows that \( \mu(i_0) < i \). Therefore from \( y_2 = a_{i_0} > a_i \) it follows that \( a_{i_0} > b_i \), and this contradicts the fact that \( a_{i_0} = y_2 \in [a_i, b_i] \). So in this case \( i_0 > k_m \geq n_0 \), and the claim for \( s = 3 \) follows, since \( y_3, y_2 \in [a_{i_0}, b_{i_0}] \).

If \( y_3 \notin \{a_n \mid n = 1, 2, 3, \ldots\} \), then \( y_3 = y_4 = y_5 = \cdots, \) and therefore for each \( s \in \{4, 5, 6, \ldots, j\} \), \( y_s = y_{s-1} = y_3 \in [a_{i_0}, b_{i_0}] \).

In the rest of the proof we consider the case \( y_3 = a_{i_1} \) for some positive integer \( i_1 \). If \( i_1 \leq k_{m-1} \), then \( \mu(i_1) \in \{n_0, \mu(1), \mu(2), \mu(3), \ldots, \mu(k_{m-1})\} \), and therefore \( \mu(i_1) \leq k_m \).

Since \( k_m < i_0 \), it follows that \( \mu(i_1) < i_0 \). Therefore from \( y_3 = a_{i_1} > a_{i_0} \) it follows that \( a_{i_1} > b_{i_0} \), and this contradicts the fact that \( a_{i_1} = y_3 \in [a_{i_0}, b_{i_0}] \). So in this case \( i_1 > k_{m-1} \geq n_0 \), and the claim follows for \( s = 4 \), since \( y_4, y_3 \in [a_{i_1}, b_{i_1}] \).

Repeating this reasoning \( m \) times proves that for each \( s \in \{2, 3, 4, \ldots, j\} \) there is a positive integer \( \ell > n_0 \) such that \( y_s, y_{s-1} \in [a_{i_\ell}, b_{i_\ell}] \).
It follows that
\[
d(y_j, a_i) \leq d(y_j, y_{j-1}) + \ldots + d(y_3, y_2) + d(y_2, a_i) \leq (j-1) \frac{\varepsilon}{m} < \varepsilon,
\]
since \(y_s, y_{s-1} \in [a_\ell, b_\ell]\) for each \(s\), for some \(\ell > n_0\). Therefore \(\frac{y_s-a_i}{2^n} < \frac{\varepsilon}{2}\).

**Theorem 3.26** Let \(f : [0, 1] \to 2^{[0,1]}\) be the comb function with respect to any admissible sequence \(\{(a_n, b_n)\}_{n=1}^\infty\). Then \(\lim \left\{\left[0, 1\right], f\right\}_k\) is a dendrite.

**Proof.** We show that \(\lim \left\{\left[0, 1\right], f\right\}_k\) is homeomorphic to the inverse limit of an inverse sequence of dendrites with monotone bonding functions, which is by [45, p. 180] a dendrite, and therefore \(\lim \left\{\left[0, 1\right], f\right\}_k\) is a dendrite, too.

More specifically we prove that the inverse limit \(\lim \left\{D_n, f_n\right\}_n\) is homeomorphic to \(\lim \left\{D_n, f_n\right\}_n\), where \(f_n : D_{n+1} \to D_n\) is the mapping defined in Definition 3.14 and that each \(f_n\) is monotone.

For fixed \(x = (x_1, x_2, x_3, \ldots, x_m, a_{n+1}^\infty), x_m \neq a_{n+1},\) and fixed \(k\), let
\[
B_k(x) = \{(x_1, x_2, x_3, \ldots, x_m, a_n^k, t^\infty), t \in [a_{n+1}, b_{n+1}]\}.
\]

Then each
\[
S(x) = \bigcup_{k=1}^\infty B_k(x),
\]
is the star with the center \(x\) and beams \(B_k(x), k = 1, 2, 3, \ldots\)

Using Remark 3.12 we see that

1. \(f_n^{-1}(x) = \{x\}\) for each \(x \in D_n \setminus \text{Cl}(D_{n+1} \setminus D_n)\), and

2. \(f_n^{-1}(x)\) is the star \(S(x)\) for each \(x \in D_n \cap \text{Cl}(D_{n+1} \setminus D_n)\).

Therefore \(f_n : D_{n+1} \to D_n\) is monotone for each \(n\), and by [45, p. 180]
\[
\lim \left\{D_n, f_n\right\}_n
\]
is a dendrite.

Next we show that by
\[
F(x_1, x_2, x_3, \ldots) = \lim_{n \to \infty} x_n
\]
a homeomorphism

\[ F : \lim_{n \to \infty} \{ D_n, f_n \} \to \lim_{n \to \infty} \{ [0, 1], f \} \]

is defined.

1. First we show that \( F : \lim_{n \to \infty} \{ D_n, f_n \} \to \lim_{n \to \infty} \{ [0, 1], f \} \) is a well defined function. Take any point \((x_1, x_2, x_3, \ldots) \in \lim_{n \to \infty} \{ D_n, f_n \} \subseteq \prod_{i=1}^{\infty} D_i\). If there is a positive integer \(n_0\), such that \(x_n = x_{n_0}\) for each \(n \geq n_0\), then \(\lim_{n \to \infty} x_n = x_{n_0}\) and \(x_{n_0} \in D_{n_0} \subseteq \lim_{n \to \infty} \{ [0, 1], f \} \) for each \(n \geq n_0\). Therefore \(F(x_1, x_2, x_3, \ldots) \in \lim_{n \to \infty} \{ [0, 1], f \}\). If there is no such \(n_0\), then let \(i_1 < i_2 < i_3 < \ldots\) be the sequence of all such integers that \(x_i \neq x_{i+1}\) for each \(n\). Then \(x_{i+1} = x_{i+1} \in f(\cdot)(x_i)\) is the star \(S(x_i) \subseteq D_{i+1}\) with center \(x_i\). Therefore \(x_i\) is of the form \(x_i = (y_1, y_2, y_3, \ldots, y_m, a_i, a_{i+1})\).

Similarly, \(x_{i+1}\) is of the form \(x_{i+1} = (z_1, z_2, z_3, \ldots, z_{m+1}, a_{i+1}, a_{i+2})\).

Since \(x_{i+1} \in S(x_i)\), it follows that \(m_n < m_{n+1}\) and \(y_i = z_i\) for each \(i = 1, \ldots, m_n\). From \(m_n < m_{n+1}\) for each \(n\), it follows that \(m_n \geq n\) for each \(n\). Therefore \(D(x_n, x_{n+1}) \leq \frac{1}{2^{m_n}} \leq \frac{1}{2^n}\). It follows that the sequence \(\{x_n\}\) is a Cauchy sequence in \(\text{Cl}(\bigcup_{n=1}^{\infty} D_n)\), and hence by Theorem 3.19 it converges to a point in \(\lim_{n \to \infty} \{ [0, 1], f \}\).

2. We show that \(F\) is continuous.

Take any \(x = (x_1, x_2, x_3, \ldots) \in \lim_{n \to \infty} \{ D_n, f_n \}\) and any \(\varepsilon > 0\). Choose a positive integer \(k\) (given by Lemma 3.25), such that

\[
\text{diam} \left( \bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(p) \right) < \varepsilon
\]

for each \(p \in D_k\).
Let $B = \{ z \in \lim_{n \to \infty} \{ [0, 1], f \}_{n=1}^{\infty} \mid d(z, F(x)) < \varepsilon \}$, and let

$$V = P_k^{-1}(B \cap D_k),$$

where $P_k : \lim_{n \to \infty} \{ D_n, f_n \}_{n=1}^{\infty} \to D_k$ is the projection map to the $k$-th factor. Since $B \cap D_k$ is open in $D_k$, $V$ is open in $\lim_{n \to \infty} \{ D_n, f_n \}_{n=1}^{\infty}$. Since $x \in \lim_{n \to \infty} \{ D_n, f_n \}_{n=1}^{\infty}$ and $x_k \in D_k$, it follows from the definition of $F$ that $F(x) \in \text{Cl}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(x_k))$. From the definition of functions $f_j$ it follows that $x_k \in \text{Cl}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(x_k))$. Therefore $d(x_k, F(x)) < \varepsilon$ hence $x_k \in B$. It follows that $x_k \in B \cap D_k$, and thus $x \in V$. Let $y = (y_1, y_2, y_3, \ldots) \in V$. It follows that $y_k \in B$, and therefore $d(y_k, F(x)) < \varepsilon$. Since $F(y), y_k \in \text{Cl}(\bigcup_{n \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_n)^{-1}(y_k))$, it follows that $d(y_k, F(y)) < \varepsilon$. Hence,

$$d(F(x), F(y)) \leq d(F(x), y_k) + d(y_k, F(y)) < 2\varepsilon.$$

Therefore $F$ is continuous.

3. We show that $F$ is a surjection. Let

$$y = (y_1, y_2, y_3, \ldots) \in \lim_{n \to \infty} \{ [0, 1], f \}_{n=1}^{\infty}$$

be arbitrarily chosen. We define a sequence $\{ x_n \}_{n=1}^{\infty}$, such that

(a) for each $n$, $x_n \in D_n$,

(b) for each $n$, $f_n(x_{n+1}) = x_n$,

(c) $\lim_{n \to \infty} x_n = y$.

If $y \notin D_n$ for each $n$, then by Remark 3.20 $y$ is of the form $y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots)$, where for each $\ell$, it holds that $a_{i_\ell} < a_{i_{\ell+1}} \leq b_\ell$ and that $k_\ell$ is a positive integer. In this case we define

$$x_n = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_m}^{k_m}, a_{i_{m+1}}^{k_{m+1}}),$$

where $i_\ell \leq n$ for each $\ell = 1, 2, 3, \ldots, m$, and $i_{m+1} > n$. If $y \in D_m$ for some $m$, then define $x_n = y$ for $n \geq m$ and $x_n = (f_n \circ \ldots \circ f_{m-1})(y)$ for $n < m$. 


4. Finally we show that $F$ is an injection. Let $x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ be any points in $\lim_{n=1}^{\infty} \{D_n, f_n\}$ such that $x \neq y$. Let $k$ be a positive integer such that $x_k \neq y_k$. Since $x_k, y_k \in D_k$, it follows that

$$x_k = (a_1^{q_1}, a_2^{q_2}, a_3^{q_3}, \ldots, a_j^{q_j}, t^\infty)$$

and

$$y_k = (a_1^{r_1}, a_2^{r_2}, a_3^{r_3}, \ldots, a_m^{r_m}, s^\infty),$$

where $i_1, i_2, \ldots, i_j, \ell_1, \ell_2, \ldots, \ell_m \leq k$, $t \in (a_i, b_i)$ and $s \in (a_{\ell_m}, b_{\ell_m})$, by 1. from Remark 3.12. Let $q = q_1 + q_2 + q_3 + \ldots + q_j$, $r = r_1 + r_2 + r_3 + \ldots + r_m$. Assume that $q \leq r$. Also, let $n$ be the smallest integer such that $p_n(x_k) = p_n(y_k)$. If $n \leq q$ then for each $z_1 \in \text{Cl}(\bigcup_{i \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(x_k))$ and each $z_2 \in \text{Cl}(\bigcup_{i \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(y_k))$ by Lemma 3.24 it follows that $p_n(z_1) = p_n(x_k)$ and $p_n(z_2) = p_n(y_k)$, and therefore

$$D(z_1, z_2) \geq \frac{d(p_n(x_k), p_n(y_k))}{2^n}.$$ 

Since

$$F(x) \in \text{Cl} \left( \bigcup_{i \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(x_k) \right)$$

and

$$F(y) \in \text{Cl} \left( \bigcup_{i \geq k} (f_k \circ f_{k+1} \circ \ldots \circ f_i)^{-1}(y_k) \right)$$

it follows that $F(x) \neq F(y)$. If $n > q$, then $y_k$ is of the form

$$y_k = (a_1^{q_1}, a_2^{q_2}, a_3^{q_3}, \ldots, a_j^{q_j}, a_i^p, a_j^{r_{j+1}}, a_{j+2}^{r_{j+2}}, a_{j+3}^{r_{j+3}}, \ldots, a_{\ell_m}^{r_m}, s^\infty),$$

since $r \geq q$.

We consider several cases.

Case 1. If $p \geq 1$, then $n = q + 1$, since $p_{q+1}(x_k) = t$ and $p_{q+1}(y_k) = a_i$, and
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by Lemma 3.24, $p_n(F(x)) = p_n(x_k) = t \neq a_{i_j} = p_n(y_k) = p_n(F(y))$, hence $F(x) \neq F(y)$.

Case 2. If $p + r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m = 0$, then $n = q + 1$ and by Lemma 3.24, $p_n(F(x)) = p_n(x_k) = t \neq s = p_n(y_k) = p_n(F(y))$, hence $F(x) \neq F(y)$.

Case 3. If $p = 0$ and $r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m > 0$ and if there is a positive integer $i \leq k$ such that $t = a_{i_j}$, then $F(x) = x_k$ and $n \leq r + 1$, and it follows that $p_n(F(x)) = p_n(x_k) \neq p_n(y_k) = p_n(F(y))$, where the last equality follows by Lemma 3.24.

Case 4. If $p = 0$ and $r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m > 0$ and if there is a positive integer $i > k$ such that $t = a_{i_j}$, then $n = q + 1$ since $p_{q+1}(y_k) = a_{j+1}$ and $l_{j+1} \leq k$, while $p_{q+1}(x_k) = t = a_i$, $i > k$. Therefore $p_n(F(x)) = p_n(x_k) = a_i \neq a_{i_{j+1}} = p_n(y_k) = p_n(F(y))$, by Lemma 3.24.

Case 5. If $p = 0$ and $r_{j+1} + r_{j+2} + r_{j+3} + \ldots + r_m > 0$ and if $t \neq a_i$ for each positive integer $i$, then $F(x) = x_k$ and $n = q + 1 < r + 1$, and we continue as in Case 3.

Since $F : \lim\{K_n, f_n\}^\infty_{n=1} \to \lim\{[0, 1], f\}^\infty_{n=1}$ is a continuous bijection from a compact space onto a metric space, it is by [44, p. 167] a homeomorphism. \qed

The following example shows that the conditions of Theorem 3.26 are not sufficient to guaranty that the corresponding inverse limit is homeomorphic to Ważewski’s universal dendrite.

**Example 3.27** Let for each positive integer $n$, $(a_n, b_n) = \left(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right)$. By Theorem 3.26, $\lim\{[0, 1], f_{(a_n, b_n)}\}^\infty_{n=1}$ is a dendrite. Since $a_n < b_n < a_{n+1}$ for each positive integer $n$, using Lemma 4.9 and Remark 3.12 (3), we see that $x \in \lim\{[0, 1], f_{(a_n, b_n)}\}^\infty_{n=1}$ is a ramification point if and only if there is a positive integer $m$, such that $x = (a_{m}^\infty)$. Therefore the set of all ramification points is not dense in $\lim\{[0, 1], f_{(a_n, b_n)}\}^\infty_{n=1}$.
Hence, by Theorem 3.6, \( \lim_{k \rightarrow \infty} \{(0, 1), f_{(a_n, b_n)}\}_{k=1}^{\infty} \) is not homeomorphic to Ważewski’s universal dendrite.

![Graph of the comb function](image)

Figure 3.10: Graph of the comb function \( f \) defined in example 3.27 and the corresponding inverse limit

In Theorem 3.31 we show that with the additional condition that the set \( \{a_n \mid n = 1, 2, 3, \ldots\} \) is dense in \([0, 1]\), it follows that the inverse limit \( \lim_{k \rightarrow \infty} \{(0, 1), f\}_{k=1}^{\infty} \) is homeomorphic to Ważewski’s universal dendrite. In Theorem 3.32 we show that in fact this additional condition characterizes inverse limits \( \lim_{k \rightarrow \infty} \{(0, 1), f\}_{k=1}^{\infty} \) that are homeomorphic to Ważewski’s universal dendrite.

First we prove the following lemmas.

**Lemma 3.28** Let \( f : [0, 1] \rightarrow 2^{[0,1]} \) be any comb function with respect to an admissible sequence \( \{(a_n, b_n)\}_{n=1}^{\infty} \). Let

\[
y \in \lim_{k \rightarrow \infty} \{(0, 1), f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n
\]

be arbitrarily chosen. Then for each \( x \in \bigcup_{n=1}^{\infty} D_n \), the uniquely determined arc \( L \) from \( x \) to \( y \) satisfies the condition

\[
L \setminus \{y\} \subseteq \bigcup_{n=1}^{\infty} D_n.
\]

**Proof.** By Remark 3.20 \( y \) is of the form \( y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots) \), where \( a_{i_{\ell}} < a_{i_{\ell+1}} \leq b_{i_{\ell}} \) for each \( \ell \). We use the same sequence \( \{x_n\}_{n=1}^{\infty} \) as in the proof of surjectivity of \( F \) in
the proof of Theorem 3.26, i.e.

\[ x_n = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_m}^{k_m}, a_{i_{m+1}}^{\infty}) \in D_n, \]

where \( i_\ell \leq n \) for each \( \ell = 1, 2, 3, \ldots, m \), and \( i_{m+1} > n \). Since \( D_{n+1} \) is a dendrite, there is a unique arc \([x_n, x_{n+1}]\) from \( x_n \) to \( x_{n+1} \) in \( D_{n+1} \) if \( x_n \neq x_{n+1} \). If \( x_n = x_{n+1} \), let \([x_n, x_{n+1}]\) denote \( \{x_n\} \). Then \( A = \bigcup_{n=1}^{\infty} [x_n, x_{n+1}] \cup \{y\} \) is an arc from \( x_1 \) to \( y \), since \([x_n, x_{n+1}] \setminus \{x_n\} \in D_{n+1} \setminus D_n \) and since \( \lim_{n \to \infty} x_n = y \), as shown in the above mentioned proof of Theorem 3.26. Obviously

\[ A \setminus \{y\} \subseteq \bigcup_{n=1}^{\infty} D_n. \]

Next, take the unique arc \( B \) from \( x_1 \) to \( x \) in \( \bigcup_{n=1}^{\infty} D_n \) (the existence of such an arc follows from the fact that there is an integer \( m \) such that \( x_1, x \in D_m \)). Then \( \text{Cl}(\big((A \setminus B) \cup (B \setminus A)\big)) \) is an arc from \( x \) to \( y \) in \( \lim_{k=1}^{\infty} \{[0, 1], f(a_{i_k}, b_{i_k})\}_{k=1}^{\infty} \). Since \( \lim_{k=1}^{\infty} \{[0, 1], f(a_{i_k}, b_{i_k})\}_{k=1}^{\infty} \) is a dendrite, it follows that \( \text{Cl}(\big((A \setminus B) \cup (B \setminus A)\big)) = L \). Obviously

\[ L \setminus \{y\} = \text{Cl}(\big((A \setminus B) \cup (B \setminus A)\big)) \setminus \{y\} \subseteq \bigcup_{n=1}^{\infty} D_n. \]

Lemma 3.29 Let \( f : [0, 1] \to 2^{[0, 1]} \) be any comb function with respect to an admissible sequence \( \{(a_n, b_n)\}_{n=1}^{\infty} \). Then each

\[ y \in \lim_{k=1}^{\infty} \{[0, 1], f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n \]

is an endpoint of \( \lim_{k=1}^{\infty} \{[0, 1], f\}_{k=1}^{\infty} \) (and hence it is not a ramification point).

Proof. Assuming that \( y \) is not an endpoint, using Lemma 3.28, one easily gets a simple closed curve in \( \lim_{k=1}^{\infty} \{[0, 1], f\}_{k=1}^{\infty} \). This contradicts the fact that \( \lim_{k=1}^{\infty} \{[0, 1], f\}_{k=1}^{\infty} \) is a dendrite by Theorem 3.26. □
Lemma 3.30 Let $f : [0, 1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$. Let $x \in \lim \{[0, 1], f\}_{k=1}^{\infty}$. The following statements are equivalent.

1. $x$ is a ramification point in $\lim \{[0, 1], f\}_{k=1}^{\infty}$.

2. $x$ is a ramification point in $D_n$ for some positive integer $n$.

Proof. It is obvious that if there is a positive integer $n$, such that $x$ is a ramification point in $D_n$, then $x$ is a ramification point in $\lim \{[0, 1], f\}_{k=1}^{\infty}$ (since $D_n \subseteq \lim \{[0, 1], f\}_{k=1}^{\infty}$). Suppose that $x$ is a ramification point in $\lim \{[0, 1], f\}_{k=1}^{\infty}$. Since no point of

$$\lim \{[0, 1], f\}_{k=1}^{\infty} \setminus \bigcup_{n=1}^{\infty} D_n$$

is a ramification point in $\lim \{[0, 1], f\}_{k=1}^{\infty}$, by Lemma 3.29, it follows that $x \in D_{n_0}$ for some positive integer $n_0$. Let $[x, x_i], i = 1, 2, 3$, be any three arcs in $\lim \{[0, 1], f\}_{k=1}^{\infty}$, such that $[x, x_1] \cup [x, x_2] \cup [x, x_3]$ is a simple triod. Without loss of generality we may assume that $x_i \in \bigcup_{n=1}^{\infty} D_n$, $i = 1, 2, 3$, since if $x_i \notin \bigcup_{n=1}^{\infty} D_n$, we may replace $[x, x_i]$ by $[x, y_i]$, where $y_i \in (x, x_i)$, by Lemma 3.28. For each $i = 1, 2, 3$ let $n_i$ be a positive integer such that $x_i \in D_{n_i}$. Let $n = \max\{n_0, n_1, n_2, n_3\}$. Obviously $[x, x_1] \cup [x, x_2] \cup [x, x_3]$ is a simple triod in $D_n$, and therefore $x$ is a ramification point in $D_n$. $\square$

Theorem 3.31 Let $f : [0, 1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^{\infty}$ such that the set $\{a_n \mid n = 1, 2, 3, \ldots\}$ is dense in $[0, 1]$. Then $\lim \{[0, 1], f\}_{k=1}^{\infty}$ is homeomorphic to Ważewski’s universal dendrite.

Proof. By Theorem 3.26, $D = \lim \{[0, 1], f\}_{k=1}^{\infty}$ is a dendrite. We show that the set of ramification points of $D$ is dense in $D$ and that each ramification point of $D$ is of infinite order in $D$, and therefore by Theorem 3.6 $D$ is homeomorphic to Ważewski’s universal dendrite.

Let $y = (y_1, y_2, y_3, \ldots) \in D$ be arbitrarily chosen, such that $y$ is not a ramification point. We show that there is a sequence of ramification points $\{z_n\}_{n=1}^{\infty}$ in $D$, such that $\lim_{n \to \infty} z_n = y$. 


If $y \in D_n$ for some positive integer $n$, then by Remark 3.12 (1), (3) (taking into account that by Lemma 4.9 $y$ is not a ramification point in $D_\ell$ for each $\ell$) there are a positive integer $m$ and a real number $t \in [0, 1] \setminus \{a_1, a_2, a_3, \ldots\}$, such that

$$y = (y_1, y_2, y_3, \ldots, y_{m-1}, t^\infty),$$

where $y_{m-1} = a_k$ for some $k \leq n$, and $t \in (a_k, b_k]$. Since the set $\{a_n \mid n = 1, 2, 3, \ldots\}$ is dense in $[0, 1]$, there is a strictly increasing sequence $\{i_n\}_{n=1}^\infty$ of positive integers, such that $\lim_{n \to \infty} a_{i_n} = t$ and $a_{i_n} \in (a_k, b_k]$. Therefore

$$\{(y_1, y_2, y_3, \ldots, y_{m-1}, a_{i_n}^\infty)\}_{n=1}^\infty$$

is a sequence of ramification points in $D$, which converges to $y$.

If $y \in D \setminus \bigcup_{n=1}^\infty D_n$, then by Remark 3.20

$$y = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots),$$

where for each $\ell$ it holds that $k_\ell > 0$ and $a_{i_\ell} < a_{i_{\ell+1}} \leq b_{i_\ell}$. Then the sequence $\{z_n\}_{n=1}^\infty$, where

$$z_n = (a_{i_1}^{k_1}, a_{i_2}^{k_2}, a_{i_3}^{k_3}, \ldots, a_{i_{n-1}}^{k_{n-1}}, a_{i_n}^\infty)$$

for each $n$, is a sequence of ramification points in $D$, which converges to $y$.

Next we show that each of the ramification points is of infinite order in $D$. Let $x \in D$ be any ramification point. Then by Lemma 4.9 and Remark 3.12 (3) there are positive integers $m$ and $j$, such that $p_k(x) = a_j$ for each positive integer $k \geq m$. Without loss of generality we may assume that $p_k(x) \neq a_j$ for each $k < m$.

Since

$$x \in f_{j-1}^{-1}(x) \subseteq D$$

and $f_{j-1}^{-1}(x)$ is a star with the center $x$ by Lemma 3.16, it follows that $x$ is of infinite order in $D$. \qed

**Theorem 3.32** Let $f : [0, 1] \to 2^{[0,1]}$ be any comb function with respect to an admissible sequence $\{(a_n, b_n)\}_{n=1}^\infty$. Then $\lim\{[0, 1], f\}_{k=1}^\infty$ is homeomorphic to Ważewski’s universal dendrite if and only if the set $\{a_n \mid n = 1, 2, 3, \ldots\}$ is dense in $[0, 1]$. 

Proof. Taking Theorem 3.31 into account it suffices to prove that if the set 
\{a_n \mid n = 1, 2, 3, \ldots\} is not dense in \([0, 1]\), then \(\lim_{k \to \infty} \{[0, 1], f\}_{k=1}^{\infty}\) is not homeomorphic to Ważewski’s universal dendrite. If there is an interval \(J = (a_j, a_k) \subseteq [0, 1]\) containing no \(a_n\), let \(t = \frac{a_j + a_k}{2}\) and \(\delta = \frac{a_k - a_j}{2}\). For any ramification point \(x\) of \(\lim_{k \to \infty} \{[0, 1], f\}_{k=1}^{\infty}\), 
\[D(x, (t^\infty)) \geq \frac{d(p_1(x), t)}{2} > \delta,\]
since \(p_1(x) = a_n\) for some \(n\). Therefore the open ball in \(\lim_{k \to \infty} \{[0, 1], f\}_{k=1}^{\infty}\) centered at \((t^\infty)\) with the radius \(\delta\) contains no ramification points and hence by Theorem 3.6, \(\lim_{k \to \infty} \{[0, 1], f\}_{k=1}^{\infty}\) is not homeomorphic to Ważewski’s universal dendrite. □

**Theorem 3.33** There is a comb function \(f\) such that \(\lim_{k \to \infty} \{[0, 1], f\}_{k=1}^{\infty}\) is homeomorphic to Ważewski’s universal dendrite.

Proof. Let \(\{a_n \mid n \in \mathbb{N}\}\) be any dense subset of \((0, 1)\) with \(a_i \neq a_j\) for all \(i \neq j\). Inductively we define sequence \(\{b_n\}_{n=1}^{\infty}\) in such a way that \(\{(a_n, b_n)\}_{n=1}^{\infty}\) would be admissible which would by Theorem 3.31 guaranty that \(\lim_{k \to \infty} \{[0, 1], f\}_{k=1}^{\infty}\) is homeomorphic to Ważewski’s universal dendrite. For each positive integer \(n\), let
\[b_n = \frac{1}{2} \left( a_n + \min\{1, a_i \mid i < n, a_i > a_n\} \right).\]
First we show that \(\lim_{n \to \infty} (b_n - a_n) = 0\). Let \(\varepsilon > 0\) be arbitrary; choose a positive integer \(k\) such that \(\frac{1}{k} < \varepsilon\). For each \(j \leq k\) choose \(i_j\), such that \(a_{i_j} \in \left(\frac{j-1}{k}, \frac{j}{k}\right)\), and let \(n_0 = \max\{i_j \mid j = 1, 2, 3, \ldots, k\}\). For any \(n > n_0\) let \(a < b\) be two consecutive elements of the set \(\{0, 1, a_{i_j} \mid j = 1, 2, 3, \ldots, k\}\) such that \(a_n \in (a, b)\). Then \(b_n - a_n \leq \frac{a_n + b}{2} - a_n = \frac{b - a}{2} < \frac{b - a}{2} < \varepsilon\).
Since for each positive integer \(n\) for each \(m \geq n\) it holds that if \(a_m < a_n\), then \(b_m < \frac{1}{2}(a_m + a_n) < a_n\), it follows that the sequence \(\{(a_n, b_n)\}_{n=1}^{\infty}\) is admissible. □
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4

COMPACTIFICATIONS OF A RAY

4.1 An irreducible smooth compactification of a ray

As another important apparatus for constructing new examples of continua, we introduced the compactifications of rays. By using the fact that each continuum can be embedded into a Hilbert cube, we prove that for each continuum \( K \) there is an irreducible smooth compactification of a ray which contains a topological copy of \( K \). In particular, for any continuum \( K \) we give a compactification \( X \) of a ray with \( K \) and show that \( X \) is an irreducible and smooth continuum. We will use the following lemma.

**Lemma 4.1** Let \( K \in C(Q) \) be a nondegenerate continuum in a Hilbert cube \( Q \) and let \( a, b \), where \( a \neq b \) be any points in \( K \). Then for each \( \varepsilon > 0 \) there is an arc \( A_{ab} \subseteq Q \) from \( a \) to \( b \), such that \( d_H(A_{ab}, K) < \varepsilon \).

**Proof.** See [2, p. 111].

**Theorem 4.2** Given any continuum \( K \), there is an irreducible smooth compactification of a ray with \( K \).

**Proof.** Let \( K \) be an arbitrary continuum and \( f : K \rightarrow Q \) an embedding of \( K \) into a Hilbert cube \( Q \). Denote \( Y' = f(K) \). We accomplish the proof in four steps.
1. Let \( a = (a_1, a_2, a_3, \ldots) \) and \( b = (b_1, b_2, b_3, \ldots) \) be any points in \( Y' \). By Lemma 4.1, for each positive integer \( n \), there is an arc \( L'_n \subseteq Q \) from \( a \) to \( b \), such that \( d_H(L'_n, Y') < \frac{1}{n} \). Let \( (L'_n)^\infty_{n=1} \) be a sequence of arcs \( L'_n \) from \( a \) to \( b \) in \( Q \), such that \( d_H(L'_n, Y') < \frac{1}{n} \) for each positive integer \( n \).

2. Let \( v : Y' \to \{0\} \times Q \) be the embedding defined by

\[
v(y_1, y_2, y_3, \ldots) = (0, y_1, y_2, y_3, \ldots),
\]

for each \((y_1, y_2, y_3, \ldots) \in Y'\). Denote \( Y = v(Y') \). We show that there is a sequence of arcs \((L_n)^\infty_{n=1}\), that limits to \( Y \), such that \( L_n \cap Y = \emptyset \) for each positive integer \( n \).

Clearly, \( \{L'_n\}^\infty_{n=1} \) is a sequence of arcs that is convergent to \( Y' \). Since \( L'_n \subseteq Q \) is an arc, there is a homeomorphism \( h : [0, 1] \to L'_n, \; h(t) = (x_1^n(t), x_2^n(t), x_3^n(t), \ldots) \), such that \( L'_n = \{(x_1^n(t), x_2^n(t), x_3^n(t), \ldots) \mid t \in [0, 1]\} \subseteq Q \). For each positive integer \( n \) we define \( L_n = \{(\frac{1}{n}, x_1^n(t), x_2^n(t), x_3^n(t), \ldots) \mid t \in [0, 1]\} \subseteq \{\frac{1}{n}\} \times Q \) and \( h' : L'_n \to L_n \) by \( h'(x_1^n(t), x_2^n(t), x_3^n(t), \ldots) = (\frac{1}{n}, x_1^n(t), x_2^n(t), x_3^n(t), \ldots) \). Since \( L'_n \) is an arc for each positive integer \( n \) and while \( h' : L'_n \to L_n \) is a homeomorphism, \( L_n \) is obviously an arc too.

One can easily see that \( L_n \cap Y = \emptyset \) for each positive integer, since \( L_n \subseteq \{\frac{1}{n}\} \times Q \) and \( Y \subseteq \{0\} \times Q \).

Figure 4.1: Sketch of the sequence of arcs \( L_n \) from the proof of the Theorem 4.2
4.1 An irreducible smooth compactification of a ray

We still need to prove that \((L_n)_{n=1}^\infty\) is convergent with limit \(Y\) with respect to the Hausdorff metric. More precisely, we show that for each \(\varepsilon > 0\), there is a positive integer \(n_0\), such that for each \(n > n_0\) it holds that \(d_H(L_n, Y) < \varepsilon\).

Since \((L'_n)_{n=1}^\infty\) is convergent with limit \(Y'\), for each \(\varepsilon_1 > 0\) there exists a positive integer \(n_1\), such that for each \(n > n_1\) it holds that \(d_H(L'_n, Y') < \varepsilon_1\), i.e.:

(a) For each \(t \in [0, 1]\), there exists \(z = (z_1, z_2, z_3, \ldots) \in Y'\) such that

\[
d((x^n_1(t), x^n_2(t), x^n_3(t), \ldots), (z_1, z_2, z_3, \ldots)) < \varepsilon_1,
\]

respectively:

\[
\sup_{i \in 1, 2, 3, \ldots} \left\{ \frac{|x^n_i(t) - z_i|}{2^i} \right\} < \varepsilon_1.
\]

(b) For each \(z \in Y'\), there exists \(t(z) \in [0, 1]\) such that

\[
d((z_1, z_2, z_3, \ldots), (x^n_1(t(z)), x^n_2(t(z)), x^n_3(t(z)), \ldots)) < \varepsilon_1,
\]

respectively:

\[
\sup_{i \in 1, 2, 3, \ldots} \left\{ \frac{|z_i - x^n_i(t(z))|}{2^i} \right\} < \varepsilon_1.
\]

Let \(\varepsilon > 0\) and let \(n_2\) be a positive integer, such that \(\frac{1}{n_2} < \varepsilon\) and let \(n_0 = \max \{n_1, n_2\}\).

First we show, that for each \(x = (\frac{1}{n}, x^n_1(t), x^n_2(t), x^n_3(t), \ldots) \in L_n\) there is an \(y \in Y\), such that \(d(x, y) < \varepsilon\). Take \(y = (0, z_1, z_2, z_3, \ldots)\), with \((z_1, z_2, z_3, \ldots) \in Y'\), such that \(d((x^n_1(t), x^n_2(t), x^n_3(t), \ldots), (z_1, z_2, z_3, \ldots)) < \varepsilon_1\), Then it is easy to see, that

\[
d(x, y) = \sup_{i \in 1, 2, 3, \ldots} \left\{ \frac{1}{n}, \frac{|x^n_i(t) - z_i|}{2^i+1} \right\} < \varepsilon
\]

holds true for each positive integer \(n, n > n_0\).

It remains to show, that for each \(y \in Y\) there is an \(x \in L_n\) such that \(d(x, y) < \varepsilon\).

Let \(y = (0, y_1, y_2, y_3, \ldots) \in Y\). Take \(x = (\frac{1}{n}, x^n_1(t(y)), x^n_2(t(y)), x^n_3(t(y)), \ldots) \in Y\), with \(d((y_1, y_2, y_3, \ldots), (x^n_1(t(y)), x^n_2(t(y)), x^n_3(t(y)), \ldots)) < \varepsilon_1\). Then it is easy to see, that

\[
d(x, y) = \sup_{i \in 1, 2, 3, \ldots} \left\{ \frac{1}{n}, \frac{|y_i - x^n_i(t(y))|}{2^i+1} \right\} < \varepsilon
\]
holds true for each positive integer $n$, $n > n_0$.

3. In the following step we glue arcs $L_n$ into a ray $R$ as follows.

Note that for each positive integer $n$, $L_n$ is an arc from the point $(\frac{1}{n}, a_1, a_2, a_3...)$ to the point $(\frac{1}{n}, b_1, b_2, b_3...)$. Let $A_i$ be an arc from an endpoint of $L_i$ to an endpoint of $L_{i+1}$, defined by

$$A_{2n-1} = \{(1 - t)(\frac{1}{2n-1}, b_1, b_2, b_3, ...)+ t(\frac{1}{2n}, b_1, b_2, b_3, ...) \mid t \in [0, 1]\},$$

$$A_{2n} = \{(1 - t)(\frac{1}{2n}, a_1, a_2, a_3, ...)+ t(\frac{1}{2n+1}, a_1, a_2, a_3, ...) \mid t \in [0, 1]\},$$

for each positive integer $n$.

Furthermore, let

$$R = L_1 \cup A_1 \cup L_2 \cup A_2 \cup ... = \bigcup_{i=1,2,3,...} (L_i \cup A_i).$$

Then $R$ is a continues image of $[0, \infty)$ by the pasting lemma (see [44, p. 108]).

4. Finally, we show that $X$ defined by $X = \text{Cl}(R)$ is the desired compactification. More precisely, we show that $X$ is smooth compactification of a ray,
that is irreducible. Obviously $X$ is a continuum. First we show, that $X$ is a compactification of $R$ with remainder $Y$. While $X$ is compact it is enough to prove that $X = R \cup Y$. Since $R \subseteq \text{Cl}(R)$ and $Y \subseteq \text{Cl}(R)$ it easily follows that $R \cup Y \subseteq \text{Cl}(R)$. From the definition of $R$ it follows that
\[
\text{Cl}(R) = \text{Cl}(\bigcup_{i=1,2,3,...} L_i) \cup (\bigcup_{i=1,2,3,...} A_i) \subseteq \text{Cl}(\bigcup_{i=1,2,3,...} L_i) \cup \text{Cl}(\bigcup_{i=1,2,3,...} A_i).
\]
Since
\[
\text{Cl}(\bigcup_{i=1,2,3,...} L_i) \subseteq R
\]
and
\[
\text{Cl}(\bigcup_{i=1,2,3,...} A_i) = (\bigcup_{i=1,2,3,...} A_i) \cup \{(0, a_1, a_2, a_3, ...)\} \cup \{(0, b_1, b_2, b_3, ...)\} \subseteq R \cup Y
\]
it follows that $\text{Cl}(R) \subseteq R \cup Y$.

Next we prove that $X$ is irreducible. Let $p = (1, a_1, a_2, a_3, ...)$ and let $x \in X \setminus R$ be arbitrarily chosen. Since the only subcontinuum of $X$ that contains $\{p, x\}$ is whole $X$, $X$ is irreducible (between $p$ and $x$).

Finally, we show that $X$ is smooth, by proving that $X$ is smooth in $p = (1, a_1, a_2, a_3, ...)$. 

(a) If $x \in R$, then there is an uniquely determined arc $px$ from $p$ to $x$, which is a subcontinuum of $X$. If $x \in X \setminus R$, then the unique subcontinuum of $X$, which is irreducible between $p$ and $x$, is $X$. Therefore for each $x \in X$ there is a unique subcontinuum $px$ of $X$ that is irreducible between $p$ and $x$.

(b) It remains to prove that for each sequence of points $x_n \in X$, which is convergent with limit $x \in X$, the sequence of irreducible continua $px_n$ is convergent with the limit continuum $px$.

If $x \in R$, this is obviously true. So, let $x \in X \setminus R$. Without loss of generality we can assume, that either $x_n \in R$ for each positive integer $n$, or $x_n \in X \setminus R$ for each positive integer $n$. If $x_n \in X \setminus R$ for each positive integer $n$, then it follows from $px_n = X = px$, that $px_n$ limits to $px$. 
Suppose $x_n \in R$ for each positive integer $n$. In that case we show that $\text{Cl}(\bigcup_{n=1,2,3,...} px_n) = px$. Since $px = X$ it follows that $\text{Cl}(\bigcup_{n=1,2,3,...} px_n) \subseteq px$.

We prove that for each $\varepsilon > 0$ there is a positive integer $n$, such that $X \subseteq \bigcup_{y \in px_n} K_\varepsilon(y)$, where $K_\varepsilon(y) = \{ x \in Q \mid d(x,y) < \varepsilon \}$.

Suppose that there exists $\varepsilon > 0$, such that for each positive integer $n$, there is an $z_n \in X \setminus \bigcup_{y \in px_n} K_\varepsilon(y)$. Without loss of generality, we suppose that $s$ is the limit point of the sequence $(z_n)_{n=1}^\infty$ and consider the following two cases:

i. Suppose $(z_n)_{n=1}^\infty \subseteq R$. If $s \in R$, then there are positive integers $m$ and $n_0$, such that $(z_n)_{n=n_0}^\infty \subseteq px_m$. It follows that $z_n \in px_n$ for some $n > \max\{n_0, m\}$, which is a contradiction. Suppose $s \not\in R$ and let $m$ be a positive integer, such that $d_H(L_m, Y) < \varepsilon$. Since $s \in Y$ and $\lim_{n \to \infty} z_n = s$ it holds that $(z_n)_{n=n_0}^\infty \subseteq \bigcup_{y \in L_m} K_\varepsilon(y)$. Next, let $k > m$ and $n > n_0$ be positive integers such that $x_n \in L_k \cup A_k$. It follows that $\bigcup_{y \in L_m} K_\varepsilon(y) \subseteq \bigcup_{y \in px_n} K_\varepsilon(y)$ and therefore $z_n \in \bigcup_{y \in px_n} K_\varepsilon(y)$, which is a contradiction.

ii. Suppose $(z_n)_{n=1}^\infty \subseteq X \setminus R$. Let $n_0$ and $m$ be a positive integers, such that $(z_n)_{n=n_0}^\infty \subseteq K_\varepsilon(s)$ and $d_H(L_m, Y) < \varepsilon$. As before, let $k > m$ and $n > n_0$ be positive integers such that $x_n \in L_k \cup A_k$. It follows that $\bigcup_{y \in L_m} K_\varepsilon(y) \subseteq \bigcup_{y \in px_n} K_\varepsilon(y)$ and therefore $z_n \in \bigcup_{y \in px_n} K_\varepsilon(y)$, which is again a contradiction.

□
4.2 Span of compactifications of a ray with remainders having span zero

By studying compactifications, a question about their span appears. Remember the famous Lelek’s problem about whether the converse of the statement that chainable continua have span zero, also holds true, which was rejected in 2011 by L. C. Hoehn. We will show that if $X$ is a compactification of a ray such that the remainder of $X$ has span zero, then $X$ has span zero too. The Hoehn’s example is the only known nonchainable continuum with span zero at the time. Our result could be helpful when constructing an uncountable family of pairwise nonhomeomorphic nonchainable span zero continua.

Suppose that $X$ is a compactification of a ray $[0, \infty)$ such that the remainder of $X$ has span zero. To prove that $X$ has span zero too, we only need to prove that $X$ has surjective span zero. See the proposition below.

**Proposition 4.3** Let $X = [0, \infty) \cup Y$ be a compactification of a ray $[0, \infty)$ such that $\sigma(Y) = 0$. Then $\sigma(X) = 0$ if and only if $\sigma^*(X) = 0$.

**Proof.** It is a well known fact, that $\sigma^*(X) \leq \sigma(X)$. Therefore from $\sigma(X) = 0$ it easily follows that $\sigma^*(X) = 0$. To show that converse also holds, it is enough to prove that if $\sigma^*(X) = 0$, then for each subcontinuum $Z$ of $X$ it holds that $\sigma^*(Z) = 0$. Let $Z$ be a subcontinuum of $X$. If $Z$ is a subcontinuum of the remainder $Y$, then $\sigma^*(Z) = 0$ since $\sigma^*(Y) \leq \sigma(Y) = 0$. Obviously the same argument holds true if $Z$ is a subcontinuum of the ray $[0, \infty)$. If $Z$ is a subcontinuum of $X$ and it is not a subspace neither of $Y$, neither of $[0, \infty)$, then it must be homeomorphic to the whole $X$. Since having a span zero is a topological property it easily follows that $\sigma^*(Z) = 0$. \[\square\]

In this section we will help ourselves with a discretization of span. Before proving the main result, we need to state some propositions first.
Proposition 4.4 Suppose that $0 < \varepsilon < \frac{c}{8}$, $X$ is a continuum and there exist onto functions $f : \{1, \ldots, n\} \to A$, $g : \{1, \ldots, m\} \to B$ such that

1. $A, B \subseteq X$,
2. $d_H(A, X) < \frac{\varepsilon}{8}$, $d_H(B, X) < \frac{\varepsilon}{8}$,
3. $d(f(i), f(i + 1)) < \frac{\varepsilon}{4}$ for each $i$,
4. $d(g(i), g(i + 1)) < \frac{\varepsilon}{4}$ for each $i$,
5. $d(f(i), g(i)) > c$ for each $i$.

Then there exist onto functions $\hat{f}, \hat{g} : \{1, \ldots, k\} \to C$ such that

1. $C \subseteq X$,
2. $d_H(C, X) < \frac{\varepsilon}{2}$
3. $d(\hat{f}(i), \hat{f}(i + 1)) < \varepsilon$ for each $i$,
4. $d(\hat{g}(i), \hat{g}(i + 1)) < \varepsilon$ for each $i$,
5. $d(\hat{f}(i), \hat{g}(i)) > c - \varepsilon$ for each $i$.

Proof. Without loss of generality suppose that $n \leq m$. Since $d_H(A, X) < \frac{\varepsilon}{8}$ and $d_H(B, X) < \frac{\varepsilon}{8}$, it follows that $d_H(A, B) < \frac{\varepsilon}{4}$. Since $d_H(A, X) < \frac{\varepsilon}{8}$, it holds that $\{K_{\frac{\varepsilon}{6}}(f(i))\}_{i=1}^n$ is an open cover for $X$. Therefore for each $i \in \{1, \ldots, m\}$ there is a $j_i \in \{1, \ldots, n\}$ such that $g(i) \in K_{\frac{\varepsilon}{6}}(f(j_i))$. Let $k = n$, $\hat{g}(i) = f(j_i)$ and $C = \{\hat{g}(i)\}_{i=1}^k$. Notice that $d(\hat{g}(i), g(i)) < \frac{\varepsilon}{6}$.

First we show that for every $t \in \{1, \ldots, k\}$ there exists $i_t \in \{1, \ldots, k\}$ such that $K_{\frac{\varepsilon}{6}}(f(t)) \cap K_{\frac{\varepsilon}{6}}(\hat{g}(i_t)) \neq \emptyset$.

Notice that $B \subseteq \{K_{\frac{\varepsilon}{6}}(\hat{g}(i))\}_{i=1}^k$. Suppose on the contrary that there exists $t$ such that $K_{\frac{\varepsilon}{6}}(f(t)) \cap K_{\frac{\varepsilon}{6}}(\hat{g}(i)) = \emptyset$ for all $i$. Then $K_{\frac{\varepsilon}{6}}(f(t)) \cap B = \emptyset$ and it follows that

$$d_H(B, X) \geq d(B, f(t)) \geq \frac{\varepsilon}{6}$$
which it a contradiction.

It follows that $K_r(f(t)) \subseteq K_r(\hat{g}(i))$ for each $t$ and hence $\{K_r(\hat{g}(i)) \mid i = 1, ..., k\}$ is an open cover for $D$. Thus $d_H(C, X) < \frac{\varepsilon}{2}$. Now define

$$\hat{f}(t) = \begin{cases} f(t) & \text{if } f(t) \in C \\ \hat{g}(i) & \text{otherwise.} \end{cases}$$

Note that $d(\hat{f}(t), f(t)) < \frac{\varepsilon}{3}$. Since $C \subseteq \{f(i)\}_{i=1}^k$, it follows that $\hat{f}, \hat{g} : \{1, ..., k\} \rightarrow C$ are both onto. Now we show that (3) – (5) hold.

(3) By the definition of $\hat{g}$ and since $d(g(i), g(i+1)) < \frac{\varepsilon}{4}$, it follows that

$$d(\hat{g}(i), \hat{g}(i+1)) \leq d(\hat{g}(i), g(i)) + d(g(i), g(i+1)) + d(g(i+1), \hat{g}(i+1)) < \frac{\varepsilon}{6} + \frac{\varepsilon}{4} + \frac{\varepsilon}{6} < \varepsilon.$$

(4) By the definition of $\hat{f}$ it is easy to see that

$$d(\hat{f}(i), \hat{f}(i+1)) < d(\hat{f}(i), f(i)) + d(f(i), f(i+1)) + d(f(i+1), \hat{f}(i+1)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{4} + \frac{\varepsilon}{3} < \varepsilon.$$

(5) Finally,

$$d(\hat{f}(t), \hat{g}(t)) \geq d(f(t), g(t)) - d(\hat{f}(t), f(t)) - d(g(t), \hat{g}(t)) > c - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} > c - \varepsilon$$

for each $t$. 

\[\square\]
Theorem 4.5 Let $X$ be a continuum and suppose that there exists a $c > 0$ such that for every $\varepsilon > 0$ there exists a finite subset $B_{\varepsilon}$ of $X$ such that

1. $d_H(B_{\varepsilon}, X) < \frac{\varepsilon}{2}$

2. there exist onto maps $f_{\varepsilon}, g_{\varepsilon}: \{1, ..., m_{\varepsilon}\} \to B_{\varepsilon}$ such that
   
   (a) $d(f_{\varepsilon}(i), f_{\varepsilon}(i + 1)) < \varepsilon$ for each $i$,
   
   (b) $d(g_{\varepsilon}(i), g_{\varepsilon}(i + 1)) < \varepsilon$ for each $i$,
   
   (c) $d(f_{\varepsilon}(i), g_{\varepsilon}(i)) > c$ for each $i$.

Then $\sigma^*(X) \geq c$.

Proof. Let $n$ be a positive integer and let $\varepsilon = \frac{1}{n}$. Then, by hypothesis, there exists a finite subset $B_n$ of $X$ such that $d_H(B_n, X) < \frac{1}{2n}$ and there exist onto maps $f_n, g_n: \{1, ..., m_n\} \to B_n$ such that

1. $d(f_n(i), f_n(i + 1)) < \frac{1}{n}$ for each $i$,
2. $d(g_n(i), g_n(i + 1)) < \frac{1}{n}$ for each $i$,
3. $d(f_n(i), g_n(i)) > c$ for each $i$.

Let $E_n = \{(f_n(i), g_n(i))| i = 1, ..., m_n\}$ be a subset of $X \times X$ and $F = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n$.

Let $x \in F$ be arbitrarily chosen and let $j_1 < j_2 < ...$ be an increasing sequence of positive integers such that for each $n$ it holds that $d(x, (f_{j_n}(i_n), g_{j_n}(i_n))) < \frac{1}{n}$ for some $i_n$. Denote

$$E = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_{j_n}.$$ 

We show that $E$ is subcontinuum of $X \times X$ with $\pi_1(E) = \pi_2(E) = X$ and such that $d(x, y) \geq c$ for each $(x, y) \in E$.

1. First we will show that $E$ is subcontinuum of $X \times X$. $E$ is obviously nonempty and compact subspace of $X \times X$. Therefore we will prove only that $E$ is connected. Let $F_k = \bigcup_{n=k}^{\infty} E_{j_n}$. We will show that $F_k$ is $(d, \frac{2}{k})$-chainable for
each positive integer \( k \). To see that, let \( k \) be arbitrarily chosen and \( a, b \) any points in \( F_k \). Let’s find a \((d, \frac{2}{k})\)-chain from \( a \) to \( b \). By the definition of \( F_k \) there exist \( l, n \geq k \) such that \( a = (f_{jl}(i), g_{jl}(i)) \) for some \( i \in \{1, \ldots, m_{jl}\} \) and \( b = (f_{jn}(j), g_{jn}(j)) \) for some \( j \in \{1, \ldots, m_{jn}\} \). Without loss of generality we can assume, that \( l < n \). Again, by the definition of \( F_k \), there exist positive integers \( i_l, i_n \) such that

\[
\begin{align*}
d((f_{jl}(i_l), g_{jl}(i_l)), (f_{jn}(i_n), g_{jn}(i_n))) & \leq d((f_{jl}(i_l), g_{jl}(i_l)), x) \\
& \quad + d(x, (f_{jn}(i_n), g_{jn}(i_n))) \\
& < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}.
\end{align*}
\]

It follows from 1. and 2. that

\[
\begin{align*}
d((f_n(i), g_n(i)), (f_n(i + 1), g_n(i + 1))) &= d(f_n(i), f_n(i + 1)) \\
& \quad + d(g_n(i), g_n(i + 1)) \\
& < \frac{1}{n} + \frac{1}{n}.
\end{align*}
\]

Therefore there is a \((d, \frac{1}{n})\)-chain from \( a = (f_{jl}(i), g_{jl}(i)) \) to \( (f_{jn}(i_l), g_{jn}(i_l)) \) and \( \frac{1}{j_l} < \frac{1}{k} \), there is a \((d, \frac{2}{k})\)-chain from \( a \) to \( (f_{jl}(i_l), g_{jl}(i_l)) \). Likewise, since there is a \((d, \frac{1}{j_n})\)-chain from \( (f_{jn}(i_n), g_{jn}(i_n)) \) to \( b = (f_{jn}(j), g_{jn}(j)) \) and \( \frac{1}{j_n} < \frac{1}{k} \), there is a \((d, \frac{2}{k})\)-chain from \( (f_{jn}(i_n), g_{jn}(i_n)) \) to \( b \). So, it follows that we have found a \((d, \frac{2}{k})\)-chain from \( a \) to \( b \), which means that \( F_k = \bigcup_{n=k}^{\infty} E_{jn} \) is \((d, \frac{2}{k})\)-chainable for each \( k \). Suppose that there exist \( U \) and \( V \) with \( \text{Cl}(U) \cap \text{Cl}(V) \neq \emptyset \) which separate \( E = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_{jn} \). Let \( \alpha = d(\text{Cl}(U), \text{Cl}(V)) \) and \( k_0 \) a positive integer such that \( \frac{2}{k_0} < \alpha \). Obviously there is no \((d, \frac{2}{k_0})\)-chain from a point in \( U \) to a point in \( V \), which is a contradiction. Consequently \( E = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_{jn} \) is connected.

2. Next, we will show that \( \pi_1(E) = \pi_2(E) = X \). Obviously \( \pi_1(E) \subseteq X \), so we must prove that \( X \subseteq \pi_1(E) = \{x \in X \mid \text{there exists } y \in X \text{ such that } (x, y) \in E\} \). Suppose that \( X \not\subseteq \pi_1(E) \) and let \( x \in X \setminus \pi_1(E) \). This means that for
each \( y \in X \) there is a \( k_y \) such that \((x, y) \notin F_{k_y}\). So, there is a \( k_y \) and an open subset \( U_y \) of \( X \times X \) such that \((x, y) \in U_y \subseteq X \times X \setminus F_{k_y}\). Since \( \{x\} \times X \) is compact, there are \( y_1, \ldots, y_l \in X \) such that \( \{x\} \times X \subseteq U_{y_1} \cup \ldots \cup U_{y_l} \). Let \( N = U_{y_1} \cup \ldots \cup U_{y_l} \). \( N \) is an open set containing the slice \( \{x\} \times X \) and \( X \) is compact. By the Lemma 1.6 there exists a tube \( W \) in \( X \times X \) containing \( \{x\} \times X \) and contained in \( N \). Let \( V = \pi_1(W) \). Hence \( V \) is an open set in \( X \), such that \( x \in V \) and \( V \cap \pi_1(F_k) = \emptyset \). Let \( r > 0 \) be such that \( K_r(x) \subseteq V \) and let \( n_0 \geq k \) be such that \( \frac{1}{2j_{n_0}} < r \). Since \( d_H(X, B_{j_{n_0}}) < \frac{1}{2j_{n_0}} \), it holds that for \( x \in X \) there is \( x_0 \in B_{j_{n_0}} \) such that \( d(x, x_0) < \frac{1}{2j_{n_0}} \). So, there exists a positive integer \( i \) such that \( f_{j_{n_0}}(i) = x_0 \), and hence, \( d(x, f_{j_{n_0}}(i)) < \frac{1}{2j_{n_0}} < r \). This means that \( f_{j_{n_0}}(i) \in V \), which is a contradiction. Therefore \( \pi_1(E) = X \). The proof of equality \( \pi_2(E) = X \) is the same.

3. Finally we will prove that \( d(x, y) \geq c \) for each \((x, y) \in E\). Let \((x, y) \) be any point in \( E \). By the definition of \( E \), \((x, y) \in \bigcup_{n=k}^{\infty} E_{jn}\) for each positive integer \( k \). This means that for each \( k \) and for each open subset \( U \) of \( X \times X \) with \((x, y) \in U \) it holds that \( U \cap \bigcup_{n=k}^{\infty} E_{jn} \neq \emptyset \). So, for each positive integer \( k \) and for each open subset \( U \) of \( X \times X \) with \((x, y) \in U \), there is an \( n_k \geq k \) such that \( U \cap E_{jn_k} \neq \emptyset \).

For each \( k \) let \( U_k = K((x, y), \frac{1}{k}) \) and \((x_k, y_k) \in U_k \cap E_{jn_k} \). It follows that \( \lim_{x \to \infty} (x_k, y_k) = (x, y) \) and that for each \( k \), \((x_k, y_k) = (f_{jn_k}(i_k), g_{jn_k}(i_k)) \) for some \( i_k \). Therefore \( \lim_{x \to \infty} (x_k, y_k) = (x, y) \) and \( d(x_k, y_k) \geq c \) for each \( k \), which means that \( d(x, y) \geq c \).

\[ \square \]

**Theorem 4.6** Let \( X \) be a continuum and suppose that \( \sigma^*(X) \geq 2c \). Then for every \( 0 < \lambda < \frac{c}{2} \) there exists a finite subset \( C \) of \( X \) and onto maps \( f, g : \{1, \ldots, k\} \to C \) such that

1. \( d_H(C, X) < \frac{\lambda}{2} \),
2. \( d(f(i), f(i+1)) < \lambda \) for each \( i \),
3. \( d(g(i), g(i+1)) < \lambda \) for each \( i \),
4. \( d(f(i), g(i)) > c \) for each \( i \).
Proof. Let \( X \) be a continuum, such that \( \sigma^*(X) \geq 2c \) and let \( 0 < \lambda < \frac{c}{2} \) be arbitrarily chosen. By the definition of \( \sigma^*(X) \), there is a subcontinuum \( Y \) of \( X \times X \) with \( \pi_1(Y) = \pi_2(Y) = X \) and \( d(x, y) > c + \lambda \) for each \((x, y) \in Y\). Let \( Z = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \) be a \((d, \varepsilon)-chain\) in \( Y \), such that \( d_H(Z, Y) < \frac{\lambda}{8} \). We define \( A = \pi_1(Z) \subseteq X \) and \( B = \pi_2(Z) \subseteq X \). Since \( d_H(Z, Y) < \frac{\lambda}{8} \) it holds that \( d_H(A, X) < \frac{\lambda}{8} \) and \( d_H(B, X) < \frac{\lambda}{8} \).

Next, let \( f': \{1, \ldots, m\} \to A \) and \( g': \{1, \ldots, m\} \to B \) be defined as

\[
f'(i) = x_i,
\]

\[
g'(i) = y_i.
\]

Since \( A = \{x_i \mid i \in \{1, \ldots, m\}\} \) and \( B = \{y_i \mid i \in \{1, \ldots, m\}\} \) it follows that \( f' \) and \( g' \) are both onto. Clearly,

\[
d(f'(i), f'(i + 1)) = d(x_i, x_{i+1}) \leq d((x_i, y_i), (x_{i+1}, y_{i+1})) < \lambda.
\]

and similarly

\[
d(g'(i), g'(i + 1)) = d(y_i, y_{i+1}) \leq d((x_i, y_i), (x_{i+1}, y_{i+1})) < \lambda.
\]

Since \( d(f'(i), g'(i)) = d(x_i, y_i) \in Z \subseteq Y \), it follows that \( d(f'(i), g'(i)) > c + \lambda \). Thus by Proposition 4.4, there exist a finite subset \( C \) of \( X \) and onto maps \( f, g : \{1, \ldots, k\} \longrightarrow C \) such that

1. \( d_H(C, X) < \frac{\lambda}{2} \),
2. \( d(f(i), f(i + 1)) < \lambda \) for each \( i \),
3. \( d(g(i), g(i + 1)) < \lambda \) for each \( i \),
4. \( d(f(i), g(i)) > (c + \lambda) - \lambda = c \) for each \( i \).

\( \square \)
Proposition 4.7 Let \([a, b]\) be an arc with metric \(d\) and \(\varepsilon > 0\). Then there exists \(\delta > 0\) such that if \(f : \{1, \ldots, m\} \rightarrow [a, b]\) is a function with the following properties:

1. \(\text{diam}([a, f(1)]) < \frac{\varepsilon}{2}\),
2. \(\text{diam}([f(m), b]) < \frac{\varepsilon}{2}\),
3. \(d(f(i), f(i + 1)) < \delta\) for each \(i\),

then \(d_H(\{f(i)\}_{i=1}^{m}, [a, b]) < \varepsilon\).

Proof. Let \([a, b]\) be an arc and \(\varepsilon > 0\). There exist \(a = a_0 < a_1 < \ldots < a_p = b\) in \([a, b]\) such that \(\text{diam}([a_i, a_{i+1}]) < \frac{\varepsilon}{2}\). Let \(\delta_i = d([a, a_i], [a_{i+1}, b])\) for \(i \in \{1, \ldots, p - 2\}\) and let \(\delta\) be an arbitrarily positive chosen number, such that \(\delta < \min\{\delta_i\}\). We show that if \(f : \{1, \ldots, m\} \rightarrow [a, b]\) is a function with properties 1., 2. and 3. above, then \(d_H(\{f(i)\}_{i=1}^{m}, [a, b]) < \varepsilon\) holds true.

Let \(i_1\) be the largest index such that \(a_{i_1} \leq f(1)\) and \(i_2\) be the smallest index such that \(a_{i_2} \geq f(m)\).

Claim: For every \(i \in \{i_1, \ldots, i_2\}\) there exists \(j_i\) such that \(f(j_i) \in [a_i, a_{i+1}]\).

Suppose on the contrary that there exists an \(i'\) such that \(\{f(j)\}_{j=1}^{m} \cap [a_{i'}, a_{i'+1}] = \emptyset\). Since \(f(1) \in [a, a_{i_1+1}]\) and \(f(m) \in [a_{i_2-1}, b]\) it follow that there is a \(j'\) such that \(f(j') \in [a, a_{i'}]\) and \(f(j' + 1) \in [a_{i'+1}, b]\). However, then \(d(f(j'), f(j' + 1)) \geq \delta_{i'} > \delta\), which is a contradiction.

Pick any \(x \in [a, b]\). We have three cases:

Case 1: \(x \in [a, a_{i_1}] \subseteq [a, f(1)]\).
Then \(d(x, f(1)) \leq \text{diam}([a, f(1)]) \leq \frac{\varepsilon}{2}\).

Case 2: \(x \in [a_{i_2}, b] \subseteq [f(m), b]\).
Then \(d(x, f(m)) \leq \text{diam}([f(m), b]) \leq \frac{\varepsilon}{2}\).
Case 3: $x \in [a_{i_1}, a_{i_2}]$.
Then there exist $i'$ and $j_i$ such that $x, f(j_i) \in [a_{i'}, a_{i'+1}]$. It follows that $d(x, f(j_i)) < \frac{\varepsilon}{2}$. Hence, $d_H(\{f(i)\}_{i=1}^m, [a, b]) < \varepsilon$. □

Proposition 4.8 Suppose that $I$ is an arc, $d$ the metric on $I$ and $h : I \rightarrow [0, 1] \subseteq \mathbb{R}$ a homeomorphism. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $a_0, b_0, a_1, b_1 \in I$ such that

1. $h(a_0) \leq h(b_0),$
2. $h(b_1) \leq h(a_1),$
3. $d(a_0, a_1) < \delta,$
4. $d(b_0, b_1) < \delta,$

then $d(a_0, b_0) < \varepsilon$.

Proof. By uniform continuity, there exists $\gamma > 0$ such that if $x, y \in [0, 1]$, with $|x - y| < \gamma$, then $d(h^{-1}(x), h^{-1}(y)) < \varepsilon$. Likewise, there exists $\delta > 0$ such that if $a, b \in I$ with $d(a, b) < \delta$, then $|h(a) - h(b)| < \frac{\varepsilon}{2}$. Thus it follows that if $d(a_0, a_1) < \delta$ and $d(b_0, b_1) < \delta$ then $|h(a_0) - h(a_1)| < \frac{\varepsilon}{2}$ and $|h(b_0) - h(b_1)| < \frac{\varepsilon}{2}$. Since $h(a_0) \leq h(b_0)$ and $h(b_1) \leq h(a_1)$, it follows that

$$h(b_0) - \frac{\gamma}{2} < h(b_1) \leq h(a_1) < h(a_0) + \frac{\gamma}{2}.$$  

Hence, $|h(a_0) - h(b_0)| = h(b_0) - h(a_0) < \gamma$. Thus, $d(a_0, b_0) < \varepsilon$. □

Lemma 4.9 Suppose that $X = Y \cup [0, \infty)$, $c = \frac{2^*(X)}{8} > 0$ and $0 < \varepsilon < \frac{\varepsilon}{2}$. Then there exists an arc $[x, y] \subseteq [0, \infty)$, finite subset $B$ of $X$ and onto maps $f, g : \{1, \ldots, m\} \rightarrow B$ such that

1. $d_H(\{x, y\}, Y) < \varepsilon,$
2. $d_H(B, [x, y] \cup Y) < \varepsilon,$
3. \(d(f(i), f(i + 1)) < \varepsilon\) for each \(i\),

4. \(d(g(i), g(i + 1)) < \varepsilon\) for each \(i\),

5. \(d(f(i), g(i)) > 2c\) for each \(i\).

**Proof.** We must consider several steps and claims.

1. First we show that there exists \(x, y, x < y \in [0, \infty)\) such that \(d_H([x, \infty), Y) < \frac{\varepsilon}{8}\), \(d_H([x, y], Y) < \frac{\varepsilon}{8}\) and \(d_H([y, \infty), Y) < \frac{\varepsilon}{8}\).

   Let \(K_{\frac{16}{\varepsilon}}(y) = \{x \in X \mid d(x, y) < \frac{16}{\varepsilon}\}\). Obviously \(Y \subseteq \bigcup_{y \in Y} K_{\frac{16}{\varepsilon}}(y)\). Since \(Y\) is compact there exist \(y_1, \ldots, y_n \in Y\) such that \(Y \subseteq \bigcup_{i=1}^{n} K_{\frac{16}{\varepsilon}}(y_i)\). Denote \(U = \bigcup_{i=1}^{n} K_{\frac{16}{\varepsilon}}(y_i) \setminus Y\). Let \(w \in [0, \infty)\) be such that \(|w, \infty) \subseteq U\). Since \(Y \subseteq [w, \infty)\), for each \(y_i \in Y\) there is an \(x_i \in [w, \infty)\) such that \(d(y_i, x_i) < \frac{\varepsilon}{16}\).

   If \(n = 1\) let \(x = x_1\) and \(y > x\) arbitrarily chosen. In this case \([x, y] \subseteq K_{\frac{16}{\varepsilon}}(x_1)\) and \(Y \subseteq K_{\frac{16}{\varepsilon}}(x_1)\). Therefore \(d_H([x, \infty), Y) < \frac{\varepsilon}{8}\), \(d_H([x, y], Y) < \frac{\varepsilon}{8}\) and \(d_H([y, \infty), Y) < \frac{\varepsilon}{8}\) obviously hold true.

   If \(n > 1\) denote \(x = \min\{x_i \mid i = 1, \ldots, n\}, y = \max\{x_i \mid i = 1, \ldots, n\}\) in the ordering of \([0, \infty)\). Since \([x, \infty) \subseteq U, [y, \infty) \subseteq U\) and \(Y = \bigcap_{x \in [0, \infty)} [x, \infty)\), it follows that \(d_H([x, \infty), Y) < \frac{\varepsilon}{8}\) and \(d_H([y, \infty), Y) < \frac{\varepsilon}{8}\). Therefore it suffices to show that \(d_H([x, y], Y) < \frac{\varepsilon}{8}\).

   
   \(i\) Take any \(p \in Y\). We will show that there exists \(z \in [x, y]\) with \(d(p, z) < \frac{\varepsilon}{8}\).

   Since \(Y \subseteq \bigcup_{i=1}^{n} K_{\frac{16}{\varepsilon}}(y_i)\), there exists a \(y_i \in Y\) with \(d(p, y_i) < \frac{\varepsilon}{16}\) and \(x_i \in [x, y]\) with \(d(y_i, x_i) < \frac{\varepsilon}{16}\). Therefore \(d(p, x_i) < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8}\).

   \(ii\) Since \(d_H([x, \infty), Y) < \frac{\varepsilon}{8}\), for each \(z \in [x, y]\) there exists \(y \in Y\) with \(d(z, y) < \frac{\varepsilon}{8}\).

2. By Proposition 4.7, there exists \(\hat{\delta} > 0\) such that if \(f : \{1, \ldots, m\} \rightarrow [x, y]\) is a function with the following properties:

   (a) \(\text{diam}([x, f(1)]) < \frac{\varepsilon}{16}\),
   
   (b) \(\text{diam}([f(m), y]) < \frac{\varepsilon}{16}\),
   
   (c) \(d(f(i), f(i + 1)) < \hat{\delta}\) for each \(i\),

then \(d_H([f(i)]_{i=1}^{m}, [x, y]) < \frac{\varepsilon}{8}\).
3. Let \( \beta = \min\{ \frac{\varepsilon}{16}, \frac{\delta}{4}\} \). There exist \( a_x, b_x, a_y, b_y \in [0, \infty) \), \( a_x < b_x < a_y < b_y \) such that \( x \in (a_x, b_x) \), \( y \in (a_y, b_y) \), \( \text{diam}((a_x, b_x)) < \beta \) and \( \text{diam}((a_y, b_y)) < \beta \).

4. Let \( \gamma = \min\{d([0, a_x], [b_x, a_y]), d([a_y, b_y], Y), d([0, a_x], [b_y, \infty) \cup Y), d([b_x, a_y], [b_y, \infty) \cup Y]\} \).

5. By Proposition 4.8, there exists \( \delta > 0 \) such that if \( a_0, b_0, a_1, b_1 \in [0, b_y] \) with
   
   (a) \( a_0 \leq b_0 \) in the ordering on \([0, b_y]\),
   
   (b) \( b_1 \leq a_1 \) in the ordering on \([0, b_y]\),
   
   (c) \( d(a_0, a_1) < \delta \),
   
   (d) \( d(b_0, b_1) < \delta \),

   then \( d(a_0, b_0) < \varepsilon \).

6. Let \( \lambda = \frac{1}{4} \min\{\delta, \gamma, \beta\} \).

7. By Theorem 4.6 there exists a finite subset \( A \) of \( X \) and onto maps \( f, g : \{1, \ldots, m\} \rightarrow A \) such that
   
   (a) \( d_H(A, X) < \frac{\lambda}{2} \),
   
   (b) \( d(f(i), f(i + 1)) < \lambda \) for each \( i \),
   
   (c) \( d(g(i), g(i + 1)) < \lambda \) for each \( i \),
   
   (d) \( d(f(i), g(i)) > 4c \) for each \( i \).

8. Since \( A \) is finite, there exists \( z \in [0, \infty), \ z > y \) such that \( A \subseteq [0, z) \cup Y \).

\textbf{Claim:} \( A \cap [0, b_x] \neq \emptyset \) and \( A \cap ([a_y, z) \cup Y) \neq \emptyset \).

Suppose on the contrary that \( A \cap [0, b_x] = \emptyset \). Then \( A \subseteq [b_x, z] \cup Y \) and therefore

\[ d(a_x, A) \geq d(a_x, [b_x, z] \cup Y) \geq \gamma > \frac{\lambda}{2}. \]

However, this is in contradiction with \( d_H(A, X) < \frac{\lambda}{2} \). Proof that \( A \cap ([a_y, z) \cup Y) \neq \emptyset \) is similar.
9. Let

\[ F_x = \{ i \in \{1, \ldots, m\} \mid f(i) \in Y \cup [b_x, z] \}, \]
\[ F_y = \{ i \in \{1, \ldots, m\} \mid f(i) \in Y \cup [b_y, z] \}, \]
\[ G_x = \{ i \in \{1, \ldots, m\} \mid g(i) \in Y \cup [b_x, z] \}, \]
\[ G_y = \{ i \in \{1, \ldots, m\} \mid g(i) \in Y \cup [b_y, z] \}. \]

Note that

(a) if \( i \notin F_x \), then \( f(i) \in [0, b_x) \);

(b) if \( i \notin F_y \), then \( f(i) \in [0, b_y) \);

(c) if \( i \notin G_x \), then \( g(i) \in [0, b_x) \);

(d) if \( i \notin G_y \), then \( g(i) \in [0, b_y) \).

Let \( \rho(i) \in \{ i - 1, i + 1 \} \). Since

\[ d(f(i), f(\rho(i))) < \lambda \leq \min\{d([0, a_x), [b_x, z] \cup Y), d([0, a_y), [b_y, z] \cup Y)\}, \]

it follows that

(a) if \( i \in F_x \) and \( \rho(i) \notin F_x \) then \( f(\rho(i)) \in (a_x, b_x) \);

(b) if \( i \in F_y \) and \( \rho(i) \notin F_y \) then \( f(\rho(i)) \in (a_y, b_y) \).

Similarly, since \( d(g(i), g(\rho(i))) < \lambda \), it follows that

(a) if \( i \in G_x \) and \( \rho(i) \notin G_x \) then \( g(\rho(i)) \in (a_x, b_x) \);

(b) if \( i \in G_y \) and \( \rho(i) \notin G_y \) then \( g(\rho(i)) \in (a_y, b_y) \).

10. For \( x, y \in Y \cup [0, \infty) \) we say that \( x < y \) (respectively \( x \leq y \)) if \( x \in [0, b_y) \) and either \( y \in Y \cup [b_y, \infty) \) or \( x < y \) (respectively \( x \leq y \)) in the ordering of \([0, b_y)\).

**Claim:** Suppose that there exists \( i \in \{1, \ldots, m\} \) such that \( f(i) < g(i) \) and \( f(i) \in [0, b_y) \). Then \( g(i + 1) \not\leq f(i + 1) \).

Suppose on the contrary that \( g(i + 1) \leq f(i + 1) \). Then there are four cases:

**Case 1:** \( f(i + 1) \in [0, b_y) \) and \( g(i) \in [0, b_y) \).
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Then it follows from (2) that $d(g(i), f(i)) < \varepsilon < c$ which is a contradiction.

**Case 2:** $f(i + 1) \in [0, b_y)$ and $g(i) \notin [0, b_y)$.

It follows from $f(i + 1) \in [0, b_y)$ and $g(i + 1) \leq f(i + 1)$ that $g(i + 1) \in [0, b_y)$. By (9) it holds that $g(i + 1) \in (a_y, b_y)$ and hence $f(i + 1) \in (a_y, b_y)$. Thus, $d(f(i + 1), g(i + 1)) < \beta < c$ which is a contradiction.

**Case 3:** $f(i + 1) \notin [0, b_y)$ and $g(i) \in [0, b_y)$.

It follows from (9) that $f(i) \in (a_y, b_y)$ and hence $g(i) \in (a_y, b_y)$. Thus, $d(f(i), g(i)) < \beta < c$ which is a contradiction.

**Case 4:** $f(i + 1) \notin [0, b_y)$ and $g(i) \notin [0, b_y)$.

Then $f(i) \in (a_y, b_y)$ and since $d(f(i), f(i + 1)) < \gamma$ it holds that $f(i + 1) \in [b_y, z]$ and since $g(i + 1) \leq f(i + 1)$ it holds that $g(i + 1) \in [a_y, f(i + 1)]$. Therefore $d(f(i + 1), g(i + 1)) < d(f(i + 1), f(i)) + \text{diam}((a_y, b_y)) < \lambda + \beta < c$ which is again a contradiction.

For the same reason we also have:

**Claim:** Suppose that there exists $i \in \{1, ..., m\}$ such that $g(i) < f(i)$ and $g(i) \in [0, b_y)$. Then $f(i + 1) \not\leq g(i + 1)$.

11. Let $a = \min(A \cap [0, z])$ in the ordering of $[0, \infty)$, $i_a = \min\{i \in \{1, ..., m\} \mid f(i) = a\}$ and $j_a = \min\{i \in \{1, ..., m\} \mid g(i) = a\}$. It follows by (8) that $a \in [0, b_x)$. Also note that $f(i_a) < g(i_a)$ and $g(j_a) < f(j_a)$. Without loss of generality, assume $i_a < j_a$.

**Claim:** There exists $n_a \in \{i_a + 1, ..., j_a\}$ such that $f(n_a) \notin [0, b_x)$ and $g(n_a) \in [0, b_x)$.

Let $G = \{i \in \{i_a, ..., j_a\} \mid g(i) \in [0, b_x)\}$. Suppose on the contrary that if $i \in G$ then $f(i) \in [0, b_x]$. Notice that if $i \in G$ then either $f(i) > g(i)$ or $f(i) < g(i)$.

Let $n'_a = \min\{i \in G \mid f(i) > g(i)\}$. It follows that $n'_a$ exists since $f(j_a) > g(j_a)$.

Also, $i_a < n'_a$ since $f(i_a) < g(i_a)$. Consider $n'_a - 1 \in \{i_a, ..., j_a\}$. There are three cases:

**Case 1.** $f(n'_a - 1), g(n'_a - 1) \notin [0, b_x)$.

Then it follows from (9) that $f(n'_a), g(n'_a) \in (a_x, b_x)$. Hence $d(f(n'_a), g(n'_a)) < c$ by (4). However, this contradicts (7).
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Case 2. \( f(n'_a - 1) \in [0, b_x) \) and \( g(n'_a - 1) \not\in [0, b_x) \).

Then \( g(n'_a) \in (a_x, b_x) \) and since \( f(n'_a) \in [0, b_x) \) and \( g(n'_a) < f(n'_a) \) it follows that \( f(n'_a) \in (a_x, b_x) \). Again, \( d(f(n'_a), g(n'_a)) < c \) by (4) which contradicts (7).

Case 3. \( g(n'_a - 1) \in [0, b_x) \).

Then it follows that \( f(n'_a - 1) \in [0, b_x) \) and \( f(n'_a - 1) < g(n'_a - 1) \), by the definition of \( n'_a \). This contradicts the claim in (10).

Claim: There exists \( k_a \in \{i_a, ..., n_a\} \) such that \( f(k_a) \in [0, b_x) \) and \( g(k_a) \not\in [0, b_x) \).

Note that \( g(n_a) < f(n_a) \). Let \( F = \{i \in \{i_a, ..., n_a\} \mid f(i) \in [0, b_x)\} \). Suppose on the contrary that if \( i \in F \) then \( g(i) \in [0, b_x) \). Notice that if \( i \in F \) then either \( f(i) > g(i) \) or \( f(i) < g(i) \). Let \( k'_a = \max\{i \in F \mid f(i) < g(i)\} \). It follows that \( k'_a \) exists since \( f(i_a) < g(i_a) \). Also, \( k'_a < n_a \) since \( f(n_a) > g(n_a) \). Consider \( k'_a + 1 \in \{i_a, ..., n_a\} \). There are three cases:

Case 1. \( f(k'_a + 1), g(k'_a + 1) \not\in [0, b_x) \).

Then it follows from (9) that \( f(k'_a), g(k'_a) \in (a_x, b_x) \). Hence \( d(f(k'_a), g(k'_a)) < c \) by (4). However, this contradicts (7).

Case 2. \( f(k'_a + 1) \not\in [0, b_x) \) and \( g(k'_a + 1) \in [0, b_x) \).

Then \( f(n'_a) \in (a_x, b_x) \) and since \( g(k'_a) \in [0, b_x) \) and \( f(k'_a) < g(k'_a) \) it follows that \( g(k'_a) \in (a_x, b_x) \). Again, \( d(f(k'_a), g(k'_a)) < c \) by (4) which contradicts (7).

Case 3. \( f(k'_a + 1) \in [0, b_x) \).

Then it follows that \( g(k'_a + 1) \in [0, b_x) \) and \( g(k'_a + 1) < f(k'_a + 1) \), by the definition of \( k'_a \). This contradicts the claim in (10).

12. We will show that there exists \( \hat{k}_a, \hat{n}_a \in \{i_a, ..., j_a\} \) such that

(a) \( f(\hat{k}_a) \in (a_x, b_x) \) and \( g(\hat{k}_a) \in [b_x, z] \cup Y \)

(b) \( g(\hat{n}_a) \in (a_x, b_x) \) and \( f(\hat{n}_a) \in [b_x, z] \cup Y \)

(c) \( f(i), g(i) \in [b_x, z] \cup Y \) for all \( i \in \{\hat{k}_a + 1, ..., \hat{n}_a - 1\} \).

Let

\[ \hat{n}_a = \min\{i \in \{k_a, ..., n_a\} \mid f(i) \not\in [0, b_x) \) and \( g(i) \in [0, b_x) \}\]
4.2 Span of compactifications of a ray with remainders having span zero

and

\[ \hat{k}_a = \max\{i \in \{k_a, ..., \hat{n}_a\} \mid f(i) \in [0, b_x) \text{ and } g(i) \not\in [0, b_x)\}. \]

Notice that \( \hat{n}_a \) exists by the definition of \( n_a \), \( \hat{k}_a \) exists by the definition of \( k_a \), and \( i_a \leq k_a \leq \hat{k}_a < \hat{n}_a \leq n_a < j_a \). Furthermore, it follows from (10) that \( \hat{k}_a < \hat{n}_a - 1 \). Let \( \hat{k}_a < i < \hat{n}_a \). Then one of the following must be true:

(a) \( f(i), g(i) \in [0, b_x) \),

(b) \( f(i), g(i) \not\in [0, b_x) \).

Let \( M = \{i \in \{\hat{k}_a + 1, ..., \hat{n}_a - 1\} \mid f(i), g(i) \in [0, b_x)\} \). If \( M = \emptyset \) then the Claim holds, so suppose that \( M \neq \emptyset \). We have 2 cases:

**Case 1:** Suppose that \( \{\hat{k}_a + 1, ..., \hat{n}_a - 1\} \setminus M \neq \emptyset \).

Then there exists \( w \in \{\hat{k}_a + 1, ..., \hat{n}_a - 1\} \) such that \( w - 1 \in M \) or \( w + 1 \in M \).

Suppose that \( w - 1 \in M \). Then \( f(w), g(w) \not\in [0, b_x) \) but \( f(w - 1), g(w - 1) \in [0, b_x) \) and it follows from (9) that \( f(w - 1), g(w - 1) \in (a_x, b_x) \) and hence \( d(f(w - 1), g(w - 1)) < c \) which is impossible. Proof is similar if \( w + 1 \in M \).

**Case 2:** Suppose that \( \{\hat{k}_a + 1, ..., \hat{n}_a - 1\} = M \).

If \( f(\hat{n}_a - 1) < g(\hat{n}_a - 1) \) then by (10) \( f(\hat{n}_a) \not\geq g(\hat{n}_a) \) which is a contradiction.

So let \( m = \min\{i \in M \mid f(i) > g(i)\} \). Then \( f(m) \in [0, b_x) \) and \( m - 1 \geq \hat{k}_a \) and it follows that \( f(m - 1) \leq g(m - 1) \). However, this contradicts the fact that \( f(m - 1) \not\leq g(m - 1) \) by (10).

13. **Claim:** There exists \( p_a, q_a \in \{\hat{k}_a, ..., \hat{n}_a\} \) such that

(a) \( f(p_a) \in (a_y, b_y) \)

(b) \( f(i) \in (a_x, b_y) \) for all \( i \in \{\hat{k}_a, ..., p_a\} \)

(c) \( g(q_a) \in (a_y, b_y) \)

(d) \( g(i) \in (a_x, b_y) \) for all \( i \in \{q_a, ..., \hat{n}_a\} \).

First suppose that \( f(i) \in (a_x, b_y) \) for all \( i \in \{\hat{k}_a, ..., \hat{n}_a\} \). By the definition of \( \hat{k}_a \) and \( \hat{n}_a \) it holds that \( f(\hat{k}_a) < g(\hat{k}_a) \) and \( f(\hat{n}_a) > g(\hat{n}_a) \). Let \( \alpha = \min\{i \in \{\hat{k}_a, ..., \hat{n}_a\} \mid f(i) > g(i)\} \). Then \( f(\alpha) > g(\alpha) \) and \( f(\alpha - 1) < g(\alpha - 1) \). However,
this contradicts (10). Thus there exists \( p \in \{\hat{k}_a, \ldots, \hat{n}_a\} \) such that \( f(p) \not\in (a_x, b_y) \).

Let \( p_a = \min\{p \in \{\hat{k}_a, \ldots, \hat{n}_a\} \mid f(p) \not\in (a_x, b_y)\} - 1 \). Then it follows that \( f(p_a) \in (a_y, b_y) \) and \( f(i) \in (a_x, b_y) \) for all \( i \in \{\hat{k}_a, \ldots, p_a\} \). Proof is similar to find \( q_a \).

14. Let \( k = \hat{n}_a - \hat{k}_a + 1 \) and define for \( i \in \{1, \ldots, k\} \)

\[
\tilde{f}(i) = \begin{cases} 
  x & \text{if } f(\hat{k}_a + i - 1) \in (a_x, b_x) \\
  y & \text{if } f(\hat{k}_a + i - 1) \in (a_y, b_y) \\
  f(\hat{k}_a + i - 1) & \text{otherwise} 
\end{cases},
\]

and

\[
\tilde{g}(i) = \begin{cases} 
  x & \text{if } g(\hat{k}_a + i - 1) \in (a_x, b_x) \\
  y & \text{if } g(\hat{k}_a + i - 1) \in (a_y, b_y) \\
  g(\hat{k}_a + i - 1) & \text{otherwise} 
\end{cases}.
\]

Then

\[
d(\tilde{f}(i), \tilde{f}(i + 1)) \leq d(\tilde{f}(i), f(i)) + d(f(i), f(i + 1)) + d(f(i + 1), \tilde{f}(i + 1)) < \beta + 2\beta + \beta < \min\{\delta, \frac{\varepsilon}{4}\},
\]

and

\[
d(\tilde{g}(i), \tilde{g}(i + 1)) \leq d(\tilde{g}(i), g(i)) + d(g(i), g(i + 1)) + d(g(i + 1), \tilde{g}(i + 1)) < \beta + 2\beta + \beta < \min\{\delta, \frac{\varepsilon}{4}\},
\]

and

\[
d(\tilde{f}(i), \tilde{g}(i)) > d(f(i), g(i)) - d(\tilde{f}(i), f(i)) - d(\tilde{g}(i), g(i)) > 4c - \beta - \beta > 3c.
\]

15. By (2) it follows that there exist \( k_1, k_2, m_1, m_2 \in \{1, \ldots, k\} \) such that

\[
d_H(\{\tilde{f}(i)\}_{i=k_1}^{k_2}, [x, y]) < \frac{\varepsilon}{8} \text{ and } d_H(\{\tilde{g}(i)\}_{i=m_1}^{m_2}, [x, y]) < \frac{\varepsilon}{8}.
\]
4.2 Span of compactifications of a ray with remainders having span zero

16. Let $A_f = \{\tilde{f}(i)\}_{i=k_1}^{k_2} \subseteq [x, y]$ and $A_g = \{\tilde{g}(i)\}_{i=m_1}^{m_2} \subseteq [x, y]$. Then by Proposition 4.4, there exist onto functions $\hat{f}, \hat{g} : \{1, \ldots, m\} \to B \subseteq [x, y]$ such that

(a) $d_H(B, [x, y]) < \frac{\varepsilon}{2}$,
(b) $d(H(\tilde{f}(i), \tilde{f}(i+1)) < \varepsilon$ for each $i$,
(c) $d(\tilde{g}(i), \tilde{g}(i+1)) < \varepsilon$ for each $i$,
(d) $d(\tilde{f}(i), \tilde{g}(i)) > 3c - \varepsilon > 2c$ for each $i$.

Also,

$$d_H(B, [x, y] \cup Y) \leq d_H(B, [x, y]) + d_H([x, y], [x, y] \cup Y)$$
$$\leq d_H(B, [x, y]) + d_H([x, y], Y)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$ 

Proposition 4.10 Let $c > 0$, $0 < \varepsilon < \frac{c}{2}$ and $Y, W$ be continua such that $d_H(Y, W) < \frac{\varepsilon}{16}$. Suppose that there exists a finite subset $A$ of $Y \cup W$ and onto maps $f, g : \{1, \ldots, m\} \to A$ such that

1. $d_H(A, Y \cup W) < \frac{\varepsilon}{16}$,
2. $d(f(i), f(i+1)) < \frac{\varepsilon}{16}$ for each $i$,
3. $d(g(i), g(i+1)) < \frac{\varepsilon}{16}$ for each $i$,
4. $d(f(i), g(i)) > 2c$ for each $i$.

Then there exists $C_\varepsilon \subseteq Y$ and onto maps $f_\varepsilon, g_\varepsilon : \{1, \ldots, n\} \to C_\varepsilon$ such that

1. $d_H(C_\varepsilon, Y) < \frac{\varepsilon}{2}$,
2. $d(f_\varepsilon(i), f_\varepsilon(i+1)) < \varepsilon$ for each $i$,
3. $d(g_\varepsilon(i), g_\varepsilon(i+1)) < \varepsilon$ for each $i$,
4. $d(f_\varepsilon(i), g_\varepsilon(i)) > c$ for each $i$. 
Proof. First we show that $d_H(A, Y) < \frac{\varepsilon}{16}$.

(i) We will show that for every $i \in \{1, \ldots, m\}$ there exists a $y \in Y$ such that $d(f(i), y) < \frac{\varepsilon}{16}$. If $f(i) \in Y$, we can take $y = f(i)$. So suppose $f(i) \in W \setminus Y$. Since $d_H(W, Y) < \frac{\varepsilon}{16}$, there exists a $y \in Y$ such that $d(f(i), y) < \frac{\varepsilon}{16}$.

(ii) Now we prove that for each $y \in Y$ there is an $i \in \{1, \ldots, m\}$ with $d(y, f(i)) < \frac{\varepsilon}{16}$. Let $y \in Y$. Since $d_H(A, W \cup Y) < \frac{\varepsilon}{16}$ it holds that $(W \cup Y) \subseteq \bigcup_{i=1}^{m} K_{\frac{\varepsilon}{16}}(f(i))$. Therefore there exists an $i \in \{1, \ldots, m\}$ such that $y \in K_{\frac{\varepsilon}{16}}(f(i))$, which means that $d(y, f(i)) < \frac{\varepsilon}{16}$.

Since $d_H(A, Y) < \frac{\varepsilon}{16}$, for each $i \in \{1, \ldots, m\}$ we can find a $x_i \in Y$ such that $d(f(i), x_i) < \frac{\varepsilon}{16}$ and $y_i \in Y$ such that $d(g(i), y_i) < \frac{\varepsilon}{16}$. Denote $A_{\varepsilon} = \{x_i\}_{i=1}^{m} \subseteq Y$ and $B_{\varepsilon} = \{y_i\}_{i=1}^{m} \subseteq Y$ and let $f'_{\varepsilon} : \{1, \ldots, m\} \rightarrow A_{\varepsilon}, g'_{\varepsilon} : \{1, \ldots, m\} \rightarrow B_{\varepsilon}$ be defined as $f'_{\varepsilon}(i) = x_i$ and $g'_{\varepsilon}(i) = y_i$. Clearly $f'_{\varepsilon}$ and $g'_{\varepsilon}$ are both onto. We will end this proof by using the Proposition 4.4. Therefore we need to show, that $A_{\varepsilon}, B_{\varepsilon}, f'_{\varepsilon}$ and $g'_{\varepsilon}$ have the required properties.

1. $d_H(A_{\varepsilon}, Y) < \frac{\varepsilon}{8}, d_H(B_{\varepsilon}, Y) < \frac{\varepsilon}{8}$.

   Obviously for each $x_i \in A_{\varepsilon}$ there is a $y \in Y$ such that $d(x_i, y) < \frac{\varepsilon}{8}$. We will prove that for each $y \in Y$ there exists a $x_i \in A_{\varepsilon}$, with $d(y, x_i) < \frac{\varepsilon}{8}$. Let $y \in Y$. Because $d_H(A, Y) < \frac{\varepsilon}{16}$, there is an $i$ such that $d(y, f(i)) < \frac{\varepsilon}{16}$. So, $d(y, x_i) < d(y, f(i)) + d(f(i), x_i) < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8}$ and therefore $d_H(A_{\varepsilon}, Y) < \frac{\varepsilon}{8}$.

   With the same argument we show that $d_H(B_{\varepsilon}, Y) < \frac{\varepsilon}{8}$.

2. By the definitions above it is easy to see that

   $d(f'_{\varepsilon}(i), f'_{\varepsilon}(i+1)) = d(x_i, x_{i+1}) 
   \leq d(x_i, f(i)) + d(f(i), f(i+1)) + d(f(i+1), x_{i+1}) 
   < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} < \frac{\varepsilon}{4}.$
3. Similarly,
\[
d(g'_e(i), g'_e(i + 1)) = d(y_i, y_{i+1}) \\
\leq d(y_i, f(i)) + d(f(i), f(i + 1)) + d(f(i + 1), y_{i+1}) \\
< \frac{\varepsilon}{16} + \frac{\varepsilon}{16} + \frac{\varepsilon}{16} < \frac{\varepsilon}{4}.
\]

4.
\[
d(f'_e(i), g'_e(i)) = d(x_i, y_i) \\
> d(f(i), g(i)) - d(f(i), x_i) - d(g(i), y_i) \\
> 2c - \frac{\varepsilon}{16} - \frac{\varepsilon}{16} = 2c - \frac{\varepsilon}{8} > c - \varepsilon.
\]

By Proposition 4.4 there exist onto functions \(f_\varepsilon, g_\varepsilon : \{1, \ldots, n\} \to C_\varepsilon\) such that

1. \(C_\varepsilon \subseteq Y\)
2. \(d_H(C_\varepsilon, Y) < \frac{\varepsilon}{2}\)
3. \(d(f_\varepsilon(i), f_\varepsilon(i + 1)) < \varepsilon\) for each \(i\)
4. \(d(g_\varepsilon(i), g_\varepsilon(i + 1)) < \varepsilon\) for each \(i\)
5. \(d(f_\varepsilon(i), g_\varepsilon(i)) > c\) for each \(i\).

Finally we state the announced theorem.

**Theorem 4.11** Let \(X = [0, \infty) \cup Y\) be a compactification of the ray \([0, \infty)\) such that \(\sigma(Y) = 0\). Then \(\sigma(X) = 0\).

**Proof.** By Proposition 4.3 it is enough to prove that \(\sigma^*(X) = 0\). Suppose that \(2\sigma^*(X) = c > 0\) and let \(0 < \varepsilon < \frac{c}{2}\). By Lemma 4.9, there exists an arc \(W\) and onto maps \(f, g : \{1, \ldots, m\} \to A \subseteq X\) such that
1. $d_H(W, Y) < \frac{\varepsilon}{16}$

2. $d_H(A, W \cup Y) < \frac{\varepsilon}{16}$

3. $d(f(i), f(i + 1)) < \frac{\varepsilon}{16}$ for each $i$,

4. $d(g(i), g(i + 1)) < \frac{\varepsilon}{16}$ for each $i$,

5. $d(f(i), g(i)) > 2c$ for each $i$.

It follows from Theorem 4.5 and Proposition 4.10 that $\sigma^*(Y) \geq c > 0$ which is a contradiction. □

**Corollary 4.12** There is more than one nonchainable span zero continuum.

**Proof.** Take the nonchainable span zero continuum, which was constructed by L. Hoehn in [25]. Any compactification of a ray with the Hoehn’s continuum is nonchainable span zero continuum by Theorem 4.11. □
5
OPEN PROBLEMS

Inverse limits in the category $CHU$

In Section 2.4 we investigated inverse limits in the category $CHU$ of compact Hausdorff spaces with u.s.c. functions. With a counterexample we showed that the inverse limits with u.s.c. set-valued bonding functions together with the projections are not necessarily inverse limits in $CHU$, but there is still no characterization of them. Therefore the following question is reasonable.

**Question 5.1** Do there exist the inverse limits in the category $CHU$? If they do, what is their characterization?

Construction of $D_n$ with generalized inverse limits for each positive integer $n$

In Chapter 3 we construct the Ważewski’s universal dendrite as an inverse limit with one set-valued bonding function. Remember, that Ważewski’s universal dendrite $D$ is a dendrit, which set of ramification points is dense in $D$ and each of its ramification points is of infinite order in $D$. Therefore $D$ is often denoted by $D_\infty$. Similary, for each positive integer $n$, $D_n$ denotes a dendrite $D$, which set of ramification points is dense in $D$ and each of its ramification points is of order $n$ in $D$. In [10] one can find the construction of $D_3$ as an inverse limit with one set-valued bonding function. The following question appears.

**Question 5.2** How to construct $D_n$ as an inverse limit with one set-valued bonding function for each positive integer $n$?
Construction of uncountable many nonhomeomorphic continua with span zero

Remember the famous Lelek’s problem about whether the converse of the statement that chainable continua have span zero, also holds true, which was rejected in 2011 by L. C. Hoehn. The Hoehn’s example is the only known nonchainable continuum with span zero at the time. In chapter 3 we showed that if $X$ is a compactification of a ray such that the remainder of $X$ has span zero, then $X$ has span zero too. We already know how to construct uncountable many nonhomeomorphic compactifications of the ray with the closed arc as remainder. See [1]. The question is, if we can construct uncountable many nonhomeomorphic compactifications of the ray with a span zero continuum as remainder.

**Question 5.3** How to construct an uncountable family of pairwise nonhomeomorphic nonchainable span zero continua?


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Author was born on February 10th 1986 in Murska Sobota. She finished the primary school Ivana Cankarja in Ljutomer and continued to Gimnazija Franca Miklošiča, also in Ljutomer. After the high school, she enrolled in the study of Educational Mathematics at the Faculty of Education, University of Maribor. In 2009 she graduated at the Faculty of Natural Sciences and Mathematics, University of Maribor with diploma thesis "Smooth continua" under supervision of dr. Iztok Banič. After that she continued postgraduate study of Mathematics, also at the Faculty of Natural Sciences and Mathematics and started to work as an assistant for Mathematics at the Faculty of Civil Engineering, University of Maribor.

Her personal bibliography for the period 2009-2013 includes:

**Journal Publications**


**Participation at international conferences**

1. 46th Annual Spring Topology and Dynamics Conference, Mexico city, Mexico 2012
   Lecture title:*Comb functions and generalized inverse limits.*
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3. International Conference on Topology and its Applications, Nafpaktos, Greece 2010
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