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## New Goods and the Size Distribution of Firms\*

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### ABSTRACT

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This paper describes a simple model of aggregate and firm growth based on the introduction of new goods. An incumbent firm can combine labor with blueprints for goods it already produces to develop new blueprints. Every worker in the economy is also a potential entrepreneur who can design a new blueprint from scratch and set up a new firm. The implied firm size distribution closely matches the fat tail observed in the data when the marginal entrepreneur is far out in the tail of the entrepreneurial skill distribution. The model produces a variance of firm growth that declines with size. But the decline is more rapid than suggested by the evidence. The model also predicts a new-firm entry rate equal to only 2.5% per annum, instead of the observed rate of 10% in U.S. data.

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\*Luttmer, University of Minnesota and Federal Reserve Bank of Minneapolis. This is a report on ongoing research. The second half of this paper mimics parts of Luttmer [2006b]. Comments welcome. The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

## 1. INTRODUCTION

Most firms produce more than one good, and large firms tend to produce many different goods. Producing certain goods can make it easier for a firm to introduce new goods. The rate at which a firm introduces new goods is likely to depend on the number of different goods it already produces.

In this paper, a firm can combine labor with the blueprint for any of the goods it currently produces to develop blueprints for new goods, at a stochastic rate. These rates are taken to be independent across goods. New firms can be set up by entrepreneurs who develop blueprints for new goods from scratch using only their time. Every worker in the economy is a potential entrepreneur, but the frequency with which workers can come up with ideas for new goods varies across workers. The entrepreneurial skill distribution is assumed to have unbounded support so that there will always be some workers who find it optimal to become entrepreneurs.

Consumers in the economy have Dixit-Stiglitz preferences over many differentiated goods, and firms are monopolists in the markets of the various goods they produce. Goods are produced using a linear labor-only technology. The technology with which labor is converted into goods becomes more productive over time at an exogenous rate that is common to all producers. In equilibrium, the amount of labor used optimally by a firm to produce a particular good is constant over time, as is the amount of labor used per good to develop new goods. The per capita labor supply is constant. Along a balanced growth path, the number of differentiated goods produced in the economy must therefore grow at the same rate as the population.

The shape of the firm size distribution depends critically on how many new goods are introduced by existing firms, and how many are introduced by entrepreneurs who develop a new good and set up a new firm. If few workers have sufficient entrepreneurial skill to become an entrepreneur, then most of the new goods are introduced by existing firms and the growth rate of existing firms is close to the population growth rate. There will then be many incumbent firms that have had a long time to grow, and the resulting size distribution will have the very fat tail observed in the data.

This paper complements the model described in Luttmer [2006a], in which every firm is the monopolist producer of a single good and becomes more productive at a firm-specific idiosyncratic rate. In the model, firm productivity follows an exogenously specified geometric Brownian motion. The economy-wide distribution of productivity improves over time as a result of selection and imitation by entrants. Stationarity of the firm size distribution is induced by the spillover that arises from the ability

of entrants to imitate. The resulting firm size distribution also closely matches the observed distribution. The implied variance of firm growth rates is constant, while here it is inversely proportional to size. The data indicate something in between these two extremes. One possible interpretation is that the mechanisms described here and in Luttmer [2006a] are both needed to account for the empirical evidence.

Klette and Kortum [2003] build a model of firm growth along the lines of the quality ladder model of Grossman and Helpman [1991]. The stochastic process of the number goods produced by an individual firm obtained here for an economy with differentiated commodities is the same as in Klette and Kortum [2003]. The quality ladder environment has a constant number of goods. In the presence of entry, the average firm therefore has to decline in size. This rules out size distributions that are sufficiently fat tailed. Here, instead, the growth rate of the average firm can be positive if there is positive population growth and incumbent firms account for a sufficient fraction of the new goods introduced. This is essential to replicate the fat tail of the observed firm size distribution.

Section 2 describes the economy. The balanced growth path is characterized in Section 3. The stochastic process of firm size and the resulting stationary distribution are analyzed in Section 4. The parameters of the size distribution are estimated in Section 5 and the implications of these parameters for the research and development productivity of firms and for the skill distribution of entrepreneurs are discussed. Section 6 concludes.

## 2. GROWTH IN VARIETY

### 2.1 Consumers

Time is continuous and indexed by  $t$ . There is a continuum of consumers alive at any point in time. The population size at time  $t$  is  $H_t = He^{\eta t}$ , and the population growth rate  $\eta$  is non-negative. There is a representative consumer with preferences over rates of dynastic consumption  $\{C_t\}_{t \geq 0}$  of a composite good, defined by the utility function

$$\left( \mathbb{E}_0 \left[ \int_0^\infty \rho e^{-\rho t} [C_t e^{-\eta t}]^{1-\gamma} dt \right] \right)^{1/(1-\gamma)}.$$

The discount rate  $\rho$  and the intertemporal elasticity of substitution  $1/\gamma$  are positive. The composite good is made up of a continuum of differentiated commodities. Preferences over these commodities are additively separable and symmetric. This implies that all commodities trading at the same price are consumed at the same rate. Let  $c_t(p)$  be

consumption at time  $t$  of a commodity that trades at a price  $p$ . In equilibrium, there will be a measure  $N_t$  of commodities that are available at time  $t$ , defined on the set of commodity prices. The composite good is defined as

$$C_t = \left[ \int c_t^\beta(p) N_t(dp) \right]^{1/\beta}.$$

where  $\beta \in (0, 1)$ . Consumers choose  $c_t(p)$  to minimize the cost of acquiring  $C_t$ . This implies that

$$pc_t(p) = P_t C_t^{1-\beta} c_t^\beta(p),$$

where  $P_t$  is the price index

$$P_t = \left[ \int p^{-\beta/(1-\beta)} N_t(dp) \right]^{-(1-\beta)/\beta}.$$

The price elasticity of the demand is  $-1/(1-\beta) < -1$  for all commodities.

Every consumer is endowed with a flow of one unit of labor per unit of time that can be sold for wages or be used as an entrepreneur. Consumers also own claims to firms. Contingent claims markets are complete and the representative consumer faces a standard present-value budget constraint. There is no aggregate uncertainty. Optimal consumption growth satisfies the Euler condition

$$r_t = \rho + \gamma \left[ \frac{DC_t}{C_t} - \eta \right], \quad (1)$$

where  $r_t$  is the interest rate expressed in units of the composite good.

## 2.2 Commodity Producers

Producing any particular commodity requires a unique commodity-specific blueprint. The owner of such a blueprint can use  $l$  units of labor to produce  $Z_t l$  units of the associated differentiated commodity. The resulting revenues are  $C_t^{1-\beta} (Z_t l)^\beta$ , measured in units of the composite good. Labor can be hired for wages  $w_t$ . The commodity producer solves

$$w_t R_t = \max_l C_t^{1-\beta} (Z_t l)^\beta - w_t l$$

at any time  $t$ . The optimal net revenues  $R_t$  are a constant markup over labor inputs,

$$R_t = \left( \frac{1}{\beta} - 1 \right) L_t, \quad (2)$$

and the associated labor inputs are

$$L_t = \left( \frac{\beta Z_t}{w_t} \right)^{\beta/(1-\beta)} \frac{\beta C_t}{w_t}. \quad (3)$$

Labor productivity is assumed to be the same across all producers and evolves according to  $Z_t = Z e^{\theta t}$ .

Market clearing implies that  $c_t(p_t) = Z_t L_t$  for every commodity. Hence all producers charge the same price  $p_t$  and the number of commodities can be written as  $N_t = N_t(p_t)$ . The definition of the composite commodity then implies that aggregate consumption is given by

$$C_t = Z_t L_t N_t^{1/\beta}. \quad (4)$$

Together with (3) this implies that wages are

$$w_t = \beta Z_t N_t^{(1-\beta)/\beta}, \quad (5)$$

in any equilibrium.

## 2.3 New Blueprints and Commodities

The introduction of new commodities requires new blueprints. There are two technologies for producing new blueprints. One combines the blueprint for an existing commodity with labor to produce a different blueprint. The other uses the input of an entrepreneur only to develop a new blueprint from scratch. There is no technology for copying existing blueprints.

### 2.3.1 Combining Existing Blueprints and Labor

Any existing blueprint can be combined with a flow of  $I_t$  units of labor to produce a blueprint for a new commodity. The new blueprint is completed following an exponentially distributed waiting time with mean  $\mu_t = f(I_t)$ . The production function  $f$  is strictly increasing and concave, with an unbounded marginal product at zero. Blueprints depreciate, in one-hoss-shay fashion, at some non-negative rate  $\lambda$ .

A blueprint allows its owner to jointly produce a commodity and new blueprints. The price of a blueprint  $s_t$  must satisfy the Bellman equation

$$r_t s_t = \max_{\mu_t \leq f(I_t)} \{w_t [R_t - I_t] + (\mu_t - \lambda)s_t + D s_t\},$$

and a transversality condition. The first-order condition for labor used to develop new blueprints is

$$\mu_t = f(I_t), \frac{w_t}{s_t} \geq Df(I_t), \text{ w.e. if } I_t > 0. \quad (6)$$

Not surprisingly, the search intensity for new blueprints is high when the price of a blueprint is high relative to the wage. Solving the Bellman equation forward for  $s_t$  yields

$$s_t = \int_t^\infty \exp\left(-\int_t^v [r_u + \lambda - \mu_u] du\right) w_v [R_v - I_v] dv. \quad (7)$$

This is simply the present value of all the net revenues that will flow from a particular blueprint and the sequences of future blueprints that will be produced from it. Given a time path for interest rates  $r_t$ , wages  $w_t$  and net revenues  $R_t$ , (6) and (7) determine the growth rate  $\mu_t = f(I_t)$  and the value  $s_t$  of a blueprint.

### 2.3.2 Entrepreneurs

All consumers in the economy are potential entrepreneurs who can attempt to design new blueprints from scratch. Consumers are heterogeneous and must choose to either supply one unit of labor at wage  $w_t$  or become an entrepreneur. A type- $x$  consumer can generate a new blueprint after an exponentially distributed waiting time  $x$ . An entrepreneur with a newly designed blueprint at time  $t$  can sell this blueprint at a price  $s_t$  and then go on to design the next blueprint, or work for the wage  $w_t$ .<sup>1</sup> All consumers with  $xs_t > w_t$  choose to be entrepreneurs and all agents with  $xs_t < w_t$  choose to work at wage  $w_t$ . The cross-sectional distribution of productivities has support  $(0, \infty)$  and is denoted by  $S$ . Hence, a fraction  $S(w_t/s_t)$  of low-productivity consumers works for firms, and a fraction  $1 - S(w_t/s_t)$  become entrepreneurs. The labor endowment of a type- $x$  consumer has market value  $\max\{w_t, xs_t\}$ , and so the shape of  $S$  will be reflected in the distribution of earnings at a point in time, and in the distribution of wealth across consumers if entrepreneurial skills are permanent.

### 2.3.3 Aggregate Innovation

New commodities are introduced by entrepreneurs at a rate  $\nu_t$ , measured in new commodities per unit of time as a fraction of the existing number of commodities at time  $t$ .

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<sup>1</sup>A more realistic scenario might be if the entrepreneur would, at least for while, be more efficient than others at using the newly developed blueprint to produce the commodity or design new blueprints based on it. This would generate a type of match similar to the one between firms and managers analyzed in Holmes and Schmitz [1995].

This rate is given by

$$\nu_t N_t = H_t \int_{w_t/s_t}^{\infty} x S(dx). \quad (8)$$

This follows since there are  $H_t$  potential entrepreneurs,  $x$  is the rate at which an individual entrepreneur generates a new blueprint, and only those consumers with  $x > w_t/s_t$  choose to act as entrepreneurs. Together with investment by owners of existing blueprints, entrepreneurial activity generates a flow of new blueprints equal to

$$DN_t = (\nu_t + \mu_t - \lambda)N_t. \quad (9)$$

The initial stock of blueprints is given by  $N_0 = N$ .

## 2.4 Equilibrium

A fraction  $S(w_t/s_t)$  of the  $H_t$  consumers in the economy use their unit of time to supply labor. The amount of labor used by owners of existing blueprints to produce new blueprints is  $I_t$  per blueprint and production takes  $L_t$  units of labor per commodity. There are  $N_t$  blueprints and commodities. The labor market clearing condition is therefore

$$H_t S\left(\frac{w_t}{s_t}\right) = N_t [I_t + L_t] \quad (10)$$

at any time  $t$ .

Consumer wealth equals the present value of labor and entrepreneurial income, together with the value of the stock of blueprints  $s_t N_t$ . In equilibrium, this must add up to the present value of aggregate consumption  $C_t$ . This restriction implies a transversality condition for  $s_t N_t$ . The equilibrium is determined by this transversality condition, together with the other equilibrium conditions (1)-(10), as well as an initial condition for  $N_0$  and the exogenous evolution of  $Z_t$ .

## 3. BALANCED GROWTH

A balanced growth path for this economy is a competitive equilibrium in which  $C_t = C e^{(\kappa+\eta)t}$  and  $w_t = w e^{\kappa t}$  for some equilibrium growth rate  $\kappa$ , and in owners of existing blueprints develop new blueprints at a constant rate  $\mu_t = \mu$ .

Because the marginal product of  $f$  at zero is infinite,  $\mu$  and  $I = f^{-1}(\mu)$  must be positive. Hence (6) implies that  $s_t/w_t$  is constant, and thus  $s_t = s e^{\kappa t}$ . Constant consumption growth rate implies a constant interest rate  $r_t = r$ , where  $r = \rho + \gamma\kappa$ . The fact that  $w_t$  and  $s_t$  both grow at the rate  $\kappa$  implies that  $R_t = R$ , by (7). Because of (2),

labor inputs can also be written as  $L_t = L$ . Since  $Z_t$ ,  $w_t$  and  $C_t$  grow at the respective rates  $\theta$ ,  $\kappa$  and  $\kappa + \eta$ , (4)-(5) imply that  $N_t = Ne^{\eta t}$  and

$$\kappa = \theta + \left(\frac{1-\beta}{\beta}\right)\eta, \quad (11)$$

along any balanced growth path. The number of commodities must grow at the same rate as the population, and this growth adds to productivity growth with an elasticity  $(1-\beta)/\beta$ .

Along any balanced growth path, the first-order condition (6) and the present-value restriction (7) together with  $R = L(1-\beta)/\beta$  simplify to  $\mu = f(I)$ ,  $w/s = Df(I)$  and

$$\frac{1}{Df(I)} = \frac{\left(\frac{1-\beta}{\beta}\right)L - I}{r - \kappa + \lambda - f(I)}, \quad (12)$$

as long as  $f(I) < r - \kappa + \lambda$ . This condition relates investment  $I$  in new blueprints to the amount of labor  $L$  used to produce a commodity. This relation depends on the precise properties of the production function  $f$ . In the following, suppose that  $I + [r - \kappa + \lambda - f(I)]/Df(I)$  is increasing in  $I$  for  $f(I) < r - \kappa + \lambda$ . Then (12) implies that  $L$  is increasing in  $I$ . In this case, the equilibrium value of a blueprint rises with  $I$  because more labor is used to generate more revenues, and not so much because those revenues are discounted at a lower rate.

The labor market clearing condition (10) reduces to

$$\frac{H}{N} = \frac{I + L}{S(Df(I))}. \quad (13)$$

Using (12) to eliminate  $L$  from the right-hand side of (13) gives an expression that is increasing in  $I$ . If  $\mu = f(I)$  is high, then blueprint owners must be using a lot of labor to produce commodities and design new blueprints. On the other hand many consumers will also choose to be entrepreneurs, because  $s/w = 1/Df(I)$  is high. Under such circumstances, the labor market can only clear if there are relatively few commodities. Thus (12)-(13) imply a negatively sloped relation between  $N$  and  $I$ . The assumption that the marginal product  $Df(I)$  is unbounded at zero implies that  $N$  goes to infinity as  $I$  goes to zero along the equilibrium relation implied by (12)-(13).

The result that the number of commodities grows at a rate  $\eta$  along a balanced growth path implies that investment in new blueprints by entrepreneurs and incumbent firms must satisfy

$$\eta + \lambda = \frac{H}{N} \int_{Df(I)}^{\infty} xS(dx) + f(I), \quad (14)$$



by (8) and (9). Since  $S$  has unbounded support, this condition ensures that  $f(I) < \eta + \lambda$  in any equilibrium. The right-hand side of (14) is increasing  $I$ . Blueprint owners produce more blueprints and more consumers choose to be entrepreneurs when  $s/w = 1/Df(I)$  and thus  $I$  is high. The rate at which new blueprints must be produced along a balanced growth path is  $(\eta + \lambda)N$  per unit of time, and a high  $I$  can thus support a high value of  $N$ . It follows that (14) implies an increasing relation between  $N$  and  $I$  for all  $f(I) < \eta + \lambda$ . The assumption that  $S$  has unbounded support guarantees that  $N$  goes to infinity as  $f(I)$  approaches  $\eta + \lambda$  from below. Therefore,  $N$  must go to zero as  $I$  goes to zero along (14).

Note that  $r$  follows from the Euler equation (1) and (11). The levels of aggregate consumption  $C$  and wages  $w$  are implied by (4) and (5), given  $L$  and  $N$ . The equilibrium conditions (12)-(14) are shown in Figure I. They determine  $I$ ,  $L$ , and  $N$ , and hence  $\mu = f(I)$  and  $s/w = 1/Df(I)$ . The first part of the following proposition now follows.

**PROPOSITION 1** *Suppose  $r > \kappa + \eta$ , that  $I + [\eta + \lambda - f(I)]/Df(I)$  is increasing in  $I$  for all  $I$  such that  $f(I) < \eta + \lambda$ , and that entrepreneurial skill distribution  $S$  is continuous with support  $(0, \infty)$ . If  $r > \eta$ , then there is a unique balanced growth path defined by (12)-(14). Suppose  $f(I) = g(AI)$  and  $S(x) = G(x/B)$ , where  $g$  is a production function and  $G$  a distribution function. Across balanced growth paths,*

- (i) *the elasticities of  $[N, C/H, s/w]$  with respect to  $H$  are  $[1, (1 - \beta)/\beta, 0]$ ;*
- (ii) *the elasticities of  $[N, C/H, s/w]$  with respect to  $A = B$  are  $[1, (1 - \beta)/\beta, -1]$ ;*
- (iii) *if consumers are endowed with  $J$  units of labor, then the elasticities of  $[N, C/H, s/w]$  with respect to  $J$  are  $[1, 1/\beta, 0]$ ;*
- (iv)  *$\mu = f(I)$  converges to  $\eta + \lambda$  from below as  $B$  goes to zero.*

*There is no effect on  $\mu = f(I)$  in (i)-(iii).*

Strictly concave constant-elasticity production functions satisfy the condition given in this proposition. Note that  $Z$  does not appear in the equilibrium conditions (12)-(14), and hence per capita consumption has a unit elasticity with respect to labor productivity in commodity production, by (4). This and the per capita consumption elasticity reported in (i) matches (11). Labor-augmenting technological progress in blueprint production that is neutral across entrepreneurs and blueprint owners, as in (ii), has the

same effect as a population increase on the number of commodities and per capita consumption. In both cases it is possible to sustain a larger number of commodities, and the increased variety implies an increase in per capita consumption. But this type of technological progress in blueprint production lowers the per-commodity inputs  $L$  and  $I$ , and this lowers  $s/w$ , thereby reducing the dispersion of earnings  $\max\{1, xs/w\}$ . Population growth leaves the distribution of earnings unchanged. An increase in per capita labor supply, as in (iii), is equivalent to a population increase combined with a proportional increase in labor productivity in commodity production.

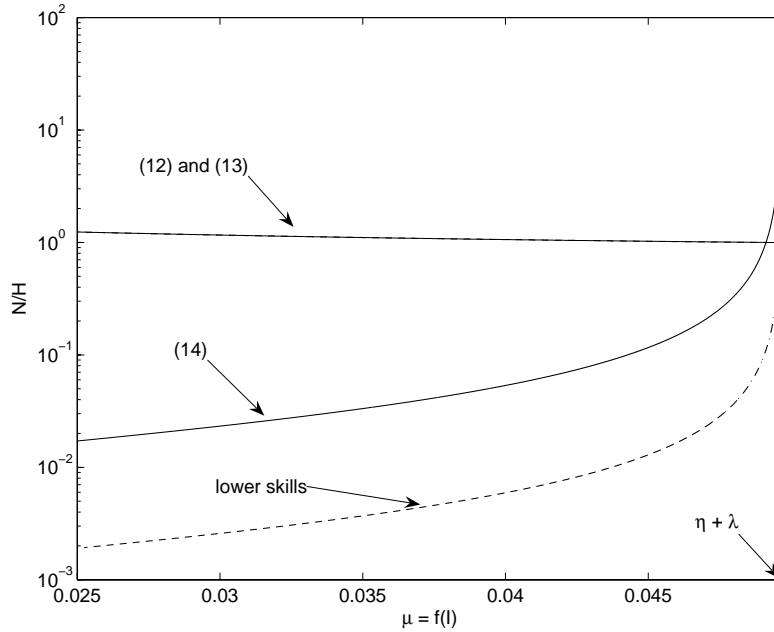


FIGURE I. Equilibrium Conditions for  $N/H$  and  $\mu = f(I)$ .

Result (iv) of Proposition 1 is illustrated in Figure I for functional forms and parameters discussed in Section 5. As  $B$  goes to zero the fraction of consumers  $G(Df(I)/B)$  who are not entrepreneurs goes to 1 for any fixed  $I$ . Thus the curve defined by the equilibrium conditions (12)-(13) converges to one defined by (12) and  $N/H = 1/(I + L)$ . In Figure I,  $B$  is reduced by a factor three without a perceptible effect on the relation (12)-(13). Entrepreneurs are largely irrelevant for labor market clearing when  $B$  is small. But  $N/H$  as a function of  $\mu = f(I)$  converges to zero along (14) for any  $\mu = f(I) < \eta + \lambda$ . For any fixed level  $I < f^{-1}(\eta + \lambda)$  of investment by blueprint owners, the number of commodities that is sustainable along a balanced growth path converges to zero as entrepreneurs become less and less efficient in coming up with new blueprints. As a result, the equilibrium  $\mu = f(I)$  shown in Figure I converges to the asymptote

at  $\eta + \lambda$ . Existing blueprint owners will then account for essentially all growth in the number of blueprints and commodities.

#### 4. THE SIZE DISTRIBUTION OF FIRMS

The economy described so far has consumers who are also workers and entrepreneurs. Everyone can own blueprints and there are no firms.

##### 4.1 Transaction Costs

Consider an entrepreneur who has just developed a new blueprint. To hire labor to produce the associated commodity and develop further blueprints, the entrepreneur can set up a firm at no cost. This defines a firm entry. Claims to firms can be traded freely. But there is a potentially very small cost involved in firms hiring entrepreneurs to develop new blueprints from scratch, in selling blueprints to firms, and in merging firms. There are no cost advantages to any of these transactions, and so they will not occur in equilibrium.<sup>2</sup>

A firm will therefore only gain new commodities through its use of the technology for combining existing blueprints with labor, at a rate  $\mu_t$  per commodity already owned by the firm. The firm only loses commodities as its blueprints become obsolete, at a rate  $\lambda$  per commodity. A firm that has lost all its commodities is shut down and exits. In this environment, firms differ only by the number of commodities they produce, and this number can be used to measure the size of a firm. In the following, the distribution of firm size is derived assuming that the economy is on a balanced growth path.

The measure of firms with  $n$  commodities at time  $t$  is denoted by  $M_{n,t}$ . Since every commodity is produced by one and only one firm,

$$N_t = \sum_{n=1}^{\infty} nM_{n,t}. \quad (15)$$

Over time, the change in the number of firms with one commodity is

$$DM_{1,t} = \lambda 2M_{2,t} + \nu N_t - (\lambda + \mu)M_{1,t}. \quad (16)$$

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<sup>2</sup>Of course these transactions do occur in the data. This is a familiar and important failure of the type of model described in this paper. Chatterjee and Rossi-Hansberg [2006] provide an interesting model of firm size in which adverse selection makes it difficult for firms to hire entrepreneurs or buy their projects.

where  $\mu$  and  $\nu = \eta + \lambda - \mu$  are the values of  $\mu_t$  and  $\nu_t$  along the balanced growth path. The number of firms with one commodity increases because firms with two commodities lose one, or because of entry. The number declines because firms with one commodity gain or lose a commodity. Similarly, the number of firms with  $n \in \mathbb{N} \setminus \{1\}$  commodities evolves according to

$$DM_{n,t} = \lambda(n+1)M_{n+1,t} + \mu(n-1)M_{n-1,t} - (\lambda + \mu)nM_{n,t}. \quad (17)$$

The joint dynamics of  $N_t$  and  $\{M_{n,t}\}_{n=1}^{\infty}$  is fully described by  $DN_t = \eta N_t$  and (16)-(17). By construction, (16)-(17) ensure that (15) holds if it holds at the initial date, which is assumed to be the case.

## 4.2 The Stationary Distribution

A stationary distribution of firm size exists if (16)-(17) has a solution for which  $M_{n,t}/N_t$  is constant over time. Since  $N_t$  grows at a rate  $\eta$ , this means that  $DM_{n,t} = \eta M_{n,t}$  for all  $n \in \mathbb{N}$ . Given that  $N_t$  and  $M_{n,t}$  grow at the common rate  $\eta$ , one can define

$$P_n = \frac{M_{n,t}}{\sum_{n=1}^{\infty} M_{n,t}}$$

for all  $n \in \mathbb{N}$ . This is the fraction of firms that produce  $n$  commodities. Analytically more convenient is the fraction of all commodities produced by firms of size  $n$ , which is given by

$$Q_n = \frac{nM_{n,t}}{\sum_{n=1}^{\infty} nM_{n,t}}$$

for all  $n \in \mathbb{N}$ . The mean number of commodities per firm can be written in terms of the two stationary distributions  $\{P_n\}_{n=1}^{\infty}$  and  $\{Q_n\}_{n=1}^{\infty}$  as

$$\frac{\sum_{n=1}^{\infty} nM_{n,t}}{\sum_{n=1}^{\infty} M_{n,t}} = \sum_{n=1}^{\infty} nP_n = \left( \sum_{n=1}^{\infty} \frac{1}{n} Q_n \right)^{-1}.$$

The numerator of the left-hand side adds up to the total measure of commodities in the economy. This is finite by construction. Hence the mean firm size is well defined and finite by construction. The right-hand side is the reciprocal of the mean number of firms per commodity, provided that this mean is calculated against the distribution of commodities by size of firm producing the commodity.

Recall that  $\mu - \lambda = \eta - \nu$ . Using this, (16) can now be written as

$$\eta Q_1 = \lambda Q_2 + \eta - (\mu - \lambda) - (\lambda + \mu)Q_1, \quad (18)$$

and (17) implies that

$$\frac{1}{n}\eta Q_n = \lambda Q_{n+1} + \mu Q_{n-1} - (\lambda + \mu)Q_n, \quad (19)$$

for  $n \in \mathbb{N} \setminus \{1\}$ . Condition (15) corresponds to the requirement that the fractions  $Q_n$  add up to one,

$$\sum_{n=1}^{\infty} Q_n = 1. \quad (20)$$

Any sequence  $\{Q_n\}_{n=1}^{\infty} \subset [0, 1]$  that satisfies (18)-(20) defines a stationary size distribution  $\{P_n\}_{n=1}^{\infty}$  via  $Q_n \propto P_n/n$ . Note that (18)-(20) only depend on the parameters  $\mu/\lambda$  and  $\eta/\lambda$ .

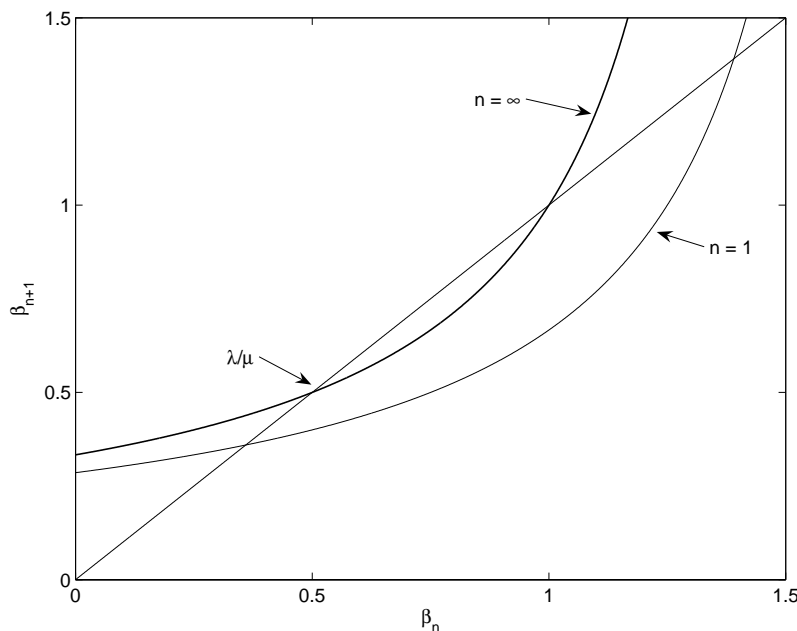


FIGURE II. The Dynamics of  $\{\beta_n\}_{n=1}^{\infty}$ .

The equation (19) is a second-order difference equation in  $\{Q_n\}_{n=1}^{\infty}$ . It comes with two boundary conditions, (18) and (20). To solve (18)-(20), it is convenient to reduce (19) to a first-order equation in the variables

$$Z_{n+1} = \frac{1}{\beta_{n+1}} [Q_n - \beta_{n+1}Q_{n+1}], \quad (21)$$

for all  $n \in \mathbb{N}$  and some sequence  $\{\beta_n\}_{n=1}^{\infty}$ . Set  $\beta_2 = 1/(1 + (\eta + \mu)/\lambda)$ . Then the initial condition (18) translates into

$$Z_2 = \frac{1}{\lambda} [\eta - (\mu - \lambda)] \quad (22)$$

Also, (19) can be written as

$$Z_{n+1} = \left( \frac{\mu\beta_n}{\lambda} \right) Z_n \quad (23)$$

for all  $n \in \mathbb{N} \setminus \{1\}$  if

$$\beta_{n+1} = \left( 1 + \frac{\eta + \mu n}{\lambda n} - \frac{\mu\beta_n}{\lambda} \right)^{-1}. \quad (24)$$

From the definition of  $\beta_2$ , observe that the recursion (24) holds for all  $n \in \mathbb{N}$  if initialized by  $\beta_1 = 0$ . The recursion (24) is depicted in Figure II for the case  $\mu > \lambda$ . Note in particular that the curve defined by (24) shifts upwards as  $n$  increases. Using this observation and the diagram, as well as an analogous diagram for  $\lambda > \mu$ , one can verify that  $\{\beta_n\}_{n=1}^\infty$  converges monotonically from  $\beta_1 = 0$  to  $\min\{1, \lambda/\mu\}$ .

The sequence  $\{Z_n\}_{n=2}^\infty$  is completely determined by (22)-(23). Observe from (21) that  $Q_n = \beta_{n+1}(Q_{n+1} + Z_{n+1})$ . The boundary condition (20) together with the fact that  $\beta_n \leq 1$  implies that  $Q_K \prod_{n=1}^K \beta_n$  must converge to zero as  $K$  becomes large. Thus one can iterate forward to obtain the solution for  $\{Q_n\}_{n=1}^\infty$ . The following proposition presents this solution and provides upper and lower bounds  $Q_n$  when  $n$  is large.

**PROPOSITION 2** *Suppose that  $\mu, \lambda, \eta$  and  $\nu = \eta - (\mu - \lambda)$  are positive. Define the sequence  $\{\beta_n\}_{n=1}^\infty$  by the recursion (24) and the initial condition  $\beta_1 = 0$ . This sequence is monotone and converges to  $\min\{1, \lambda/\mu\}$ . The solution to (18)-(20) is given by*

$$Q_n = \frac{\nu}{\lambda} \sum_{k=n+1}^{\infty} \left( \prod_{m=n+1}^k \beta_m \right) \left( \frac{\mu\beta_k}{\lambda} \right)^{-1} \prod_{m=2}^k \frac{\mu\beta_m}{\lambda}. \quad (25)$$

Take any  $\varepsilon > 0$ . If  $\mu > \lambda$  then

$$\frac{\nu}{(1 + \varepsilon)\mu - \frac{\lambda}{1+\varepsilon}} \leq \left( \prod_{m=2}^n \frac{\mu\beta_m}{\lambda} \right)^{-1} Q_n \leq \frac{\nu}{\mu - \lambda} \quad (26)$$

for all large enough  $n$ . If  $\mu < \lambda$  then

$$\frac{\nu}{(1 + \varepsilon)\lambda - \frac{\mu}{1+\varepsilon}} \leq \left( \prod_{m=2}^n \frac{\mu\beta_m}{\lambda} \right)^{-1} Q_n \leq \frac{\nu}{\lambda - \mu} \quad (27)$$

for all large enough  $n$ .

Note that the balanced growth path described in Proposition 1 implies  $\nu > 0$ . The proof of Proposition 2 in Appendix A shows that the solution (25) satisfies (20). The distribution  $\{P_n\}_{n=1}^\infty$  follows immediately from  $P_n \propto Q_n/n$ .

### 4.3 The Right Tail

As shown in (26)-(27),

$$Q_n \sim \prod_{k=2}^n \frac{\mu\beta_k}{\lambda} \quad (28)$$

for all large enough  $n$ . When  $\lambda > \mu$ , the properties of this product are quite different from what they are when  $\mu < \lambda$ . If  $\lambda > \mu$ , then  $Q_n$  is bounded above by a multiple of the geometrically declining sequence  $(\mu/\lambda)^n$ . On the other hand, if  $\mu > \lambda$  then  $\mu\beta_n/\lambda \uparrow 1$ , and hence the right-hand side of (28) declines at a rate that is slower than any given geometric rate. The proof of Proposition 2 shows that the right-hand side of (28) is nevertheless summable. The following proposition gives a further characterization of the right tail of the distribution.

**PROPOSITION 3** *Suppose that  $\eta > 0$ ,  $\mu > \lambda$  and  $\eta > \mu - \lambda$ . Then the right tail probabilities of the stationary firm size distribution satisfy*

$$\limsup_{K \rightarrow \infty} K^z \sum_{n=K}^{\infty} P_n = 0$$

for any  $z$  smaller than the tail index  $\zeta = \eta/(\mu - \lambda)$ .

The proof is in Appendix B. This proposition implies that

$$\ln \left( \sum_{n=K}^{\infty} P_n \right) \sim c - \zeta \ln(K) \quad (29)$$

for some constant  $c$ . The limiting tail index  $\zeta = 1$  associated with Zipf's law arises when the entry rate  $\nu = \eta - (\mu - \lambda)$  converges to zero.

### 4.4 Entry and Exit Rates

The entry rate  $\nu N_t$  represents the rate per unit of time at which new commodities are introduced by new firms. Each new firm starts with one commodity, and so  $\nu N_t$  is also the number of new firms that enters per unit of time. The firm entry rate  $\phi$  as a fraction of the number of incumbent firms is therefore equal to  $\nu N_t$  divided by the number of firms in the economy,

$$\sum_{n=1}^{\infty} M_{n,t} = N_t \times \sum_{n=1}^{\infty} \frac{1}{n} Q_n.$$

This implies a firm entry rate equal to

$$\phi = \frac{\nu}{\sum_{n=1}^{\infty} \frac{1}{n} Q_n}. \quad (30)$$

That is, the firm entry rate is equal to the rate  $\nu$  at which new commodities are introduced by new firms, times the average number of commodities per firm. The only firms that exit in this economy are one-commodity firms. There are  $N_t Q_1$  such firms, and they each exit at a rate  $\lambda$ . The balance  $[\phi - \lambda Q_1]N_t$  of firms entering and exiting per unit of time must equal  $\eta$  times the total number of firms. On a per-commodity basis, this gives

$$\underbrace{\nu}_{\text{gross entry per commodity}} = \underbrace{\eta \sum_{n=1}^{\infty} \frac{1}{n} Q_n}_{\text{net entry per commodity}} + \underbrace{\lambda Q_1}_{\text{exit per commodity}}. \quad (31)$$

This can be verified mechanically by adding up (18) and (19) over all  $n$ . In terms of  $\{P_n\}_{n=1}^{\infty}$  one can summarize (30)-(31) more concisely as

$$\phi = \nu \sum_{n=1}^{\infty} n P_n = \eta + \lambda P_1.$$

Eliminating the mean number of commodities per firm from (30) and (31) yields  $\phi/\eta = 1/(1 - (\lambda/\nu)Q_1)$  and then the mean firm size is  $\phi/\nu = (\eta/\nu)(1 - (\lambda/\nu)Q_1)$ . Together with the expression for  $Q_1$  implied by (25) this gives explicit solutions for the firm entry rate and the mean firm size.

## 4.5 Special Cases

As will be shown in Section 5, the empirically relevant firm size process is one with large  $\mu$  and  $\lambda$ , where  $\mu > \lambda$  and where  $\nu = \eta - (\mu - \lambda)$  is positive but very close to zero. These conditions are violated in some well-known special cases of the firm size process (15)-(17). Taking the limit  $\nu \downarrow 0$  gives a tractable special case that is very close to what is observed in U.S. data.

### 4.5.1 The Logarithmic Series Distribution

Suppose there is no population growth,  $\eta = 0$ , and so no growth in the aggregate number of commodities. Then  $\mu < \lambda$  and hence the size distribution must have a geometrically declining right tail. The sequence  $\{\beta_n\}_{n=1}^{\infty}$  is simply  $\beta_1 = 0$  and  $\beta_n = 1$  for all larger  $n$ . The transition (19) simplifies to  $[Q_{n+1} - Q_n] = (\mu/\lambda)[Q_n - Q_{n-1}]$ . Clearly,  $Q_{n+1} - Q_n \propto (\mu/\lambda)^n$ , and then  $Q_n \propto (\mu/\lambda)^n$  as well. The resulting size distribution  $\{P_n\}_{n=1}^{\infty}$  is R.A. Fischer's logarithmic series distribution

$$P_n = \frac{\frac{1}{n} (\mu/\lambda)^n}{\ln \left( \frac{1}{1 - \mu/\lambda} \right)}.$$



This is the distribution that arises in Klette and Kortum [2004]. The mean of this distribution is easy to compute, and  $\nu = \lambda - \mu$ . The resulting firm entry rate (30) equals the exit rate  $\lambda P_1$ . This can be written as

$$\frac{\eta}{\lambda} = \frac{\mu/\lambda}{\ln\left(\frac{1}{1-\mu/\lambda}\right)}.$$

This ratio ranges from 1 to 0 as  $\mu/\lambda$  ranges from 0 to 1. To obtain a size distribution with a right tail that decays at a slow geometric rate one needs  $\mu$  close to  $\lambda > \mu$ . This implies  $\phi/\lambda$  close to 0. High observed entry rates then imply high values of  $\lambda$ .

#### 4.5.2 The Yule Process

Consider the case  $\lambda = 0$  and  $\mu \in (0, \eta)$ . In this scenario, firms can only grow. A stationary size distribution arises because not all firms have had the same time to grow, and the population of firms itself grows. The entry rate is  $\nu = \eta - \mu$ , and the resulting stochastic process is known as the Yule process. It was used by Simon [1955] as a model for various skewed empirical distributions, including the city size distribution. The difference equations (18)-(19) simplify to  $Q_1 = (\eta - \mu)/(\eta + \mu)$  and

$$Q_n = \left(\frac{n}{n + \frac{\eta}{\mu}}\right) Q_{n-1},$$

for all  $n \in \mathbb{N} \setminus \{1\}$ . Working out this recursion and using  $P_n \propto Q_n/n$  gives

$$P_n = \frac{\eta}{\mu} \frac{\Gamma(n)\Gamma\left(1 + \frac{\eta}{\mu}\right)}{\Gamma\left(n + 1 + \frac{\eta}{\mu}\right)}$$

where  $\Gamma$  is the gamma function. The right tail probabilities are

$$\sum_{n=K}^{\infty} P_n = \frac{\Gamma(K)\Gamma\left(1 + \frac{\eta}{\mu}\right)}{\Gamma\left(K + \frac{\eta}{\mu}\right)}$$

Since  $K^{1-a}\Gamma(K+a)/\Gamma(K+1)$  converges to 1 as  $m$  becomes large, these tail probabilities behave like  $K^{-\eta/\mu}$  for large  $K$ , as predicted by Proposition 3.

The size distribution has a mean  $1/(1 - \mu/\eta)$ , and so the formula for the firm entry rate (30) reduces to  $\phi = \eta$ , as expected. The limiting distribution as  $\mu \uparrow \eta$ , or  $\nu \downarrow 0$ , is  $P_n = 1/[n(n+1)]$ . This is the discrete analog of the Pareto distribution that corresponds

to Zipf's law. The right tail probabilities of this limiting distribution are exactly  $1/n$ , and so the distribution has no well defined mean.

One can verify that the stationary distribution would be  $P_n = (\eta/\mu)(1 + \eta/\mu)^{-n}$  if individual firms gain commodities at a rate  $\mu$  instead of  $\mu n$ . This corresponds to an environment in which a fraction  $(\mu/\eta)/\sum_{n=1}^{\infty} nP_n$  of the flow  $\eta N_t$  of new commodities is introduced by incumbent firms and the remaining fraction  $1/\sum_{n=1}^{\infty} nP_n$  by new firms. By itself, population growth is not enough to generate the heavy right tail. It needs to be combined with geometric growth of the individual firms.

### 4.5.3 The Limiting Case $\nu \downarrow 0$

As before, assume that  $\eta > 0$  and  $\lambda > 0$ . Letting  $\nu$  approach zero from above implies that  $\eta = \mu - \lambda$  in the limit. In this limit, the recursion (19) for  $Q_n \propto nP_n$  becomes

$$(\mu - \lambda)P_n = \lambda(n + 1)P_{n+1} + \mu(n - 1)P_{n-1} - (\lambda + \mu)nP_n,$$

for all  $n \in \mathbb{N} \setminus \{1\}$ . This can also be written as

$$P_n = \frac{\lambda}{\mu} (P_{n+1} + X_{n+1}) \quad (32)$$

together with

$$X_{n+1} = \left( \frac{n-1}{n+1} \right) X_n \quad (33)$$

for all  $n \in \mathbb{N} \setminus \{1\}$ . Iterating on (33) gives  $X_{n+1} = (2/[n(n+1)])X_2$ . Since the  $P_n$  have to add up to 1, it must be that  $P_n \rightarrow 0$ . The fact that  $\lambda/\mu < 1$  then implies that we can solve (32) forward. The result is

$$P_n = \frac{1}{\ln(\mu/\eta)} \sum_{k=n}^{\infty} \frac{(\lambda/\mu)^{k+1-n}}{k(k+1)}.$$

where we have used  $\eta/\mu = 1 - \lambda/\mu$ . The right tail probabilities are

$$\sum_{n=K}^{\infty} P_n = \frac{1}{\ln(\mu/\eta)} \sum_{n=K}^{\infty} \frac{1}{n} \left( \frac{\lambda}{\mu} \right)^{n+1-K}, \quad (34)$$

and these satisfy

$$\lim_{K \rightarrow \infty} K \sum_{n=K}^{\infty} P_n = \lim_{K \rightarrow \infty} \frac{1}{\ln(\mu/\eta)} \sum_{m=0}^{\infty} \frac{K}{K+m} \left( \frac{\lambda}{\mu} \right)^{m+1} = \frac{1}{\ln(\mu/\eta)} \frac{1}{\mu/\lambda - 1}$$

by the dominated convergence theorem. Thus the right tail probabilities behave like  $1/K$ , and the log right tail probabilities expressed as a function of  $\ln(K)$  must asymptote to a straight line with slope  $-1$ . The distribution does not have a finite mean.

At the same time as the rate  $\nu$  at which commodities are introduced by entrepreneurs who set up new firms converges to zero, the number of firms per commodity also goes to zero. But the entry rate of new firms satisfies  $\phi = \eta + \lambda P_1$ , and this converges to a positive value. A calculation yields  $\phi = \lambda / \ln(\mu/\eta)$ . Together with  $\eta = \mu - \lambda$  this gives

$$\frac{\phi}{\eta} = \frac{\mu/\eta - 1}{\ln(\mu/\eta)}. \quad (35)$$

This allows one to infer  $\mu$  and  $\lambda = \mu - \eta$  simply from the ratio of the firm entry rate  $\phi$  and the population growth rate  $\eta$ .

## 5. U.S. EMPLOYER FIRMS

U.S. Internal Revenue Service statistics contain more than 26 million corporations, partnerships and non-farm proprietorships. Business statistics collected by the U.S. Census consist of both non-employer firms and employer firms. In 2002 there were more than 17 million non-employer firms, many with very small receipts, and close to 6 million employer firms. In the following, Census data on employer firms assembled by the U.S. Small Business Administration (SBA) will be considered. For employer firms, part-time employees are included in employee counts, as are executives. But proprietors and partners of unincorporated business are not (Armington [1998, p.9]). This is likely to create significant biases in measured employment for small firms.

Figures III and IV show the 2002 numbers of firms in the right and left tails of the size distribution of U.S. employer firms. In the data, size is measured by number of employees. The number of commodities produced by a firm is inferred by dividing the number of employees by an estimate of  $L + I$  that is described below. Employer firms reported to have 0 to 4 employees during the observation period in March 2002 are interpreted to have  $L + I$  to 4 employees. The tail index for this data is  $\zeta \approx 1.06$ —note from (29) that  $\zeta$  does not depend on the units in which firm size is measured. U.S. population growth is around 1% per annum. These two numbers imply that incumbent firms introduce new commodities at an average net annual rate of  $\mu - \lambda = \eta/\zeta \approx .94\%$ . Since  $\zeta$  is so close to 1, this is only slightly below the population growth rate. Observe that  $\nu = \eta - (\mu - \lambda) = \eta(1 - 1/\zeta) \approx .057\%$ . The rate at which commodities are introduced by new firms is only about .057% per annum.

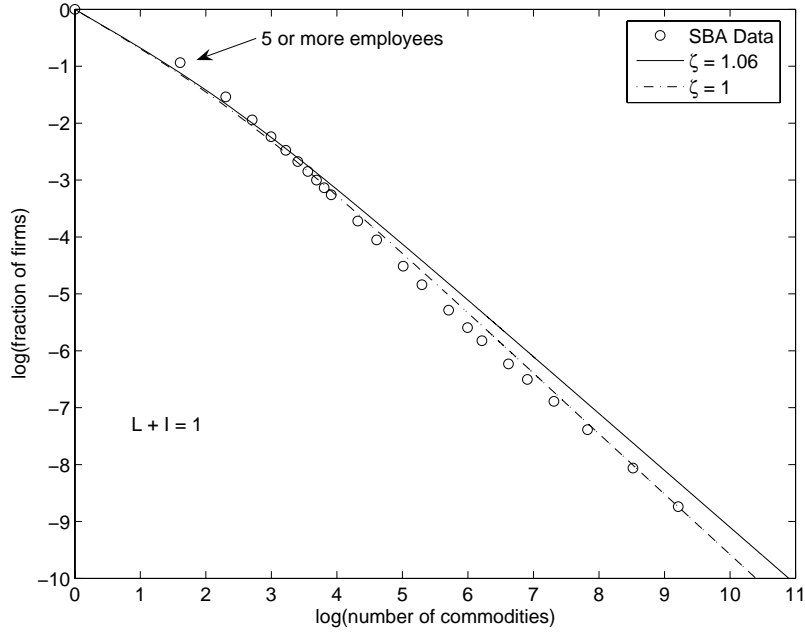


FIGURE III. Right Tail of the Size Distribution.

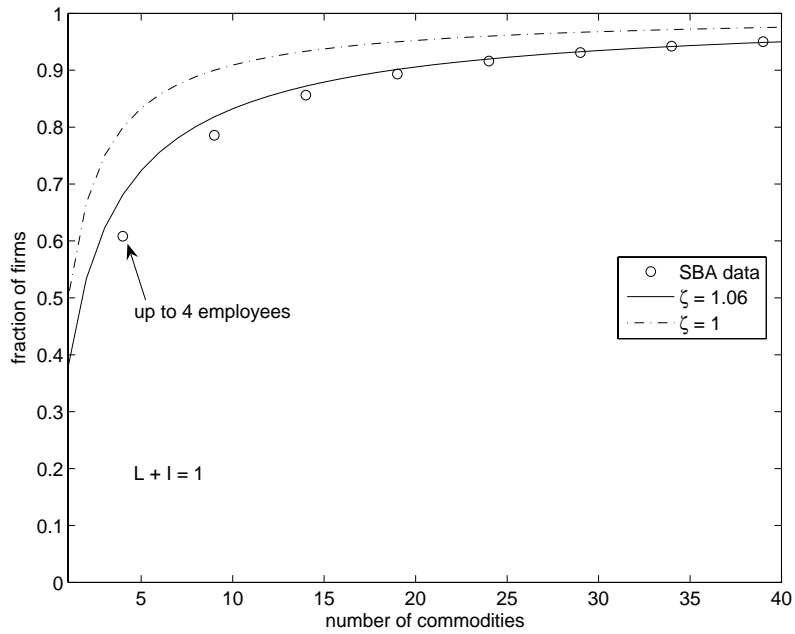


FIGURE IV. Left Tail of the Size Distribution.

Figures III and IV also show fitted size distributions for the case  $\zeta = 1.06$  implied by the data and for the limiting case of  $\zeta = 1$ . When measured by employment, the size distribution depends not only on  $\mu/\eta$  and  $\lambda/\eta$ , but also on the number of employees per commodity  $L + I$ . The fitted distribution for  $\zeta = 1.06$  is obtained by choosing  $\mu/\eta$ ,  $\lambda/\eta$

and  $L + I$  to approximate the empirical distribution show in Figure IV, subject to the restriction  $\zeta = \eta/(\mu - \lambda) = 1.06$ . The resulting estimates are  $L + I = 1$ ,  $\lambda = .03969$  and  $\mu = .04912$ . Given  $\zeta$ , the ratio  $\mu/\lambda$  mainly affects the shape of the left tail of the distribution. Alternative choices for  $L + I$  give rise to horizontal shifts by  $-\ln(L + I)$  of the curve representing the data in Figure III. The estimate  $L + I = 1$ , one employee per commodity, suggests that the firm size distribution is best approximated with very narrowly defined, highly differentiated commodities.

### 5.1 Inferring Productivity and Entrepreneurial Skill Parameters

The equilibrium conditions shown in Figure I arise from the iso-elastic production function  $f(I) = g(AI) = (AI)^\alpha$  with  $\alpha \in (0, 1)$ , and the skill distribution  $S(x) = G(x/B)$ , where  $G$  is the Fréchet distribution  $G(z) = e^{-1/z^2}$ . The labor share parameter is  $\alpha = .7$  and  $\beta = .9$ , implying a 10% markup of price over unit labor costs. The interest rate is  $r = .04$ , in accord with commonly used estimates based on U.S. data. The stationary firm size distribution with size measured by employment identifies  $\mu/\eta$ ,  $\lambda/\eta$  and  $L + I$ . The shape restriction  $f(I) = g(AI)$  together with the equilibrium condition (12) then identify  $A$  and  $I$  and  $L$  separately. The implied value of a new blueprint is only  $s/w = 1.6$ . As a result, the marginal entrepreneur must be coming up with an average of about two new blueprints every three years. Using (13) to eliminate  $N/H$  from (14) gives

$$\frac{\eta - (\mu - \lambda)}{I + L} = \frac{\int_{w/s}^{\infty} xS(dx)}{S(w/s)} \quad (36)$$

The left-hand side of (36) equals .00057, implying that the marginal entrepreneur must be very far out in the tail of the entrepreneurial skill distribution. This restriction identifies the scale parameter  $B$ , given the shape restriction  $S(x) = G(x/B)$  and the productivity of the marginal entrepreneur  $w/s$  that was inferred from the present-value condition (12). The stationary size distribution therefore identifies the scale parameters  $A$  and  $B$ , given  $\beta$  and the functions  $g$  and  $G$ .

Clearly, one has to go beyond firm size data alone to infer the preference parameter  $\beta$  and the shapes of the production function  $f$  and the entrepreneurial skill distribution  $S$ .

### 5.2 Over-Identifying Restrictions

The average number of commodities per firm implied  $\mu/\eta$  and  $\lambda/\eta$  is  $1/[\sum_{n=1}^{\infty} Q_n/n] = 44.2$ . Combining the stationary distribution with the observed population growth rate

gives the  $\mu$  and  $\lambda$  reported above, and then  $\nu \approx .00057$ . The resulting entry rate of new firms is then  $\phi \approx .00057 \times 44.2 \approx .025$ , by (30). The  $\zeta = 1$  approximation (35) gives essentially the same estimate. This 2.5% entry rate of new firms is well below what can be observed directly. In the SBA data, the number of new employer firms that enter and survive during a year is about 10% of the number of incumbent firms. Ignoring even the fact that firms may enter and exit within a year, the parameters inferred from the stationary distribution and the population growth rate only account for about a quarter of the observed entry rate. According to these parameters, firms move up and down the firm size distribution at fairly low rates, implying a puzzle: why is there so much entry? Jovanovic [1982] is a classic answer that is abstracted from in the model described here.

In the SBA employer firm data for 2002, roughly 6 million employer firms employ about 110 million employees. If this is taken to be an estimate of aggregate employment then another estimate of  $L + I$  is

$$L + I = \frac{\text{firms}}{\text{commodities}} \times \frac{\text{employees}}{\text{firms}} = \frac{110}{44.2 \times 6} = .42, \quad (37)$$

This is half of the estimate  $L + I = 1$  derived from fitting the stationary size distribution in Figure III. Thus the fitted distribution implies an average firm that is too large. To some extent this can be attributed to the fact that the average firm size is extremely sensitive to small changes in  $\zeta > 1$  near the asymptote 1. The mean of the size distribution behaves like  $1/(\zeta - 1)$ . Values of  $\zeta$  that are slightly higher than  $\zeta = 1.06$  imply significantly fewer commodities per firm than the estimated 44.2, and hence a higher estimate of  $L + I$  than (37). A related source of discrepancy between (37) and the  $L + I = 1$  estimated in Figures III and IV is the fact that the empirical size distribution necessarily has a finite support. Because the estimated tail index  $\zeta$  is so close to 1, truncating the estimated distribution can lead to large reductions in its mean.<sup>3</sup>

Over short intervals of time, the variance of firm growth is  $(\lambda + \mu)/n$  for a firm with  $n$  commodities. The estimated standard deviation of a firm with  $n$  commodities is therefore  $\sqrt{(\lambda + \mu)/n} \approx .29801/\sqrt{n}$ . Hence the standard deviation of the growth rate of a small firm with 9 employees and commodities is only about 10% per annum. The standard deviation of the returns on the stocks of the much larger firms traded on the NYSE is about 30% per annum. Leverage and other factors probably make these returns more volatile than the standard deviation of firm growth. But the 10% standard deviation for a firm with fewer than 10 employees is likely to be too low.

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<sup>3</sup>Gabaix [2005] and Atkeson and Burstein [2006a] emphasize the sensitivity of predictions of models of this type for aggregates and employment levels in the right tail of the distribution.

As emphasized by Klette and Kortum [2004], the empirical evidence suggests that the variance of firm growth rates declines more slowly than  $1/n$ . Hymer and Pashigian [1962] compared standard deviations of firm growth rates across size quartiles and found that firms in the largest quartile were significantly more volatile than predicted by the  $1/n$  rule. More recently, Stanley et al [1996] and Sutton [2002] find that the variance of the growth rate of Compustat firms behaves like  $1/n^{1/3}$ . Tentative interpretations are given in Stanley et al [1996] and Sutton [2002, 2006].

## 6. CONCLUDING REMARKS

U.S. firm data exhibit (i) high entry rates, (ii) growth rate standard deviations that decline with firm size at a rate that is slower than one over the square root of firm size, and (iii) a size distribution with many small firms and a very long right tail. Luttmer [2006a] and the current paper provide alternative interpretations of (iii). Both papers require large amounts of randomness to deal with (i). The two papers seem to be on opposite sides of the data when it comes to (ii). Finding tractable equilibrium models of this phenomenon is task for further research.

Skewed firm size distributions are interpreted as reflecting skewed productivity distributions in Hopenhayn [1992], Klette and Kortum [2004], Lentz and Mortensen [2006], Luttmer [2006a] and Atkeson and Burstein [2006b]. Furthermore, the continuous reallocation of resources across firms plays a crucial role in generating aggregate growth. In Lucas [1978] and Gabaix and Lanier [2006] an assignment problem relates the firm size and talent distributions. By contrast, in Simon and Bonini [1958] and Ijiri and Simon [1964] firm size does not affect aggregate outcomes because all firms operate under constant returns to scale. Similarly, firms in the economy described here and in Luttmer [2006b] are all equally productive. The relative importance of each of these interpretations of the firm size distribution remains to be sorted out.

## A PROOF OF PROPOSITION 2

Write the candidate solution (25) as

$$\left( \prod_{m=2}^n \frac{\mu\beta_m}{\lambda} \right)^{-1} Q_n = \frac{\nu}{\lambda} \sum_{k=n+1}^{\infty} \left( \prod_{m=2}^k \beta_m \right) \left( \frac{\mu\beta_k}{\lambda} \right)^{-1} \prod_{m=2}^k \frac{\mu\beta_m}{\lambda}$$

Since  $\beta_n \leq \min\{1, \lambda/\mu\}$  this implies the upper bounds in (26) and (27). Take some  $\varepsilon > 0$ . The lower bounds rely on  $\beta_n \uparrow \min\{1, \lambda/\mu\}$ . If  $\mu > \lambda$ , then eventually  $\beta_n \geq (\lambda/\mu)(1+\varepsilon)$ , and this gives the lower bound in (26). If  $\mu < \lambda$ , then  $1/(1+\varepsilon) \leq \beta_n$  for all large enough  $n$ , and this implies (27). Thus the sums defining  $\{Q_n\}_{n=1}^\infty$  converge and (26) and (27) hold. By construction, the candidate solution satisfies (18)-(19). It remains to prove the adding-up condition (20), which ensures that  $\{Q_n\}_{n=1}^\infty$  is in fact a probability distribution.

Define  $F_1 = 1$  and

$$F_n = n \prod_{k=2}^n \frac{\mu\beta_k}{\lambda},$$

for all  $n \in \mathbb{N} \setminus \{1\}$ . Note from the bounds (26)-(27) that the sequence  $\{Q_n\}_{n=1}^\infty$  is summable if and only if  $\{F_n/n\}_{n=1}^\infty$  is summable. Define

$$X_n = n \left( \frac{\mu\beta_n}{\lambda} - 1 \right)$$

for all  $n \in \mathbb{N}$ . The recursion (24) is equivalent to

$$X_{n+1} = \left( 1 + \frac{1}{n} \right) \left( \frac{X_n - \frac{\eta}{\lambda}}{\frac{\eta+n\mu}{n\lambda} - \frac{1}{n}X_n} \right) \quad (38)$$

Observe from this that  $X_{n+1} < -1$  if and only if  $X_n$  satisfies

$$X_n < \frac{\eta - (\mu - \lambda)}{\lambda} - 1 \quad (39)$$

Since  $\eta > \mu - \lambda$ , this is true if  $X_n \leq -1$ . But this follows by induction starting from  $X_1 = -1$ . The fixed point for the  $n = \infty$  version of (38) is  $-\eta/(\mu - \lambda) < -1$ . One can verify that  $X_n$  converges to this fixed point starting from  $X_1 = -1$ . The fact that  $\lim_{n \rightarrow \infty} X_n < -1$  implies that  $\{F_n/n\}_{n=1}^\infty$  is summable, by Raabe's test. The inequality  $X_{n+1} < 1$  is equivalent to  $F_{n+1} < F_n$ , and so  $F_n \downarrow F_\infty$  for some  $F_\infty \geq 0$ . Therefore

$$\sum_{n=2}^K \frac{1}{n} F_n \geq F_\infty \sum_{n=1}^K \frac{1}{n}$$

for all  $K$ . Since the left-hand side is summable, it must be that  $F_\infty = 0$ .

Write (18) as  $\eta Q_1 = \lambda [Q_2 - Q_1] + \eta - (\mu - \lambda) - \mu Q_1$  and (19) as

$$\eta Q_n = \lambda n [Q_{n+1} - Q_n] - \mu n [Q_n - Q_{n-1}],$$



for  $n \in \mathbb{N} \setminus \{1\}$ . Adding up gives over all  $n$  gives

$$\eta \sum_{n=1}^{\infty} Q_n = \eta - (\mu - \lambda) + \lambda \sum_{n=1}^{\infty} n [Q_{n+1} - Q_n] - \mu \left( Q_1 + \sum_{n=2}^{\infty} n [Q_n - Q_{n-1}] \right). \quad (40)$$

Note that  $n[Q_{n+1} - Q_n] = (n+1)Q_{n+1} - nQ_n - Q_{n+1}$  and  $n[Q_n - Q_{n-1}] = nQ_n - (n-1)Q_{n-1} - Q_{n-1}$ , and observe that the candidate solution (25) satisfies  $\lim_{n \rightarrow \infty} nQ_n = 0$ , since  $F_\infty = 0$ . Using summation-by-parts for the two sums on the right-hand side of (40) one obtains

$$\sum_{n=1}^{\infty} n [Q_{n+1} - Q_n] = Q_1 + \sum_{n=2}^{\infty} n [Q_n - Q_{n-1}] = - \sum_{n=1}^{\infty} Q_n. \quad (41)$$

Together with  $\eta > \mu - \lambda$ , (40) and (41) imply that the sequence  $\{Q_n\}_{n=1}^{\infty}$  adds up to 1.

### B PROOF OF PROPOSITION 3

Recall that  $P_n \sim F_n/n^2$  and define  $R_K = \sum_{n=K}^{\infty} F_n/n^2$ . Observe

$$K^z R_K = \sum_{n=K}^{\infty} \left( \frac{K}{n} \right)^z n^{z-1} \prod_{k=2}^n \frac{\mu \beta_k}{\lambda} \leq \sum_{n=K}^{\infty} n^{z-1} \prod_{k=2}^n \frac{\mu \beta_k}{\lambda}.$$

If the sum on the right-hand side is finite, then  $\lim_{K \rightarrow \infty} K^z R_K = 0$ . A sufficient condition for the sum to converge is a version of Raabe's test,  $\lim_{n \rightarrow \infty} X_n > 1$ , where now

$$X_n = n \left( \left[ 1 - \frac{1}{n} \right]^{z-1} \frac{\lambda}{\mu \beta_n} - 1 \right).$$

The recursion (24) for  $\beta_n$  is equivalent to

$$X_{n+1} = \left( 1 + \frac{1}{n} \right) \left( A_n + \left( 1 - \frac{1}{n} \right)^{z-1} \left[ \frac{\eta}{\mu} + \frac{\lambda X_n - A_n}{\mu \left( 1 + \frac{1}{n} X_n \right)} \right] \right)$$

where

$$A_n = n \left( \left[ 1 - \frac{1}{n} \right]^{z-1} - 1 \right).$$

Observe that  $\lim_{n \rightarrow \infty} A_n = 1 - z$ . The limiting recursion for  $X_n$  is therefore

$$X_{n+1} - [1 - z] \approx \frac{\eta}{\mu} + \frac{\lambda}{\mu} (X_n - [1 - z]),$$

and this has the unique fixed point  $1 - z + \zeta$ . One can verify that  $X_n$  converges to this fixed point starting from  $X_1 = -1$ . Thus  $z < \zeta$  guarantees convergence.

C THE LIMITING CASE  $\nu \downarrow 0$

Write

$$P_n = \xi \sum_{k=n}^{\infty} \frac{(\lambda/\mu)^{k+1-n}}{k(k+1)},$$

and note that

$$\begin{aligned} \frac{1}{\xi} \sum_{n=K}^{\infty} P_n &= \sum_{n=K}^{\infty} \sum_{k=n}^{\infty} \frac{(\lambda/\mu)^{k+1-n}}{k(k+1)} \\ &= \sum_{m=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^{m+1} \sum_{n=K}^{\infty} \frac{1}{(m+n)(m+n+1)} = \sum_{m=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^{m+1} \frac{1}{m+K}. \end{aligned}$$

For  $K = 1$  this gives:

$$\frac{1}{\xi} \sum_{n=1}^{\infty} P_n = \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\lambda}{\mu}\right)^m = -\ln\left(1 - \frac{\lambda}{\mu}\right),$$

and hence  $\xi = -\ln(1 - \lambda/\mu) = \ln(\eta/\mu)$ . Using the fact that  $\eta = \mu - \lambda$ , one can write the entry rate as

$$\eta = \eta + \lambda P_1 = \eta + \frac{\lambda}{-\ln\left(1 - \frac{\lambda}{\mu}\right)} \sum_{k=1}^{\infty} \frac{(\lambda/\mu)^k}{k(k+1)} = \lambda \xi,$$

which is the result reported in (35).

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