Optimal debt contracts and diversity of opinions : an extreme case of bunching.

G. Carlier [*], L. Renou [†]

9th January 2003

Abstract

This paper studies optimal menus of debt contracts such as secured debentures or bonds, in the presence of diversity of opinions between borrowers and lenders. We first characterize incentive compatible contracts, then prove the existence of optimal debt contracts. Finally, we are able to explicitly characterize such optimal menus within a specific case: we notably show that borrowers optimally offer at most two contracts, which is an extreme case of bunching.

Keywords: debt contracts, heterogeneity of beliefs, multidimensional screening, bunching.

JEL classification: C7, D8, G3.

[*]Université Bordeaux I, MAB, UMR CNRS 5466 and Université Bordeaux IV, GRAPE, UMR CNRS 5113, Av. L. Duguit, 33608, Pesac, France, Guillaume.Carlier@math.u-bordeaux.fr.

[†]European University Institute, Via dei Roccettini 9, 50016 San Domenico di Fiesole, Italy, ludovic.renou@iue.it.

1 Introduction

The present paper draws upon three lines of research. First, since the seminal work of Townsend [29], the *costly state verification* (henceforth, CSV) literature has received a great deal of attention. A CSV problem is a situation where investors can only observe the project return by incurring a cost while the entrepreneur observes it freely. CSV models have notably been useful to provide a rationale to debt-like financial contracts with costly liquidation (see for instance [11], [32], [33]). To further explain certain features of real-world financial contracts such as coupon payments, financial covenants, and maturity structure of debt, the basic CSV model has been extended along several lines: moral hazard (see [4], [16]), multi-period contracting problem (see [3], [9]), stochastic monitoring (see [5], [21]) or macroeconomic models with aggregate shocks (see [6], [18]). However, a common feature of all these contributions is to maintain the common prior assumption, which brings us to the second line of research.

Gul ([13]) and Morris ([22]) have recently challenged the common prior assumption arguing that none of the arguments in support of the common prior assumption (expression of rationality, all difference in beliefs result from difference in information or impossibility of normative approach) are compelling. Relaxing the common prior assumption brings new and interesting results. In Harrison and Kreps [14], risk neutral traders have heterogenous prior beliefs, infinite endowment and can retrade their assets but cannot sell them short. Harrison and Kreps then show that the price of an asset is always greater than or equal to each individual's expected value of future payments of that asset. Their model thus illustrates speculative phenomena, i.e. investors buy stocks now so as to sell them later for more than they think they are actually worth. Another interesting contribution is found in Varian [31]. In an Arrow-Debreu single good economy with heterogenous prior beliefs, Varian has shown that the higher the heterogeneity of beliefs, the lower the equilibrium asset prices if risk aversion does not decline too rapidly with increasing income. A measure of the diversity of opinions is thus a good indicator for *ex-post* returns. Empirical works are consistent with this theoretical finding (see [12]). These two contributions are illuminating examples of the role of heterogeneity of beliefs in asset pricing theory. For a broader perspective, see the excellent survey of [22] and all references therein.

Along this line of research, the purpose of our paper is to characterize optimal menus of debt contracts in the presence of diversity of opinions, that is, we relax the common prior assumption in an otherwise standard CSV model. More specifically, within the class of *secured simple debt contracts*, we derive optimal menus of contracts that a borrower offers to a lender. The borrower and the lender have different beliefs and beliefs are private information. Thus the borrower faces an adverse selection problem and wishes to discriminate among the different beliefs, so-called epistemic types, of the lender.

Quite naturally, the third line of literature we follow here is the multidimensional screening literature, see for instance [1], [2], [20], [25] and [26] for an excellent survey. An important result of this literature is that bunching is a robust phenomenon. First, perfect screening might be ruled out by dimensionality considerations, this is bunching of the first type in the terminology of [25]. Second, there typically exists a conflict between rent extraction and second-order compatibility constraint implying that several types will be offered the same contract. This is an example of bunching of the second type (see [25]). This sharply contrasts with unidimensional screening problems where perfect discrimination of types is rather the rule than the exception (however, see [15] for the possibility of bunching in unidimensional problems).

This paper provides a characterization of the set of incentive compatible debt contracts as well as a proof of the existence of optimal menus of debt contracts in the full generality of the contracting problem. Besides these general results, we construct an example in order to provide an explicit characterization of the optimal menus of debt contracts. Notably, we show that there always exist optimal menus of debt contracts with at most two contracts offered. This is an extreme case of bunching.

In section 2, we present the model. Section 3 is devoted to the characterization of incentive compatible contracts. Section 4 addresses the existence of admissible SDC incentive compatible contracts. Finally in section 5, we completely solve the problem in a simple case.

2 The model

2.1 The framework

The framework is adapted from Renou [23]. We consider a static, two-period economy with a single good used for investment and consumption. In this economy, there are a unique "borrower/entrepreneur" and a unique "lender/investor". The borrower has no initial endowment, but he does have access to an individual-specific, high return investment project, described below. Both the borrower and the lender are assumed to be risk-neutral and to care only about second period consumption.

Investment can only occur in the first period using one or more of the following two technologies. First, there is a *commonly available, riskless technology* whereby one unit invested in the first period yields $r > 1$ units of output in the second period. Second, there is a *stochastic technology* that converts current investment into future output. This project requires *exactly one unit* of fund to be undertaken in period one and produces ω units in the second period. The project return ω is drawn from an unknown probability law, explained later.

Moreover, the return of the project is not freely observable in the sense that ω is *costlessly* observable only by the entrepreneur. However, there exists a technology which can be used by the investor to verify, in the second period, the realization ω of the project. This state verification technology is costly to use, requiring a utility cost of γ in the second period.

Finally, only the entrepreneur is endowed with access to the high-return investment technology and ownership of this investment cannot be traded. The lender is endowed with 1 unit of the good in the first period. Up to this point, our framework is similar to a standard *costly state verification* problem, see for instance [29], [32] and [33].

2.2 Beliefs

This section constitutes the main point of departure from a standard CSV model. We therefore present assumptions on beliefs in detail and give intuitions beyond the formulation adopted.

The lender and the borrower conceive possible futures captured by a random variable θ, whose realizations influence the project return ω. The random variable θ is discrete and takes values in $\Theta := \{\theta_1, \ldots, \theta_N\}$. One might interpret a realization of θ as a summary of *exogenous factors* such as consumers' confidence or taste, demand factors, weather, macroeconomic policies, etc. Moreover, we define g_n as the probability density of ω conditional on $[\theta = \theta_n]$, $n = 1, \ldots, N$. The density g_n has full support on $[a_n, b_n]$ a compact subset of $\mathbb{R}_{++}$.

We impose three assumptions on beliefs. First, we assume that it is *common knowledge* that the return ω conditional on $[\theta = \theta_n]$ is drawn from g_n. We also suppose that the expected return to undertake the risky project is strictly greater than the safe return, that is, $E_n := \int \omega g_n(\omega)\, d\omega > r$ for $n = 1, \ldots, N$.

Second, the borrower and the lender have different opinions about the likelihood of these exogenous factors. For example, the borrower and the lender might have a different perception of the consumers' taste for a new product (rhubarb wine) or a new selling method (Internet). These exogenous factors are certainly difficult to evaluate or to forecast, explaining that the borrower and the lender form different beliefs. Beliefs are subjective or personalistic. It is in this sense that we capture the diversity of opinions in the economy. Formally, we define p_n (resp. q_n) as the subjective probability of θ_n for the lender (resp. the borrower), and $p := (p_1, \ldots, p_{N-1})$ a point in $\Sigma_{N-1} := \left\{ (p_1, \ldots, p_{N-1}) \in \mathbb{R}_+^{N-1} : \sum_{n=1}^{N-1} p_n \leq 1 \right\}$, the simplex of dimension $N-1$. Similarly, q is the vector of subjective beliefs for the borrower. We call the vector p of subjective beliefs the *epistemic type* of the lender.

Third, we assume that epistemic types are not observable. Furthermore, the borrower believes that the lender epistemic type is drawn from a non-degenerate probability measure with density ρ with respect to Lebesgue on Σ_{N-1}. The borrower thus faces an adverse selection problem.

2.3 Secured financial contracts

The first period comprises two stages. In a first stage, the borrower publicly announces a financial contract specifying repayments in various contingencies. We must think of the contract as a promise to repay:
- a fixed interest rate,
- a variable interest rate contingent on the project return.

In a second stage, the investor accepts or not to fund the borrower. If the lender is indifferent between accepting or rejecting, he accepts. In the second period, the investment return is realized. Monitoring takes place or not (we only consider deterministic monitoring), and repayments occur. When monitoring takes place, it is perfect in the sense that the true realization of the return is disclosed. Let us define a menu of secured debt contracts.

Definition 1 *A menu of secured debt contracts is a triple of mappings* $(R(\cdot,\cdot), C(\cdot), M(\cdot))$ *where*

- $R : \Omega \times \Sigma_{N-1} \to \mathbb{R}_+$, $(\omega, p) \mapsto R(\omega, p)$ *is the variable repayment to epistemic type* p *when the return project is* ω, *with* $\Omega := \cup_n [a_n, b_n]$,

- $R(\omega, p) \leq \omega$, $\forall \omega \in \Omega$, $\forall p \in \Sigma_{N-1}$,

- $M : \Sigma_{N-1} \to 2^{\Omega}$, $p \mapsto M(p)$ *an open subset of* Ω *which determines the monitoring states for epistemic type* p,

- $C : \Sigma_{N-1} \to \mathbb{R}$, $p \mapsto C(p)$ *the secured repayment to epistemic type* p.

Definition 2 *A menu of contracts* $(R(\cdot,\cdot), C(\cdot), M(\cdot))$ *is truthtelling if and only if* $\forall p \in \Sigma_{N-1}$, $\forall \omega \in \Omega$, $\forall \omega' \notin M(p)$, *:*

$$R(\omega, p) \leq R(\omega', p).$$

A menu of contract is thus said to be truthtelling when the borrower has no incentive to misreport the project return. It is a principle of sincerity, balance sheets and other financial informations disclosed by the entrepreneur have to correctly reflect the financial situation of the firm. Without loss of generality (see [30]), we restrict ourselves to the class of truthtelling secured debt contracts.

If a menu of contracts $(R(\cdot,\cdot), C(\cdot), M(\cdot))$ is truthtelling then there exists a *threshold* function $\overline{R} : \Sigma_{N-1} \to \mathbb{R}$ such that for all $p \in \Sigma_{N-1}$, $M(p) := \left\{\omega : R(\omega, p) < \overline{R}(p)\right\}$. Obviously, in non-monitoring states, i.e., $\Omega \setminus M(p)$, the borrower repays the less possible i.e., $R(\omega, p) = \overline{R}(p)$ since the lender does not verify the return realized. The set of all truthtelling secured debt contracts is therefore fully specified by the triple of mappings $\left(R(\cdot,\cdot), C(\cdot), \overline{R}(\cdot)\right)$.

The so-called *simple debt contracts* play a special role in both theory and practice. A simple debt contract (henceforth, SDC) is defined by a threshold $\overline{\omega}$ such that the repayment is $R(\omega) = \omega$ for $\omega < \overline{\omega}$ and $R(\omega) = \overline{\omega}$, otherwise. Assuming a common prior, Gale and Hellwig ([11]) and Williamson ([32]) have shown that simple debt contracts are optimal among the class of secured debt contracts in the sense that simple debt contracts minimize the resources destroyed in the monitoring process.

Furthermore, real-world counterparts of (secured) SDCs are (secured) *debentures or bonds*, which are increasingly used by corporations. A debenture is a fixed-interest security issued by limited companies in return for loan. Debenture interest must be paid whether the company makes a profit or not and in the event of non-payment, debentures holders can force liquidation.[1] In what follows, we restrict ourselves to the class of secured SDC because of their similarities with secured debentures and bonds. However, it is worth pointing out that in the presence of heterogeneity of beliefs, the class of secured SDC might not be optimal among the class of secured debt contracts.

Definition 3 *A menu of secured SDC is a pair of mappings* $(\overline{\omega}(\cdot), C(\cdot)) : \Sigma_{N-1} \to \Omega \times \mathbb{R}$.

From Definition 3, a secured SDC comprises two distinct parts. First, there is a secured repayment C independent of the outcome ω of the project,

[1] See the *International Dictionary of Finance*, The Economist book, 1989.

and second, a contingent repayment. In non-monitoring states $\Omega \setminus M$, the entrepreneur promises to repay a return $\overline{\omega}$ and in monitoring states M, the investor monitors the corporation and *seizes all the outcome*. Thus our class of contracts well and truly features the same characteristics as secured debentures.

Several remarks are in order. First, the contract does not depend on θ since we assume that it is not contractible.[2] Second, the class of contracts we consider is richer than the standard class of SDC (see for instance [11], [17], [32]) since the entrepreneur can possibly offer a secured part C to the investor, while a borrower is not allowed to offer such a secured part in the standard class of SDC.

3 Incentive compatible secured simple debt contracts

Let $(\overline{\omega}, C)$ be a contract, we define the expected payoff $U(p, \overline{\omega}, C)$ of the lender of epistemic type p as

$$\begin{aligned} U(p, \overline{\omega}, C) &:= \sum_{n=1}^{N} p_n \left(\int_{a_n}^{\overline{\omega}} (\omega - \gamma) g_n(\omega) \, d\omega + \int_{\overline{\omega}}^{b_n} \overline{\omega} g_n(\omega) \, d\omega \right) + C \\ &= \sum_{n=1}^{N-1} p_n (z_n - z_N)(\overline{\omega}) + z_N(\overline{\omega}) + C \end{aligned}$$

with, for $n = 1, \ldots, N-1$,

$$z_n(\overline{\omega}) := \left(\int_{a_n}^{\overline{\omega}} (\omega - \gamma) g_n(\omega) \, d\omega + \int_{\overline{\omega}}^{b_n} \overline{\omega} g_n(\omega) \, d\omega \right). \tag{1}$$

We denote $Z : \Omega \to \mathbb{R}^{N-1}$ the map

$$\overline{\omega} \mapsto Z(\overline{\omega}) := ((z_1 - z_N)(\overline{\omega}), \ldots, (z_{N-1} - z_N)(\overline{\omega})),$$

and $V : \Sigma_{N-1} \to \mathbb{R}$ the potential associated with the menu $(\overline{\omega}(.), C(.))$ defined by

$$V(p) := U(p, \overline{\omega}(p), C(p)). \tag{2}$$

Definition 4 *Let $(\overline{\omega}(\cdot), C(\cdot))$ be a menu of secured SDC, and define V by (2). The menu $(\overline{\omega}(\cdot), C(\cdot))$ is:*

[2] This assumption is justified if we interpret a realization of θ in a sufficiently broad sense. Indeed it would be legally difficult to enforce a contract based on consumers' confidence or taste since these characteristics are not easily verifiable and thus enforceable.

1. *incentive compatible if for all* $(p,\, p') \in \Sigma_{N-1} \times \Sigma_{N-1}$,

$$V(p) \geq U(p, \overline{\omega}(p'), C(p')), \tag{3}$$

2. *individually rational if for all* $p \in \Sigma_{N-1}$,

$$V(p) \geq r. \tag{4}$$

The next proposition characterizes incentive compatible contracts in terms of the associated potential V. In the sequel, we denote $\partial_n V := \frac{\partial V}{\partial p_n}$.

Proposition 1 *A menu of secured SDC* $(\overline{\omega}(\cdot), C(\cdot))$ *is incentive compatible if and only if the potential* V *defined by (2) satisfies:*

1. V *is convex,*

2. $\partial_n V(p) = (z_n - z_N)(\overline{\omega}(p))$ *a.e. for* $n = 1, \ldots, N-1$.

Proof. *Necessity.* Assume $(\overline{\omega}, C)$ is incentive compatible. Then for all $p \in \Sigma_{N-1}$,

$$V(p) = \sup_{p' \in \Sigma_{N-1}} U(p, \overline{\omega}(p'), C(p')), \tag{5}$$

and U is linear in p, therefore V is convex as the supremum of convex functions. Hence V is differentiable a.e.. For a.e. $p \in \Sigma_{N-1}$, the Envelope theorem then yields

$$\nabla V(p) = \frac{\partial}{\partial p} U(p, \overline{\omega}(p), C(p)),$$

meaning $\partial_n V(p) = (z_n - z_N)(\overline{\omega}(p))$ a.e. for $n = 1, \ldots, N-1$.

Sufficiency. Let V be a potential satisfying *1-2* of Proposition 1, and define

$$C(p) = V(p) - \sum_{n=1}^{N-1} p_n (z_n - z_N)(\overline{\omega}(p)) - z_N(\overline{\omega}(p)).$$

Then by convexity of V, for all $p, p' \in \Sigma_{N-1}^2$,

$$\begin{aligned} V(p) &\geq V(p') + (p - p') \nabla V(p') \\ &\geq V(p') + \sum_{n=1}^{N-1} (p_n - p'_n)(z_n - z_N)(\overline{\omega}(p')) \\ &\geq \sum_{n=1}^{N-1} p_n (z_n - z_N)(\overline{\omega}(p')) + z_N(\overline{\omega}(p')) + C(p'), \end{aligned}$$

so that $(\overline{\omega}, C)$ is an incentive-compatible contract. $\square$

Proposition 1 gives a very standard characterization of the set of incentive compatible contracts (for similar results see [7], [24] and [25]). A contract $(\overline{\omega}, C)$ is incentive compatible if the potential associated with $(\overline{\omega}, C)$ is convex and its gradient belongs to the image of Z. Formally,

$$\nabla V \in \{Z(\overline{\omega}),\ \overline{\omega} \in \Omega\}.$$

This set is a manifold of dimension one and thus one might expect that the borrower discriminates different epistemic types in only one dimension. We can interpret condition *2*) of proposition 1 as a first-order condition of incentive compatibility whereas the convexity of V is a second order one.

Two additional remarks are worth making. First, we do not impose (for the moment) a condition of single-crossing type. Observe that if $N = 2$, i.e., epistemic types are unidimensional, the Spence-Mirlees condition reads:

$$\frac{\partial^2 U(p, \overline{\omega}, C)}{\partial p_1 \partial \overline{\omega}} = (z_1 - z_2)'(\overline{\omega})$$

is of constant sign, implying that $(z_1 - z_N)(\cdot)$ is injective. In higher dimensions, Carlier [8] generalizes the Spence-Mirlees condition[3] to the injectivity of Z. As in the unidimensional case, an explicit characterization of incentive compatible contracts crucially rests upon the injectivity of Z. Without such an injectivity condition, no explicit characterization could be provided. Second, the linearity of U in p is not doubtful in our context since p represents probabilistic beliefs.

4 Optimal contracts: the problem

Without loss of generality, we assume that the borrower offers a contract that is accepted. For otherwise, the project is not undertaken. Moreover, we impose the additional constraint that $\overline{\omega}(\Sigma_{N-1}) \subseteq [\underline{\delta}, \overline{\delta}]$, a compact subset of the real line.[4] Hence, the borrower's program consists of maximizing its total expected profit over the set of *admissible* contracts, i.e. contracts that are incentive compatible, individually rational and satisfy $\overline{\omega}(\Sigma_{N-1}) \subseteq [\underline{\delta}, \overline{\delta}]$. This program can be written as follows :

$$\sup\{\Pi(\overline{\omega}, C) : (\overline{\omega}, C) \text{ is admissible}\}, \tag{6}$$

[3]See also the generalized Spence-Mirlees condition of McAfee and McMillan ([20]).

[4]This is without loss of generality. To see this, suppose that for some epistemic types p, $\overline{\omega}(p) < a = \min_n a_n$, then $V(p) = \overline{\omega}(p) + C(p)$. The borrower can increase $\overline{\omega}(p)$ to a, decrease $C(p)$ by $a - \overline{\omega}(p)$, leaving all the constraints satisfied and yielding the same payoff. A similar argument applies to the case $\overline{\omega}(p) > b = \min_n b_n$. Thus, one might take $[\underline{\delta}, \overline{\delta}] \supseteq \Omega$ a compact subset of the real line without loss of generality.

where

$$\Pi\left(\overline{\omega},C\right):=\int_{\Sigma_{N-1}}\left(\sum_{n=1}^{N}q_n\int_{\overline{\omega}(p)}^{b_n}\left(\omega-\overline{\omega}\left(p\right)\right)g_n\left(\omega\right)d\omega-C\left(p\right)\right)\rho\left(p\right)dp. \quad (7)$$

The mathematical formulation of the borrower problem is a constrained variational problem.

Proposition 2 *The borrower's program* (6) *is equivalent to the following problem:*

$$\inf\left\{J(\overline{\omega},V)\ :\ (\overline{\omega},V)\in\mathcal{A}\right\}, \quad (8)$$

where

$$\begin{aligned}J(\overline{\omega},V):=&\int_{\Sigma_{N-1}}\left(V(p)-p\cdot\nabla V(p)-z_N(\overline{\omega}(p))\right)\rho(p)dp\\&-\int_{\Sigma_{N-1}}\left(\sum_{n=1}^{N}q_n\int_{\overline{\omega}(p)}^{b_n}\left(\omega-\overline{\omega}\left(p\right)\right)g_n\left(\omega\right)d\omega\right)\rho(p)dp,\end{aligned} \quad (9)$$

and

$$\mathcal{A}:=\left\{(\overline{\omega},V)\ :\ V\ \textit{convex},\ V\geq r,\ \overline{\omega}\in[\underline{\delta},\overline{\delta}]\ \textit{and}\ \nabla V(p)=Z(\overline{\omega}(p))\ \textit{a.e.}\ \right\}. \quad (10)$$

Proof. The proof directly follows from the results of section 3. □

This problem is a non-standard optimal control problem with the state variable V and the control variable $\overline{\omega}$. The state variable is subject to two constraints: the participation constraint and convexity. The latter is of global nature and must be understood as a second-order condition of incentive compatibility. The next result establishes existence of a solution to program (8).

Theorem 1 *Program (8) admits at least one solution.*

The proof is given in the appendix.

Now that we know that there exists a solution to program (8), we are interested in characterizing optimal menus. Do they feature bunching? Where? How does the borrower discriminate among different epistemic types? Is the participation constraint binding at the bottom? While all these questions unfortunately cannot be answered in the full generality of our problem, the next section completely solves the problem in a specific case.

5 Characterization of optimal contracts and bunching

This section is devoted to explicitly solving the borrower's program in a particular example, namely in the *uniform* case where the law of the return conditional to each realization of θ is uniform. We do not know whether the problem can be explicitly solved in other cases,[5] this is doubtful yet, due to complexity of the constraints in general.

5.1 Assumptions

From now on, we suppose that for each realization θ_n of θ, the density of the return ω conditional to $[\theta = \theta_n]$, g_n, is uniform on $[\theta_n - \varepsilon, \theta_n + \varepsilon]$, that is

$$g_n := \frac{1}{2\varepsilon}\mathbf{1}_{[\theta_n-\varepsilon,\theta_n+\varepsilon]}.$$

Note that with this specification, the variance of the return does not depend on θ. We assume furthermore that $\theta_1 < \theta_2 < \ldots \theta_n < \ldots < \theta_N$ (means are ordered), that $\theta_1 \geq \varepsilon$ (the return is almost surely nonnegative) and $\theta_N - \theta_1 < 2\varepsilon$ (the intersection of the supports is nonempty).

For the sake of tractability, we impose bounds on feasible contracts as follows : we restrict our attention to menus of contracts $p \mapsto (\overline{\omega}(p), C(p))$ such that $\overline{\omega}(p) \in [\underline{\delta}, \overline{\delta}]$ for all $p \in \Sigma_{N-1}$, and we assume

$$[\underline{\delta}, \overline{\delta}] \subset [\theta_N - \varepsilon, \theta_1 + \varepsilon] = \bigcap_n [\theta_n - \varepsilon, \theta_n + \varepsilon]. \tag{11}$$

An explicit computation then yields that for $\overline{\omega} \in [\underline{\delta}, \overline{\delta}]$ and for $n = 1, \ldots, N$, $z_n(\overline{\omega})$, as defined in (1), is given by:

$$z_n(\overline{\omega}) = \frac{1}{2\varepsilon}\left[-\frac{1}{2}\overline{\omega}^2 + \overline{\omega}(\theta_n + \varepsilon - \gamma) - \frac{1}{2}(\theta_n - \varepsilon)^2 + \gamma(\theta_n - \varepsilon)\right]. \tag{12}$$

It thus follows that for $n = 1, \ldots, N-1$:

$$(z_n - z_N)(\overline{\omega}) = \alpha_n \overline{\omega} + \beta_n, \tag{13}$$

with

$$\alpha_n := \frac{1}{2\varepsilon}(\theta_n - \theta_N), \tag{14}$$

[5] We have tried other cases in which the law of the return conditional to each realization of θ is triangular, exponential or Gaussian, but it gives rise to untractable partial differential equations.

and

$$\beta_n := \frac{1}{2\varepsilon}\left(\theta_n - \theta_N\right)\left[\gamma + \varepsilon - \frac{1}{2}\left(\theta_n + \theta_N\right)\right]. \tag{15}$$

The fact that $(z_n - z_N)(.)$ is linear (on $[\underline{\delta}, \overline{\delta}]$) follows from the uniform assumption and assumption (11). This will considerably simplify the analysis of the next sections. Notably, as has already been pointed out, our specification satisfies the *generalized single-crossing condition* of [8]. More precisely, the mapping $Z : [\underline{\delta}, \overline{\delta}] \to \mathbb{R}^{N-1}$ with

$$\overline{\omega} \mapsto Z\left(\overline{\omega}\right) := \left(\alpha_1 \overline{\omega} + \beta_1, \ldots, \alpha_{N-1}\overline{\omega} + \beta_{N-1}\right),$$

is easily seen to be injective. On the contrary, Z is not injective on the entire Ω. Thus our bound conditions, that is, $\overline{\omega}(p) \in [\underline{\delta}, \overline{\delta}]$ for all $p \in \Sigma_{N-1}$, plays the role of a single-crossing condition, and without such an assumption, we are not able to solve the problem.

For further use, note that each α_n is negative, α_n increases with n and that the points (α_n, β_n), $n = 1, \ldots, N-1$, lie on a parabola. More precisely, $\beta_n = P(\alpha_n)$ where

$$P(\alpha) := \alpha\left(\gamma + \varepsilon - \theta_N - \varepsilon\alpha\right). \tag{16}$$

5.2 The geometry of admissible contracts

Our first step in solving the borrower's program consists in characterizing the set of admissible contracts in our uniform specification. We recall that an admissible contract is a menu $\{(\overline{\omega}(p), C(p)), p \in \Sigma_{N-1}\}$, which is incentive compatible, individually rational, and satisfies the bound conditions $\overline{\omega}(p) \in [\underline{\delta}, \overline{\delta}]$ for all $p \in \Sigma_{N-1}$.

5.2.1 Incentive compatibility

As in section 2, with any contract $(\overline{\omega}(.), C(.))$ we associate its potential V defined by:

$$V(p) := U\left(p, \overline{\omega}(p), C(p)\right), \text{ for all } p \in \Sigma_{N-1} \tag{17}$$

Proposition 3 *Let $(\overline{\omega}(.), C(.))$ be any contract and let V be the potential associated to $(\overline{\omega}(.), C(.))$ by formula (17), then $(\overline{\omega}(.), C(.))$ is admissible if and only if:*

- *V is convex on Σ_{N-1},*
- *V satisfies (in the a.e. sense on Σ_{N-1}) the linear system :*

$$\frac{1}{\alpha_1}\left(\partial_1 V - \beta_1\right) = \ldots = \frac{1}{\alpha_{N-1}}\left(\partial_{N-1} V - \beta_{N-1}\right) \tag{18}$$

- $\frac{1}{\alpha_1}(\partial_1 V - \beta_1) \in [\underline{\delta}, \overline{\delta}]$ *a.e.,*
- $V \geq r$.

Proof. This is an immediate consequence of Proposition 1 and of the particular form of $z_n - z_N$ given by (13). $\square$

To shorten notations, denote by $x \cdot y$ the usual inner product of x and y in $\mathbb{R}^{N-1}$, and simply write α (respectively β) instead of $(\alpha_1, \cdots, \alpha_{N-1})$ (respectively $(\beta_1, \cdots, \beta_{N-1})$).

Due to the linearity of system (18), characteristics are simply hyperplanes and the previous conditions can be easily simplified. Although it is a simple application of the method of characteristics we state and prove the following:

Proposition 4 *V satisfies the requirements of Proposition 3 if and only if $V \geq r$, and there exists a function $f : [\alpha_1, 0] \to \mathbb{R}$ such that:*

$$V(p) = f(\alpha \cdot p) + \beta \cdot p, \textit{ for all } p \in \Sigma_{N-1}, \tag{19}$$

and

- *f is convex,*
- *$f'(t) \in [\underline{\delta}, \overline{\delta}]$ for a.e. $t \in [\alpha_1, 0]$.*

Proof. Assume first that V is Lipschitz, satisfies (18) and define $W(p) := V(p) - \beta \cdot p$, then we have

$$\frac{1}{\alpha_1}\partial_1 W = \ldots = \frac{1}{\alpha_{N-1}}\partial_{N-1} W. \tag{20}$$

Let p and p' be two points of Σ_{N-1} such that $\alpha \cdot p = \alpha \cdot p'$. We have:

$$W(p) - W(p') = (p - p') \cdot \int_0^1 \nabla W(p' + t(p - p'))dt.$$

Since (20) implies that $\nabla W(p' + t(p-p'))$ is colinear to α, we obtain $W(p) = W(p')$. Hence $W(p)$ only depends on $\alpha \cdot p$. Hence, if V satisfies the requirements of Proposition 3, there exists a function f such that (19) is satisfied, since the convexity of V is equivalent to that of f. Finally, since $\frac{1}{\alpha_1}(\partial_1 V(p) - \beta_1) = f'(\alpha \cdot p)$, we get the desired characterization. $\square$

5.2.2 Individual rationality

In the previous paragraph, we have shown that feasible contracts are *unidimensional* in the sense that they only depend on $\alpha \cdot p$ (heuristically, this is not surprising since the dimension of the adverse selection problem is $N-1$ whereas we consider only two components in the class of contracts). Roughly speaking, the only instrument of the borrower is the unknown function f of Proposition 4. To make the borrower's program tractable, however, and to determine optimal contracts we still have to express the participation constraint in terms of the function f.

Proposition 5 *Let $f : [\alpha_1, 0] \to \mathbb{R}$ be a Lipschitz function such that $f' \geq \underline{\delta}$, then*

$$\min_{p \in \Sigma_{N-1}} f(\alpha \cdot p) + \beta \cdot p \geq r$$

if and only if $f(\alpha_1) \geq r - \beta_1$.

Proof. Let us write the first condition in the proposition as

$$f(x) + y \geq r, \text{ for all } (x, y) \in \mathcal{C}, \tag{21}$$

where $\mathcal{C} := \{(\alpha \cdot p, \beta \cdot p),\ p \in \Sigma_{N-1}\}$. It is direct to check that $\mathcal{C}$ is the convex hull of the points $(0,0) = (0, P(0))$ and $(\alpha_n, \beta_n)_{n=1,\cdots,N-1} = (\alpha_n, P(\alpha_n)_{n=1,\cdots,N-1}$ where P is the concave quadratic function defined by (16). Defining for all $x \in [\alpha_1, 0]$

$$\Gamma(x) := \min\{y : (x, y) \in \mathcal{C}\},$$

then condition (21) can be rewritten as:

$$f(x) + \Gamma(x) \geq r \text{ for all } x \in [\alpha_1, 0] \tag{22}$$

We claim that Γ is linear, more precisely

$$\Gamma(x) = \frac{P(\alpha_1)}{\alpha_1} x \text{ for all } x \in [\alpha_1, 0]$$

First, since $(0,0)$ and $(\alpha_1, P(\alpha_1))$ belong to $\mathcal{C}$, which is convex, $(x, \frac{P(\alpha_1)}{\alpha_1} x) \in \mathcal{C}$ for all $x \in [\alpha_1, 0]$ so that $\Gamma(x) \leq \frac{P(\alpha_1)}{\alpha_1} x$. Second, let us prove the converse inequality i.e. $y \geq \frac{P(\alpha_1)}{\alpha_1} x$ for all $(x, y) \in \mathcal{C}$. Since it is a linear inequality it is enough to check it at the vertices of $\mathcal{C}$, for $(x, y) = (0, 0)$ or $(x, y) = (\alpha_1, \beta_1)$ there is nothing to check, for the vertices $(\alpha_n, P(\alpha_n))$ the desired inequality follows from the concavity of P and the fact that $\alpha_n \in [\alpha_1, 0]$.

Using (16) and since $\alpha_1 < 0$, we have:

$$\frac{P(\alpha_1)}{\alpha_1} = \gamma + \varepsilon - \theta_N - \varepsilon\alpha_1 \geq \gamma + \varepsilon - \theta_N.$$

But since $f' \geq \underline{\delta} \geq \theta_N - \varepsilon$, we obtain that $f + \Gamma$ is nondecreasing, hence achieves its minimum at $\alpha = \alpha_1$. It thus follows that the condition $f + \Gamma \geq r$ is equivalent to $f(\alpha_1) + \Gamma(\alpha_1) \geq r$, and since $\Gamma(\alpha_1) = P(\alpha_1) = \beta_1$ the proof is achieved. $\square$

5.3 Reformulation of the problem

In our uniform case, for any admissible contract, $(\overline{\omega}(.), C(.))$, the borrower's expected profit whose general expression is given by (6) can be computed as:

$$\Pi(\overline{\omega}, C) = \int_{\Sigma_{N-1}} \left[\frac{1}{4\varepsilon}\overline{\omega}^2 - \frac{1}{2\varepsilon}\left(\sum_{n=1}^{N} \theta_n q_n + \varepsilon\right)\overline{\omega} - C\right]\rho(p)dp + \frac{1}{4\varepsilon}\sum_{n=1}^{N}(\theta_n + \varepsilon)^2 q_n. \tag{23}$$

On the one hand, defining the potential V associated with $(\overline{\omega}(.), C(.))$ by (17) and the function f as in Proposition 4, we have for all p:

$$\overline{\omega}(p) = f'(\alpha \cdot p), \tag{24}$$

$$V(p) = f(\alpha \cdot p) + \beta \cdot p, \tag{25}$$

and, on the other hand,

$$V(p) = \sum_{n=1}^{N-1} p_n(\alpha_n \overline{\omega}(p) + \beta_n) + z_N(\overline{\omega}(p)) + C(p), \tag{26}$$

so that:

$$C(p) = f(\alpha \cdot p) - (\alpha \cdot p) f'(\alpha \cdot p) - z_N(f'(\alpha \cdot p)). \tag{27}$$

Note that C only depends on $\alpha \cdot p$, and slightly abusing notations we shall write in the sequel $C(\alpha \cdot p)$ instead of $C(p)$. Using (27) and replacing z_N by its expression, we get for all $t \in [\alpha_1, 0]$:

$$C(t) = \frac{1}{4\varepsilon} f'(t)^2 - \left[t + \frac{1}{2\varepsilon}(\theta_N + \varepsilon - \gamma) \right] f'(t) + f(t) + \frac{1}{4\varepsilon}(\theta_N - \varepsilon)(\theta_N - \varepsilon - 2\gamma) \tag{28}$$

Let us define the probability measure μ on $[\alpha_1, 0]$ as the image of $\rho(p)dp$ by the linear form $p \mapsto \alpha \cdot p$, that is for every continuous function on $[\alpha_1, 0]$, φ:

$$\int_{\alpha_1}^{0} \varphi(t) d\mu(t) = \int_{\Sigma_{N-1}} \varphi(\alpha \cdot p)\rho(p) dp.$$

Substitution of (24) and (28) in (23) enables us to write $\Pi = \Pi(\overline{\omega}, C)$ as a function of f:

$$\Pi = \int_{\alpha_1}^{0} \left[\left(t + \frac{1}{2\varepsilon}(\theta_N - \sum_{n=1}^{N} \theta_n q_n - \gamma) \right) f'(t) - f(t) \right] d\mu(t) + k, \tag{29}$$

where k is the constant

$$k := \frac{1}{4\varepsilon} \left(\sum_{n=1}^{N} (\theta_n + \varepsilon)^2 q_n - (\theta_N - \varepsilon)(\theta_N - \varepsilon - 2\gamma) \right).$$

At this point it is worth pointing out that Π is linear with respect to f; once again this is very specific to the uniform assumption (quadratic terms in f' vanish) and assumption (11). This linearity will of course dramatically simplify the structure of optimal contracts.

Let us assume for simplicity that μ has a density g_μ[6] and denote by G_μ the cumulative function of μ. Then an integration by parts in (29) yields:

$$\Pi = -f(\alpha_1) + \int_{\alpha_1}^{0} \left[G_\mu(t) - 1 + \left(t + \frac{1}{2\varepsilon}(\theta_N - \sum_{n=1}^{N} \theta_n q_n - \gamma) \right) g_\mu(t) \right] f'(t) dt + k$$

Using Propositions 4 and 5, the borrower's program is equivalent to maximizing the previous quantity (linear in f) in the set of functions

$$\left\{ f : [\alpha_1, 0] \to \mathbb{R},\ f \text{ is convex },\ f' \in [\underline{\delta}, \overline{\delta}] \text{ a.e. and } f(\alpha_1) \geq r - \beta_1 \right\}.$$

Any solution f obviously is such that the *participation constraint is binding at the bottom* : $f(\alpha_1) = r - \beta_1$ and taking $u := f'$ as new unknown (which is natural since $u(\alpha \cdot p) = f'(\alpha \cdot p) = \overline{\omega}(p)$), we have

[6]It can be shown that μ is absolutely continuous with respect to Lebesgue measure, and that its density g_μ is continous if ρ is continuous on the simplex. Moreover, it can be checked that this density vanishes at endpoints $g_\mu(0) = g_\mu(\alpha_1) = 0$. These facts are proved in the appendix.

Proposition 6 *The borrower's program is equivalent to*

$$\max_{u \in K} L(u) \tag{30}$$

where L is the linear form

$$L(u) = \int_{\alpha_1}^{0} h(t)u(t)dt,$$

with

$$h(t) := G_\mu(t) - 1 + \left(t + \frac{1}{2\varepsilon}(\theta_N - \sum_{n=1}^{N} \theta_n q_n - \gamma)\right) g_\mu(t), \tag{31}$$

and

$$K := \left\{u : [\alpha_1, 0] \to [\underline{\delta}, \overline{\delta}],\ u \textit{ is nondecreasing } \right\}.$$

The proof immediately follows from previous computations and propositions 4 and 5. Note that K is convex and compact, say for instance in the weak (or even strong) L^p topology ($p \in (1, +\infty)$). And since h is continuous and bounded, the maximum of (30) is achieved.

5.4 The geometry of optimal solutions

Since the borrower program is a simple linear program, the maximum is achieved in at least one extreme point of K. Moreover Krein-Millman's Theorem (see [28]) and compactness of K in L^p imply that the set of solutions of (30) (which is a *face* of K) is the closed convex hull of the set of extreme points which also solve (30). We shall therefore focus on solutions in the set of extreme points of K.

The next result characterizes extreme points of K: these are the nondecreasing functions which take values only in $\{\underline{\delta}, \overline{\delta}\}$. Although it is quite classical, we give a proof for the sake of completeness. In the sequel, without loss of generality we normalize nondecreasing functions so as to be right-continuous.

Lemma 1 *The set of extreme points of K,* $\mathrm{ext}(K)$ *is given by:*

$$\mathrm{ext}(K) = \{\underline{\delta}\mathbf{1}_{[\alpha_1,t)} + \overline{\delta}\mathbf{1}_{[t,0]}, t \in [\alpha_1, 0]\}.$$

Proof. First it is obvious that if u is of the form $\underline{\delta}\mathbf{1}_{[\alpha_1,t)} + \overline{\delta}\mathbf{1}_{[t,0]}$, u is an extreme point of K.

To prove the converse inclusion let us proceed as follows. Let $u \in \mathrm{ext}(K)$. Define a and b by $a = u(\alpha_1)$, $a + b = u(0)$ ($\underline{\delta} \le a \le a + b \le \overline{\delta}$) and define

$$K_{a,b} := \{v \in K \,:\, v(\alpha_1) = a,\, v(0) = a + b\}.$$

$K_{a,b}$ can also be parametrized with probability measures as follows :

$$K_{a,b} = \{v \,:\, v(t) = a + b \int_{\alpha_1}^{t} d\nu, \text{ for some probability measure } \nu \text{ on } [\alpha_1, 0]\}$$

Hence we write u in the form:

$$u(t) = a + b \int_{\alpha_1}^{t} d\nu. \tag{32}$$

Obviously u is an extreme point of $K_{a,b}$. We claim that this implies that ν in (32) is a Dirac mass δ_t for some $t \in [\alpha_1, 0]$. If not, ν would not be an extreme point of the set of probability measures (see Meyer [19]), hence there would exist probabilities ν_1 and ν_2 with $\nu_1 \neq \nu_2$ and $\nu = \frac{1}{2}(\nu_1 + \nu_2)$. Defining for $i = 1, 2$:

$$u_i(t) = a + b \int_{\alpha_1}^{t} d\nu_i,$$

we would have $u = \frac{1}{2}(u_1 + u_2)$ with $u_1 \neq u_2$ and $(u_1, u_2) \in K^2$, a contradiction with the extremality of u. We have therefore proved that u is of the form $u = a\mathbf{1}_{[\alpha_1, t)} + (a + b)\mathbf{1}_{[t,0]}$ for some $t \in [\alpha_1, 0]$. Finally, it is easy to prove that extremality of u implies that either $t \in (\alpha_1, 0)$ and $(a, a + b) = (\underline{\delta}, \overline{\delta})$, or u is constant with value $\underline{\delta}$ or $\overline{\delta}$. This ends the proof. $\square$

As a first easy consequence, we have:

Corollary 1 *Program (30) admits at least one solution which only takes values $\underline{\delta}$ and $\overline{\delta}$.*

The economic interpretation of this result is that there always exists an optimal menu of contracts with at most two contracts. Moreover, let us define

$$F(t) := \underline{\delta} \int_{\alpha_1}^{t} h(s)\, ds + \overline{\delta} \int_{t}^{0} h(s)\, ds \text{ for all } t \in [\alpha_1, 0],$$

so that the extreme function $u = \underline{\delta}\mathbf{1}_{[\alpha_1, t)} + \overline{\delta}\mathbf{1}_{[t,0]}$ is a solution of (30) if and only if t maximizes F. Finding the solutions of (30) that belong to $\mathrm{ext}(K)$ reduces then to solve the one dimensional problem:

$$\max\{F(t),\ t \in [\alpha_1, 0]\}. \tag{33}$$

Let us denote by A the set of solutions of (33). Since h is continuous, A is a nonempty compact subset of $[\alpha_1, 0]$. The set of solutions of (30), hence of optimal contracts, is fully determined by A as expressed by the following statement:

Proposition 7 *The set of solutions of (30) is the closed convex hull (say in the L^p topology, $p \in (1, +\infty)$) of $\{\underline{\delta}\mathbf{1}_{[\alpha_1,t)} + \overline{\delta}\mathbf{1}_{[t,0]},\ t \in A\}$.*

If the set A where F achieves its maximum is not reduced to a singleton, say $(t, t') \in A^2$ with $t < t'$ then both step functions $\underline{\delta}\mathbf{1}_{[\alpha_1,t)} + \overline{\delta}\mathbf{1}_{[t,0]}$ and $\underline{\delta}\mathbf{1}_{[\alpha_1,t')} + \overline{\delta}\mathbf{1}_{[t',0]}$ are solutions of (30), and any convex combination of those step functions is also optimal. Taking convex combinations amounts to adding an intermediate third value to the function. In the case where a menu with three or more contracts yields the same profit as a simpler menu, the borrower is more likely to offer the simplest one (remember we abstract from writing cost). For simplicity, we only consider those simplest menus.

Since h is explicitly given by (31), we may be more precise. Note first that F is differentiable and its derivative can be computed explicitly:

$$F'(t) = (\underline{\delta} - \overline{\delta})h(t) = (\underline{\delta} - \overline{\delta}) \left(G_\mu(t) - 1 + \left(t + \frac{1}{2\varepsilon}(\theta_N - \sum_{n=1}^{N} \theta_n q_n - \gamma) \right) g_\mu(t) \right). \tag{34}$$

Hence, since $g_\mu(0) = g_\mu(\alpha_1) = 0$, $G_\mu(0) = 1$ and $G_\mu(\alpha_1) = 0$ we have $F'(\alpha_1) = (\overline{\delta} - \underline{\delta}) > 0$, and $F'(0) = 0$. This proves indeed that $\alpha_1 \notin A$, i.e. the constant function $u \equiv \overline{\delta}$ is not a solution of (30). Note also that if the condition

$$\theta_N - \sum_{n=1}^{N} \theta_n q_n \leq \gamma \tag{35}$$

is satisfied, then F is increasing, hence $A = \{0\}$ and the only optimal menu of contracts is a single contract such that $\overline{\omega} \equiv \underline{\delta}$.

Proposition 8

- *The constant function $u \equiv \overline{\delta}$ is not a solution of (30),*
- *If (35) is satisfied, then (30) admits as unique solution the constant function $u \equiv \underline{\delta}$.*
- *If, in addition, g_μ is nonincreasing in a neighborhood of 0[7], and that (35) does not hold then $u \equiv \underline{\delta}$ is not a solution of (30), hence every solution of (30) takes at least two values.*

[7]It can be checked that these assumptions are satisfied if ρ is smooth and strictly positive. These fact are proved in the appendix.

Proof. Only the last statement has not been established yet. Assume then that:

$$\theta_N - \sum_{n=1}^{N} \theta_n q_n > \gamma.$$

It is enough to prove that F does not achieve its maximum at 0. For $\alpha_1 < t < 0$ compute

$$\frac{F'(t)}{(\bar{\delta} - \underline{\delta}) g_\mu(t)} = \left(\frac{1 - G_\mu(t)}{g_\mu(t)} - \left(t + \frac{1}{2\varepsilon} (\theta_N - \sum_{n=1}^{N} \theta_n q_n - \gamma) \right) \right)$$

Since

$$0 \le \frac{1 - G_\mu(t)}{g_\mu(t)} = \frac{\int_t^0 g_\mu}{g_\mu(t)},$$

our assumption implies that for t close to 0

$$0 \le \frac{1 - G_\mu(t)}{g_\mu(t)} \le -t \text{ hence } \lim_{t \to 0^-} \frac{F'(t)}{(\bar{\delta} - \underline{\delta}) g_\mu(t)} < 0.$$

This implies that $F'(t)/g_\mu(t) < 0$ for t close to 0 so that $0 \notin A$. This ends the proof. $\square$

5.5 Economic interpretation

In this paragraph, we summarize and interpret the results obtained in the above example, paying special attention to bunching.

Our first result (proposition 4) is that the borrower is able to discriminate epistemic types of lenders in only one dimension. The only dimension in which screening may occur can be interpreted in terms of the expected return of the project $\sum_{n=1}^{N} \theta_n p_n$. More precisely, we have shown that two epistemic types p and p' such that $\alpha \cdot p = \alpha \cdot p'$ are offered the same contract and a simple computation shows that this condition is equivalent to $\sum_{n=1}^{N} \theta_n p_n = \sum_{n=1}^{N} \theta_n p'_n$. Note that this property only follows from the incentive compatibility constraint. Unidimensional discrimination reflects the fact that perfect screening is ruled out by dimensionality considerations. Indeed, in our model, the type space is $N-1$ dimensional while the instrument space is unidimensional. Hence the dimensionality of the problem implies that perfect discrimination is impossible : *bunching of the first type* occurs (see [25]).

Second, Proposition 4 implies that if a menu $(\overline{\omega}(.), C(.))$ is incentive compatible then $\overline{\omega}(.)$ is nondecreasing in the reduced type $\alpha \cdot p$, this feature allows a natural interpretation. The more optimistic or confident the lender (i.e. the higher $\alpha \cdot p$) the higher $\overline{\omega}$, i.e. the more often monitoring occurs. Moreover, Proposition 5 expresses that the participation constraint is binding at the bottom, i.e. the most pessimistic lender (with epistemic type assigning probability one to θ_1) receives no informational rent.

Third, since the profit is linear with respect to the instrument $\overline{\omega}$ with our specifications, the borrower's program turns out to be of linear programming type. Solving such a problem amounts to finding extreme points of the admissible set. This argument together with Lemma 1 implies that there always exist very degenerate optimal menus. Our second important result indeed establishes the existence of optimal menus with at most two contracts. Put differently, optimal menus of contracts always exist where the borrower offers at most two contracts. Therefore, our specific case highlights an extreme case of bunching of the *second type,* in the terminology of [25].

Finally, Proposition 8 gives a necessary and sufficient condition under which it is optimal to offer a single contract $\overline{\omega} \equiv \underline{\delta}$ (complete bunching). Condition (35) means that the difference between the highest expected return θ_N and the expected return from the borrower's viewpoint $\sum \theta_n q_n$ is less than the monitoring cost γ. If (35) holds, then offering the *low* contract $\overline{\omega} \equiv \underline{\delta}$ is optimal. Put differently, if opinions are not too diverse, no discrimination occurs and the monitoring set is as small as possible. If (35) is violated, then optimal menus include both the *low* $(\overline{\omega} \equiv \underline{\delta})$ and the *high* contract $(\overline{\omega} \equiv \overline{\delta})$. In that case the epistemic type space is split into two regions: optimistic types with high expected value of the return who get the high contract, and pessimistic types with low expected value of the return who get the low contract.

6 Appendix

6.1 Proof of Theorem 1

First, note that obviously, $\mathcal{A}$ defined in (10) is nonempty and the value of program (8) is finite.
Second, note that the set

$$B := \{Z(\overline{\omega}) : \overline{\omega} \in [\underline{\delta}, \overline{\delta}]\}$$

is compact, hence there exists a constant M such that for every $(\overline{\omega}, V) \in \mathcal{A}$, V is M-Lipschitz. Let us write the functional J as follows:

$$J(\overline{\omega}, V) := \int_{\Sigma_{N-1}} \left(V(p) - p \cdot \nabla V(p)\right)) \rho(p) dp - \int_{\Sigma_{N-1}} h(\overline{\omega}(p)) \rho(p) dp,$$

where

$$h(\overline{\omega}) := z_N(\overline{\omega}) + \sum_{n=1}^{N} q_n \int_{\overline{\omega}}^{b_n} (\omega - \overline{\omega})\, g_n(\omega)\, d\omega.$$

Let $(\overline{\omega}_n, V_n)_n \in \mathcal{A}^{\mathbb{N}}$ be a minimizing sequence of (8). Since $h(\overline{\omega}_n)$ is uniformly bounded and ∇V_n is uniformly bounded (bounded for the L^∞ norm), we deduce that $\int V_n \rho$ is bounded, which, using the fact that the V_n are uniformly Lipschitz, implies that V_n is uniformly bounded. Using Ascoli's Theorem, and taking a subsequence if necessary, again denoted V_n, we may assume that V_n converges uniformly to some function V. Since the V_n are convex functions, we also have a.e. convergence of ∇V_n to ∇V (see Rockafellar [27]). Obviously V is convex, $V \geq r$, $\nabla V \in B$, a.e. and

$$\lim_n \int_{\Sigma_{N-1}} \left(V_n - p \cdot \nabla V_n\right)) \rho = \int_{\Sigma_{N-1}} \left(V - p \cdot \nabla V\right)) \rho. \tag{36}$$

Since $\nabla V \in B$, a.e. then for all $p \in \Sigma_{N-1}$, $\partial V(p) \cap B \neq \emptyset$ (where ∂V stands for the subdifferential of V). Define then for all p the two set-valued maps:

$$p \mapsto \Phi_V(p) := \{\overline{\omega} \in [\underline{\delta}, \overline{\delta}] : Z(\overline{\omega}) \in \partial V(p)\},$$

and

$$p \mapsto \Psi_V(p) := \{\overline{\omega} \in \Phi_V(p) :\ h(\omega) = \max_{\overline{\omega}' \in \Phi_V(p)} h(\omega')\}.$$

Notice that both Φ_V and Ψ_V are nonempty compact valued and have a closed graph. Let $\overline{\omega}$ be a measurable selection of Ψ_V (cf. Ekeland-Temam [10] Th.1.2 ch VIII for the existence of such a measurable selection). By construction $(\overline{\omega}, V) \in \mathcal{A}$ and one can prove (see [7] for details) that

$$\int_{\Sigma_{N-1}} h(\overline{\omega}(p)) \rho(p) dp \geq \mathrm{limsup}_n \int_{\Sigma_{N-1}} h(\overline{\omega}_n(p)) \rho(p) dp \tag{37}$$

Finally, (36) and (37) prove that $(\overline{\omega}, V)$ is a solution of (8).

6.2 Computation and properties of the density g_μ

In the sequel for $x := (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}$ we shall simply write $x := (x_1, x')$. Taking a test-function $\varphi \in C^0([\alpha_1, 0], \mathbb{R})$, using the change of variables $p = (p_1, p') \mapsto (\alpha \cdot p, p')$ and Fubini's formula we get:

$$\int_{\Sigma_{N-1}} \varphi(\alpha \cdot p)\rho(p)dp = -\int_{\alpha_1}^{0} \frac{\varphi(t)}{\alpha_1} \left(\int_{\Sigma_t} \rho(\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p') dp' \right) dt,$$

where:

$$\Sigma_t := \left\{ p' \in \Sigma_{N-2} : (\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p') \in \Sigma_{N-1} \right\}.$$

This proves that μ is absolutely continuous with respect to Lebesgue's measure on $[\alpha_1, 0]$ and admits the density:

$$g_\mu(t) = -\frac{1}{\alpha_1} \int_{\Sigma_t} \rho(\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p') dp'.$$

g_μ is trivially positive since α_1 is negative. We shall write g_μ in a more compact way as:

$$g_\mu(t) = \int_{\Sigma_t} \sigma(t, p') dp' \text{ with } \sigma(t, p') = -\frac{1}{\alpha_1} \rho(\frac{t}{\alpha_1} - \sum_{n=2}^{N-1} \frac{\alpha_n}{\alpha_1} p_n, p').$$

If ρ is continuous so is g_μ (use Lebesgue dominated convergence theorem and the fact that, if t_n converge to t, the indicator function of Σ_{t_n} converges to that of Σ_t a.e. with respect to the $N-2$ dimensional Lebesgue's measure). Since $\Sigma_0 = \Sigma_{\alpha_1} = 0_{\mathbb{R}^{N-2}}$, we have $g_\mu(0) = g_\mu(\alpha_1) = 0$. Moreover, if ρ is strictly positive on the interior of Σ_{N-1}, so is g_μ on $(\alpha_1, 0)$.

Lemma 2 *If $\rho \in C^1(\Sigma_{N-1}, \mathbb{R})$ and ρ is strictly positive on Σ_{N-1}, then there exists a neighborhood of 0 on which g_μ is nonincreasing.*

Proof. First, note that Σ_t can be written as

$$\Sigma_t = \left\{ p' \in \Sigma_{N-2} : \sum_{n=2}^{N-1} \alpha_n p_n \geq t \text{ and } \sum_{n=2}^{N-1} (1 - \frac{\alpha_n}{\alpha_1}) p_n \leq 1 - \frac{t}{\alpha_1} \right\}.$$

For t sufficiently close to 0, the requirements $p' \in \mathbb{R}_+^{N-2}$ and $\sum \alpha_n p_n \geq t$ imply $\sum_{n=2}^{N-1} (1 - \frac{\alpha_n}{\alpha_1}) p_n \leq 1 - \frac{t}{\alpha_1}$ and $\sum_{n=2}^{N-1} p_n \leq 1$ so that there exists t_0 such that for $t_0 \leq t < 0$

$$\Sigma_t = \left\{ p' \in \mathbb{R}_+^{N-2} : \sum_{n=2}^{N-1} \alpha_n p_n \geq t \right\} = -tA_0, \tag{38}$$

where A_0 is the (fixed) set

$$A_0 := \left\{ p' \in \mathbb{R}_+^{N-2} : \sum_{n=2}^{N-1} \alpha_n p_n \geq -1 \right\}.$$

Hence, for $t \in (t_0, 0)$, $g_\mu(t)$ can be computed as

$$g_\mu(t) = (-t)^{N-2} \int_{A_0} \sigma(t, -tq) dq.$$

Differentiating this expression yields:

$$\dot{g_\mu}(t) = -(N-2)(-t)^{N-3} \int_{A_0} \sigma(t, -tq) dq$$

$$+(-t)^{N-2} \int_{A_0} (\partial_t \sigma(t, -tq) - \partial_q \sigma(t, -tq) \cdot q) dq$$

so that $\dot{g_\mu}(t) < 0$ for sufficiently close to 0. $\square$

References

[1] M. Armstrong, Mulitproduct nonlinear pricing, *Econometrica*, **64**(1), 1996, 51-75.

[2] S. Basov, Hamiltonian Approach to Multidimensional Screening Problem, *J. of Math. Econ.*, **36**, 2001, 77-94.

[3] P. Bolton and D.S. Sharfstein, A Theory of Predation Based on Agency Problems in Financial Contracting., *Amer. Econ. Rev.*, 80, 1990, 93-106.

[4] J. Boyd and B.D. Smith, The Equilibrium Allocation of Investment Capital in the Presence of Adverse Selection and Costly State Verification, *Econ Theory*, **3(3)**, 1993, 427-51.

[5] J. Boyd and B.D. Smith, How Good Are Standard Debt Contracts? Stochastic versus Nonstochastic Monitoring in a Costly State Verification Environment, *J. Bus.*, **67(4)**, 1994, 539-61.

[6] J. Boyd and B.D. Smith, The Use of Debt and Equity in Optimal Financial Contracts, *J. of Finan. Intermediation*, **8**, October 1999, 270-316.

[7] G. Carlier, A general existence result for the principal-agent problem with adverse selection, *J. of Math. Econ.*, **35**(1), 2001, 129-50.

[8] G. Carlier, Duality and existence for a class of mass transportation problems and economic applications, to appear in *Adv. In Math. Economics.*

[9] C. Chang., The Dynamic Structure of Optimal Debt Contracts, *J. of Econ. Theory*, **52**, 1990, 68-86

[10] I. Ekeland, R. Temam, *Convex Analysis and Variational Problems*, North-Holland (1972).

[11] D. Gale and M. Hellwig, Incentive-Compatible Debt Contracts: The One-Period Problem, *Rev. of Econ. Stud.*, **52(4)**, 1985, 647-663.

[12] Gragg and Makiel, *Expectations and the structure of prices*, Chicago Universitry Press, 1982.

[13] F. Gul, A Comment on Aumann's Bayesian View, *Econometrica*, 66(4), 1998, 923-27.

[14] M. Harrison and D. Kreps., Speculative Investor behavior in a stock market with heterogenous expectation, *Quart. J. Of Econ.*, **92**, 1978, 323-36

[15] B. Jullien, Participation Constraints in Adevrse Selction Models., *J. of Econ. Theory*, **93(1)**, 2000, 1-47.

[16] J.A. Kahn, Debt, Asymmetric Information, and Bankruptcy, Rochester Center for Economic Research, 1990, working paper no. 96, August.

[17] S. Krasa and A. Villamil, Monitoring the Monitor: An Incentive Structure for a Financial Intermediary, *J. of Econ Theory*, **57(1)**, 1992, 197-221.

[18] S. Krasa and A. Villamil, A Theory of Optimal Bank Size, *Oxford Econ.Pap.*, **44(4)**, 1992, 725-49.

[19] P.A. Meyer *Probabilités et Potentiel*, Hermann, 1966.

[20] R. McAfee and J. McMillan, Multidimensional Incentive Compatibility and Mechanism Design, *J. of Econ. Theory*, **46(2)**, 1988, 335-54.

[21] D. Mookherjee and I. Png, Optimal Auditing, Insurance, and Redistribution, *Quart. J. of Econ.*, **104(2)**, 1989, 399-415.

[22] S. Morris, The Common Prior Assumption in Economic Theory, *Econ. Philos.*,.**53**, 1995, 227-253.

[23] L. Renou, Beliefs about beliefs and endogeneous formation of a multi-lender coalition in a CSV model, Mimeo, http://iue.it/Personal/Researchers/renou, 2002.

[24] J.C. Rochet, The taxation Principle and Multi-Time Hamilton-Jacobi Equations, *J. of Math. Econ.*, **14**, 1985, 113-128.

[25] J.C. Rochet and P. Chone, Ironing, Sweeping and Multidimensional Screening, *Econometrica*, **66**(4), 1998, 783-826.

[26] J.C. Rochet and L. Stole, The Economics of multidimensional screening, presented in Seattle, Washington at the World Congress of the Econometric Society, August 15, 2000.

[27] R. T. Rockafellar, *Convex Analysis*, Princeton University Press (1970).

[28] H.L. Royden, *Real Analysis*, Third Edition, Prentice Hall (1988).

[29] R. Townsend, Optimal Contracts and Competitive Markets with Costly State Verification, *J. of Econ. Theory*, **21(2),** 1979, 265-93.

[30] R. Townsend, Information Constrained Insurance: The Revelation Principle Extended, *J. of Monet. Econ.*, **21(3),** 1998, 411-450.

[31] H. Varian., Divergence in Opinion in Financial Markets: a Note., *J. of Finance*, **40**, 1985, 309-17

[32] S.D. Williamson, Costly Monitoring, Financial Intermediation and Equilibrium Credit Rationing, *J. of Monet. Econ.*, **18(2)**, 1986, 159-179.

[33] S.D. Williamson., "Costly Monitoring, Loan Contract and Equilibrium Credit Rationing," *Quart. J. Econ.*, **102(1),** 1987, 135-145.