Seasonal Harvests and Commodity Prices: Some analytical results

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Abstract:
This paper characterises the analytical solution to a two period competitive storage model of commodity prices with periodic harvest distributions, extending the analytical results in Chambers and Bailey (1996). We present a sufficient condition is for the general model such that stocks are never depleted in the harvest period. The partial analytical solution for the price function in the harvest period is derived under the assumption of a linear demand function. This solution provides the conditions under which stock-outs in the nonharvest period can become an absorbing state.

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Key Words: Competitive storage model; periodic harvests, heterogeneous distributions.

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1. Introduction

Recent extensions to the competitive storage model of commodity prices of (Deaton and Laroque (DL), (1992)) have incorporated periodic harvest distributions (Chambers and Bailey (CB) 1996, Osborne, 2003) to allow both for different harvest sizes across time and periods in which a harvest does not occur, but the commodity is traded and consumed. It is difficult to characterise the resulting equilibrium price functions since analytical solutions are not available and so the literature has relied on numerical solutions. This note presents analytical results which characterise the equilibrium price functions at low levels of stock in a two period model. The periodicity is represented as the harvest occurring every second period. Firstly, a sufficient condition is derived so that stock-outs never occur in a period when there is a harvest. Secondly, the partial analytical solution for the price function in the harvest period is derived when the demand function is assumed to be linear. This implies that a stock-out can reoccur in every nonharvest period even at the maximum possible harvest. Hence, depending on the parameters of the model, stockouts in the nonharvest period can become an absorbing state. This property of the model arises from the common assumption of a linear demand function. Section 2 briefly presents the competitive storage model of Chambers and Bailey (1996), upon which the analysis is based. Section 3 presents the sufficient condition such that storage is always profitable in the harvest period. Section 4 presents the analytical solution and other results for the linear demand case. Section 5 concludes.

2. The Competitive Storage Model with Periodic Disturbances

Following CB (1996) assume that time periods *(*denoted *t)* can be grouped into "epochs", with different time period types (denoted *i*) within each epoch. The simplest representation is to assume two time periods within each epoch, one when the harvest occurs (*i*=*h*, the *harvest* period) which is followed by a period with no harvest, (*i*=*n*, the *nonharvest* period). An equally valid representation is a "large" harvest followed by a "small" harvest. Uncertainty arises from the harvest realisation (z) and from demand shocks (v). These two elements cannot be separately identified and are denoted $w^i \equiv z + v$. The following assumptions are made regarding the relative production in each season, consumer behaviour and the capital market:

(i) w^h has compact support $W^h \equiv \left[\underline{w}^h, \overline{w}^h\right]$ and cumulative distribution function Q^h, with expectation denoted $\omega^h \equiv \int_{W^h} w^h Q^h\left(dw^h\right)$.

(ii) w^n has compact support $W^n \equiv \left[\underline{w}^n, \overline{w}^n\right]$ and cumulative distribution function Q^n, with expectation denoted $\omega^n \equiv \int_{W^n} w^n Q^n\left(dw^n\right)$.

(iii) $-\infty < \underline{w}^n < \underline{w}^h$ and $\overline{w}^n < \overline{w}^h < \infty$.

(iv) Q^h exhibits First Order Stochastic Dominance over Q^n, (although this does not require that the supports of the two distributions do not overlap).

(v) Consumer demand is represented by the function $D(p)$ with the inverse denoted $P(q) \equiv D^{-1}(q)$.

(vi) The range of the demand function $D(p)$ is bounded above, such that $D(0)$ is defined, and that the range of $P(q)$ is $\left[\underline{w}^n, +\infty\right[$.

(vii) The capital market interest rate is *r*, and the rate of wastage caused by storage is δ; the discounted cost of storage θ is such that $0 < \theta \equiv (1-\delta)/(1+r) < 1$.

Denote the equilibrium spot price as p_t and the equilibrium price function in any time period of type *i*, for *i* = *h*,*n*, as $f^i(x_t)$, where x_t is current stock of the commodity given by the harvest and any inventory carried into the period. Assume each equilibrium price function is non-negative, non-increasing and continuous. Consumers and risk neutral speculators jointly determine demand for a commodity, although consumers behave passively in the market. Speculators form a (rational) expectation of price in the following period of type *j*, for *j* = *n*,*h*. If this expected price is 'low', then all stock is sold to consumers in *t*, and speculators have zero demand for inventory. A "stockout" occurs in the period, and the spot price is given by the inverse demand function, $p_t = f^i(x_t) = P(x_t)$. If the expected price is "high" i.e. greater than the price from selling all current stock to consumers, then speculators demand the commodity for storage. In this case, an amount $D\left(f^i(x_t)\right)$ is sold to consumers, the (positive) inventory level is then $x_t - D\left(f^i(x_t)\right)$ and the available stock in the next period is $x_{t+1} \equiv w_{t+1}^j + (1-\delta)\left(x_t - D\left(f^i(x_t)\right)\right)$. The level of inventory is chosen to equate the spot price and the discounted expectation of price in the next time period given by

$$p_t = f^i(x_t) = \theta \int_{W^j} f^j\left[w_{t+1}^j + (1-\delta)\left(x_t - D\left(f^i(x_t)\right)\right)\right] Q^j(dw).$$

CB prove the existence of unique stationary price functions f^i, which are continuous, non-negative, and non-increasing that satisfy

$$p_t = f^i(x) = \max\left\{\theta \int_{W^j} f^j\left[w + (1-\delta)\left(x - D\left(f^i(x)\right)\right)\right] Q^j(dw), P(x)\right\}. \quad (1)$$

Thus, in equilibrium, the spot price is either the expected future price or the price given by the demand curve, whichever is greater[2].

Let p^{*j} denote the discounted expected price if no inventories are carried over from period *i*, which is given by

$$p^{*j} = \theta E f^j(w) \equiv \theta \int_{W^j} f^j(w)\, Q^j(dw). \tag{2}$$

Following DL (1992), p^{*j} is "the current price at which, with no inventory demand, a unit held into the next period would make zero expected profit", and is the price at which the solution switches between the two possible regimes given in (1). The equilibrium price function is above the demand function when the price from selling all current stock is less than this critical price i.e.

$$p_t = f^i(x_t) > P(x_t) \text{ when } P(x_t) < p^{*j}.$$

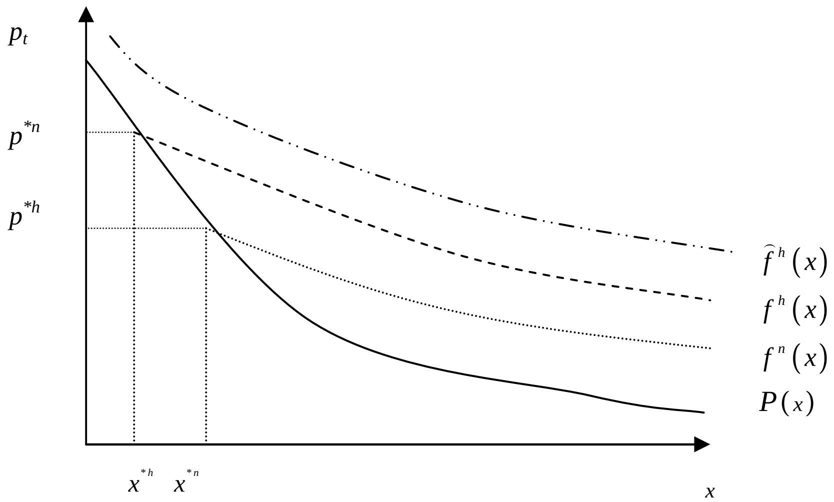
In this case, carrying inventories is profitable, $p_t < p^{*j}$, the equilibrium price function is given by the first term in (1), and $p_{t+1} = f^j(x_{t+1})$. Alternatively, $p_t = f^i(x_t) = P(x_t)$, when $P(x_t) > p^{*j}$, in which case no inventories are held, there is a stock-out and $p_{t+1} = f^j(w_{t+1})$. In summary, as the amount of inventory carried between the periods varies, the equilibrium price function $f^i(x)$ switches between the discounted expected price (positive inventories) and the demand function (zero inventories), at the critical price p^{*j}. This critical price also defines the maximum value of the *stock* for which the equilibrium price function is the demand function. For values of the stock below x^{*i}, where $x^{*i} \equiv P^{-1}(p^{*j})$, the period *i* price function is given by the demand curve.

Figure 1 portrays a possible solution to the model (similar to Figure 2 in CB). Above the critical prices p^{*n} and p^{*h}, the functions $f^h(x)$ and $f^n(x)$, are the demand function, respectively. Below these prices the functions lie above the demand function. In the example shown, the functions do not intersect and $f^h(x) \geq f^n(x)$. CB (1996) characterise the relationship between the critical prices in each period type. By strengthening the assumption of FOSD to one of nonoverlapping supports of the distributions (i.e. $-\infty < \underline{w}^n < \overline{w}^n < \underline{w}^h < \overline{w}^h < \infty$), they show that $p^{*n} > p^{*h}$ i.e. the minimum price at which it is not profitable to carry inventory into the nonharvest period (p^{*n}) is greater than the minimum price at which is not profitable to carry inventory into the harvest period, (p^{*h}). However they assert that even this result is

[2] In the terminology of CB: harvest corresponds to odd and nonharvest corresponds to even. The results of CB apply for any number of period types within each epoch.

Figure 1

Possible solution to two period model with periodic disturbances



insufficient to identify the relationship between the price functions themselves, and that any further analytical results would require additional assumptions on the disturbances and/or the demand function. The remainder of this note addresses this issue.

3. A Sufficient Condition for Positive Storage in the Harvest Period.[3]

In this section, Proposition 1 presents a sufficient condition for the *harvest* period price function to lie above the demand curve for all values of the stock. Then, stockouts never occur in the harvest period and therefore inventory is always carried into the nonharvest period.[4]

Proposition 1.

$P\left(\underline{w}^h\right) < \theta E^n\left[P\left(w^n\right)\right]$ is a sufficient condition for $f^h(x) > P(x)$, over all x in $\left[\underline{w}^h, +\infty\right[$.

□

Proof. See Appendix.

This condition requires that the price given by the inverse demand curve at the worst harvest must be less than the discounted expectation of the inverse demand curve over the nonharvest period distribution. If satisfied, $f^h(x) = P(x)$ is never an optimal solution in the harvest

[3] The stationarity of the functions allows us to omit the time subscript in the remainder of the paper.
[4] Osborne (2003) shows that a stockout is impossible in a period prior to one where there is no harvest or demand uncertainty, which trivially implies that $f^h(x)$ is never the demand function. Thus this analysis is most applicable when the distribution of the harvests is heterogeneous across periods i.e. small and large harvests alternate. Then the conditions under which a stockout occurs in the harvest period is non-trivial.

period and inventory, $x - D\left(f^h(x)\right)$, is always positive. Then, the price function is given by the discounted expectation of next period price

$$f^h(x) = \theta \int_{W^n} f^n\left[w + (1-\delta)\left(x - D\left(f^h(x)\right)\right)\right] Q^n(dw). \qquad (3)$$

This result allows us to order the price functions over a restricted range of the stock. For values of the stock *x*, where $x \in [\underline{w}^n, x^{*n}]$ (recall $x^{*n} \equiv P^{-1}(p^{*h})$), the price function in the nonharvest period is the demand function, $f^n(x) = P(x)$. Hence, if the condition in Proposition 1 is satisfied, for any given value of *x* within that range, $f^h(x) > f^n(x)$; the harvest period function is always greater than the nonharvest period function. In Figure 1, $\hat{f}^h(x)$ is a possible harvest period price function if Proposition 1 is true i.e. $\hat{f}^h(x) > P(x)$ for all *x*, and hence $\hat{f}^h(x) > f^n(x)$ when $f^n(x) = P(x)$. However, this ordering of the price functions levels of the stock below x^{*n}, does not imply that spot prices are higher in either the harvest or nonharvest period, since the level of stock will not be constant over time. Neither does it imply that this ordering will be maintained at values of the stock greater than x^{*n}, where it is possible that the functions will cross.[5]

The characterisation of the equilibrium price functions implied by Proposition 1, extends the results in CB. However their assumption of nonoverlapping supports of the distributions implies $P\left(\underline{w}^h\right) < P\left(\overline{w}^n\right)$, which is clearly more stringent than $P\left(\underline{w}^h\right) < E^n\left[P\left(w^n\right)\right]$. Note however that since $\theta < 1$, for a sufficiently small value of θ the condition in Proposition 1, $P\left(\underline{w}^h\right) < \theta E^n\left[P\left(w^n\right)\right]$, is *more* demanding than $P\left(\underline{w}^h\right) < P\left(\overline{w}^n\right)$. Conversely, for θ close to 1 it is *less* demanding. Therefore, although it is not possible to compare the restrictiveness of the conditions in general, there are values of θ for which Proposition 1 is satisfied under weaker conditions than the assumption in CB.

In the case where a harvest occurs every period, DL prove in Theorem 2 that the limit distribution of inventories has a compact support and price follows a renewal process when the condition $P(\overline{w}) < \theta E P(w)$ is satisfied. The proof requires that inventories are positive in some time periods but become zero in finite time i.e. a stock-out occurs with probability equal to one and the inventory does not become infinite. In the periodic case, the analogous condition for such a proof is $P\left(\overline{w}^h\right) < \theta E^n P\left(w^n\right)$. This ensures that at the maximum harvest, inventories are non-zero between harvest and nonharvest periods, but are

[5] Numerical analysis (not shown here) reveals that even if Proposition 1 is true, the price functions will meet for some values of the parameters at stock levels greater than x^{*n}.

depleted with probability equal to one in the nonharvest period. Given $P\left(\bar{w}^h\right) < P\left(\underline{w}^h\right)$, because $P(.)$ is a decreasing function, the condition is satisfied trivially when the condition in Proposition 1 holds. Thus the solution to the heterogeneous harvest case shares important properties of the simpler nonperiodic case.

4. Linear Demand Case

The result in Proposition 1 provides a characterisation of the price functions in the general case. However, as suggested in CB, it is necessary to make additional assumptions about functional form to gain further knowledge of the price functions. In this section the partial analytical solution to (1) for the harvest period is presented, based on the assumption that the demand function is linear its argument.

Proposition 2.

Assume that the inverse demand function is given by $P(x) = ax + b$, where $a < 0$, $b > 0$, and $\bar{w}^h < -b/a$. From (1), the nonharvest period equilibrium price function is the demand function, for all $x \in [\underline{w}^n, x^{*n}]$, i.e. $p_t = f^n(x) = P(x)$.

(i) Consider a level of stock in the harvest period $x \in [x^{*h}, \tilde{x}]$ such that $x - D\left(f^h(x)\right) > 0$, and $w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right) \le x^{*n}$, for all $w^n \in W^n$. Then the equilibrium price function in the harvest period is linear and is given by the expectation of the demand function over the nonharvest period distribution, discounted by θ, i.e.

$$f^h(x) = \theta \int_{W^n} P\left[w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right)\right] Q^n\left(dw^n\right),$$

which can be written as

$$f^h(x) = \alpha x + \beta, \qquad (4)$$

where

$$\alpha = \frac{\theta(1-\delta)}{1+\theta(1-\delta)} a > a\,; \; \beta = \frac{\theta}{1+\theta(1-\delta)} a\omega^n + \frac{\theta + \theta(1-\delta)}{1+\theta(1-\delta)} b\,;$$

(ii) The maximum level of the stock in the harvest period, $\tilde{x}$, such that a stockout always occurs in the nonharvest period i.e. $w^n + (1-\delta)\left(\tilde{x} - D\left(f^h(\tilde{x})\right)\right) \le x^{*n}$, for all $w^n \in W^n$, is defined as

$$\tilde{x} = \frac{1}{a}\left\{\left(\theta + \frac{1}{1-\delta}\right)\left(p^{*h} - P\left(\bar{w}^n\right)\right) - \left(b - \theta P\left(\omega^n\right)\right)\right\}. \qquad (5)$$

(iii) The maximum harvest leads to a stockout in the nonharvest period (i.e. $\bar{w}^h < \tilde{x}$), if

$$P\left(\overline{w}^h\right)-\theta P\left(\omega^n\right)>\left(\theta+\frac{1}{1-\delta}\right)\left(p^{*h}-P\left(\overline{w}^n\right)\right) \tag{6}$$

□

Proof. See Appendix.

The analytical solution for the harvest period price function given in (4) is defined over levels of the harvest period stock such that a stockout does not occur in the harvest period. All of the inventory carried into the nonharvest period is depleted (a stock-out always occurs in the non-harvest), irrespective of the realisation of the demand shock w^n. The maximum level of the harvest period stock for which (4) applies, $\tilde{x}$, is defined in (5), and is determined by the parameters of the demand function, the wastage rate δ, and the distribution of nonharvest period disturbances. The term $p^{*h}-P\left(\underline{w}^n\right)$ is the difference between the price at which a unit of inventory would make zero expected profit and the maximum possible price in the nonharvest period. The second term, $b-\theta P\left(\omega^n\right)$, is the difference between the intercept of the linear inverse demand function and the inverse demand curve at the mean of the nonharvest period disturbances, discounted by the storage cost[6].

Furthermore, if the condition in (6) is satisfied, then once a stockout occurs in the nonharvest period, and even if the maximum harvest occurs in the next harvest period, there will be a stockout in the nonharvest period. Thus stockouts become permanent in the nonharvest period. In this case, price follows a renewal process in each harvest "cycle" because stocks are depleted in every nonharvest period. In addition since $\tilde{x}$ is an increasing function of δ, a higher wastage rate increases the range of harvest over which this "absorbing state" characteristic dominates.

To be more precise, whenever $\underline{w}^h < x^{*h}$, and for the lowest values of the harvest period stock, i.e $x\in\left[\underline{w}^h, x^{*h}\right]$, the solution is the demand curve, $f^h(x)=P(x)$. For $x\in\left[x^{*h},\tilde{x}\right]$, the $f^h(x)$ is the discounted expectation of the demand curve given in (4), and there is a stockout in the nonharvest period. Finally, when $x>\tilde{x}$, the harvest period solution is the discounted expectation of the nonharvest period function given by (3), the function in the nonharvest period is not the demand curve, and we do not know how to characterise the analytical solution for $f^h(x)$. Figure 2a presents the numerical solution to the model where the parameter values are chosen to show these three regimes of the equilibrium price function (δ

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[6] In the linear case $\int_{W^n} P\left(w^n\right)Q^n\left(dw^n\right)=P\left(\int_{W^n} w^n Q^n\left(dw^n\right)\right)=P\left(\omega^v\right)$, and this simplification is used in Proposition 3.

Figure 2a

Numerical solution with linear demand function

$a = -1.5$, $b = 5$, $w^n \sim U[0.4, 0.85]$, $w^h \sim U[0.8, 1.25]$, $\delta = 15\%$, $r = 3\%$, $\theta = 0.825$.

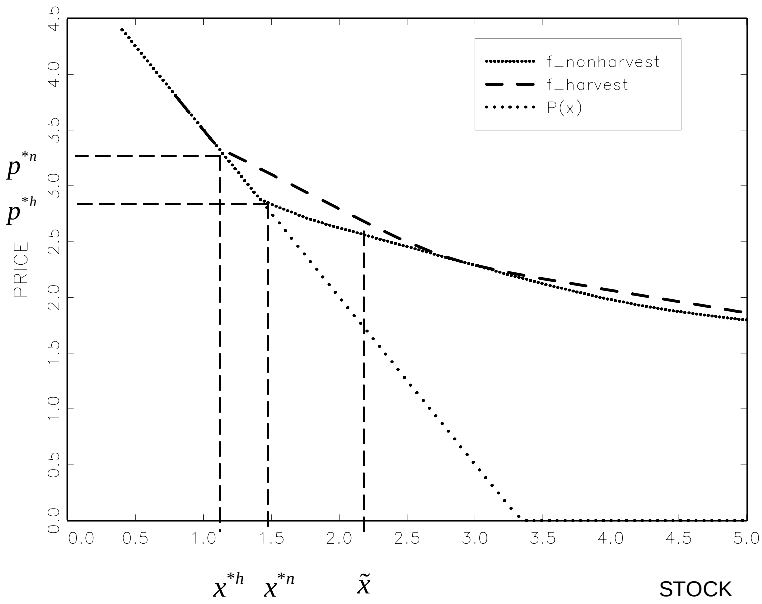
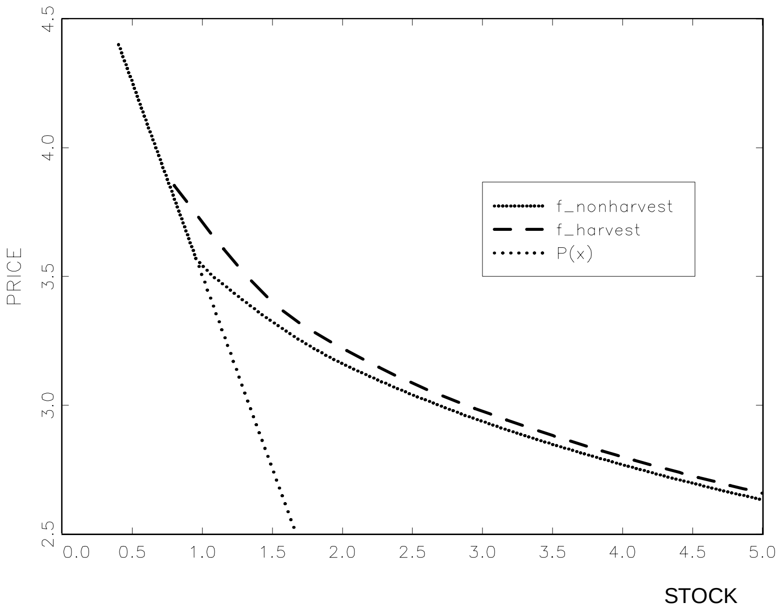


Figure 2b

$a = -1.5$, $b = 5$, $w^n \sim U[0.4, 0.85]$, $w^h \sim U[0.8, 1.25]$, $\delta = 1\%$, $r = 3\%$, $\theta = 0.94$.



is exaggerated and set equal to 15%). Note that Q^h exhibits FOSD over Q^n, the supports of the distributions overlap and $\omega^n > 0$. For this combination of the parameters, $\tilde{x} = 2.23$, which is greater than $\bar{w}^h = 1.25$. Thus (6) is satisfied implying that stock-outs are an "absorbing state". Note also that the price functions cross in this case. In Proposition 3 we specialise Proposition 1 to the linear demand case.

Proposition 3.

If

$$\underline{w}^h > \theta\omega^n - (1-\theta)\frac{b}{a}, \tag{7}$$

then

$$f^h(x) > P(x), \text{ for all } x \in \left[\underline{w}^h, +\infty\right[.$$

□

Proof. See Appendix

The condition then takes a simpler form, (7). If it holds then $\underline{w}^h > x^{*h}$ and the solution given in (4) applies for the range of harvest period stock *x*, $x \in \left[\underline{w}^h, \tilde{x}\right]$. Furthermore, storage is always positive at the end of the harvest period and the harvest period function lies strictly above the nonharvest period function when it equals the demand function. The condition requires that the *minimum* of the harvest distribution, is greater than a weighted average of the *mean* of the nonharvest period distribution, ω^n, and the stock value at which the price given by the demand curve is zero, $-b/a$.

This lower bound is increasing in the mean of the nonharvest distribution ω^n, the slope of the inverse demand function *a*, and the wastage rate δ. Thus a higher ω^n or δ, or a more elastic demand function all imply that (7) is harder to satisfy and stockouts are more probable in the harvest period, cet. par. The effect of the elasticity of demand is consistent with the numerical analysis in DL.

The condition in Proposition 3 is less restrictive than the nonoverlapping support assumption of CB, ($\underline{w}^h > \overline{w}^n$), for values of θ such that $\underline{\theta} \equiv \dfrac{\overline{w}^n + b/a}{\omega^n + b/a} < \theta < 1$. The value of $\underline{\theta}$ in the numerical example is 0.92, implying that (7) is weaker than $\underline{w}^h > \overline{w}^n$ for wastage rates up to 5% when the interest rate is 3%.

The numerical solution to the model is presented in Figure 2b using the previous parameter values but for a lower wastage rate, (δ = 1%), which ensures Proposition 3 is satisfied. Hence the harvest period price function is above the demand curve for all values of the stock, showing that even at the lowest possible harvest $\underline{w}^h$, speculators always demand the commodity for inventory. With the lower wastage rate, (6) is not satisfied and stockouts in the nonharvest period are not an absorbing state, ($\overline{w}^h = 1.25$ and $\tilde{x} = 0.96$).

5. Conclusion

This note has characterised the equilibrium price functions in a two period competitive storage model of commodity prices, when harvests have periodic distributions. We present a sufficient condition for stockouts not to occur in the harvest period.
Assuming a linear demand function allows us to derive the analytical solution for the harvest period price function over an interval of the range of the stock. In the linear demand case, we characterise the condition ((6) above) such that the stock reaching the harvest period is permanently zero.

The conditions underlying the characterisation of the equilibrium price functions above are not directly comparable to the assumption made in CB of nonoverlapping support of the periodic distributions. However, if the wastage and interest rates are low enough, our results hold under weaker conditions.

6. References.

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7. Appendix

Proof of Proposition 1

All stock is consumed in the harvest period and no inventory is carried into the nonharvest period when $p^{*n} \leq p_t$, in which case

$$p_t = f^h(x) = P(x),\ p_{t+1} = f^n(w_{t+1}),$$

from (1), where $p^{*n} = \theta \int_{W^n} f^n(w)\, Q^n(dw)$. This implies $p^{*n} \leq P(x)$ i.e. the discounted expected price in the nonharvest period is less than the price attainable from selling all current stock. Conversely, stock is always carried into the nonharvest period and $P(x)$ is never the solution in the harvest period if

$$\max(P(x)) < p^{*n} \qquad \text{(A1)}$$

i.e. the highest possible price in the current period is less than the discounted expected price in the nonharvest period. By definition $P(x) \leq f^n(x)$ in the nonharvest period, therefore $\theta \int_{W^n} P(w)\, Q^n(dw) \leq \theta \int_{W^n} f^n(w)\, Q^n dw \equiv p^{*n}$. Since $\arg \max_{w \in [\underline{w}^h, +\infty[} P(w) = \underline{w}^h$, a sufficient condition for (A1) is

$$P\left(\underline{w}^h\right) < \theta \int_{W^n} P(w)\, Q^n dw \equiv \theta \mathrm{E}^n\left[P\left(W^n\right)\right]. \quad \text{(A2)}$$

If (A2) is satisfied, holding stocks until the nonharvest period is always profitable.

Proof of Proposition 2

(i) Recall (1): the solution in the harvest period is given by

$$f^h(x) = \max\left\{\theta \int_{W^n} f^n\left[w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right)\right] Q^n\left(dw^n\right), P(x)\right\}$$

If $f^h(x) = P(x)$, then the analytical result is directly available. If $f^h(x) > P(x)$, then ,

$$f^h(x) = \theta \int_{W^n} f^n\left[w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right)\right] Q^n\left(dw^n\right).$$

For all values of the stock x in the nonharvest period, such that $x \in [\underline{w}^n, x^{*n}]$, then, $f^n(x) = P(x)$, $w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right) \leq x^{*n}$ and

$$f^h\left(w^h\right) = \theta \int_{W^n} P\left[w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right)\right] Q^n\left(dw^n\right) \quad \text{(A3)}$$

Then for any draw of the harvest in the nonharvest period, w^n, the solution is given as

$$P\left[w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right)\right] = a\left\{w^n + (1-\delta)\left(x - \left(\frac{f^h(x) - b}{a}\right)\right)\right\} + b.$$

Simplifying and integrating over the nonharvest period distribution gives

$$\int_{W^n} P\left[w^n + (1-\delta)\left(x - D\left(f^h(x)\right)\right)\right] Q^n\left(dw^n\right)$$

$$= a \int_{W^n} w^n\, Q^n dw^n + b + (1-\delta)\left(ax - f^h(x) + b\right). \quad \text{(A4)}$$

Denote $\omega^n = \int_{W^n} w^n Q^n\left(dw^n\right)$. Substituting (A4) into (A3) and solving for $f^h(x)$ gives

$$f^h(x) = \frac{\theta}{1+\theta(1-\delta)}\left(a\omega^n + b\right) + \frac{\theta(1-\delta)}{1+\theta(1-\delta)}(ax + b).$$

Denoting

$$\alpha = \frac{\theta(1-\delta)}{1+\theta(1-\delta)} a, \text{ and } \beta = \frac{\theta}{1+\theta(1-\delta)} a\omega^n + \frac{\theta + \theta(1-\delta)}{1+\theta(1-\delta)} b,$$

gives Proposition 2(i).

$$f^h(x) = \alpha x + \beta. \quad \text{(A5)}$$

Clearly $\dfrac{\theta(1-\delta)}{1+\theta(1-\delta)} < 1$ giving $\alpha > a$, because $a < 0$.

(ii) Let $\tilde{x}$ denote the maximum value of the stock in the harvest period, such that $f^h(\tilde{x}) = \alpha\tilde{x} + \beta$ i.e.

$$w^n + (1-\delta)\left(\tilde{x} - D\left(f^h(\tilde{x})\right)\right) \le x^{*n}. \qquad \text{(A6)}$$

Then for all $w^n \in \left[\underline{w}^n, \overline{w}^n\right]$,

$$f^n\left(w^n + (1-\delta)\left(\tilde{x} - D\left(f^h(\tilde{x})\right)\right)\right) = P\left(w^n + (1-\delta)\left(\tilde{x} - D\left(f^h(\tilde{x})\right)\right)\right).$$

Assume (A6) holds exactly at the maximum harvest in the nonharvest period $\overline{w}^n$, i.e.

$$\overline{w}^n + (1-\delta)\left(\tilde{x} - D\left(f^h(\tilde{x})\right)\right) = x^{*n}.$$

Substituting the inverse demand function, the equilibrium price in the harvest period from (4), using $x^{*n} = P^{-1}\left(p^{*h}\right)$, and simplifying gives

$$a\overline{w}^n + (1-\delta)\left(a\tilde{x} - \left((\alpha\tilde{x} + \beta) - b\right)\right) = a\left(\frac{p^{*h} - b}{a}\right),$$

and solving for $\tilde{x}$ gives

$$\tilde{x} = \frac{p^{*h} - \left(a\overline{w}^n + b\right)}{(a-\alpha)(1-\delta)} - \frac{(b-\beta)}{(a-\alpha)}. \qquad \text{(A7)}$$

Noting

$$a - \alpha = \frac{a}{1+\theta(1-\delta)}; \; b - \beta = \frac{b(1-\theta) - a\theta\omega^n}{1+\theta(1-\delta)},$$

this simplifies further to

$$\tilde{x} = \frac{1}{a}\left\{\frac{1+\theta(1-\delta)}{1-\delta}\left(p^{*h} - \left(a\overline{w}^n + b\right)\right) - \left(b - \theta\left(a\omega^n + b\right)\right)\right\}. \qquad \text{(A8)}$$

Noting $P(x) = ax + b$, gives the result in Proposition 4.

(iii) To show $\overline{w}^h < \tilde{x}$, substitute (A8) and $P(x) = ax + b$, which gives

$$a\overline{w}^h + b > \left(\theta + \frac{1}{1-\delta}\right)\left(p^{*h} - P\left(\overline{w}^n\right)\right) + \theta P\left(\omega^n\right)$$

and with some simple manipulation this becomes,

$$P\left(\overline{w}^h\right) - \theta P\left(\omega^n\right) > \left(\theta + \frac{1}{1-\delta}\right)\left(p^{*h} - P\left(\overline{w}^n\right)\right)$$

.

Proof of Proposition 3

by substitution.