# The Price of Size and Financial Market Allocations

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**Abstract**

If size is a scarce productive resource, then there is a positive equilibrium price of size. In this paper we articulate a model of financial contracting with lenders heterogeneous in size. Absent size premia, borrowers prefer contracting with large lenders due to costly state verification and limited communication among lenders. Our main results are (i) In a competitive equilibrium, financial contract is nonlinear in that the expected rate of return on loans is an increasing function of loan size; (ii) Lenders do not divide their loan assets, giving rise to endogenous asset indivisibility; (iii) The total social surplus under a nonlinear contract is less than that under a linear structure which does not reflect the price of size; (iv) The market value of a firm increases with the average size of its debt. So is the market value of a firm's debt. The market value of a firm's equity, however, is invariant with respect to its average size of debt.

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# 1 Introduction

An enterprise might involve one or more transactions. If each transaction entails a cost, then it is desirable to reduce the number of transactions, or equivalently, to increase the average size of individual transactions. Examples of this sort abound. A consumer who wants to buy ten tomatoes of a given quality will not go to ten grocery stores and buy one from each. Most likely, she will buy all ten tomatoes from just one store to save shopping time. An entrepreneur who needs one million dollars to fund her project will not borrow from one million lenders with one dollar from each lender, if writing a contract with each lender entails some annoying paper work. More likely she will try to get the money from one or several lenders. In this case, getting a million one-dollar loans is not the same as getting a one-million-dollar loan. Size is productive in the sense that increasing the average size of individual deals reduces total transaction costs and therefore generates more surplus for the enterprise.

The above two examples have distinct features besides the objects of transaction. Consider the first example. Virtually all grocery stores are able to satisfy customers who demand an usual quantity of tomatoes. That is, sizes of transaction that suppliers can provide exceeds sizes demanded by virtually all consumers. Thus any grocery store cannot charge a premium for being able to provide ten tomatoes in a transaction. Size is not scarce and does not command a positive price. In the second example, it is possible that sizes of loans that can be provided by lenders fall short of sizes demanded by borrowers. Size becomes a scarce productive resource and will command a positive price. That is, lenders who are able to provide large loans will receive a higher rate of return on their loans. One might argue that there are financial intermediaries such as banks that have sufficient amount of resources to meet large borrowers' needs. But banks themselves have to secure funds from many smaller ultimate lenders. Thus financial intermediation does not eliminate the problem of scarce sizes.

We shall consider a more interesting situation in this paper, where the amount of transaction cost is determined endogenously. In particular, we take as point of departure the financial contracting environment of Gale and Hellwig (1985) and Williamson (1986), who in turn adopted Townsend (1979)'s costly state verification (CSV) framework[1]. In that environment the probability of insolvency and therefore the expected monitoring cost is determined by solving an optimal contract problem that maximizes borrower's expected profit such that lenders receive some required expected rates of return on their loans.

Our analysis is centered around the following themes that did not concern previous researchers. First, we consider a contracting environment with lenders and borrowers both heterogeneous in size and distinguish a nonlinear contract structure from a linear one. A linear contract offers a uniform expected rate of return for all lenders. In most applications of the CSV framework, financial contracts are restricted to be linear. This restriction is innocuous if there is no size differentials, but become problematic when heterogeneity arises. In our model loan size is scarce and borrowers compete for large loans, so that linear contract is not viable in a competitive equilibrium. The equilibrium contract is

[1] See Dowd (1992) for an excellent survey on financial contracting in the CSV environment.

nonlinear in that it offers an expected rate of return schedule that is increasing in loan size. This gives rise to positive equilibrium price of size. Interestingly, as a result of the positive price of size, lenders do not divide their loan assets, which endogenously generates asset indivisibility.

Second, we analyze implications of the price of size for the efficiency of financial market allocations and for the equilibrium market value of firms. For the former, we compare the total social surplus under the equilibrium nonlinear contract structure with that under a linear structure. For the latter, we examine how the average size of a firm's debt affects the market value of its asset, debt, and equity.

The main results on the second theme are (1) total social surplus under nonlinear contract structure, which reflects the price of size, is less than that under a linear structure which does not reflect the price of size, a result that appears to be paradoxical; and (2) the market value of a firm increases with the average size of its debt. So is the market value of a firm's debt. The market value of a firm's equity, however, is invariant with respect to the average size of its debt.

Our analysis is carried out in an environment with direct lending and borrowing, where the issue of financial intermediation is abstracted away. As mentioned earlier, financial intermediation does not eliminate the problem of scare sizes, especially when the portfolios of financial intermediaries are themselves risky from the perspectives of depositors [see Krasa and Villamil (1992) for an example[2]]. An environment with direct lending and

[2] In Krasa and Villamil (1992), the financial intermediary cannot fully diversify its portfolio and is subject to bankruptcy risk since there are only a finite number of borrowers in the economy.

borrowing is worth investigating, as such financial relationships do prevail in addition to intermediation. And in the U.S. economy, corporate bond markets are not dominated by banking.[3].

The rest of the paper is organized as follows. Section 2 lays out the basic structure of the model, followed by the analysis of financial contract in Section 3. In Section 4 we discuss the price of size that arises in competitive equilibrium and its implications for lenders' behavior. Section 5 then copes with the issue of how size premia affect the allocative efficiency in the financial market and the value of firms. Concluding remarks are offered in the last section. All proofs are relegated to the Appendix.

## 2 The Environment

The economy is populated by entrepreneurs ("borrowers" hereafter) who have access to risky investment opportunities but have no funds to meet their investment expenditure, and lenders who have funds but do not have access to these opportunities. The number of borrowers is $\tau$. Each borrower has exactly one project at hand, with a fixed size $K > 0$. In addition, there are $\eta$ size classes of lenders. Each size class $h$, $h = 1, 2, ..., \eta$, has $N_h$ lenders who are endowed with funds in the amount $S_h$. As a matter of normalization, we order $\{S_h\}_{h=1}^{\eta}$ decreasingly, so that $S_1 > S_2 > ... > S_\eta$. In particular, $S_h = K/h$,

[3]A theory for the coexistence of direct lending and financial intermediation is provided by Cantillo (1996). In his model, the limitation of financial intermediation is due to the assumption that their loan portfolio has an undiversifiable risk that exposes banks to possible failure, a spirit similar to Krasa and Villamil (1992). This raises the cost of capital for banks. Thus cash-rich companies prefer to tap the bond market to bypass the costly intermediation. In such a setup, our size problem would be relevant for both the bond market and the banking industry.

$h = 1, ..., \eta$.[4] We assume that $S_h N_h \geq K$, $h = 1, 2, ..., \eta$, that is, the total amount of funds in any size category of lenders is sufficient to meet the investment expenditure of a single project. Some restrictions are imposed on the distribution of sizes.

*Assumption 1.*

$$\sum_{h=1}^{\eta} S_h N_h > \tau K.$$

*Assumption 2.*

$$S_h N_h < \tau K, \quad h = 1, ..., \eta.$$

Assumption 1 implies that at the aggregate level loanable funds are abundant: the total amount of lenders' funds is more than enough to cover investment expenditures of all projects. Assumption 2 indicates scarcity of sizes. It says that all the funds belonging to any size class are not sufficient to fully finance all the projects in the economy. Borrowers, therefore, have to compete for sizes if they prefer large individual loans.

We emphasize that the scarcity of size in our model reflects limited degree of communication among lenders. To some extent, lenders might form coalitions and pool their

[4] An entrepreneur might choose to adopt an evenly partitioned debt structure, that is, borrow $S_h$ units of funds from $n = K/S_h$ lenders for some $h \in \{1, ..., \eta\}$. Under this assumption, $n$ is an integer for all $h \in \{1, ..., \eta\}$. Contracting in this way exactly covers the entrepreneur's need for funds. It is one of the equilibrium properties of our model that all lenders will not divide their endowed funds when they lend to entrepreneurs. Therefore the assumption stated is technically convenient for characterizing the model's equilibrium. One can, of course, allow the endowed sizes to be continuous, so that $K/S_h$, which might not be an integer, represents the mass of lenders that a borrower contracts with. Generalization to this situation is a mere matter of technicality.

funds to make larger loans. But as far as communication is limited, each coalition formed can only include a limited number of lenders, therefore can only provide a loan that is limited is size. Thus "lenders" in our model should be reinterpreted as coalitions when communication is possible but limited. Note that we do not impose indivisibility on each lender's funds. To the contrary, we allow the funds of a lender to be perfectly divisible. As a result, it is in principle possible for a borrower to contract with a lender in some size class $h$ who lends to her $s \in (0, S_h]$ units of funds. We will show, however, that all lenders choose not to divide their funds whenever they lend to entrepreneurs — each lender loans out all her funds to at most one borrower.

The production technology is assumed to be linear. A project yields a random gross return $q\omega K$, where $q$ is a constant, $\omega$ is a unit-mean random variable with *c.d.f.* $G(\cdot)$ and *p.d.f.* $g(\cdot)$ on $\mathcal{R}^+$. We assume that the hazard rate $g(\omega)/[1 - G(\omega)]$ is an increasing function of $\omega$ — a common assumption in the incentive contract literature. Furthermore, $\omega$ is identical and independently distributed across all projects. $q$, $G(\cdot)$, and $g(\cdot)$ are common knowledge. There is, however, an informational asymmetry regarding the projects' *ex post* return. In particular, only borrowers can costlessly observe their projects' actual gross return and therefore the realizations of $\omega$, while lenders have to expend a fixed verification cost $\gamma > 0$ in order to observe the same object. These assumptions are commonplace in CSV models of financial contracting, such as Williamson (1986).

For lenders there also exists a risk-free investment opportunity which yields a constant rate of return, $R^f$. They might choose to supply their funds to entrepreneurs in the loan market, or to invest in the risk-free asset. Because of the abundance of funds (Assumption 1), in equilibrium some lenders will invest in the risk-free alternative. In addition, to ensure the existence of an equilibrium where all projects are funded, we impose a restriction on the comparison between $R^f$ and $q$:

*Assumption 3.* $R^f/q < \psi^l\left(\widehat{\omega}, S_\kappa\right)$, where $\widehat{\omega} \equiv \arg\max_{\overline{\omega}} \psi^l\left(\overline{\omega}, S_\kappa\right)$,

$$\psi^l\left(\overline{\omega}, s\right) \equiv \left[1 - G\left(\overline{\omega}\right)\right]\overline{\omega} + \int_0^{\overline{\omega}} \omega dG\left(\omega\right) - G\left(\overline{\omega}\right)\frac{\gamma}{qs},$$

and

$$\sum_{h=1}^{\kappa-1} S_h N_h < \tau K, \quad \sum_{h=1}^{\kappa} S_h N_h \geq \tau K.$$

The role played by this assumption should become clear in subsequent analysis.

We assume that all borrowers and lenders are risk neutral so that they care only about expected returns they receive. Risk neutrality arises when borrowers and lenders have linear von Neumann-Morgenstern (v.N-M) utility function, or when they have concave v.N-M utility function but there is perfect insurance. By assuming risk neutrality, we are able to highlight the price of size without distraction from considerations pertaining to risk aversion. Finally, we assume that there is limited liability that prevents borrowers' asset from being negative.

# 3 Financial Contracting

In this section we characterize the optimal form of financial contract offered by borrowers. Consider the contracting problem between a borrower and $n$ lenders each of whom wishes to lend funds in the amount $s_i$, $i = 1, ..., n$, where $\sum_{i=1}^{n} s_i = K$. The borrower and the lenders take the size-dependent required expected rates of return on loans as given by the competitive financial market. Because of the informational asymmetry and limited liability, the borrower has incentives to report a zero value of $\omega$ absent monitoring. Thus a financial contract must specify a nonempty set of $\omega$ for which monitoring occurs. Let the monitoring set be denoted by $M$. By specifying a "monitoring set", we are only considering deterministic verification. This is actually less restrictive than it appears. Krasa and Villlamil (2000) articulates a costly enforcement model that justifies deterministic monitoring when commitment is limited and enforcement is costly and imperfect[5].

To simplify matters and to conform with reality, we assume that the bankruptcy law prohibits a borrower to specify different monitoring sets for different lenders. That is, bankruptcy requires uniform insolvency. Differing monitoring sets for different lenders would imply that the borrower might be insolvent with respect to some liabilities but is solvent with respect to other liabilities, which violated the uniform insolvency requirement. In the following analysis we first establish that the optimal form of contract is a "standard debt contract".

We first define a contracting relationship.

**Definition 1.** A financing scheme (debt structure) adopted by a borrower is a set of loans from $n$ lenders, $\{s_i\}_{i=1}^{n}$, such that $\sum_{i=1}^{n} s_i = K$. A contracting relationship is a pair

[5] Krasa and Villamil (2000) also show that when there is perfect commitment, stochastic contracts are optimal. They argue, however, that for loan contracts, limited commitment and therefore simple debt seems more appropriate. See also Mookherjee and Png (1989) and Boyd and Smith (1994) on deterministic versus stochastic monitoring.

$(j, \{s_i\}_{i=1}^n)$ where $j \in \{1, ..., \tau\}$ is a borrower and $\{s_i\}_{i=1}^n$ is a financing scheme.

Associated with a contracting relationship, a financial contract specifies a monitoring set $M$, and a size-dependent payment schedule $\widetilde{P}(s_i, \omega)$, $\omega \in \mathcal{R}^+$, $i = 1, ..., n$. Obviously the borrower will report $\arg\min_{\omega \in M^c} \sum_{i=1}^n \widetilde{P}(s_i, \omega)$ when $\omega \in M^c \equiv \mathcal{R}^+ \backslash M$. Thus the actual payment is fixed for the nonmonitoring set $M^c$. Denote the size-dependent fixed payment by $P(s_i)$. Incentive compatibility requires that $\sum_{i=1}^n \widetilde{P}(s_i, \omega) < \sum_{i=1}^n P(s_i)$ for $\omega \in M$ and $\sum_{i=1}^n \widetilde{P}(s_i, \omega) \geq \sum_{i=1}^n P(s_i)$ for $\omega \in M^c$, $i = 1, ..., n$. The borrower chooses $\left\{\widetilde{P}(s_i, \omega), P(s_i)\right\}_{i=1}^n$ to maximize her expected profit, subject to the constraint that each lender at least receives her required expected rate of return $R(s_i)$, $i = 1, 2, ..., n$. The financial contract solves the following problem.

**Problem 1.**

$$\max_{\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n} qK \int \omega g(\omega)\, d\omega - \left[\int_M \sum_{i=1}^n \widetilde{P}(s_i, \omega)\, g(\omega)\, d\omega + \int_{M^c} \sum_{i=1}^n P(s_i)\, g(\omega)\, d\omega\right]$$

*s.t.*

$$\int_M \left[\widetilde{P}(s_i, \omega) - \gamma\right] g(\omega)\, d\omega + \int_{M^c} P(s_i)\, g(\omega)\, d\omega \geq R(s_i)\, s_i, \qquad i = 1, ..., n$$

Note that there are $n$ constraints in the above problem, one for each lender that the borrower contracts with. These $n$ lenders together receive payment $\sum_{i=1}^n \widetilde{P}(s_i, \omega)$ and $\sum_{i=1}^n P(s_i)$ in monitoring and nonmonitoring states, respectively, and expend verification costs $n\gamma$ when monitoring occurs. Their total expected return is $\sum_{i=1}^n R(s_i)\, s_i$. Our first proposition states that if the contract is designed optimally, then the total payment to all these lenders exhausts the borrower's revenue in monitoring states.

**Proposition 1.** The solution to Problem 1 entails $\sum_{i=1}^{n} \widetilde{P}(s_i, \omega) = q\omega K$ for $\omega \in M$.

To understand this proposition, it is useful to imagine an "aggregate lender" who lends out $\sum_{i=1}^{n} s_i$ units of funds to the borrower and demands expected return in the amount $R^n \equiv \sum_{i=1}^{n} R(s_i) s_i$. Suppose this lender receives payment $\widetilde{P}^n(\omega)$ and $P^n$ in monitoring and nonmonitoring states, respectively, and expend verification costs $n\gamma$ when monitoring occurs. It turns out that one can restrict attention to the (hypothetical) contracting problem between the borrower and the aggregate lender if the object of interest is the aggregate payment schedule. In particular, call this an "aggregate contracting problem" and its solution an "aggregate contract". One can show that if $\left(\widetilde{P}^n(\omega), P^n\right)$ is a solution to the aggregate contracting problem, then there exists $\left\{\widetilde{P}(s_i, \omega), P(s_i)\right\}_{i=1}^{n}$ such that it is a solution to Problem 1 and that $\sum_{i=1}^{n} \widetilde{P}(s_i, \omega) = \widetilde{P}^n(\omega)$, and $\sum_{i=1}^{n} P(s_i) = P^n$. Conversely, if $\left\{\widetilde{P}(s_i, \omega), P(s_i)\right\}_{i=1}^{n}$ is a solution to Problem 1, then $\left(\widetilde{P}^n(\omega), P^n\right)$ where $\widetilde{P}^n(\omega) = \sum_{i=1}^{n} \widetilde{P}(s_i, \omega)$ and $P^n = \sum_{i=1}^{n} P(s_i)$ is a solution to the aggregate contracting problem (Lemma A1, Appendix).

Applying a standard result for financial contracting in the CSV environment [Williamson (1986)] yields the conclusion that the aggregate contract entails setting the payment to the "aggregate lender" in insolvent states equal to the project's entire revenue. That is, the optimal form of contract between the borrower and the "aggregate lender" is what Gale and Hellwig (1985) called the "standard debt contract": the lender receives a fixed payment in nonmonitoring states and confiscates the entire amount of the borrower's revenue if monitoring occurs. By allowing the lender to recover the borrower's revenue to a maximum extent, the contract minimizes the fixed nondefault payment and therefore the default probability, as well as the expected monitoring costs. By the equivalence between Problem 1 and the aggregate contracting problem, we conclude that the optimal form of financial contract with possibly heterogeneous lenders entails setting the total payment to all lenders in insolvent states equal to the project's entire revenue. Proposition 1 is thus a generalization of the result for conventional settings of homogeneous lenders to the current setting of heterogeneous loan sizes. The "standard debt contract" structure applies optimally to the current setting.

An alternative way to characterize the solution to Problem 1 is to define a cutoff value for the idiosyncratic risk $\omega$, denoted by $\overline{\omega}$. With this cutoff value, the monitoring set $M$ is simply $[0, \overline{\omega})$. The total fixed payment can be expressed as $P^n = qK\overline{\omega}$, while the total payment for nonmonitoring states is given by $q\omega K$, $\omega \in M$. Under this characterization, the borrower's expected profit is $qK\psi^b(\overline{\omega})$, where

$$\psi^b(\overline{\omega}) \equiv 1 - \int_0^{\overline{\omega}} \omega dG(\omega) - [1 - G(\overline{\omega})]\overline{\omega}. \tag{1}$$

Note that $\psi^b(\overline{\omega})$ does not depend directly on the per lender monitoring cost $\gamma$ and the number of lenders $n$. The sum of lenders' expected returns, net of monitoring costs, is given by $qK\psi^l(\overline{\omega}, s)$, where

$$\psi^l(\overline{\omega}, s) \equiv [1 - G(\overline{\omega})]\overline{\omega} + \int_0^{\overline{\omega}} \omega dG(\omega) - G(\overline{\omega})\frac{\gamma}{qs}. \tag{2}$$

Here $s \equiv K/n$ is the average loan size of the lenders that the borrower contract with. Multiplying the last term on the right-hand side of (2) by $qK$ gives $n\gamma G(\overline{\omega})$, which is the total expected monitoring costs incurred by all lenders. Note that when measured as a fraction of the total expected revenue of the project $qK$, lenders' share $\psi^l(\overline{\omega}, s)$ depends on the average size $s$, and does not depend separately on the number of lenders in the contracting relationship, $n$, and the project size $K$. It suffices to consider the following simple problem.

**Problem 2.**

$$\max_{\overline{\omega}} qK\psi^b(\overline{\omega})$$

*s.t.*

$$qK\psi^l(\overline{\omega}, s) \geq R^n \tag{3}$$

where $R^n \equiv \sum_{i=1}^{n} R(s_i) s_i$.

Below we list some properties of the functions $\psi^b(\overline{\omega})$ and $\psi^l(\overline{\omega}, s)$ which will be useful in later analysis.

*i.* $\psi^b(\overline{\omega})$ is decreasing and convex in $\overline{\omega}$:

$$\frac{\partial \psi^b(\overline{\omega})}{\partial \overline{\omega}} = -[1 - G(\overline{\omega})] < 0,$$
$$\frac{\partial^2 \psi^b(\overline{\omega})}{\partial \overline{\omega}^2} = g(\overline{\omega}) > 0.$$

*ii.*

$$\begin{aligned} & \frac{\partial \psi^l(\overline{\omega}, s)}{\partial \overline{\omega}} \\ = \;& [1 - G(\overline{\omega})]\left[1 - \frac{\gamma}{qs}\frac{g(\overline{\omega})}{1 - G(\overline{\omega})}\right] \\ \gtreqless \;& 0 \text{ for } \overline{\omega} \lesseqgtr \widehat{\omega} \end{aligned}$$

where $\widehat{\omega}$ satisfies

$$\frac{g(\widehat{\omega})}{1-G(\widehat{\omega})}=\frac{qs}{\gamma}.$$

This property holds since the hazard rate $g(\overline{\omega})/[1-G(\overline{\omega})]$ is increasing in $\overline{\omega}$ by assumption.

*iii.*

$$\psi^b(\overline{\omega})+\psi^l(\overline{\omega},s)=1-G(\overline{\omega})\frac{\gamma}{qs}<1$$

$$\frac{\partial}{\partial\overline{\omega}}\left[\psi^b(\overline{\omega})+\psi^l(\overline{\omega},s)\right]=-g(\overline{\omega})\frac{\gamma}{qs}<0$$

Here $\psi^b(\overline{\omega})+\psi^l(\overline{\omega},s)$ is the total surplus per unit of the gross return of a project. The fact that $\psi^b(\overline{\omega})+\psi^l(\overline{\omega},s)<1$ indicates a deadweight loss due to costly state verification and a positive monitoring probability. The deadweight loss (agency cost) for the entire project is $n\gamma G(\overline{\omega})$, so that deadweight loss per unit of the gross return of a project is given by $G(\overline{\omega})\frac{\gamma}{qs}$. The higher the monitoring threshold $\overline{\omega}$, the higher the deadweight loss, and the lower the total surplus.

*iv.*

$$\frac{\partial\psi^l(\overline{\omega},s)}{\partial s}=G(\overline{\omega})\frac{\gamma}{qs^2}>0$$

An increase in the average loan size, $s$, in a contracting relationship raises lenders' total expected return, other things equal. Since the borrower's expected profit does not directly depend on $s$, total surplus of the project also increases with the average size of loan.

In the sequel we shall only be concerned with values of $\overline{\omega}$ that lie within the region $[0,\widehat{\omega})$, where $\psi^l(\overline{\omega},s)$ is monotonically increasing in $\overline{\omega}$. This in effect rules out the possibility of credit rationing [for details, see Williamson (1986)]. Examining Problem 2 reveals some interesting features of its solution. These are stated in Lemma 1, where we use $\overline{R} \equiv R^n/(ns)$ to denote the average expected rate of return for the $n$ lenders with average loan size $s$ who contract with the borrower.

**Lemma 1.** Let $\varpi^*$ be the solution to Problem 2 given $\overline{R}$ and $s$. Then $\varpi^*$ satisfies

$$q\psi^l(\varpi^*, s) = \overline{R}. \tag{4}$$

$\varpi^*$ is increasing in $\overline{R}$ and decreasing in $s$, while $\psi^b(\varpi^*)$ is decreasing in $\overline{R}$ and increasing in $s$.

# 4 Equilibrium Price of Size

We now define a competitive equilibrium for the model economy.

**Definition 2.** A competitive equilibrium is an expected rate of return schedule on loans, $R(s)$, where $s$ is loan size, a set of contracting relationships each associated with a monitoring threshold $\varpi^*$, such that

*i.* for each contracting relationship, $\varpi^*$ solves Problem 2, i.e., satisfies (4) given $R(s)$ and the adopted financing scheme,

*ii.* each borrower does not strictly prefer a financing scheme that differs from the one in her contracting relationship,

*iii.* lenders choose their lending behavior optimally, and

*iv.* the loan market clears, that is, the supply of funds equals demand $\tau K$.

The following proposition states a property that a competitive equilibrium must satisfy.

**Proposition 2.** Let $n_j$ be the number of lenders that borrower $j$, $j = 1, ..., \tau$, contracts with in a competitive equilibrium. Let $\overline{s}_j \equiv K/n_j$ denote the average size of borrower $j$'s debt. And Let $\overline{R}_j \equiv \sum_{i=1}^{n_j} R(s_i) s_i/K$ be the associated average expected rate of return. Then

$$\overline{R}_j > (=) \ \overline{R}_k \ \text{ if } \ \overline{s}_j > (=) \ \overline{s}_k, \ \ j, k \in \{1, ..., \tau\}.$$

The intuition behind the result in Proposition 2 is as follows. If to the contrary, a financing scheme with a larger average size of debt does not require a larger average expected rate of return, then the monitoring threshold will be lower with this contracting relationship. This is because lenders' share of a project's expected return, net of expected monitoring costs, is larger when the average size of loans they make is larger. If they do not require a larger expected rate of return, then the monitoring threshold can be lowered since lenders' share, already raised by the larger average loan size, is an increasing function of the monitoring threshold. But lower values for the monitoring threshold is associated with larger expected profits for a borrower. In equilibrium all financing schemes must entail the same monitoring threshold, otherwise profitable and feasible deviations exist for some borrowers. Therefore a higher average expected rate of return must be there to compensate for the monitoring-cost-reducing benefits of a larger average size of debt.

The above reasoning indicates that there is a price that borrowers must pay for being able to utilize a financing scheme with a larger average size of debt. It is in this sense that we say there is a positive price of size, or size premium. For any pair of different average loan sizes $(\overline{s}_j, \overline{s}_k)$, the premium is given by $\overline{R}_j - \overline{R}_k$. Put in another way, the average expected rate of return schedule is an increasing function of average loan size.

Proposition 2 also implies that a linear contract structure, where the expected rate of return schedule $R(s)$ is a constant [so that $R(s)\,s$ is a linear function of $s$], is not viable in equilibrium. With a linear contract structure, the average expected rate of return on loans is fixed regardless of its average size, which violates Proposition 2.

The notion of the price of size can be sharpened once we restrict attention to evenly partitioned debt structures, where every borrower contract with a set of lenders who all lend to her the same amount of funds. With these simple debt structures, we can directly identify the expected rate of return schedule, $R(s)$, not just the average expected rates of return. The following proposition states properties of the function $R(s)$. Over there we will in fact construct a competitive equilibrium. The existence of equilibrium is thus proved.

**Proposition 3.** A competitive equilibrium exists. In one equilibrium, each borrower adopts an evenly partitioned debt structure, i.e., for each $j \in \{1, ..., \tau\}$,

$$s_i = \overline{s}_j = s, \;\; i = 1, ..., n_j, \quad \text{for some } s \in (0, S_1].$$

And the expected rate of return schedule is such that

$$R(s) > (=) \;\; R(s') \quad \text{if} \;\; s > (=) \;\; s', \;\; s, s' \in (0, S_1].$$

In particular,

$$R(s) - R(s') = \gamma G(\varpi^*)\left(\frac{1}{s'} - \frac{1}{s}\right). \tag{5}$$

Here $\varpi^* \in (0, \widehat{\omega})$ satisfies

$$q\psi^l(\varpi^*, S_\kappa) = R^f, \tag{6}$$

where $\kappa$ is such that

$$\sum_{h=1}^{\kappa-1} S_h N_h < \tau K, \quad \sum_{h=1}^{\kappa} S_h N_h \geq \tau K, \quad 0 < \kappa < \eta.$$

When financing schemes are restricted to evenly partitioned ones, the average expected rate of return on a set of loans coincides with the expected rate of return on an individual piece of loan. It is then an implication of Proposition 2 that the expected rate of return schedule $R(s)$ is an increasing function of loan size $s$. In a more direct sense we observe a positive price for size: a larger size of loan commands a higher expected rate of return.. For any pair of different loan sizes $(s, s') \in (0, S_1] \times (0, S_1]$, the size premium is given by $R(s) - R(s')$. In equilibrium the monitoring threshold $\varpi^*$ is the same for all contracting relationships. Setting $q\psi^l(\varpi^*, s)$ equal to $R(s)$ and using the expression for $\psi^l$ allows us to derive the formula for the size premium given in (5).

The equilibrium monitoring threshold $\varpi^*$, in turn, is determined by the market clearing condition for loans. Having the risk-free alternative in mind, lenders in size class $h \in \{1, ..., \eta\}$ such that $R(S_h) < R^f$ will choose to invest in the risk-free asset. Only lenders in size class $h \in \{1, ..., \eta\}$ such that $R(S_h) \geq R^f$ will supply their funds to the loan market. Given that the expected rate of return schedule $R(s)$ is monotonically increasing in $s$, it must be the case that lenders who are endowed with large funds lend to entrepreneurs while those with small funds invest in the risk-free asset. Let $\kappa$ be the cutoff value for the size class index $h$ as described in Proposition 3. Lenders in this size class obtains an expected rate of return equal to the risk-free rate $R^f$. Assumption 3 ensures the existence of a competitive equilibrium where all projects are fully funded.

An implication of Proposition 3 is that the total expected return that a lender with loan size $s$ receives, $R(s)\,s$, is a nonlinear function of $s$. This is because the expected rate of return $R(s)$ is nonconstant with respect to $s$. Because of this we say that the equilibrium financial contract is nonlinear. As already shown, only nonlinear contract structures are viable in a competitive equilibrium. We can further investigate the consequences of financial contracts being nonlinear. Proposition 4 states an implication of an increasing expected rate of return schedule on the behavior of lenders.

**Proposition 4.** In a competitive equilibrium where the expected rate of return schedule $R(s)$ is monotonically increasing in $s$, each lender lends all of her funds to one borrower if she does not invest in the risk-free asset.

The result that lenders choose not to divide the loans they can make is a direct consequence of the positive price of size. If a lender divided a sum of funds into two parts and lent them to two borrowers separately, then both parts would command lower expected rates of return than the original sum of funds. Thus, the presence of a positive price for loan size gives rise to endogenous asset indivisibility on the part of lenders. Proposition

4 therefore provides a case *against* portfolio diversification. Note that we have chosen to abstract from risk aversion to highlight the mechanism that leads to such asset indivisibility.

Wallace (2000) presents a random matching model of exchange and production that is based on asset indivisibility, whereby an inverse link between transaction velocities and yields is produced. The indivisibility of asset is, however, exogenously given in his model. Wallace also assumes that assets differ in size, which is identified with the magnitude of an exogenous real dividend on a unit of an asset. Our model endogenously delivers, in a competitive financial contracting context, an expected rate of return schedule that is increasing in the size of loan assets, which in turn, endogenously generates asset indivisibility.

# 5 Social Surplus and the Value of Firms

In this section we turn to a discussion of the efficiency property of competitive equilibrium. In particular, we compare the total social surplus in a competitive equilibrium where the financial contract is nonlinear to the total social surplus under a hypothetical linear contract structure. For the latter structure, we consider allocations where only lenders in size class $h \in \{1, ..., \kappa\}$ supply their funds to the loan market.[6]

**Proposition 5.** The total social surplus in a competitive equilibrium with a monotonically increasing expected rate of return schedule for loans is less than the total social surplus under a linear contract structure where the expected rate of return is fixed at $R^f$

[6] Any slight deviation from the linear contract structure to some nonlinear structure, where the expected rate of return is a (slightly) increasing function of loan size, would generate the result that only large lenders lend to entrepreneurs. It is thus not arbitrary to consider this kind of allocations.

for all $s \in (0, S_1]$.

Proposition 5 implies that a social planner who wants to maximize total social surplus subject to the informational constraints and the lenders' individual rationality constraints would design a linear contract structure which sets $R(s) = R^f$ for all $s$. This result seems to be counter-intuitive. A linear contract structure does not reflect the scarcity and price of size, while an equilibrium nonlinear contract structure does. Yet the total social surplus under the former structure is higher than the latter. The reason is as follows. With the linear contract structure, there are no size premia to offset the benefits of adopting larger average loan sizes, this allows monitoring thresholds to be smaller for financing schemes with larger average sizes of debt. Under the nonlinear contract structure, a positive price of size is required to offset the benefits of large loans, which keeps the monitoring threshold from declining when the average loan size is raised. But larger monitoring thresholds are associated with lower social surplus. Thus loans that are costly to borrowers are also costly from a social point of view. Therefore the linear contract structure appears to be more efficient than the nonlinear one.

We now discuss the market value of firms in a competitive equilibrium, assuming that each firm has exactly one project, and that each entrepreneur is the owner-manager of a firm. As in Modigliani and Miller (1958), the market value of a firm is the sum of the market value of equity and the market value of debt. Let $s$ represent the average size of a firm's debt. The market value of a firm's equity[7] is given by $qK\psi^b(\overline{\omega}^*)$, the mar-

[7] Here "equity" refers to inside equity, which is simply the firm's ownership held by the entrepreneur.

ket value of its debt is given by $qK\psi^l(\overline{\omega}^*, s)$, and the market value of the firm is simply $qK\left[\psi^b(\overline{\omega}^*) + \psi^l(\overline{\omega}^*, s)\right]$. It is well known that with costly verification the Modigliani-Miller theorem fails to hold: capital structure matters for the firm's value as debt and equity financing differ in a nonneutral way. In the CSV environment, debt is the optimal instrument for external finance. This feature is obviously inherited by our present model. Our model, however, brings forth new observations. Given that debt is the optimal instrument for external finance, the value of a firm varies with the average size of its debt, a result stated in Proposition 6.

**Proposition 6.** In a competitive equilibrium, we have

*i.* The market value of a firm increases with the average size of its debt, $s$:

$$\frac{\partial}{\partial s}\left[qK\left(\psi^b(\overline{\omega}^*) + \psi^l(\overline{\omega}^*, s)\right)\right] > 0.$$

*ii.* The market value of a firm's equity is invariant with respect to the average size of its debt:

$$\frac{\partial}{\partial s}\left[qK\psi^b(\overline{\omega}^*)\right] = 0.$$

*iii.* The market value of a firm's debt increases with the average size of its debt:

$$\frac{\partial}{\partial s}\left[qK\psi^l(\overline{\omega}^*, s)\right] > 0.$$

Results *i-iii* in the above proposition are consistent with each other. Because firms seek to maximize the value of their equities, only when the market-determined expected rate of return increases with loan size can the arbitrage opportunities for firms be eliminated.

Thus, adopting a financing scheme with a larger average size of debt is associated with an increase in the total value of the firm's debt (Result *iii*), and the market value of a firm's equity is invariant with respect to the average size of its debt (Result *ii*) only if Result *iii* holds. Result *i*, that the market value of a firm increases with the average size of its debt, is simply a corollary of Result *ii* and *iii*. It is because of the scarcity of loan sizes that the differences in the value of a firm's debt for different average loan sizes can not be arbitraged away by the heterogenous lenders.

## 6 Conclusions

Our analysis has been conducted in a model where projects are financed solely by direct lending, and financial intermediation is not considered. The reality is that direct lending and intermediated lending coexist. Thus it is of interest to investigate the implications of size differentials in a direct lending context, just as it will be of interest to carry out a similar study in a context of financial intermediation. One might wonder whether the problem caused by the scarcity of sizes will largely disappear when projects are funded by large financial intermediaries such as large banks. We argue that this is not so. Banks themselves need to raise funds from depositors and investors, many of them being small compared to the size of banks. Hence the price of scare size remains relevant in this context. The relevance is especially prominent when there are needs to "monitor the monitor" [Krasa and Villamil (1992)]. If banks are subject to risks that cannot be fully diversified, then the kind of agency cost problem between banks and ultimate borrowers applies equally well to the relationship between banks and their liability holders. The agency cost problem gives rise to endogenously determined transaction costs that justify a price for size in our model. According to our theory, large deposits should enjoy a higher expected rate of return over small deposits. This is indeed what we observe in the banking data, though it is obvious that we cannot attribute all of the rate of return differential to their size differential. Other factors, such as liquidity, count.

It should be noted that the paper has abstracted from several realistic features of financial markets and participants in such markets. For example, the risk neutrality assumption, together with the positive price of size, ensures that lenders will not diversify their loan portfolio. Taking risk aversion into account would certainly modify this result. The essential point, however, is that even with risk aversion, the degree of portfolio diversification would be substantially deterred by the tendency to concentrate lending induced by the positive return to size.

In the discussion of the implications of the price of size for the market value of firms, we argue that the size structure of debt is a determinant of the market value of firm. This conclusion can be extended to a more general level. The point is that characteristics of debt and the composition of these characteristics in a firm's liability structure can influence the market value of the firm.

We end the paper by suggesting two potential applications of the theory. First, the price of size provides incentives for agents to agglomerate capital. This might generate interesting dynamics for the accumulation of financial and physical capital. Second, wealth inequality among agents can be magnified by the price of size because large investors receive higher rates of return on their investment. Whether size premia have significant implications for the dynamics of wealth inequality deserves future exploration.

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# Appendix. Proof of Propositions and Lemmas

The proof of Proposition 1 requires first establishing a lemma that concerns the following problem.

**Problem A1.**

$$\max_{\widetilde{P}^n(\omega),P^n} qK\int \omega g(\omega)\,d\omega - \left[\int_M \widetilde{P}^n(\omega)\,g(\omega)\,d\omega + \int_{M^c} P^n g(\omega)\,d\omega\right]$$

*s.t.*

$$\int_M \left[\widetilde{P}^n(\omega) - n\gamma\right] g(\omega)\,d\omega + \int_{M^c} P^n g(\omega)\,d\omega \geq R^n,$$

where $R^n \equiv \sum_{i=1}^n R(s_i)\,s_i$.

Problem A1 is derived by consolidating the $n$ constraints in Problem 1 into a single aggregate constraint. Lemma A1 states the relation between the solutions to Problem 1 and Problem A1.

**Lemma A1.** If $\left(\widetilde{P}^n(\omega), P^n\right)$ is a solution to Problem A1, then there exists $\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n$ such that it is a solution to Problem 1 and that $\sum_{i=1}^n \widetilde{P}(s_i,\omega) = \widetilde{P}^n(\omega)$, $\sum_{i=1}^n P(s_i) = P^n$. Conversely, if $\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n$ is a solution to Problem 1, then $\left(\widetilde{P}^n(\omega), P^n\right)$ where $\widetilde{P}^n(\omega) = \sum_{i=1}^n \widetilde{P}(s_i,\omega)$ and $P^n = \sum_{i=1}^n P(s_i)$ is a solution to Problem A1.

**Proof.** Suppose $\left(\widetilde{P}^n(\omega), P^n\right)$ is a solution to Problem A1. Let $M = \left\{\omega : \widetilde{P}^n(\omega) < P^n\right\}$, and

$$\begin{aligned}
\widetilde{P}(s_i,\omega) &= \widetilde{P}^n(\omega)\,\frac{R(s_i)\,s_i + \gamma\int_M g(\omega)\,d\omega}{R^n + n\gamma\int_M g(\omega)\,d\omega}, \\
P(s_i) &= P^n\,\frac{R(s_i)\,s_i + \gamma\int_M g(\omega)\,d\omega}{R^n + n\gamma\int_M g(\omega)\,d\omega}, \quad i = 1,...,n
\end{aligned}$$

Then $\widetilde{P}^n(\omega) < P^n$ if and only if $\widetilde{P}(s_i,\omega) < P(s_i)$, $i = 1, ..., n$, implying that $\left(\widetilde{P}^n(\omega), P^n\right)$ and $\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n$ generate the same monitoring set. We also have $\sum_{i=1}^n \widetilde{P}(s_i,\omega) = \widetilde{P}^n(\omega)$, $\sum_{i=1}^n P(s_i) = P^n$. Furthermore, $\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n$ satisfy the constraints in Problem 1 for all $i \in \{1, ..., n\}$. Since Problem A1 is less constrained than Problem 1, as $\left(\widetilde{P}^n(\omega), P^n\right)$ attains the maximum in Problem A1, $\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n$ so constructed must attain the maximum in Problem 1. Therefore this $\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n$ is a solution to Problem 1.

Suppose $\left\{\widetilde{P}(s_i,\omega), P(s_i)\right\}_{i=1}^n$ is a solution to Problem 1. According to the preceding paragraph, the maximum attained equals the maximum attained in Problem A1. Let $\widetilde{P}^n(\omega) = \sum_{i=1}^n \widetilde{P}(s_i,\omega)$ and $P^n = \sum_{i=1}^n P(s_i)$. By summing over all constraints in Problem 1, one can see that $\left(\widetilde{P}^n(\omega), P^n\right)$ satisfies the constraint in Problem A1. Furthermore, it attains the maximum in Problem A1. Therefore $\left(\widetilde{P}^n(\omega), P^n\right)$ so constructed is a solution to Problem A1. *Q.E.D.*

**Proof of Proposition 1.**

First note that the solution to Problem A1 entails $\widetilde{P}^n(\omega) = q\omega K$ for $\omega \in M$. Since Problem A1 is identical to the problem considered in Williamson (1986). The proof therein applies. Then Lemma A1 implies that the solution to Problem 1 entails $\sum_{i=1}^n \widetilde{P}(s_i,\omega) = q\omega K$ for $\omega \in M$. *Q.E.D.*

**Proof of Lemma 1.**

First note that constraint (3) must bind in an optimal solution to Problem 2. That is,

$qK\psi^l(\overline{\omega}^*, s) = R^n$, which can be rewritten as

$$q\psi^l(\overline{\omega}^*, s) = \overline{R}. \tag{4}$$

Recall that $\psi^l(\overline{\omega}^*, s)$ is increasing in both arguments. Thus holding other things fixed, an increase in $\overline{R}$ raises $\overline{\omega}^*$. An increase in $s$, other things equal, would raise the left side of (4) if $\overline{\omega}^*$ did not change. Hence to maintain the equality of (4) $\overline{\omega}^*$ must decline. This proves the conclusion that $\overline{\omega}^*$ is increasing in $\overline{R}$ and decreasing in $s$. Since $\psi^b(\overline{\omega})$ is decreasing in $\overline{\omega}$, we have that $\psi^b(\overline{\omega}^*)$ is decreasing in $\overline{R}$ and increasing in $s$. *Q.E.D.*

**Proof of Proposition 2.**

Consider borrower $j, k \in \{1, ..., \tau\}$. Denote their financing schemes by $\{s_i\}_{i=1}^{n_j}$ and $\{s_i\}_{i=1}^{n_k}$, respectively. In a competitive equilibrium we must have $\overline{\omega}_j^* = \overline{\omega}_k^*$, otherwise a profitable opportunity exists for some lender. Suppose $\overline{\omega}_j^* < \overline{\omega}_k^*$. Then $\psi^b\left(\overline{\omega}_j^*\right) > \psi^b(\overline{\omega}_k^*)$. It is feasible for borrower $k$ to adopt the financing scheme $\{s_i\}_{i=1}^{n_j}$. She can simply offer an expected rate of return $R(s_i) + \varepsilon$ to $s_i$, $i = 1, ..., n_j$, where $\varepsilon$ is a positive constant arbitrarily close to zero. Borrower $k$'s expected profit will increase from $\psi^b(\overline{\omega}_k^*)$ to a level arbitrarily close to $\psi^b\left(\overline{\omega}_j^*\right)$. In equilibrium such profitable opportunity does not exist. This establishes our claim that $\overline{\omega}_j^* = \overline{\omega}_k^*$.

According to (4), we have

$$\begin{aligned} q\psi^l\left(\overline{\omega}_j^*, \overline{s}_j\right) &= \overline{R}_j, \\ q\psi^l\left(\overline{\omega}_k^*, \overline{s}_k\right) &= \overline{R}_k. \end{aligned}$$

Since $\psi^l$ is monotonically increasing in its second argument, we have the stated conclusion. *Q.E.D.*

**Proof of Proposition 3.**

Consider a set of contracting relationships where for each borrower $j \in \{1, ..., \tau\}$, $s_i = \overline{s}_j = s$, $i = 1, ..., n_j$, for some $s \in (0, S_1]$. With this set of contracting relationships, we have $\overline{R}_j = R(s)$, $s \in (0, S_1]$, $j = 1, ..., \tau$. According to Proposition 2, for these contracting relationships and the expected rate of return schedule to constitute a competitive equilibrium, we must have

$$R(s) > (=)\ R(s') \quad \text{if} \quad s > (=)\ s'.$$

Let $\kappa$ satisfy

$$\sum_{h=1}^{\kappa-1} S_h N_h < \tau K, \quad \sum_{h=1}^{\kappa} S_h N_h \geq \tau K, \quad 0 < \kappa < \eta.$$

Determine $\overline{\omega}^* \in (0, \widehat{\omega})$ implicitly by

$$q\psi^l(\overline{\omega}^*, S_\kappa) = R^f.$$

Then using (4), we have

$$q\psi^l(\overline{\omega}^*, s) = R(s), \quad q\psi^l(\overline{\omega}^*, s') = R(s'), \quad s, s' \in (0, S_1].$$

But,

$$\psi^l(\overline{\omega}, s) \equiv [1 - G(\overline{\omega})]\,\overline{\omega} + \int_0^{\overline{\omega}} \omega dG(\omega) - G(\overline{\omega})\frac{\gamma}{qs}, \quad s \in (0, S_1].$$

Thus,

$$R(s) - R(s') = \gamma G(\overline{\omega}^*)\left(\frac{1}{s'} - \frac{1}{s}\right), \quad s, s' \in (0, S_1]. \tag{5}$$

We now show that the set of contracting relationships and the expected rate of return schedule stated in the Proposition indeed constitute a competitive equilibrium. We shall argue that no borrower will prefer to deviate from her current financing scheme. Rewrite (5) as

$$R(s) = R(K) - \frac{\gamma G(\overline{\omega}^*)}{K}\left(\frac{K}{s} - 1\right), \quad s \in (0, S_1].$$

Consider any financing scheme (not necessarily evenly partitioned) where the number of lenders is $n$ and the average expected rate of return is $\overline{R}$. Denote this financing scheme by $\{s_i\}_{i=1}^n$. Then $\overline{R} = R(s)$ with $s = K/n$. To see this, note that

$$\begin{aligned}
\overline{R} &\equiv \sum_{i=1}^{n} \frac{s_i}{K} R(s_i) \\
&= \sum_{i=1}^{n} \frac{s_i}{K}\left[R(K) - \frac{\gamma G(\overline{\omega}^*)}{K}\left(\frac{K}{s_i} - 1\right)\right] \\
&= R(K) - \frac{\gamma G(\overline{\omega}^*)}{K}\left(\frac{K}{s} - 1\right) \\
&= R(s), \quad s = K/n.
\end{aligned}$$

Therefore any financing scheme where the number of lenders is $n$ are equally costly. It prescribes the same monitoring threshold $\overline{\omega}^*$ and yields the same expected profit, $qK\psi^b(\overline{\omega}^*)$, for any borrower. Now consider any two financing schemes where the numbers of lenders and the average expected rates of return are $(n, \overline{R})$ and $(n', \overline{R}')$, respectively. The way we derive Equation (5) implies that these two financing schemes give the same expected profit, $qK\psi^b(\overline{\omega}^*)$, for any borrower. Finally, the loan market clears with $\overline{\omega}^*$ determined by (6). *Q.E.D.*

**Proof of Proposition 4.**

Since lenders are risk neutral, a lender in size class $h$, $1 \leq h \leq \kappa$, solves the following problem,

$$\max_{m, \{x_i\}_{i=1}^m} \sum_{i=1}^{m} R(x_i)\, x_i$$

*s.t.*

$$\sum_{i=1}^{m} R(x_i)\, x_i \leq S_h.$$

Given that $R(\cdot)$ is monotonically increasing, the solution obviously entails $m = 1$ and $x_1 = S_h$. *Q.E.D.*

**Proof of Proposition 5.**

Under the linear contract structure the monitoring threshold, $\overline{\omega}_h^f$, for a contracting relationship with the average loan size being $S_h$, $h \in \{1, ..., \kappa\}$, is given by

$$q\psi^l\left(\overline{\omega}_h^f, S_h\right) = R^f.$$

Since $\psi^l$ is monotonically increasing in both its first and second arguments, we have $\overline{\omega}_1^f < \overline{\omega}_2^f < ... < \overline{\omega}_\kappa^f = \overline{\omega}^*$.

Under the equilibrium nonlinear contract, for $S_h$, $h \in \{1, ..., \kappa\}$, the common monitoring threshold is $\overline{\omega}^*$. But $\partial\left[\psi^b(\overline{\omega}) + \psi^l(\overline{\omega}, s)\right]/\partial\overline{\omega} < 0$, $s \in (0, S_1]$. Thus $\psi^b(\overline{\omega}^*) + \psi^l(\overline{\omega}^*, S_h) < \psi^b\left(\overline{\omega}_h^f\right) + \psi^l\left(\overline{\omega}_h^f, S_h\right)$, $h = 1, ..., \kappa - 1$. *Q.E.D.*

**Proof of Proposition 6.**

Since $\overline{\omega}^*$ is independent of $s$, $\psi^b(\overline{\omega}^*)$ is independent of $s$, and $\psi^l(\overline{\omega}^*, s)$ increases with $s$, we have the stated conclusions. *Q.E.D.*