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A Stochastic Discount Factor Approach to Asset Pricing Using Panel Data

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A Stochastic Discount Factor Approach to Asset Pricing Using Panel Data[*]

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Abstract

Using the Pricing Equation, in a panel-data framework, we construct a novel consistent estimator of the stochastic discount factor (SDF) mimicking portfolio which relies on the fact that its logarithm is the "common feature" in every asset return of the economy. Our estimator is a simple function of asset returns and does not depend on any parametric function representing preferences, making it suitable for testing different preference specifications or investigating intertemporal substitution puzzles.

1 Introduction

We derive a novel consistent estimator of the stochastic discount factor (SDF) mimicking portfolio that takes seriously the consequences of the Pricing Equation established by Harrison and Kreps (1979), Hansen and Richard (1987), and Hansen and Jagannathan (1991), where asset prices today are a function of their expected discounted future payoffs. If the Pricing Equation is valid for all assets at all times, it can serve as a basis to construct an estimator of the SDF mimicking portfolio in a panel-data framework when the number of assets and of time periods are sufficiently large.

We start with an exact taylor expansion of the Pricing Equation to derive the determinants of the *logarithm* of asset returns. The identification strategy employed to recover the *logarithm* of the mimicking portfolio relies on one of its basic properties – it is a "common feature" of every asset return of the economy; see Hansen and Singleton (1983) and Engle and Kozicki (1993). Under plausible restrictions on the behavior of asset returns, we show how to construct a consistent estimator of the SDF mimicking portfolio which is a simple function of the arithmetic and geometric averages of asset returns alone. This allows to study intertemporal asset pricing without the need to characterize preferences or the use of consumption data; see Hansen and Jagannathan (1991) and Campbell(1993) for similar alternatives.

Our approach is related to the recent financial econometrics literature on the SDF (Rosenberg and Engle (2002) and Chen and Ludvigson (2004)), to work on common factors in macroeconomics and finance (Chamberlain and Rothschild (1983), Connor and Korajzcyk (1986), Forni et al. (2000), Lettau and Ludvigson (2001), Stock and Watson (2002), Bai and Ng (2002, 2004), Bai (2005), and Pesaran (2005)), and to the recent work of Mulligan (2002) on cross-sectional aggregation. It is also related to Long's concept of numeraire portfolio as discussed in Bajeux-Besnainou and Portait (1997).

The next Section presents basic theoretical results and a discussion of our assumptions. Section 3 contains the main result and Section 4 a brief discussion of it.

2 Economic Theory and Econometric Setup

2.1 Economic Theory, Econometrics, and Basic Assumptions

Harrison and Kreps (1979), Hansen and Richard (1987), and Hansen and Jagannathan (1991) describe a general framework to asset pricing, associated to the stochastic discount factor (SDF), which relies on the Pricing Equation:

$$\mathbb{E}_t \left\{ M_{t+1} R_{i,t+1} \right\} = 1, \qquad i = 1, 2, \ldots, N, \tag{1}$$

where $\mathbb{E}_t(\cdot)$ denotes the conditional expectation given the information available at time t, M_{t+1} is the stochastic discount factor, and $R_{i,t+1}$ is the gross return of the i-th asset in $t+1$. N is the number of assets in the economy.

The SDF contains all sources of aggregate risk in the economy but does not contain idiosyncratic risk. It is also the only source of risk that matters for pricing assets. Individual-asset risk only matters for pricing if it is correlated with the SDF. Existence of M_{t+1} is obtained under mild conditions, but uniqueness requires complete markets. Under incomplete markets, there still exists a unique discount factor M_{t+1}^* – the SDF mimicking portfolio – which is an element of the payoff space and prices all traded securities. There is an infinite number of SDFs pricing assets, but all can be decomposed as $M_{t+1} = M_{t+1}^* + \nu_{t+1}$, with $\mathbb{E}_t(\nu_{t+1} R_{i,t+1}) = 0$. Hence, we can think of the SDF mimicking portfolio as a "SDF generator." Since the economic environment we deal with is that of incomplete markets, it only makes sense to devise econometric techniques to estimate the unique SDF mimicking portfolio M_{t+1}^*. This is exactly the goal of this paper. Nevertheless, we use the words SDF,

SDF mimicking portfolio, and mimicking portfolio interchangeably throughout the paper.

Assumption 1: The Pricing Equation (1) holds.

Assumption 2: The mimicking portfolio obeys $M_t^* > 0$.

To construct a consistent estimator for M_t^* we consider a second-order taylor expansion of the exponential function around x, with increment h, as follows:

$$e^{x+h} = e^x + he^x + \frac{h^2 e^{x+\lambda(h) \cdot h}}{2}, \text{ with } \lambda(h) : \mathbb{R} \to (0,1) . \tag{2}$$

For a generic function, $\lambda(\cdot)$ depends on x and h, but not for the exponential function. Indeed, dividing (2) by e^x, we get:

$$e^h = 1 + h + \frac{h^2 e^{\lambda(h) \cdot h}}{2}, \tag{3}$$

showing that $\lambda(\cdot)$ depends only on h, being straightforward to get a closed-form solution for $\lambda(h)$. To connect (3) with the Pricing Equation (1), we let $h = \ln(M_t^* R_{i,t})$ to obtain:

$$M_t^* R_{i,t} = 1 + \ln(M_t^* R_{i,t}) + \frac{[\ln(M_t^* R_{i,t})]^2 \, e^{\lambda(\ln(M_t^* R_{i,t})) \cdot \ln(M_t^* R_{i,t})}}{2}. \tag{4}$$

It is important to stress that (4) is not an approximation but an exact relationship. The behavior of $M_t^* R_{i,t}$ is governed solely by that of $\ln(M_t^* R_{i,t})$, which motivates our next assumption.

Assumption 3: Let $\mathbf{R}_t = (R_{1,t}, R_{2,t}, \dots R_{N,t})'$ be an $N \times 1$ vector stacking all asset returns in the economy. The vector process $\{\ln(M_t^* \mathbf{R}_t)\}$ is assumed to be covariance stationary with finite first and second moments[1].

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[1] See Hamilton (1994, chapter 10) for a precise definition.

4

It is useful to define a stochastic process collecting the higher-order term in (4):

$$z_{i,t} \equiv \frac{1}{2} \times [\ln(M_t^* R_{i,t})]^2 \, e^{\lambda(\ln(M_t^* R_{i,t})) \cdot \ln(M_t^* R_{i,t})}.$$

Notice that $z_{i,t}$ is a function of $\ln(M_t^* R_{i,t})$ alone and that $z_{i,t} \geq 0$ for all (i,t). Taking the conditional expectation of both sides of (4), imposing the Pricing Equation and rearranging terms, gives:

$$\mathbb{E}_{t-1}(z_{i,t}) = -\mathbb{E}_{t-1}\left\{\ln(M_t^* R_{i,t})\right\}. \tag{5}$$

This is an important result, since it allows characterizing the first moment of $z_{i,t}$ using solely the first moment of $\ln(M_t^* R_{i,t})$, without resorting explicitly to $\lambda(\ln(M_t^* R_{i,t}))$. Non-negativity of $z_{i,t}$ implies $\mathbb{E}_{t-1}(z_{i,t}) \equiv \gamma_{i,t}^2 \geq 0$, which motivates the notation $\gamma_{i,t}^2$. Define now the forecast errors $\varepsilon_{i,t} = \ln(M_t^* R_{i,t}) - \mathbb{E}_{t-1}\left\{\ln(M_t^* R_{i,t})\right\}$ and let $\boldsymbol{\gamma}_t^2 \equiv \left(\gamma_{1,t}^2, \gamma_{2,t}^2, ..., \gamma_{N,t}^2\right)'$ and $\boldsymbol{\varepsilon}_t \equiv (\varepsilon_{1,t}, \varepsilon_{2,t}, ..., \varepsilon_{N,t})'$. From the definition of $\boldsymbol{\varepsilon}_t$ and (5) we have:

$$\ln(M_t^* \mathbf{R}_t) = \mathbb{E}_{t-1}\{\ln(M_t^* \mathbf{R}_t)\} + \boldsymbol{\varepsilon}_t = -\boldsymbol{\gamma}_t^2 + \boldsymbol{\varepsilon}_t. \tag{6}$$

Denoting $\mathbf{r}_t = \ln(\mathbf{R}_t)$, with elements denoted by $r_{i,t}$, and $m_t^* = \ln(M_t^*)$, we write (6) as:

$$r_{i,t} = -m_t^* - \gamma_{i,t}^2 + \varepsilon_{i,t}, \qquad i = 1, 2, \ldots, N. \tag{7}$$

System (7) shows that the (log of the) SDF (m_t^*) is a common feature, in the sense of Engle and Kozicki (1993), of all (logged) asset returns[2]. For any two economic series, a common feature exists if it is present in both of them and can be removed by linear combination. Here, subtracting any two (logged) returns eliminates the term m_t^*. It is interesting to note

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[2]We could have derived (7) following Blundell, Browning, and Meghir (1994, p. 60, eq. (2.10)), by writing the Pricing Equation as $M_{t+1} R_{i,t+1} = u_{i,t+1}$, $i = 1, 2, \ldots, N$. They argue that $\mathbb{E}_t\{\ln(u_{i,t+1})\}$ is a function of higher-order moments of $\ln(u_{i,t+1})$. Our expansion in (4) makes clear the exact way in which $\mathbb{E}_t\{\ln(u_{i,t+1})\}$ depends on these higher-order moments and our consistency proof follows directly from exploiting this relationship.

that the idea of common features had been used in finance much earlier than its formal characterization by Engle and Kozicki. Indeed, Hansen and Singleton (1983) proposed a dynamic version of (7) – a VAR model for logged returns and consumption growth – the latter being the basic variable in m_t^*. Explicit in their VAR are common-feature restrictions identical to the ones in (7), i.e., that $r_{i,t} - r_{j,t}$ eliminates $\mathbb{E}_{t-1}(m_t^*)$; see Hansen and Singleton (1983, p. 255, equation 15).

Our strategy to obtain a consistent estimator for M_t^* starts with averaging (7) across i to obtain a consistent estimator for m_t^*. By inspecting (7), one immediately (and correctly) suspects that we need $\plim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$ to hold. However, if we decompose $\varepsilon_{i,t}$ as:

$$\varepsilon_{i,t} = [m_t^* - \mathbb{E}_{t-1}(m_t^*)] + [r_{i,t} - \mathbb{E}_{t-1}(r_{i,t})] = q_t + v_{i,t}, \tag{8}$$

where $q_t = [m_t^* - \mathbb{E}_{t-1}(m_t^*)]$ is the factor (feature) innovation, $v_{i,t} = [r_{i,t} - \mathbb{E}_{t-1}(r_{i,t})]$ is the data innovation, and $\varepsilon_{i,t}$ is the factor-model error innovation, it becomes clear that it is impossible to apply a weak law-of-large-numbers (WLLN) simultaneously to $\varepsilon_{i,t}$ and $v_{i,t}$, since q_t has no cross-sectional variation. However, to get $\plim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$, we need,

$$\plim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} v_{i,t} = -q_t. \tag{9}$$

To see restriction (9) in a more familiar setting, consider a K-factor model in $\widetilde{r_{i,t}}$ and $\widetilde{m_t^*}$, which are respectively demeaned versions of $r_{i,t}$ and m_t^*:

$$\widetilde{r_{i,t}} = \sum_{k=1}^{K} \beta_{i,k} f_{k,t} + \eta_{it}, \text{ and, } \widetilde{m_t^*} = \sum_{k=1}^{K} \beta_k f_{k,t}, \tag{10}$$

where $f_{k,t}$ are zero-mean pervasive factors and, as is usual in factor analysis, $\plim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \eta_{i,t} = 0$[3]. Then (9) translates into the following assumption:

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[3]One may argue that, if there exists a factor representation for the level variables $R_{i,t}$ and M_t^*, then the

Assumption 4 (Identification Condition): We assume that $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \beta_{i,k} = -\beta_k$, for $k = 1, 2, \cdots, K$.

2.2 Discussing Our Assumptions

Assumption 1 is present in virtually all studies in finance and macroeconomics dealing with asset pricing and intertemporal substitution. It is equivalent to the "law of one price." Assumption 2 is required because we need to take logs of M_t^* to have an explicit common-feature representation. Weak stationarity in Assumption 3 controls the degree of time-series dependence in the data. It is completely justified on empirical grounds. It is well known that asset returns also display signs of conditional heteroskedasticity, which can coexist with weak stationarity; see Engle (1982), among others.

Assumption 4 is a weaker version of second-moment identification restrictions commonly used in the factor-model literature, e.g., Chamberlain and Rothschild (1983, pp. 1284-5), Connor and Korajzcyk (1986, p. 376), Stock and Watson (2002, p. 1168), Bai and Ng (2002, pp. 196-7), and Bai (2005, p. 10). For instance, Bai characterizes weak cross-sectional dependence for factor-model errors as $\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |\mathbb{E}(\varepsilon_{i,t}\varepsilon_{j,t})| < \infty$, which implies $\lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\mathbb{E}(\varepsilon_{i,t}\varepsilon_{j,t})| = 0$, a sufficient condition for $\lim_{N\to\infty} \text{VAR}\left(\frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t}\right) = 0$ and for $\text{plim}_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$. Here, we use Assumption 4, which, in a large economy setting, implies that the equally-weighted portfolio (logs) is well diversified, a necessary and sufficient condition for $\text{plim}_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$ to hold[4]; see an identical condition in Pesaran (2005, pp. 6-7) when equal weights $(1/N)$ are considered.

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m_t^* equation in (10) must include an additional measurement error term. The latter must be independent of factors $f_{k,t}$, if one wants to reconcile the pricing properties of the level and the logarithmic representations.

[4]Note that $\text{plim}_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$ is implied but does not imply $\lim_{N\to\infty} \text{VAR}\left(\frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t}\right) = 0$.

3 Main Result

Theorem 1 *If the vector process $\{\ln(M_t^* \mathbf{R}_t)\}$ satisfies assumptions 1 to 4, the realization of the SDF mimicking portfolio at time t, denoted by M_t^*, can be consistently estimated as $N, T \to \infty$ using:*

$$\widehat{M_t^*} = \frac{\overline{R}_t^G}{\frac{1}{T}\sum_{j=1}^{T}\left(\overline{R}_j^G \overline{R}_j^A\right)},$$

where $\overline{R}_t^G = \prod_{i=1}^{N} R_{i,t}^{-\frac{1}{N}}$ and $\overline{R}_t^A = \frac{1}{N}\sum_{i=1}^{N} R_{i,t}$ are respectively the geometric average of the reciprocal of all asset returns and the arithmetic average of all asset returns.

Proof. Because $\ln(M_t^* \mathbf{R}_t)$ is weakly stationary, for every one of its elements $\ln(M_t^* R_{i,t})$, there exists a Wold representation of the form:

$$\ln(M_t^* R_{i,t}) = \mu_i + \sum_{j=0}^{\infty} b_{i,j}\varepsilon_{i,t-j} \tag{11}$$

where, for all i, $b_{i,0} = 1$, $|\mu_i| < \infty$, $\sum_{j=0}^{\infty} b_{i,j}^2 < \infty$, and $\varepsilon_{i,t}$ is white noise. Taking the unconditional expectation of (5), in light of (11), leads to $\gamma_i^2 \equiv \mathbb{E}(z_{i,t}) = -\mathbb{E}\{\ln(M_t^* R_{i,t})\} = -\mu_i$, which are well defined and time-invariant under Assumption 3. Taking conditional expectations of (11), using $\varepsilon_{i,t} = \ln(M_t^* R_{i,t}) - \mathbb{E}_{t-1}\{\ln(M_t^* R_{i,t})\}$, yields:

$$r_{i,t} = -m_t^* - \gamma_i^2 + \varepsilon_{i,t} - \sum_{j=1}^{\infty} b_{i,j}\varepsilon_{i,t-j}, \qquad i = 1, 2, \ldots, N. \tag{12}$$

We consider now a cross-sectional average of (12):

$$\frac{1}{N}\sum_{i=1}^{N} r_{i,t} = -m_t^* - \frac{1}{N}\sum_{i=1}^{N}\gamma_i^2 + \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t} - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\infty} b_{i,j}\varepsilon_{i,t-j}, \tag{13}$$

and examine convergence in probability of $\frac{1}{N}\sum_{i=1}^{N} r_{i,t} + m_t^*$ using (13). For the deterministic term $-\frac{1}{N}\sum_{i=1}^{N}\gamma_i^2$, because every term $\ln(M_t^* R_{i,t})$ has a finite unconditional mean uniformly

8

bounded in i, $\mu_i = -\gamma_i^2$, the limit of their average must be finite, i.e.,

$$-\infty < \lim_{N\to\infty} -\frac{1}{N}\sum_{i=1}^{N}\gamma_i^2 \equiv -\gamma^2 < 0.$$

Compute now $\frac{1}{N}\sum_{i=1}^{N}\widetilde{r_{i,t}} + \widetilde{m_t^*}$ using (10):

$$\frac{1}{N}\sum_{i=1}^{N}\widetilde{r_{i,t}} + \widetilde{m_t^*} = \sum_{i=1}^{N}\sum_{k=1}^{K}\left(\frac{1}{N}\beta_{i,k} + \beta_k\right)f_{k,t} + \frac{1}{N}\sum_{i=1}^{N}\eta_{i,t} \qquad (14)$$

$$= \sum_{k=1}^{K}\left[\left(\frac{1}{N}\sum_{i=1}^{N}\beta_{i,k}\right) + \beta_k\right]f_{k,t} + \frac{1}{N}\sum_{i=1}^{N}\eta_{i,t}. \qquad (15)$$

As $N \to \infty$, using Assumption 4, coupled with a demeaned version of (13), we obtain:

$$\operatorname*{plim}_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\widetilde{r_{i,t}} + \widetilde{m_t^*} = \operatorname*{plim}_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\eta_{i,t} = \operatorname*{plim}_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t} - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\infty}b_{i,j}\varepsilon_{i,t-j}\right) = 0,$$

showing that a WLLN applies to $\{\varepsilon_{i,t}\}_{i=1}^{N}$ and to $\{b_{i,j}\varepsilon_{i,t}\}_{i=1}^{N}$, and that

$$\frac{1}{N}\sum_{i=1}^{N}r_{i,t} + m_t^* \xrightarrow{p} -\gamma^2.$$

Hence, a consistent estimator for $e^{\gamma^2} \times M_t^* = \widetilde{M_t^*}$ is given by:

$$\widetilde{\widehat{M_t^*}} = \prod_{i=1}^{N}R_{i,t}^{-\frac{1}{N}}. \qquad (16)$$

To estimate e^{γ^2} consistently, multiply the Pricing Equation by e^{γ^2}, take the unconditional expectation and average across i to get:

$$e^{\gamma^2} = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left\{\widetilde{M_t^*}R_{i,t}\right\}.$$

As $N, T \to \infty$, a consistent estimator for e^{γ^2} using (16) is:

$$\widehat{e^{\gamma^2}} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{T} \sum_{t=1}^{T} \widetilde{\widehat{M_t^*}} R_{i,t} \right) = \frac{1}{T} \sum_{t=1}^{T} \left[\left(\prod_{i=1}^{N} R_{i,t}^{-\frac{1}{N}} \right) \left(\frac{1}{N} \sum_{i=1}^{N} R_{i,t} \right) \right] = \frac{1}{T} \sum_{t=1}^{T} \overline{R}_t^G \overline{R}_t^A.$$

We can finally propose a consistent estimator for M_t^*, as $N, T \to \infty$:

$$\widehat{M_t^*} = \frac{\widehat{\widetilde{M_t^*}}}{\widehat{e^{\gamma^2}}} = \frac{\overline{R}_t^G}{\frac{1}{T} \sum_{j=1}^{T} \overline{R}_j^G \overline{R}_j^A}. \tag{17}$$

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As a consequence of Assumption 4, $\frac{1}{N} \sum_{i=1}^{N} \widetilde{r_{i,t}} + \widetilde{m_t^*} \xrightarrow{p} 0$. This allows the use of equal weights $(1/N)$ in computing $\prod_{i=1}^{N} R_{i,t}^{-\frac{1}{N}}$ and avoids the estimation of weights in forming $\widehat{M_t^*}$. At first sight, it may look that Assumption 4 is the price to pay to have this convenient feature of (17). A potential alternative route is to postulate a factor model, for which we know that a WLLN applies to its error terms, therefore dispensing with Assumption 4:

$$\widetilde{r_{i,t}} = -\beta_i \widetilde{m_t^*} + \xi_{i,t}, \text{ where } \beta_i \equiv \frac{\text{COV}\left(\widetilde{r_{i,t}}, -\widetilde{m_t^*} \right)}{\text{VAR}\left(\widetilde{m_t^*} \right)}, \text{ and } \plim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \xi_{i,t} = 0. \tag{18}$$

Here, the factor $-\widetilde{m_t^*}$ is a latent scalar variable. Factor-model estimation traditionally employs principal-component and factor analyses. Because we are solely interested in estimating $\widetilde{m_t^*}$, and then M_t^*, we have no interest in decomposing $\widetilde{m_t^*}$ further as in (10). It is important to note that:

$$\plim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\widetilde{r_{i,t}}}{\beta_i} + \widetilde{m_t^*} = \plim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\xi_{i,t}}{\beta_i} = 0,$$

because the β_i's are bounded due to Assumption 3. If we could find consistent estimators for weights β_i's, we could follow the same steps in the proof of Theorem 1, use the Pricing

Equation, and obtain a consistent estimator for M_t^*:

$$\frac{\prod_{i=1}^N R_{i,t}^{-1/\widehat{\beta}_i \times N}}{\frac{1}{T} \sum_{j=1}^T \left[\left(\prod_{i=1}^N R_{i,j}^{-1/\widehat{\beta}_i \times N} \right) \left(\frac{1}{N} \sum_{i=1}^N R_{i,j} \right) \right]}, \tag{19}$$

where $\widehat{\beta}_i$ is a consistent estimator of β_i.

The application of principal-component and factor analyses to estimate financial models with a large number of assets was originally suggested by Chamberlain and Rothschild. Here, we show that these estimates do not have a precise structural econometric interpretation, i.e., that the first principal component of the elements of $\widetilde{\mathbf{r}}_t = (\widetilde{r_{1,t}}, \widetilde{r_{2,t}}, ..., \widetilde{r_{N,t}})'$ does not identify $\widetilde{m_t^*}$ exactly and that the respective factor loadings do not identify β_i, $i = 1, 2, \ldots, N$ exactly.

To prove it, denote by $\Sigma_{\mathbf{r}} = \mathbb{E}\left(\widetilde{\mathbf{r}}_t \widetilde{\mathbf{r}}_t'\right)$ the variance-covariance matrix of logged returns. The first principal component of $\widetilde{\mathbf{r}}_t$ is a linear combination $\alpha' \widetilde{\mathbf{r}}_t$ with maximal variance. As discussed in Dhrymes (1974), since its variance is $\alpha' \Sigma_{\mathbf{r}} \alpha$, the problem has no unique solution – we can make $\alpha' \Sigma_{\mathbf{r}} \alpha$ as large as we want by multiplying α by a constant $\kappa > 1$. Indeed, we are facing a *scale* problem, which is solved by imposing unit norm for α in a fixed N setting, i.e., $\alpha' \alpha = 1$. To understand why we need such restriction, average (18) across i, taking the probability limit to obtain:

$$\operatorname*{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \widetilde{r_{i,t}} = -\left(\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \beta_i \right) \widetilde{m_t^*} = -\overline{\beta} \cdot \widetilde{m_t^*}, \tag{20}$$

where the last equality defines notation. Equation (20) shows that we cannot separately identify $\overline{\beta}$ and $\widetilde{m_t^*}$. This is a problem for factor models and for any other estimator trying to estimate the latent variable $\widetilde{m_t^*}$. We have only one equation: the left-hand-side has observables, but the right-hand-side has two unknowns ($\overline{\beta}$ and $\widetilde{m_t^*}$). Therefore, we need an additional equation (restriction) to uniquely identify $\widetilde{m_t^*}$. Assumption 4 offers $\overline{\beta} = 1$.

We now turn to traditional factor-model identification restrictions when there is only one factor:

$$\widetilde{r_{i,t}} = -\lambda_i f_t + \mu_{i,t},$$

where f_t is a scalar factor and λ_i is its respective factor loading for the i-th asset. We now discuss the equivalence between f_t and $\widetilde{m_t^*}$ and between λ_i and β_i. When N is large, Stock and Watson (2002, p. 1168) list two critical assumptions needed for unique identification of f_t and $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$: (a) $\frac{1}{N}\Lambda'\Lambda = \frac{1}{N}\sum_{i=1}^{N}\lambda_i^2 \to 1$, and (b) $\text{VAR}(f_t) = \sigma^2 > 0$; see also Bai (2005). Assumption (a) implies that $\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\text{COV}(\widetilde{r_{i,t}},f_t)}{\sigma^2}\right]^2 \to 1$, despite the fact that there is no reason to believe that the β_i's are such that $\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\text{COV}\left(\widetilde{r_{i,t}},-\widetilde{m_t^*}\right)}{\text{VAR}\left(\widetilde{m_t^*}\right)}\right]^2 \to 1$ will hold. Because the β_i's are uniformly bounded, we know that $\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\text{COV}\left(\widetilde{r_{i,t}},-\widetilde{m_t^*}\right)}{\text{VAR}\left(\widetilde{m_t^*}\right)}\right]^2 \to c < \infty$, but there is no reason for c to be unity. This is the same *scale* problem alluded above.

Although the first principal component of the elements of $\widetilde{\mathbf{r}}_t$ will be a consistent estimator of $\widetilde{m_t^*}$ up to a scalar multiplication, and factor loadings can also be consistently estimated up to a scalar multiplication, the scale itself matters in this case, since the Pricing Equation is a structural equation and identification up to a scalar creates a problem of indeterminacy.

Looking back at (20) shows that Assumption 4 "fixes" the scale by imposing $\overline{\beta} = 1$. Hence, Assumption 4 is as restrictive as the use of principal-component and factor analyses. It just imposes a different identification restriction. Because the first principal component is only identifiable up to a scalar multiplication, we can actually "choose" this scalar to avoid estimating weights β_i. This is implicit in $\overline{\beta} = 1$, which decomposes $\frac{1}{N}\sum_{i=1}^{N}\widetilde{r_{i,t}}$ into two orthogonal components: $-\widetilde{m_t^*}$ and $\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t} - \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{\infty}b_{i,j}\varepsilon_{i,t-j}$. Therefore,

$$\lim_{N\to\infty}\mathbb{E}\left[\widetilde{m_t^*} \times \frac{1}{N}\sum_{i=1}^{N}\left(\varepsilon_{i,t} - \sum_{j=1}^{\infty}b_{i,j}\varepsilon_{i,t-j}\right)\right] = 0. \tag{21}$$

4 Discussing the Main Result

There are three important features of $\widehat{M_t^*}$: (i) it is a simple fully non-parametric estimator of the realizations of the mimicking portfolio using asset-return data alone. (ii) Although, as discussed in Cochrane (2001), no arbitrage imposes mild restrictions on preferences, $\widehat{M_t^*}$ is "preference-free," since here we made no assumptions on a functional form for preferences. Hence, it can be used to investigate whether popular preference specifications fit asset-pricing data. (iii) Asset returns used in computing $\widehat{M_t^*}$ are allowed to be heteroskedastic, which widens the application of this estimator to high-frequency data.

We can get the essence of the estimator when only a single cross-section of data is available ($T = 1$): $\widehat{M^*} = \frac{\overline{R}^G}{\overline{R}^G \overline{R}^A} = \frac{1}{\frac{1}{N} \sum_{i=1}^{N} R_i}$, i.e., it is the reciprocal of the cross-sectional average of returns. Equation (17) is just a generalization of this idea in a panel-data context. It is also worth noting that our estimator lies in the space of payoffs up to a logarithmic approximation. Using $\ln(1 + x) \simeq x$, $\ln\left(\widehat{M_t^*}\right) \simeq \widehat{M_t^*} - 1$, and $r_{i,t} \simeq R_{i,t} - 1$, which shows that $\widehat{M_t^*}$ is a linear combination of $R_{i,t}$ for large T using (17).

From an economic point-of-view, using the results in Mulligan (2002) with logarithmic utility shows that the return to aggregate capital is the reciprocal of the SDF, making our approach closely related to his. Our estimator has also the same pricing properties of the reciprocal of Long's numeraire portfolio, which Bajeux-Besnainou and Portait (1997) consider to be the only relevant variable for pricing assets. The advantage of our estimator with respect to Mulligan's is that ours filter only the common component of asset returns, instead of computing a return to aggregate capital that accumulates measurement error from national-account data.

As a final issue, we consider what economic mechanism would deliver $\plim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$. With incomplete markets, if we take a given economy and replicate it N times, letting $N \to \infty$, we should not expect $\plim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$ to hold. Of course, this happens

13

because the same structure is simply being replicated which does not entail any additional diversification of risk. However, starting again with incomplete markets, we could let an increasingly large number of diverse economies interact in such a way that new securities are being globally added allowing idiosyncratic risk to be diversified away. In the limit, we are "completing the markets" with the addition of new global securities, leading to $\text{plim}_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \varepsilon_{i,t} = 0$. If we view the limiting economy as the complete-market case, then $\widehat{M_t^*} \xrightarrow{p} M_t$, the unique SDF under complete markets.

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