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A MODELLING TOOL YIELDING CLOSED FORMS FOR ASSET PRICES

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ABSTRACT

This methodological paper presents a class of stochastic processes with appealing properties for theoretical or empirical work in finance and macroeconomics, the "linearity-generating" class. Its key property is that it yields simple exact closed-form expressions for stocks and bonds, with an arbitrary number of factors. It operates in discrete and continuous time. It has a number of economic modeling applications. These include macroeconomic situations with changing trend growth rates, or stochastic probability of disaster, asset pricing with stochastic risk premia or stochastic dividend growth rates, and yield curve analysis that allows flexibility and transparency. Many research questions may be addressed more simply and in closed form by using the linearity-generating class.

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1 Introduction

This methodological paper defines and analyzes a class of stochastic processes that has a number of attractive properties for economics and finance, the "linearity-generating" (LG) processes. It is generates closed-form solutions for the prices of stocks and bonds. It is simple and flexible, applies to an arbitrary number of factors with a rich correlation structure, and works in discrete or continuous time. These features make it an easy-to-use tool for pure and applied financial modelling.

The main advantage of the LG class is that it generates, with very little effort, tractable multifactor stock and bond models, in a way that incorporates stochastic growth rates of dividends, and a stochastic equity premium. Stock and bond prices are linear in the factors – hence the name "linearity-generating" processes.

Economically, a process is in the LG class if it satisfies two moment conditions: the expected growth rate of the stochastic discount factor (multiplied by the dividend, if one prices stocks), is linear in the factor. And, the expected growth rate of the stochastic discount factor, times the vector of factors next period, is also linear in the factors (Eq. 9-10). Given only those moments, one can price stocks and bonds (i.e., finite maturity claims on dividends). Higher order moments do not matter. In many applications, the variance of processes can be changed almost arbitrarily and the prices will not change. The fact that a few moments are enough to derive prices makes modelling easier.

Linearity-generating processes are meant to be a practical tool for several areas in economics. They are likely to be useful in: macroeconomics, with models with stochastic trend growth rate or probability of disaster; asset pricing, with models with stochastic equity premium, interest rate, or earnings growth rate.

Several literatures motivate the need for a tool such as the LG process. Many recent studies investigate the importance of long-term risk for asset pricing and macroeconomics, e.g., Bansal and Yaron (2004), Barro (2006), Croce, Lettau and Ludvigson (2006), Gabaix and Laibson (2002), Hansen, Heaton and Li (2005), Hansen and Scheinkman (2006), Julliard and Parker (2004), Lettau and Wachter (2007). The LG process offers a way to model long-term risk, while keeping a closed form for stock prices. In addition, there is debate about the existence and mechanism of the time-varying expected stock market returns, e.g., Boudoukh et al. (forth.), Campbell and Shiller (1988), Cochrane (forth.) and many others. Because of the lack of closed forms, the literature relies on simulations and approximations. The LG process offers closed forms for stocks with time-varying equity premium, which is useful for thinking about those issues.

The motivation for the LG class is inspired by the broad applicability and empirical success of the affine class identified by Duffie and Kan (1996), and further developed by Dai and Singleton (2000) and Duffie, Pan and Singleton (2000), which includes the Vasicek (1977) and the Cox, Ingersoll, Ross (1985) processes as special cases. Much theoretical and empirical work is done with the affine class. Some of this could be done with the LG class. Section 6.2 develops the link between the LG class and the affine class. The two classes give the same quantitative answers to a first order. The main advantage of the LG class is for stocks. The LG class gives a simple closed-form expression for stocks, whereas the affine class needs to express stocks as an infinite sum. Hence, while the affine class can be expected to be remain for long the central model for options and bonds, one can think that the LG class may be a auxiliary technique for bonds, but will be particularly useful for stocks.

Closed forms for stocks, or perpetuities, are not available with the current popular processes, such as the affine models those of Ornstein-Uhlenbeck / Vasicek (1977) and Cox, Ingersoll, Ross (1985), or models in the affine class (Duffie and Kan 1996). Those models simply generate infinite sum of terms.

Several papers have derived closed forms for stocks. Bakshi and Chen (1996) derive a closed form, which is an exponential-affine function of a square root process. Mamayski (2002) derives another closed form, though in a non-stationary setting. Cochrane, Longstaff and Pedro Santa (forth.) contains nice closed form solutions. We confirm results from Mele (2003, forth.), who obtains general results (particularly with one factor) for having bond and stock prices that are convex, concave, or linear in the factors. LG processes satisfy Mele's conditions for linearity. Mele, however, did not derive the closed forms for stocks and bonds in the linear case.

Linear expressions are in Bhattacharya (1978), Menzly, Santos and Veronesi (2004), Santos and Veronesi (2006), Buraschi and Jiltsov (2007).[1] Their process turns out to be to

[1] It is indeed the Menzly, Santos and Veronesi (2004) paper that alerted me to the possibility of a class

belong to the LG class (see Example 10). Indeed, we show that if processes yield linear expressions for bond prices, they belong to the LG class. In view of those earlier findings, the present paper does two things. First, it defines and analyzes the unified class that underlines disparate results of the literature (as Duffie and Kan did for affine processes). Second, it proposes what appears to be some novel processes, such as those using the "linearity-generating twist" (see Example 1, and many others).

Finally, we contribute to the vast literature on interest rate processes, by presenting a new, flexible process. The main advantage is probably that, because the LG processes are so easy to analyze, they lend themselves easily to economic analysis. As a secondary advantage, they naturally exhibit "unspanned volatility". Using the LG class, Gabaix (2007) develops a model of stocks and bonds, and Farhi and Gabaix (2007) a model of exchange rates and the forward premium puzzle.

This paper follows a productive literature that (proudly) reverse-engineers processes for preferences and payoffs, e.g., Campbell and Cochrane (1999), Cox, Ingersoll, Ross (1985), Pastor and Veronesi (2005), Ross (1978), Sims (1990), Liu (2007), and, particularly, Menzly, Santos and Veronesi (2004). Indeed, the two LG moments conditions of Definition 1 gives a recipe to "reverse-engineer" processes to ensure tractability.

Section 2 is a gentle introduction to LG processes. Section 3 presents the discrete-time version of the process, and contains the main results of the paper. Section 4 presents the continuous-time version of the LG process. The next sections are less essential. Section 5 studies the range of admissible initial conditions. Section 6 presents some additional results. Section 7 concludes.

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with closed forms for stocks. On the economic side, this article originates from a lunch with Robert Barro, who was expressing the desirability of a model with stochastic probability of disaster (Gabaix 2007). That conversation made me search for tractable ways to address this question, and led me to LG processes.

2 A simple introduction to linearity-generating processes

To motivate LG processes, this section presents a very simple, almost trivial example – the Gordon formula in discrete time.[2] We want to calculate the price:

$$P_t = E_t \left[\sum_{s=0}^{\infty} \frac{D_{t+s}}{(1+r)^s} \right]$$

of a stock with dividend growth:

$$\frac{D_{t+1}}{D_t} = 1 + g_t + \varepsilon_{t+1} \tag{1}$$

where ε_{t+1} has mean 0, and g_t is the trend growth rate of the stock, and we want it to be autocorrelated (the i.i.d. case is trivial). This is a prototypical example of stock with stochastic trend growth. Even in this example, the usual processes for g_t typically give untractable expressions, as they yield infinite sums of exponential terms.

Let us reverse engineer the process for g_t, and see if the P/D ratio can have the form:

$$\frac{P_t}{D_t} = A + Bg_t \tag{2}$$

for some constants A and B. The arbitrage equation for the stock is $P_t = D_t + \frac{1}{1+r} E_t [P_{t+1}]$ i.e.

$$\frac{P_t}{D_t} = 1 + \frac{1}{1+r} E_t \left[\frac{D_{t+1}}{D_t} \frac{P_{t+1}}{D_{t+1}} \right]. \tag{3}$$

Plugging in (1) and (2), and assuming that $E[\varepsilon_{t+1}] = E_t[\varepsilon_{t+1} g_{t+1}] = 0$, the arbitrage equation reads:

$$A + Bg_t = 1 + \frac{1}{1+r} E_t [(1+g_t)(A + Bg_{t+1})], \text{ i.e.}$$

$$A + Bg_t = 1 + \frac{A}{1+r}(1+g_t) + \frac{B}{1+r}(1+g_t) E_t [g_{t+1}] \tag{4}$$

[2]This example is so simple that it would not be surprising if it had already been done elsewhere, even though I did not find it in the previous literature. However, it seems quite certain that the class of LG processes (including the general structure with several factors, stocks bonds and continuous time), as a class, is identified and analyzed in the present paper for the first time.

If g_t is an AR(1), i.e. $E_t[g_{t+1}] = \rho g_t$, then (4) cannot hold: we have linear terms on the left-hand side, and non-linear terms on the right-hand side.

However, (4) can hold if we postulate that g_t follows the following "twisted" AR(1), with $|\rho| < 1$:

$$\text{Linearity-generating twist: } E_t[g_{t+1}] = \frac{\rho g_t}{1+g_t} \tag{5}$$

If g_t is close to 0, then to a first order, $E_t[g_{t+1}] \sim \rho g_t$, so that g_{t+1} behaves approximately like an AR(1). It's a twisted AR(1), because of the term $1+g_t$ in the denominator. However, in many applications, g_t will be within a few percentage points from 0, so materially, the twist is small (more on this later). If (5) holds, then (4) reads:

$$A + Bg_t = 1 + \frac{A}{1+r}(1+g_t) + \frac{B}{1+r}\rho g_t$$

which features only linear terms, and admits a solution. Indeed, we obtain $A = 1 + A/(1+r)$, i.e. $A = (1+r)/r$, and $B = A/(1+r) + B\rho/(1+r)$, i.e. $B = A/(1+r-\rho)$. Finally, plugging those values of A and B back in (2) gives:

$$\frac{P_t}{D_t} = \frac{1+r}{r}\left(1 + \frac{g_t}{r+1-\rho}\right) \tag{6}$$

We conclude that (6) solves (3). It is actually easy to show that the stock price satisfies (6). By induction on T, one shows that for all $T \geq 0$, $E_t[D_{t+T}] = \left(1 + \frac{1-\rho^T}{1-\rho} g_t\right) D_t$, and direct calculation yields (6).

Example 1 *(Simple stock example with LG stochastic trend growth rate) Consider a stock with dividend growth rate g_t, with $D_{t+1}/D_t = 1 + g_t + \varepsilon_{t+1}$,where ε_{t+1} has mean 0 and is uncorrelated with g_{t+1}, with the linearity-generating "twist" for the growth rate:*

$$E_t[g_{t+1}] = \frac{\rho g_t}{1+g_t}, \tag{7}$$

with price $P_t = E_t\left[\sum_{s=0}^{\infty} D_{t+s}/(1+r)^s\right]$. Suppose that, with probability 1, $\forall t, g_t > -1$. Then, *the price-dividend ratio,* P_t/D_t *is:*

$$\frac{P_t}{D_t} = \frac{1+r}{r}\left(1 + \frac{g_t}{r+1-\rho}\right). \tag{8}$$

The rest of the paper develops this systematically.

This example illustrates many general traits of LG processes.

Eq. 7 imposes just one moment conditions. Higher order moments do not matter for the price. For instance, we could have a complicated nonlinear function for the variance of the growth rate, but it would not affect the stock price. Likewise, the distribution of the noise does not matter, so that one can have jumps and the like, without changing the price.

We need restrictions on the domain of g_t. Mostly obviously, one needs $g_t > -1$. Actually, the stronger condition $g_t > \rho - 1$ is needed (section 21) gives $g_t > \rho - 1$. In particular, the variance has to go to 0 near that boundary. [3]

With the affine models of Duffie and Kan (1996), we might model: $D_{t+1}/D_t = e^{g_t}$, $g_{t+1} = \rho g_t + \varepsilon_{t+1}$. That would lead to $E_t\left[D_{t+T}/D_t\right] = e^{a(T)+b(T)g_t}$, for some functions $a(T), b(T)$, and finally: $\frac{P_t}{D_t} = \sum e^{a(T)+b(T)g_t}$ (Burnside 1998, Ang and Liu 2004). We get a infinite sum over maturities, rather than the compact expression (6). Hence, LG processes are particularly tractable for stocks.

The twisted process (7) is similar to an AR(1), $E_t g_{t+1} = \rho g_t$, up to second order terms. Hence, the behavior is likely to be close to an AR(1). To illustrate this, the Online Appendix to this paper reports the simulation of the above example, with and without the twisted terms. The values for the growth rates are quite close (within 0.1 standard deviation of each other), and hard to distinguish visually. Likewise, the associated price-dividend ratios are quite close, and despite compounding, so are the dividend processes. Of course, even if they had been quite different, this would not have been a important drawback for LG processes. We do not want to say that the true model is an AR(1), that a LG process approximates. It could as well be that the true model is a LG process, than an AR(1) model approximates. Or rather, as a model is just approximation of a complex economic reality, the respective

[3] The reason is that the function $g \mapsto \rho g/(1+g)$ has two fixed points, 0 and $\rho - 1$, and the process needs to stay on the right side of the repelling fixed point, $\rho - 1$.

advantage of LG vs affine models depends on the specific task at hand. The modeler should be able to pick whichever modelling approximation is most expedient, and LG processes offer one such choice.

We now start our systematic treatment of LG processes. As several factors are needed to capture the dynamics of stocks (Campbell and Shiller 1988, Fama and French 1996) and bonds (Litterman and Scheinkman 1991), we study the multifactor case.

3 Linearity-generating processes in discrete time

This section studies the discrete-time LG processes. We want to price an asset with dividend D_t, given a discount factor M_t.[4] The price at time t of a claim yielding a stochastic dividend D_{t+T} at maturity $T \geq 0$ is: $P_t = E\left[\sum_{T=0}^{\infty} M_{t+T} D_{t+T}\right]/M_t$. For instance, the price at t of a ("zero coupon") bond yielding 1 in T periods is: $Z_t(T) = E_t[M_{t+T}]/M_t$.

3.1 Definition and main properties

The state vector is $X_t \in \mathbb{R}^n$ ($n \in \mathbb{N}^*$) and can be generally thought of as stationary, while $M_t D_t$ generally trends, and is not stationary. The definition of the LG process is the following.

Definition 1 *The process $M_t D_t (1, X_t')_{t=0,1,2,\ldots}$, with $M_t D_t \in \mathbb{R}^*$ and $X_t \in \mathbb{R}^n$, is a linearity-generating process if there are constants $\alpha \in \mathbb{R}, \gamma, \delta \in \mathbb{R}^n, \Gamma \in \mathbb{R}^{n \times n}$, such that the following relations hold at all $t \in \mathbb{N}$:*

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t}\right] = \alpha + \delta' X_t \quad (9)$$

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_t D_t} X_{t+1}\right] = \gamma + \Gamma X_t \quad (10)$$

[4] The simplest example is $M_t = (1+r)^{-t}$. If a consumer with utility $\sum_t \delta^t U(C_t)$ prices assets, then $M_t = \delta^t U'(C_t)$. Also, some authors call "stochastic discount factor" M_{t+1}/M_t. In this context, there is no confusion.

To interpret (9), consider first the case of bonds, $D_t \equiv 1$. Eq. 9 says that the properly-defined interest is linear in the factors. When $M_t = (1+r)^{-t}$, (9) says that expected dividend growth is linear in the factors. In general, (9) means that the expected value of the (dividend augmented) stochastic discount factor growth is linear in the factors.

Condition (10) mean that, X_t follows "twisted" AR(1). It behaves in some sense like $E_t[X_{t+1}] = \gamma + \Gamma X_t$, but it is twisted by the $\frac{M_{t+1}D_{t+1}}{M_t D_t}$ term. Another useful interpretation of (10) is that it specifies the factor dynamics under the risk-neutral measure induced by $M_t D_t$.

What kinds of models are compatible with Definition 1? As the examples below show, it is not difficult to write toy economic models satisfying conditions (9)-(10), e.g. in Lucas (1978) and Campbell-Cochrane (1999) economies with exogenous consumption, dividend or marginal utility processes, or models with learning. Farhi and Gabaix (2007) and Gabaix (2007) presents a fully worked-out economic model satisfying the conditions of Definition 1.

Indeed, conditions (9)-(10) give a prescription to "reverse-engineer" macro or micro fundamentals, so as to make the model tractable: The modeler has to make sure that the endowment, technology etc. is such that (9)-(10) hold.

In addition, models that to not directly fit into the conditions of Definition 1, could be approximated by projected linearly in (9)-(10). Also, by extending the state vector, equations (9)-(10) could hold to an arbitrary degree of precision. The Online Appendix to this paper illustrates how to approximate a non-LG process with an LG process, even to an arbitrary degree of precision.

There is a more compact way to think about LG processes. Define the $(n+1) \times (n+1)$ matrix, which we will call the "generator" of the process:

$$\Omega = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix} \tag{11}$$

and the process with values in $\mathbb{R}^{n+1}$

$$Y_t := \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} = \begin{pmatrix} M_t D_t \\ M_t D_t X_{1t} \\ \vdots \\ M_t D_t X_{nt} \end{pmatrix}, \tag{12}$$

so that with vector $\nu' = (1, 0, ..., 0)$,

$$M_t = \nu' Y_t \tag{13}$$

Y_t stacks all the information relevant to the prices of the claims derived below. [5] Conditions (9)-(10) can be written:

$$E_t [Y_{t+1}] = \Omega Y_t. \tag{14}$$

Hence, the (dividend-augmented) stochastic discount factor of a LG process is simply the projection (Eq. 13) of an autoregressive process, Y_t. The tractability of LG processes comes from the tractability of autoregressive processes.

The basic pricing properties are the following two Theorems.

Theorem 1 *(Bond prices, discrete time) The price-dividend ratio of a zero-coupon equity or bond of maturity T, $Z_t(T) = E_t[M_{t+T} D_{t+T}] / (M_t D_t)$, is*

$$\begin{aligned} Z_t(T) &= \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}^T \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} & (15) \\ &= \alpha^T + \delta' \frac{\alpha^T I_n - \Gamma^T}{\alpha I_n - \Gamma} X_t \text{ when } \gamma = 0 & (16) \end{aligned}$$

where I_n the identity matrix of dimension n, and 0_n is the row vector with n zeros.

For instance, when $D_t \equiv 1$, the above Theorem can price bonds, with n factors, in closed form.

In many applications (e.g., the examples in this paper), $\gamma = 0$, which means the state variables are re-centered around 0. For instance, the state variable is the deviation of the

[5] Other assets, e.g. options, require of course to know more moments.

equity premium from its trend value.

The second main result is the most useful property of LG processes: the existence of a closed-form formula for stock prices.

Theorem 2 *(Stock prices, discrete time) Suppose that all eigenvalues of* Ω *have a modulus less than 1 (finiteness of the price). Then, the price-dividend ratio of the stock,* $P_t/D_t = E_t\left[\sum_{s=t}^{\infty} M_s D_s\right]/(M_t D_t)$, *is:*

$$P_t/D_t = \frac{1}{1-\alpha-\delta'(I_n-\Gamma)^{-1}\gamma}\left(1+\delta'(I_n-\Gamma)^{-1}X_t\right) \quad (17)$$

$$= \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \left(I_{n+1} - \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}\right)^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}. \quad (18)$$

Theorem 2 allows to generate stock prices with an arbitrary number of factors, including time-varying growth rate, and risk premia.

To make formulas concrete, consider the case where Γ is a diagonal matrix: $\Gamma \equiv Diag(\Gamma_1, ..., \Gamma_n)$, i.e.

$$\Omega = \begin{pmatrix} \alpha & \delta_1 & \cdots & \delta_n \\ \gamma_1 & \Gamma_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \gamma_n & 0 & 0 & \Gamma_n \end{pmatrix}.$$

Then, $\frac{\alpha^T I_{n+1}-\Gamma^T}{\alpha I_{n+1}-\Gamma} = Diag\left(\left(\alpha^T-\Gamma_i^T\right)/(\alpha-\Gamma_i)\right)$,[6] so that (16) and (17) read:

$$Z_t(T) = \alpha^T + \sum_{i=1}^{n} \frac{\alpha^T-\Gamma_i^T}{\alpha-\Gamma_i}\delta_i X_t^i \text{ if } \gamma = 0 \quad (19)$$

$$P_t/D_t = \frac{1+\sum_{i=1}^{n}\frac{\delta_i X_i}{1-\Gamma_i}}{1-\alpha-\sum_{i=1}^{n}\frac{\delta_i\gamma_i}{1-\Gamma_i}} \quad (20)$$

Finally, the following Propositions show that one can price claims that have dividend a linear function of $D_t X_t$. In bond applications, they show that futures price obtain in closed form.

[6] If A matrix, and $f:\mathbb{R}\to\mathbb{R}$, is analytic with $f(x)=\sum_{n=0}^{\infty} f_n x^n$ then $f(A)=\sum_{n=0}^{\infty} f_n A^n$. If $A = Diag(a_1, .., a_n)$, $f(A) = Diag(f(a_1), ..., f(a_n))$

The proofs are exactly identical to those of the previous two Theorems.

Proposition 1 *(Value of a single-maturity claim yielding $D_{t+T}\delta' X_{t+T}$). Given the LG process $M_t D_t (1, X_t)$, the price of a claim yielding a dividend $d_t = D_t \sum_{i=1}^{n} f_i X_{it} = D_t (f' X_t)$, $P_t = E_t [M_{t+T} d_{t+T}] / M_t$, is:*

$$P_t = \begin{pmatrix} 0 \\ f \end{pmatrix}' \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t \quad (21)$$

$$= f' \Gamma^T X_t D_t \text{ when } \gamma = 0. \quad (22)$$

Proposition 2 *(Value of an asset yielding $D_t \delta' X_t$ at each period) Under the conditions of Theorem 2, the price of a claim yielding a dividend $d_t = D_t \sum_{i=1}^{n} f_i X_{it} = D_t f' X_t$, $P_t = E_t \left[\sum_{s=t}^{\infty} M_s D_s \right] / M_t$ is,*

$$P_t = \begin{pmatrix} 0 \\ f \end{pmatrix}' \left(I_{n+1} - \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t = \frac{f' (I_n - \Gamma)^{-1} (\gamma + (1 - \alpha) X_t)}{1 - a - \delta' (I_n - \Gamma)^{-1} \gamma} D_t. \quad (23)$$

For instance, when $\Gamma \equiv Diag(\Gamma_1, ..., \Gamma_n)$, Eq. 22 and 23 read:

$$P_t / D_t = \sum_{i=1}^{n} f_i \Gamma_i^T X_{it} D_t \text{ if } \gamma = 0$$

$$P_t / D_t = \frac{\sum_{i=1}^{n} \frac{f_i}{1 - \Gamma_i} (\gamma_i + (1 - \alpha) X_{it})}{1 - \alpha - \sum_{i=1}^{n} \frac{\delta_i \gamma_i}{1 - \Gamma_i}}$$

3.2 Some examples

We now work out some examples. The derivations are in Appendix B.

Example 2 *A Gordon growth formula with time-varying dividend growth.*

In this example, we generalize our introductory stock example. Suppose that the interest rate is constant at r, dividend D_t, and the growth rate of dividend is:

$$\begin{aligned} \frac{D_{t+1}}{D_t} &= (1+g_*)(1+x_t)\left(1+\eta_{t+1}\right) \\ x_{t+1} &= \frac{\rho x_t}{1+x_t}+\varepsilon_{t+1} \end{aligned} \tag{24}$$

where η_t is some unimportant i.i.d. noise, greater than -1, independent of the innovation to ε_{t+1}. x_t is the deviation from the trend growth rate. If x_t was an AR(1), it would follow $E_t[x_{t+1}] = \rho x_t$. Instead, the process is slightly modified, to (24), to make the process LG. Indeed, with $M_t = (1+r)^{-t}$, and using the notation $1+R = (1+r)/(1+g_*)$, we calculate the two LG moments:

$$\begin{aligned} E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\right] &= (1+x_t)/(1+R) \\ E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}x_{t+1}\right] &= E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\right]E_t[x_{t+1}] = \frac{(1+x_t)}{1+R}\frac{\rho x_t}{1+x_t} = \frac{\rho x_t}{1+R} \end{aligned}$$

In the above equation, the $1+x_t$ terms cancel out, because of the $1+x_t$ term in the denominator of (24). We designed the process so that the LG equation (10) holds.

So $M_tD_t(1,x_t)$ is LG, with $\Omega = \begin{pmatrix} 1/(1+R) & 1/(1+R) \\ 0 & \rho/(1+R) \end{pmatrix} = \begin{pmatrix} \alpha & \delta' \\ \gamma & \Gamma \end{pmatrix}$. Hence, we apply Theorem 2, with a dimension $n=1$, $\gamma=0$, $\alpha = 1/(1+R)$, $\delta = \Gamma = \alpha\rho$. We obtain, for the price-dividend ratio, $P_t/D_t = \frac{1}{1-\alpha-\delta'(I_n-\Gamma)^{-1}\gamma}\left(1+\delta'(I_n-\Gamma)^{-1}X_t\right)$, i.e.

$$P_t/D_t = \frac{1+R}{R}\left(1+\frac{1}{1+R-\rho}x_t\right) \tag{25}$$

Hence we see how Example 1 comes from the general structure of LG processes.

Example 3 *Stock price with stochastic growth rate and stochastic equity premium*

Consider a dividend and discount factor process:

$$\begin{aligned} \frac{D_{t+1}}{D_t} &= 1 + g_t + \eta_{t+1} \\ \frac{M_{t+1}}{M_t} &= \frac{1}{1+r}\left(1 - \frac{\pi_t}{var_t\left(\eta_{t+1}\right)}\eta_{t+1}\right) \end{aligned}$$

so that g_t is the stochastic trend growth rate of the dividend, and π_t is a risk premium.[7] Decompose g_t into a fixed and a variable part, as in $g_t = g_* + \widehat{g}_t$, do the same for $\pi_t = \pi_* + \widehat{\pi}_t$, and postulate the following processes:

$$\begin{aligned} g_{t+1} &= g_* + \frac{1+g_*-\pi_*}{1+g_t-\pi_t}\rho_g\left(g_t - g_*\right) + \varepsilon^g_{t+1} \\ \pi_{t+1} &= \pi_* + \frac{1+g_*-\pi_*}{1+g_t-\pi_t}\rho_\pi\left(\pi_t - \pi_*\right) + \varepsilon^\pi_{t+1} \end{aligned}$$

where at time t ε^g_{t+1} and ε^π_{t+1} have expected values 0 and are uncorrelated with η_{t+1}.The term $\frac{(1+g_*-\pi_*)}{1+g_t-\pi_t}$ will be close to 1 in many applications. Defining: $\alpha = (1+g_*-\pi_*)/(1+r)$, the Gordon discount factor. Theorem 2 yields:

$$P_t/D_t = \frac{1+r}{r+\pi_*-g_*}\left(1 + \frac{g_t-g_*}{1-\alpha\rho_g} + \frac{\pi_t-\pi_*}{1-\alpha\rho_\pi}\right) \tag{26}$$

In the limit of small times, with $\rho_g = 1-\phi_g$, $\rho_\pi = 1-\phi_\pi$, with r and ϕ small (ϕ_g is the speed of mean-reversion of g to its trend), we obtain:

$$P_t/D_t = \frac{1}{R}\left(1 + \frac{g_t-g_*}{R+\phi_g} - \frac{\pi_t-\pi_*}{R+\phi_\pi}\right) \text{ with } R \equiv r+\pi_*-g_* \tag{27}$$

This equation nests the three main sources of variations of stock prices in a simple and natural way. Stock prices can increase because the level of dividends increases (that's the D_t terms), because the expected future growth rate of dividend increases (the $g_t - g_*$ term), or because the equity premium decreases (the $\pi_t - \pi_*$ terms). The two growth or discount factors (g_t and π_t) enter linearly, weighted by their duration (e.g., $1/(R+\phi_\pi)$), which depends on

[7] The risk premium is on the innovations to dividends. One could also have a risk premium on the innovation to dividend growth rate (as in Bansal and Yaron 2004), an exercise that we leave to the reader.

the speed of mean-reversion of the each process (parametrized by ϕ_π, ϕ_g), and the effective discount rate, R. The volatility terms do not enter in (27), and the price does not change if one changes the correlation between the instantaneous innovation in g_t and π_t.

Example 4 *A multifactor bond model with bond risk premia (in discrete time).*

There are n factors r_{it}. The stochastic discount factor is:

$$\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}\left(1 - \sum_{j=1}^{n} r_{jt}\right) \tag{28}$$

The short term rate is $r_t = 1/E_t\left[\frac{M_{t+1}}{M_t}\right] - 1 \simeq r_* + \sum r_{it}$ if the r's are small. Each factor r_{it} is postulated to evolve as:

$$r_{i,t+1} = \frac{\rho_i r_{i,t}}{1 - \sum r_{jt}} + \eta_{i,t+1} \tag{29}$$

where $E_t \eta_{i,t+1} = 0$, but the $\eta_{i,t+1}$ can otherwise have any correlation structure. This is a LG process. The bond price is:

$$Z_t(T) = \frac{1}{(1+r_*)^T}\left(1 - \sum_{i=1}^{n} \frac{1-\rho_i^T}{1-\rho_i} r_{it}\right) \tag{30}$$

This expression is quite simple, and accommodates a wide variety of specifications for the factors, Eq. 29. Furthermore, it accommodates bonds with risk premia. Just take a stochastic discount factor: $\frac{M_{t+1}}{M_t} = \frac{1}{1+r_*}\left(1 - \sum_{j=1}^{n} r_{jt}\right) + \varepsilon_{t+1}$, where $E_t \varepsilon_{t+1} = 0$, but otherwise ε_{t+1} is unspecified, and can be heteroskedastic. and postulate: $r_{i,t+1} = \frac{\rho_i r_{i,t}}{1-\sum r_{jt}} + \eta_{i,t+1} - \frac{E_t[\varepsilon_{t+1} \cdot \eta_{i,t+1}]}{E_t[M_{t+1}/M_t]}$, which means that r_{it} follow the process (29) under the risk-neutral measure. Then, Eq. 30 holds. The risk premium on the T maturity bond is:

$$\text{Risk premium} = \frac{cov\left(\varepsilon_{t+1}, Z_{t+1}(T-1)\right)}{Z_t(T)} = \frac{(1+r_*)\sum \frac{1-\rho_i^{T-1}}{1-\rho_i} cov\left(\varepsilon_{t+1}, \eta_{i,t+1}\right)}{1 - \sum \frac{1-\rho_i^T}{1-\rho_i} r_{it}}$$

Hence we easily generate an explicit yield curve. With a parametrization for $cov\left(\varepsilon_{t+1}, \eta_{i,t+1}\right)$, the above expression makes predictions for bond risk premia across maturities (see Gabaix 2007).

Example 5 *Markov chains, and some economies with learning*

There are n states. In state i the factor-augmented dividend grows at a rate G_i: $M_{t+1}D_{t+1}/\left(M_t D_t\right) = G_i$. Call $X_{it} \in \{0,1\}$, equal to 1 if the state is i, 0 otherwise. The probability of going from state j to state i is called p_{ij}. Then, $M_t D_t \left(1, X_1, ..., X_n\right)$ is a LG process. Hence, a Markov chain belongs to the LG class.[8] As many processes are (arbitrarily) well-approximated by discrete Markov chains, they are (arbitrarily) well-approximated by LG processes.

Markov chains induced by learning naturally lead to LG processes. For a complete example, the reader is encouraged to read Veronesi (2005). He finds that if X_{it} is the agents' probability estimate that the economy is in state i, under canonical models with Gaussian filtration of information, then vector X_t follows an autoregressive process. He works out the prices of stocks and bonds in an economy, and finds that they are linear function of X_t. Hence, some canonical structural with learning models naturally give rise to LG processes.

Example 6 *Flexible LG parametrization of state variables the stochastic discount factor*

Take an $n-$dimensional process X_t, such that:

$$\begin{aligned} \frac{M_{t+1}D_{t+1}}{M_t D_t} &= a + \beta' X_t + \varepsilon_{t+1} \\ X_{t+1} &= \frac{\gamma + \Gamma X_t}{a + \beta' X_t} + \eta_{t+1} - \frac{E_t\left[\varepsilon_{t+1}\eta_{t+1}\right]}{a + \beta' X_t} \end{aligned} \tag{31}$$

with $E_t\left[\varepsilon_{t+1}\right] = 0$, $E_t\left[\eta_{t+1}\right] = 0$. Then, Eq. 9-10 are satisfied. Section 5 provides conditions to ensure $M_t D_t > 0$ for all times.

To interpret (31), consider the case $\gamma = E_t\left[\varepsilon_{t+1}\eta_{t+1}\right] = 0$. Eq. 31 expresses that, when X_t is small, $E_t\left[X_{t+1}\right] = \frac{\Gamma X_t}{a+\beta' X_t} \sim \frac{\Gamma}{a} x_t$, which means that X_t follows approximately at AR(1). The corrective $1 + \beta'/a \cdot X_t$ in the denominator is often small in practice, but ensures that the process is LG.

In many applications, there is no risk premium on the factor risk, so that $E_t\left[\varepsilon_{t+1}\eta_{t+1}\right] = 0$. However, when there is a risk-premium equation (31) means that it is enough to know that the process under the "risk-neutral" measure.

[8] Veronesi and Yared (2000) and David and Veronesi (2006) have already seen that this type of Markov chain yielded prices that are linear in the factors.

We next turn to the continuous time version of what we have seen so far.

4 Linearity-generating processes in continuous time

We fix a probability space $\left(\Omega^P, \mathcal{F}, P\right)$ and an information filtration $\mathcal{F}_t$ satisfying the usual technical conditions (see, for example, Karatzas and Shreve 1991). The stochastic discount factor is M_t. For applications, we will express the results in terms of a dividend-augmented stochastic discount factor, $M_t D_t$. Often, it is better to imagine $D_t \equiv 1$.

4.1 Definition and main properties

A notation. The following notation is useful when using LG processes. For x_t, μ_t processes in a vector space V, we say $E_t\left[dx_t\right] = \mu_t dt$, or $E_t\left[dx_t\right]/dt = \mu_t$, to signify that there exists a martingale N_t with values in V such that: $x_t = x_0 + \int_0^t \mu_s ds + N_t$.

The definition in continuous time is analogous to the definition in discrete time. The vector of factors is X_t.

Definition 2 *The process $M_t D_t \left(1, X_t\right)'_{t \in \mathbb{R}_+}$, with $M_t D_t \in \mathbb{R}$ and $X_t \in \mathbb{R}^n$, is a linearity-generating process if there are constants with $a \in \mathbb{R}, b, \beta \in \mathbb{R}^n, \Phi \in \mathbb{R}^{n \times n}$, such that the following relations hold at all $t \in \mathbb{R}_+$,*

$$E_t\left[d\left(M_t D_t\right)\right] = -\left(a + \beta' X_t\right) M_t D_t dt \tag{32}$$

$$E_t\left[d\left(M_t D_t X_t\right)\right] = -\left(b + \Phi X_t\right) M_t D_t dt \tag{33}$$

The interpretation is exactly the same as for Definition 1. Eq. 32 means that the expected growth rate of $M_t D_t$ is linear in the factors. Eq. 33 means that X_t follows a twisted AR(1). Loosely speaking, it describes the process for X_t under the "risk-neutral" measure induced by $M_t D_t$.

For instance, in the case $D_t \equiv 1$ and $dM_t/M_t = -\left(a + \beta' X_t\right) dt$, Eq. 33 gives:

$$dX_t = -bdt - \left(\Phi - aI_n\right) X_t dt + \left(\beta' X_t\right) X_t dt + dN_t \tag{34}$$

where $N_t \in \mathbb{R}^n$ is a martingale. Hence, the process contains an AR(1) term, $-b-(\Phi - aI_n) X_t$, plus a "twist" quadratic term, $(\beta' X_t) X_t$. It is a "twisted" AR(1). In many applications, X_t represents a small deviation from trend, and the quadratic term $(\beta' X_t) X_t$ is small. We are agnostic about how empirically relevant the "twist" is. It could be that it is absent in the physical probability, but present under the risk-neutral measure.

So $E_t [dN_t] = 0$, but its components dN_{it}, dN_{jt} *can* be correlated. The simplest type of martingale is $dN_t = \sigma(X_t) dB_t$, for B_t a Brownian motion, but richer structures, e.g. with jumps, are allowed. As in the one-factor process, the volatility of dN_t must go to zero in some limit regions for the process to be well-defined. We defer this more technical issue until section 5.

As in the discrete-time case, we define the "generator" of the process:

$$\omega = \begin{pmatrix} \alpha & \beta \\ b & \Phi \end{pmatrix} \tag{35}$$

and the process $Y_t = \begin{pmatrix} M_t D_t \\ M_t D_t X_t \end{pmatrix} \in \mathbb{R}^{n+1}$, as in (12), which encodes the information needed for prices. Conditions (32)-(33) write more compactly as:

$$E_t [dY_t] = -\omega Y_t dt. \tag{36}$$

which is the analogue of (14). The above process leads to a LG discrete-time process with time increments Δt, with a generator $\Omega = e^{-\omega \Delta t}$.

Hence, there is a $(n+1)$ dimensional process Y_t, and a vector $\nu' = (1, 0, ..., 0)$, such that (36) holds, and $M_t = \nu' Y_t$. In other terms, there is a autoregressive process Y_t in the background, following (36). The (dividend-augmented) stochastic discount factor is the one-dimensional projection of it. LG processes are tractable, because they are the one-dimensional projection of an AR(1) process.

The next Theorem prices claims of finite maturity.

Theorem 3 *(Bond prices, continuous time). The price-dividend of a claim on a dividend* *of maturity T, $Z_t(T) = E_t[M_{t+T}D_{t+T}]/(M_tD_t)$, is:*

$$Z_t(T) = \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \exp\left[-\begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix} T\right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{37}$$

$$= e^{-aT} + \beta' \frac{e^{-\Phi T} - e^{-aT} I_n}{\Phi - a I_n} X_t \text{ when } b = 0 \tag{38}$$

where I_n the identity matrix of dimension n, and 0_n is the row vector with n zeros.

As an example, bond prices come from $D_t = 1$. In many applications, $b = 0$, which can generically be obtained by re-centering the variables.

Theorem 4 is probably the most useful of this section.

Theorem 4 *(Stock prices, continuous time). Suppose that all eigenvalues of ω have positive real part (finite stock price). Then, the price/dividend ratio of the stock, $P_t/D_t = E_t\left[\int_t^\infty M_s D_s ds\right]/(M_tD_t)$, is:*

$$P_t/D_t = \frac{1 - \beta'\Phi^{-1}X_t}{a - \beta'\Phi^{-1}b} \tag{39}$$

$$= \begin{pmatrix} 1 & 0_n \end{pmatrix} \cdot \begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} \tag{40}$$

To make things more concrete, consider the case where Φ is a diagonal matrix: $\Phi = Diag(\Phi_1, ..., \Phi_n)$, i.e.:

$$\omega = \begin{pmatrix} a & \beta_1 & \cdots & \beta_n \\ b_1 & \Phi_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ b_n & 0 & 0 & \Phi_n \end{pmatrix} \tag{41}$$

Then, $e^{-\Phi T} = Diag\left(e^{-\Phi_i T}\right)$, and then (16) and (17) read:

$$Z_t(T) = e^{-at} + \sum_{i=1}^{n} \frac{e^{-\Phi_i T} - e^{-aT}}{\Phi_i - a} \beta_i X_{it} \text{ if } b = 0 \tag{42}$$

$$P_t/D_t = \frac{1 - \sum_{i=1}^{n} \frac{\beta_i X_{it}}{\Phi_i}}{a - \sum_{i=1}^{n} \frac{\beta_i b_i}{\Phi_i}} \tag{43}$$

Finally, the following Propositions show that one can price claims that have dividend a linear function of $D_t X_t$. The proofs are exactly identical to those of the previous two Theorems.

Proposition 3 *(Value of a single-maturity claim yielding $D_{t+T} f' X_{t+T}$). Given the LG process $M_t D_t (1, X_t)$, the price of a claim yielding a dividend $d_t = D_t \sum_{i=1}^{n} f_i X_{it} = D_t (f' X_t)$, $P_t = E_t [M_{t+T} d_{t+T}] / M_t$, is:*

$$\begin{aligned} P_t &= \begin{pmatrix} 0 \\ f \end{pmatrix}' \cdot \exp\left[-\begin{pmatrix} a & \beta' \\ b & \Phi \end{pmatrix} T \right] \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t && (44) \\ &= f' e^{-\Phi T} D_t X_t \text{ when } b = 0. && (45) \end{aligned}$$

Proposition 4 *(Value of an asset yielding $D_t f' X_t$ at each period) Under the conditions of Theorem 4, the price of a claim yielding a dividend $d_t = D_t \sum_{i=1}^{n} f_i X_{it} = D_t f' X_t$, $P_t = E_t \left[\int_t^\infty M_s d_s ds \right] / M_t$, is,*

$$P_t = \begin{pmatrix} 0 \\ f \end{pmatrix}' \omega^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix} D_t = \frac{f' \Phi^{-1} (-b + a X_t)}{a - \beta' \Phi^{-1} b} D_t. \qquad (46)$$

4.2 Some examples

We start with some stock-like examples.

Example 7 *Simple stock example with LG stochastic trend growth rate, in continuous time*

We study Example 1 in continuous time. Suppose $M_T = e^{-rT}$, $D_T = D_0 \exp\left(\int_0^T g_t dt \right)$, with the continuous time limit of (5):

$$dg_t = \left(-\phi g_t - g_t^2 \right) dt + \sigma (g_t) dz_t \qquad (47)$$

In the equation above, the coefficient on g_t^2 has to be -1. Theorem 4 yields:[9]

$$P_t/D_t = \frac{1}{r}\left(1 + \frac{g_t}{r+\phi}\right). \tag{48}$$

Section 5 will present the condition $g_t \geq -\phi$ for the process to be well defined. We next generalize the example to n factors.

Example 8 *Dividend growth rate as a sum of mean-reverting processes (e.g., a slow and a fast process).*

We extend the previous example to a several factors. Suppose $M_T = e^{-rT}$, $D_T = D_0 \exp\left(\int_0^T g_t dt\right)$, with $g_t = g_* + \sum_{i=1}^n X_{it}$ and

$$E_t\left[dX_{it}\right]/dt = -\phi_i X_{it} + \left(g_* - g_t\right) X_{it} dt.$$

The growth rate g_t is a steady state value g_*, plus the sum of mean-reverting processes X_{it}. Each X_{it} mean-reverts with speed ϕ_i, and also has the quadratic perturbation $\left(g_* - g_t\right) X_{it} dt$. The price-dividend ratio is

$$P_t/D_t = \frac{1}{r - g_*}\left(1 + \sum_{i=1}^n \frac{X_{it}}{r - g_* + \phi_i}\right). \tag{49}$$

Each component X_{it} perturbs the baseline Gordon expression $1/\left(r - g_*\right)$. The perturbation is X_{it}, times the duration of X_i, discounted at rate $r - g_*$, which is the term $1/\left(r - g_* + \phi_i\right)$.[10]

[9] The result in Example 7 appear new to the literature. The Fisher-Wright process does contain a quadratic term, but it has not been applied to the pricing bonds or stocks. Also, it is more special than the LG class, because it imposes a specific functional form on the variance. Cochrane, Longstaff, and Santa-Clara (forth.) apply the Fisher-Wright process. Mele (2003, forth.) identifies a condition for the process to be linear in the factor, but does not derive stocks and bond prices such as (48). Other papers introduce different quadratic terms in stochastic process, for instance Ahn et al. (2002), and Constantidines (1992) but they do not take the form of this paper.

[10] The formula suggests the following non-LG variant. Suppose we have a process with $d\psi_t = \left(r_t \psi_t + \alpha r_t - \beta\right) dt + dN_t$,where dN_t is an adapted martingale, and is essentially arbitrary except for technical conditions. Then $V_t = \left(\psi_t + \alpha\right)/\beta$ is a solution of the perpetuity arbitrage equation: $1 - r_t V_t + E\left[dV_t\right]/dt = 0$. If the process well-defined for $t \geq 0$, then V_t is the price of a perpetuity, $V_t = E_t\left[\int_t^\infty e^{-\int_t^s r_u du} ds\right]$. For instance, with the process $d\left(1/r_t\right) = \phi\left(r_t - r^*\right) dt + dN_t$, the price of a perpetuity is: $V_t = \left(1/r_t + \phi/r_*\right)/\left(1+\phi\right)$.

Terms that mean-revert more slowly have a higher impact on the the price. Finally, Theorem 3 yields:

$$E_t\left[D_{t+T}\right] = e^{g_* T}\left(1+\sum_{i=1}^{n}\frac{1-e^{-\phi_i T}}{\phi_i}X_{it}\right)D_t.$$

Example 9 *Generalized Gordon formula, with stochastic trend in dividend growth, and stochastic equity premium, in continuous time.*

We present the continuous time version of Example 3. The stochastic discount factor M_t and the dividend process D_t follow

$$dM_t/M_t = -rdt - \frac{\pi_t}{\sigma}dz_t \text{ and } dD_t/D_t = g_t dt + \sigma dz_t$$

The price of the stock is $P_t = E_t\left[\int_t^\infty M_s D_s ds\right]/M_t$. π_t is a the stochastic equity premium, and g_t is the stochastic growth rate of dividends.

We assume that π_t and g_t follow the following LG process, best expressed in terms of their deviation from trend, $\widehat{\pi}_t = \pi_t - \pi_*$, $\widehat{g}_t = g_t - g_*$:

$$\begin{aligned} d\widehat{g}_t &= -\phi_g \widehat{g}_t dt + \left(\widehat{\pi}_t - \widehat{g}_t\right)\widehat{g}_t dt + \sigma_\gamma\left(\widehat{g}_t, \widehat{\pi}_t\right)\cdot dB_t \\ d\widehat{\pi}_t &= -\phi_\pi \widehat{\pi}_t dt + \left(\widehat{\pi}_t - \widehat{g}_t\right)\widehat{\pi}_t dt + \sigma_\pi\left(\widehat{g}_t, \widehat{\pi}_t\right)\cdot dW_t \end{aligned}$$

where the (B_t, W_t) is a Wiener process independent of z_t, that can have arbitrary time- or state-dependent correlations, and σ_γ and σ_π are vector-valued processes. We suppose that the process is defined in $[t,\infty)$. Again the processes $d\widehat{g}_t$ and $d\widehat{\pi}_t$ are to a first order linear, but with quadratic "twist" terms added, $\left(\widehat{\pi}_t - \widehat{g}_t\right)\widehat{g}_t dt$ and $\left(\widehat{\pi}_t - \widehat{g}_t\right)\widehat{\pi}_t dt$. The stock price is

$$P_t = \frac{D_t}{R}\left(1 + \frac{g_t - g_*}{R+\phi_g} - \frac{\pi_t - \pi_*}{R+\phi_\pi}\right) \text{ with } R \equiv r + \pi_* - g_* \tag{50}$$

where R is the traditional Gordon rate.[11] As in Example 3, this example nests the three sources of variation in prices, movements in dividends (D_t), in expected growth rate of dividends (g_t), and of discount factor (π_t).

[11] It is a good and simple exercise to derive the above formula directly, from the arbitrage equation $1 - \left(r + \pi_t - g_t\right)(P/D)_t + E\left[d\left(P/D\right)_t\right]/dt = 0$.

Example 10 *The aggregate model of Menzly, Santos and Veronesi (2004), and the Bhattacharya (1978) mean-reverting process, belong to the linearity-generating class.*

The following point is simple and formal. Menzly, Santos and Veronesi (MSV, 2004) rely on an Ornstein-Uhlenbeck. The inverse of their consumption-surplus ratio, y_t, follows: $E_t[dy_t] = k(\overline{y} - y_t)\,dt$. The price-consumption ratio in their economy is $V_t = y_t^{-1} E_t\left[\int_0^\infty e^{-\rho s} y_{t+s} ds\right]$. In terms of the LG process, the state variable is y_t, and $M_t = e^{-\rho t}$. We have $E_t[dM_t/dt]/M_t = -\rho dt$, and $E_t[d(M_t y_t)/dt]/M_t = -\rho y_t + k(\overline{y} - y_t)$. So $M_t(1, y_t)$ is a LG process with generator $\omega = \begin{pmatrix} \rho & 0 \\ -k\overline{Y} & \rho + k \end{pmatrix}$. The MSV pricing equation 17 comes directly from Proposition 4 of the present article, $y_t V_t = (k\overline{y} + \rho y_t)/[\rho(\rho + k)]$. Hence, in retrospect, the MSV (2004) process is tractable because it belongs to the LG class. This remark, also, immediately suggest a way to formulate the MSV paper to discrete time. In a simpler context, Bhattacharya (1978) models the dividend y_t as an Ornstein-Uhlenbeck, yielding the same closed form solution for the price.

Example 11 *A LG process where the stock price is convex (not linear) in the growth rate of dividends*

This "academic" example shows how one can obtain asset prices that are increasing in their variance, which is important in some applications (Johnson 2002, Pastor and Veronesi 2003). Consider an economy with constant discount rate r (i.e. $M_t = e^{-rt}$), and a stock with dividend $D_t = D_0 \exp\left(\int_0^t g_s ds\right)$, where[12] $dg_t = -\left(g_t^2/2 + \phi g_t\right)dt + \sqrt{k\left(G^2 - g^2\right)}dz_t$. Then, the price-dividend ratio is:

$$P_t/D_t = \frac{2(\phi + r)(2\phi + k + r) + 2(2\phi + k + r)g_t + g_t^2}{2r(\phi + r)(2\phi + k + r) - kG^2} \tag{51}$$

which is increasing in the parameter G of the volatility. In this example, the state vector is (g_t, g_t^2), which makes the price quadratic and convex in g_t. More generally, by expanding the state vector, the price could be a polynomial of arbitrary order in g.

We next present some bond-like examples. We start with a very simple example.

[12] We assume $0 < G < 2(\phi - k)$, and that the support of g_t is $(-G, G)$, with end points natural boundaries.

Example 12 *A one-factor bond model, with an always positive nominal rate.*

The following example is simply illustrates LG processes. It has just one factor, whereas multifactor models are necessary to capture the yield curve. Suppose $M_t = \exp\left(-\int_0^t r_s ds\right)$, with $r_t = r_* + \widehat{r}_t$, with

$$d\widehat{r}_t = -(\phi - \widehat{r}_t)\,\widehat{r}_t dt + dN_t$$

where $\phi > 0$, $\widehat{r}_t \leq \phi$, and N_t is a martingale, which could include a diffusive part and a jump part. The bond price is:

$$Z_t(T) = e^{-r_* T}\left(1 + \frac{e^{-\phi T} - 1}{\phi}\widehat{r}_t\right). \tag{52}$$

The independence of bond prices from volatility greatly simplifies the analysis. In particular, dN_t could have jumps, which model a decision by the central bank, or fat-tailed innovations of other kinds (Gabaix et al. 2006). One does not need to specify the volatility process to obtain the prices of bonds: only the drift part is necessary. This leaves a high margin of flexibility to calibrate volatility, for instance on interest rate derivatives, a topic we do not pursue here.

How can we ensure that the interest rate always remains positive? That is very easy (with $r_* > 0$). For instance, we could have $dN_t = \sigma(r_t)\,dz_t$, where z_t is a Brownian process, with $\sigma(r) \sim k' r^{\kappa'}$, $\kappa' > 1/2$ for r in a right neighborhood of 0, and $k' > 0$, so that the local drift at $r_t = 0$ is positive. By the usual Feller conditions on natural boundaries, the process admits a strong solution, and $r_t \geq 0$ always (Cheridito and Gabaix 2007 spell out the technical conditions). And, the bond price (52) is not changed by this assumption about the volatility process. One can indeed change the lower bound for the process (if it is less than r_*) without changing the bond price.

Section 5 will detail the conditions for the existence of the process. The interest rate needs to remains below some upper bound $\overline{r} \in (r_*, r_* + \phi]$, so as to not explode. One way is to assume that $\sigma(r) \sim k(\overline{r} - r)^{\kappa}$, for r in a left neighborhood of $\overline{r}$, $\kappa > 1/2$ and $k > 0$. Given the drift is negative around $\overline{r}$, that will ensure that $\overline{r}$ is a natural boundary, and $\{\forall t, r_t \leq \overline{r}\}$ almost surely, as detailed in Cheridito and Gabaix (2007). We next turn to the canonical LG bond case.

Example 13 *A multifactor bond model, with bond risk premia (continuous time).*

The following is Example 4 in continuous time. Suppose $dM_t/M_t = -r_t dt + dN_t$, where N_t is a martingale, and decompose the short rate in $r_t = r_* + \sum_{i=1}^n r_{it}$, with r_* a constant and:

$$E_t [dr_{it}] + \langle dr_{it}, dM_t/M_t \rangle = [-\phi_i r_{it} + (r_t - r_*) r_{it}] dt \tag{53}$$

where we use the notation $\langle dx_t, dy_t \rangle$ is the usual bracket, e.g. $\langle \sigma_1 \cdot dB_t, \sigma_2 \cdot dB_t \rangle = \sigma_1 \cdot \sigma_2$ dt.

Hence, it is enough to specify the process "under the risk-neutral measure". One does not need to separately specify the dynamics of $E_t [dr_{it}]$ and its risk premium, the $\langle dr_{it}, dM_t/M_t \rangle$ term. Only the sum matters. The process $M_t (1, r_{1t}, ..., r_{nt})$ is LG[13], and the bond price is

$$Z_t (T) = e^{-r_* T} \left(1 - \sum_{i=1}^n \frac{1 - e^{-\phi_i T}}{\phi_i} r_{it} \right). \tag{54}$$

The risk-premium at t on the $T-$maturity zero coupon, $\pi (T) := - \left\langle \frac{dZ_t(T)}{Z_t}, \frac{dM_t}{M_t} \right\rangle /dt$ can be simply expressed too.

Example 14 *Lucas economy where stocks, bonds, and a continuum of moments can be calculated.*

We consider a Lucas economy with: $\frac{dC_t}{C_t} = g_t dt + dN_t^C$, $var \left(dN_t^C \right) = \sigma^2 dt$, $dg_t = -\phi g_t dt + dN_t^g$, $\left\langle dg_t, \frac{dC_t}{C_t} \right\rangle = -g_t (g_t - A) dt$, with $A > 0$, and N_t^g, N_t^C are martingales, and $g_t \leq A$. Then:

$$\forall \alpha \leq 0, \forall T \geq 0, \ E_t \left[C_{t+T}^\alpha \right] = C_t^\alpha e^{\alpha(\alpha-1)\frac{\sigma^2}{2} t} \left(1 + \frac{1 - e^{-(\phi - \alpha A)T}}{(\phi - \alpha A)} \alpha g_t \right) \tag{55}$$

The $e^{\alpha(\alpha-1)\frac{\sigma^2}{2} t}$ term is the expected Jensen's inequality term. The novel term is the g_t term. This way, if the agent has utility $\int e^{-\rho t} C_t^{1-\gamma} / (1 - \gamma) dt$, one can calculate all the bonds prices in a Lucas economy, and the price of a claim on consumption.

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[13] The generator is $\omega = \begin{pmatrix} r_* & 1 & \cdots & 1 \\ 0 & r + \phi_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & r + \phi_n \end{pmatrix}$.

5 Conditions to keep the process well-defined

The results of this paper require that the process be defined for $t \geq 0$, and in particular that $M_t D_t > 0$, which ensures the above-derived prices are positive. This section provide simple sufficient conditions to ensure that. They are meant to be practical and easy to verify. Cheridito and Gabaix (2007) provide more abstract and general conditions.

5.1 Discrete time

We start with Example 1. We want the process to be well-defined. Write $g_{t+1} = \frac{\rho g_t + \sigma(g_t)\eta_{t+1}}{1+g_t}$, with $E_t\left[\eta_{t+1}\right] = 0$ and $\sigma(g_t) \geq 0$. First, take the case where there is no noise, $\forall t, \eta_{t+1} = 0$. The application $g \mapsto \rho g/(1+g)$ has two fixed points, an attractive one $g = 0$, and a repelling one that, $g = \rho - 1$. To ensure that the process is economically meaningful, we require that g_0 be on the right side of the repelling point, $g_0 > \rho - 1$. That will ensure (when there is no noise) that for all $t \geq 0$, $g_t > \rho - 1$, and in particular $g_t > -1$. If $g_0 < \rho - 1$, then for some time t, $g_t < -1$, not a meaningful economic outcome. In conclusion, in the deterministic growth rate case, we want to impose

$$g_t > \underline{g} = \rho - 1. \tag{56}$$

When the growth rate is stochastic, we want that for all $g_{t+1} > \rho - 1$, i.e. $g_t + \sigma(g_t)\eta_{t+1} > \underline{g}$. This is possible if η_{t+1} has a lower bound, and the volatility $\sigma(g)$ goes to 0 fast enough near the boundary $\underline{g}$, a fact formalized in the next Lemma.

Lemma 1 *(Conditions of existence of the process in the 1-dimensional, discrete time case). Consider the process in Example 1: $g_{t+1} = \frac{g_t + \sigma(g_t)\eta_{t+1}}{1+g_t}$, with $E_t\left[\eta_{t+1}\right] = 0$. Assume that (i) there is an $m > 0$ such that, almost surely, for all t, $\eta_{t+1} > -m$; and (ii) $0 \leq \sigma(g) \leq \frac{g - \underline{g}}{m}$, i.e. the volatility goes to 0 fast enough close to $\underline{g} = \rho - 1$. Suppose $g_0 > \underline{g}$. Then, almost surely, for all $t \geq 0$, $g_t > \underline{g}$, and the process is defined for all times t.*

The principle generalizes to several factors. Consider the discrete-time case where the generator Ω is of the form (**??**), with $\alpha > 0$ and $\alpha > \Gamma_i$ for all i, and $\gamma = 0$.[14] Parametrize the noise in (14) by $Y_{t+1} = \Omega Y_t + Y_{t0} u_{t+1}$, where $E_t [u_{t+1}] = 0$. The $n-$factor generalization of the criterion (56) above is the following:

Proposition 5 *(Condition to ensure a well-behaved process, with positive stochastic discount factor, discrete time) Suppose that $M_0 D_0 > 0$, and that at $t = 0$, X_0 satisfies:*

$$\textit{Condition C at time t (discrete time): } 1 + \sum_{i=1}^{n} \frac{\min (\delta_i X_{it} / \alpha, 0)}{1 - \Gamma_i / \alpha} > 0 \tag{57}$$

Suppose also that the noise u_{t+1} is bounded and goes to 0 fast enough near the boundary of (57). Then, for all times $t \geq 0$, $M_t D_t > 0$, so that the process is well-defined, and prices $E_t [M_{t+T} D_{t+T}]$ are positive. In addition, for all times t, X_t satisfies the condition (57).

The first part of the Proposition implies that, if the noise is small enough, then all prices derived above will be positive. The second part, means that if Condition C is satisfied at $t = 0$, then it will be satisfied for all future t's. This "self-perpetuating" property makes it potentially useful for applied work.

Condition C means $\delta_i X_{it}$ terms should not be too negative. It means that growth rates terms should not be too low, and interest rate terms should not be too high. This makes sense, because, in view of (19), if the terms $\delta_i X_{it}$ are too negative, then prices could be threaten to be negative.

To illustrate this, consider first Example 2. Then, $\delta = 1$, $\alpha = 1/(1 + R)$, $\Gamma_1 = \rho\alpha$, and the condition reads: $1 + \min (x_t, 0) / (1 - \rho) > 0$, i.e. $x_t > 1 - \rho$. This is exactly the condition (56) derived above. The deviation of the growth rate from trend cannot be too low.

Next, consider Example 4. Then, $\alpha = 1/(1 + r_*)$, $\delta = \alpha (-1, ..., -1)$, and $\Gamma = \alpha (\rho_1, ..., \rho_n)$. Condition C reads: $1 + \sum_{i=1}^{n} \frac{\min(-r_{it}, 0)}{1 - \rho_i} > 0$, i.e.

$$1 - \sum_{i=1}^{n} \frac{\max (r_{it}, 0)}{1 - \rho_i} > 0. \tag{58}$$

[14] This case is not very restrictive, as more general case can be reduced to it by diagonalization, if Ω is diagonalizable in $\mathbb{R}$.

This condition means that the components of the interest rate state vector cannot be too positive. Each component is weighted by its duration $1/(1-\rho_i)$, i.e. more persistent components count for more.

Finally, consider the hybrid Example 3. The condition reads: $1+\min(g_t - g_*, 0)/\left(1-\rho_g\right) - \max(\pi_t - \pi_*, 0)/(1-\rho_\pi) > 0$. This means that the growth rate should not be too low, or the risk premium should not be too high.

5.2 Continuous time

The same condition carries over to continuous time. In the case where ω is equal to (41) with $b = 0$, and $a < \Phi_i$ for all i, the condition is:

$$\text{Condition } C \text{ at time } t \text{ (continuous time): } 1 - \sum_{i=1}^{n} \frac{\max(\beta_i X_{it}/a, 0)}{\Phi_i - a} > 0 \qquad (59)$$

For instance, for the simple growth model of Example 7, we have $X_t = g_t$, $a = r$, $\beta = -1$, $\Phi = \phi + r$, so Condition C gives: $1 - \max(-g_t, 0)/\phi > 0$, i.e. $g_t > -\phi$, the continuous time limit of (56).

Likewise, for the interest rate model of Example 12, the condition is $\widehat{r}_t < \phi$. The interest rate should not be too high.

In the multi-factor model of Example 13, $a = r_*$, $\beta = 1$, $\Phi_i = r_* + \phi_i$, so the Condition is: $1 - \sum_{i=1}^{n} \max(r_{it}, 0)/\phi_i > 0$, which is just the continuous time analogue of (58).

This conclude our simple, practical sufficient conditions for processes to be well-defined. More abstract and general conditions are provided in Cheridito and Gabaix (2007).

6 Extensions

This section presents additional results and remarks on LG processes.

6.1 LG processes are the only ones that generate linearity

We show that, in a certain sense, if bond prices are linear in the factors, then they come from an LG process. To see that, let us first consider the 1-factor case. Call X_t the factor, and suppose that for $T = 1, 2$, $Z_t(T) = \alpha_T + \beta_T X_t$, for some numbers α_1, $\beta_1 \neq 0$, α_2, β_2. With $T = 1$, we get $E_t[M_{t+1}/M_t] = \alpha_1 + \beta_1 x_t$, so that condition (9) holds. Also,

$$\begin{aligned} \alpha_2 + \beta_2 X_t &= E_t\left[\frac{M_{t+2}}{M_t}\right] = E_t\left[\frac{M_{t+1}}{M_t} E_{t+1}\left[\frac{M_{t+2}}{M_{t+1}}\right]\right] = E_t\left[\frac{M_{t+1}}{M_t}(\alpha_1 + \beta_1 X_{t+1})\right] \\ &= \alpha_1(\alpha_1 + \beta_1 X_t) + \beta_1 E_t\left[\frac{M_{t+1}}{M_t} X_{t+1}\right] \end{aligned}$$

$$E_t\left[\frac{M_{t+1}}{M_t} X_{t+1}\right] = \frac{1}{\beta_1}(\alpha_2 + \beta_2 X_t - \alpha_1(\alpha_1 + \beta_1 X_t)) = a'' + b'' X_t$$

hence (10) holds. We conclude that if both the 1 and 2-period maturity bonds are affine in X_t, then $M_t(1, X_t)$ is a LG process. The next Proposition shows that the property holds with n factors[15]

Proposition 6 *(LG processes are the only processes generating linear bond prices) Suppose that there are coefficients for some coefficients* $(\alpha_T, \beta_T,)_{T\geq 0}$, *with* $\{(\alpha_T, \beta_T), T = 1, 2..\}$ *spanning* $\mathbb{R}^{n+1}$, *such that* $\forall t, T \geq 0$, $E_t[M_{t+T}/M_t] = \alpha_T + \beta_T' X_t$, *Then,* $M_t(1, X_t)$ *is a LG process, i.e. there is a matrix* Ω, *such that* $Y_t = M_t(1, X_t)'$ *follows:* $E_t[Y_{t+1}] = \Omega Y_t$.

6.2 Relation to the affine-yield class

The affine class (Duffie and Kan 1996; Dai and Singleton 2000; Duffie, Pan and Singleton 2000; Duffee 2002; Cheridito, Filipovic and Kimmel 2007) is a very important class, that contains the processes of Vasicek and Cox, Ingersoll, Ross (1985). It is a workhorse of much empirical and theoretical in asset pricing. It comprises processes of the type: $dX_t =$

[15] The property that $\{(\alpha_T, \beta_T), T = 1, 2..\}$ spans $\mathbb{R}^{n+1}$ means that $E_t[M_{t+T}/M_t] = \alpha_T + \beta_T' X_t$ is the most compact representation of the process. More precisely, if it didn't span $\mathbb{R}^{n+1}$, one could find a stricly lower dimensional process $x_t \in \mathbb{R}^m$, $m < n$, and constants A_T, B_T, such that $E_t[M_{t+T}/M_t] = A_T + B_T' x_t$. Indeed, call $\gamma_T = (\alpha_T, \beta_T')'$, and $V = Span\{\gamma_T, T \geq 0\}$. If V is a strict subset of $\mathbb{R}^{n+1}$, decompose $\mathbb{R}^{n+1} = V \oplus V^{\perp}$, call $B : V \to \mathbb{R}^{n+1}$ the natural injection, and $\langle \cdot, \cdot \rangle$ the restriction of the Euclidean product on V. Then, $\gamma'(T) Y_t = (B\gamma(T))' Y_t = \gamma(T)' B' Y_t$, so we have $Z_t(T) = \gamma(T)'(B' Y_t)$. Vector $B' Y_t$ has dimension $\dim V < n + 1$.

$(b - \Phi X_t)\, dt + w_t dz_t$, with $w_t w_t' = \sigma^2 (H_1' X_t + H_0)$, with $b, X_t \in \mathbb{R}^n$, $\Phi \in \mathbb{R}^{n \times n}$, $(H_0, H_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}, \sigma \in \mathbb{R}$, z_t is a $n-$dimensional Brownian motion. The interest rate is $r_t = r_* + \beta' (X_t - X_*)$, where $X_* = \Phi^{-1} b$, is assumed to exist. Under mild technical conditions, bond prices have the expression:

$$Z_t^{\mathrm{Aff}}(T) = \exp\left(-r_* T + \Gamma(T)' (X_t - X_*) + \sigma^2 a(T)\right),$$

where $a(T)$ and $\Gamma(T)$ satisfy coupled ordinary differential equations, that typically need to be solved numerically. This is not a problem for empirical work, but that does hinder theoretical work. The situation is simpler if $H_1 = 0$. In that case, $\Gamma(T) = \gamma(T)$, with $\gamma(T)' = \beta' \left(e^{-\Phi T} - 1\right)/\Phi$. Then: $Z_t^{\mathrm{Aff}}(T) = \exp\left(-r_* T + \gamma(T)' (X_t - X_*) + \sigma^2 a(T)\right)$. This expression can be contrasted with the expression for the LG process (38),

$$Z_t^{\mathrm{LG}}(T) = e^{-r_* T}\left(1 + \gamma(T)' (X_t - X_*)\right). \tag{60}$$

If $\gamma(T)' X_t$ is small, the two expressions are the same, up to terms of second order in $\gamma(T)' X_t$, and second order in σ. Hence, a LG process is a good approximation if the underlying process is in fact affine, and vice-versa. In most cases, the two values are likely to be close, so that existing estimates of parameters in the affine class can be used to calibrate LG processes.[16]

What are the respective merits of the LG and affine classes? First, quantitatively, they will often make close predictions, as the two models yield the same prices to a first order. Hence, for many situations, the choice of affine vs LG processes is just a matter of convenience.

In terms of the *economic* differences, as LG bond prices are independent of volatility (controlling for the covariances, see Eq. 54), LG processes generate "unspanned volatility," a relevant feature of the data, as shown by Collin-Dufresne and Goldstein (2002), Andersen and Benzoni (2007) and Joslin (2007). By contrast, affine models typically impose a tight

[16] That equivalence gives a useful way to calculate easily functionals of LG processes, that can be expressed as a linear combination of bonds. One first works with the affine process, setting volatility to 0, doing a first order Taylor expansion of terms in $(X_t - X_*)$. One gets an expression: $P_t^{\mathrm{Aff}} = a + b(X_t - X_*) + o(X_t - X_*) + o(\sigma^2)$, for some constant a, b. Then, one knows that for the corresponding LG process, the value of the asset is: $P_t^{\mathrm{LG}} = a + b(X_t - X_*)$, exactly.

link between bond prices and volatility.

On the other hand, a potential drawback of pricing bonds with the LG process, is that, in the simplest version at least, bonds have no mechanically-induced convexity in the LG framework. However, this may not be such a problem, as Joslin (2007) estimates that bond convexity plays a small role in bond prices. In addition, multifactor LG processes can have some convexity (Example 11).

Coming now to the difference in terms of *tractability*, the distinctive advantage of the LG class is for stocks. LG yield simple closed forms for stock prices. However, with the affine class, a stock price can be only be expressed $P_t^{\text{Aff}}/D_t = \sum_{t=0}^{\infty} Z_t^{\text{Aff}}(T)$ (Burnside 1998, Ang and Liu 2004). Those are infinite sums of exponentials, which is a great progress over stochastic sums, but are still not very tractable.

Beyond their advantage for stocks, LG processes have two lesser virtues. First, bond prices are quite simple, which should prove useful to theorize on bonds (Gabaix 2007). Second, LG processes allow a free functional form for the innovations dN_t, which can include jumps and non-Gaussian behavior, and a free type of heteroskedascity.

On the other hand, affine processes are the central technique to price derivatives, whereas this paper is silent about options (pricing options with LG processes is an open challenge). Finally, affine models are now well-understood, and they have been estimated. It would be very desirable to do the same for LG models (see Binsbergen and Koijen 2007)

In conclusion, LG processes have a good advantage for stocks, and affine processes have a strong advantage for options. For bonds, affine models will continue to be tremendously useful, but LG models may complement them, particularly in theoretical research.

6.3 Processes with time-dependent coefficients

It is simple to extend the process to time-dependent deterministic coefficients, i.e. in Definition 1, to have $\alpha, \delta, \gamma, \Gamma$ functions of time. With $Y_t = (M_t, M_t X_t)^\top$, this is $E_t[Y_{t+1}] = \Omega_t Y_t$, where $\Omega_t = \begin{pmatrix} \alpha_t & \delta_t' \\ \gamma_t & \Gamma_t \end{pmatrix}$. That implies $E_0[Y_T] = \prod_{t=0}^{T-1} \Omega_t Y_0$. Hence, in the zero-coupon expressions, it is enough to replace Ω^T by $\prod_{t=0}^{T-1} \Omega_t$.

6.4 Closedness under addition and multiplication

The product of two uncorrelated LG processes is LG. The product of two uncorrelated LG processes with respective dimensions d_1, d_2 (i.e., with $d_1 - 1$ and $d_2 - 1$ factor respectively) is LG, with dimension $d_1 d_2$ (i.e., with $d_1 d_2 - 1$ factors). The idea is simple, though it requires somewhat heavy notations.

We start in discrete time. Take two LG processes characterized by M_t^i, Y_t^i, Ω^i, and consider a process with discount factor $M_t = M_t^1 M_t^2$. Assume that, for any index i, j of the components, $cov\left(Y_{t+1}^{1(i)}, Y_{t+1}^{2(j)}\right) = 0$. The innovations between processes are uncorrelated, but, importantly, not necessarily independent. Then, it is easy to verify that for any vector ψ^i, $E_t\left[\left(\psi^1 Y_T^1\right)\left(\psi^2 Y_T^2\right)\right] = E_t\left[\psi^2 Y_T^2\right] E_t\left[\psi^2 Y_T^2\right]$. In particular, $E_t\left[M_T^1 M_T^2\right] = E_t\left[M_T^1\right] E_t\left[M_T^2\right]$.

Then, $M_t = M_t^1 M_t^2$ is also the stochastic discount factor of a LG process. The underlying autoregressive process is $\overline{Y}_t^1 \otimes \overline{Y}_t^2$, i.e. the vector made of the $d_1 d_2$ components $\overline{Y}_t^{1(i)} \overline{Y}_t^{2(j)}$, $i = 1...d_1, j = 1...d_2$. The corresponding generator Ω is $\Omega = \Omega^1 \otimes \Omega^2$.

The same reasoning holds in continuous time. Starting with processes M_t^i, Y_t^i, ω^i, and assuming $\left\langle dY_{t+1}^{1(i)}, dY_{t+1}^{1(2)} \right\rangle = 0$, then $M_t^1 M_t^2$ is also a pricing kernel that comes from a LG process. The underlying autoregressive process is $\overline{Y}_t^1 \otimes \overline{Y}_t^2$ (which has dimension $d_1 d_2$), and it is easy to check that the generator is: $\omega = \omega^1 \otimes I_{d_2} + I_{d_1} \otimes \omega^2$. To make the above concrete, consider the following example.

Example 15 *Stock with decoupled LG processes for the growth rate and the risk premium.*

Consider processes with $dM_t/M_t = -rt - \lambda_t dB_t$, $dD_t/D_t = g_t dt + \sigma_t dB_t$, where g_t follows: $dg_t = -\phi_g (g_t - g_*) dt - (g_t - g_*)^2 dt + dN_t^g$, and the risk premium, $\pi_t = \lambda_t \sigma_t$, follows: $d\pi_t = -\phi_\pi (\pi_t - \pi_*) dt + (\pi_t - \pi_*)^2 dt + dN_t^\pi$, where N_t^g, N_t^π are martingales. Assume that the processes dN_t^g, dN_t^π and dB_t are uncorrelated. Then, the price of a stock, $P_t = E_0\left[\int_0^\infty M_t D_t dt\right]/M_0$, is $P_t/D_t = E_t\left[\int_{s=t}^\infty \exp\left(-\int_{u=t}^s (r + \pi_u - g_u) du\right) ds\right]$. In virtue of the above properties,

$$E_t\left[\exp\left(\int_t^s -\pi_u + g_u du\right)\right] = E_t\left[\exp\left(\int_t^s -\pi_u du\right)\right] E_t\left[\exp\left(\int_t^s g_u du\right)\right].$$

For general processes, the above equation would in general require the two processes to be independent – for instance, with stochastic volatility, the respective variance processes should be independent. For LG processes, the property required is the weaker $\langle d\pi_t, dg_t \rangle = 0$ for all t's.

Using the values of the LG processes, and integrating, we obtain, with $R = r + \pi_* - g_*$,[17]

$$P_t/D_t = \frac{1}{R}\left[1 - \frac{\pi_t - \pi_*}{R+\phi_\pi} + \frac{g_t - g_*}{R+\phi_g} - \frac{\left(2R+\phi_\pi+\phi_g\right)\left(\pi_t-\pi_*\right)\left(g_t-g_*\right)}{\left(R+\phi_\pi\right)\left(R+\phi_g\right)\left(R+\phi_\pi+\phi_g\right)}\right]. \tag{61}$$

The central value is again the Gordon formula, $P_t/D_t = 1/R$. It is modified by the current level of the equity premium, and the growth rate of the stock. A stock with a currently high growth rate g_t exhibits a higher price-dividend ratio, and this is amplified when the equity premium is low, as shown by the term $(\pi_t - \pi_*)(g_t - g_*)$.

The difference between formula (61) and formula (50) is that in (61), the processes for π_t and g_t are decoupled, whereas in (50), they were coupled, i.e. in their drift term there was a term $(g_t - g_*)$. The decoupling forces the presence of a cross term $(\pi_t - \pi_*)(g_t - g_*)$ in the expression of the price. In general, one obtains simpler expressions by having one multifactor LG processes, rather than the product of many different LG processes. With n coupled factors, the stock price has $n+1$ terms, while with n decoupled factors, the stock price has 2^n terms.

The sum of two LG processes is LG. This property is quite trivial, and mentioned for completeness. Suppose two LG process M_t^i, Y_t^i, Ω^i, with $M_t^i = \nu^i Y_t^i$, for $i = 1, 2$. Call d_i the dimension of Y_t^i. Then, $M_t = M_t^1 + M_t^2$ comes from a LG process of dimension $d_1 + d_2$. Indeed, define $Y_t = (Y_t^1, Y_t^2)$, $\nu = (\nu^1, \nu^2)$, and $\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$. Then, $E_t[Y_{t+1}] = \Omega Y_t$, and $M_t = \nu' Y_t$.

7 Conclusion

Linearity-generating processes are very tractable, as they yield closed forms for stocks and bonds, and prices that are linear in factors. They are likely to be useful in several parts of

[17] Menzly, Santos and Veronesi (2004, Eq. 20) obtain a similar expression. This is natural because the LG class embeds their model, as Example 10 shows.

economics, when trend growth rates, or risk premia, are time-varying. The results of this paper suggest the following research directions.

Most importantly, LG processes allow the construction of paper and pencil tractable general equilibrium models, with closed forms for stocks and bonds. Indeed they suggest a way to "reverse engineer" the processes for endowments and technology, so that the model is tractable. Gabaix (2007) and Farhi and Gabaix (2007) present such models.

Second, it would be good to use the flexibility and closed forms of LG processes for empirical work. In an ambitious new paper, Binsbergen and Koijen (2007) take up this task, and demonstrate the fruitfulness of having closed forms for multifactor models. Also, since the LG processes are defined by moment conditions (Eq. 9-10), they lend themselves to estimation and testing by GMM techniques.

Third, LG processes suggest a new way to linearize models. Given a model, one could do a Taylor expansion expressing moments $E_t\left[M_{t+1}D_{t+1}/M_tD_t\right]$ and $E_t\left[M_{t+1}D_{t+1}X_{t+1}/M_tD_t\right]$ as a linear function of the factors, thereby making Eq. 9-10 hold to a first order approximation. The projected model is then in the LG class, and its asset prices are approximations of the prices of the initial problem. Hence the LG class offers a way to derive linear approximations of the asset prices of more complicated models. The Online Appendix to this paper studies such an example, where a non-LG process can be approximated by an LG process to an arbitrary degree of precision.

Fourth, LG processes can be enriched by a decision variable, and offer a way to do multifactor, closed-form dynamic programming. I explore those issues in ongoing research.

I conclude that LG processes might be a useful addition to the economist's toolbox.

Appendix A. Matrix Algebra

Derivations with LG processes often use the following Lemmas, which are standard.

Lemma 2 *With $a \in \mathbb{R}, b, c \in \mathbb{R}^n$, and $d \in \mathbb{R}^{n\times n}$, suppose that d is invertible and that the real number $a - b'd^{-1}c$ is $\neq 0$. Then the $(n+1)\times(n+1)$ matrix $\begin{pmatrix} a & b' \\ c & d \end{pmatrix}$ is invertible,* *and its inverse is:*

$$\begin{pmatrix} a & b' \\ c & d \end{pmatrix}^{-1} = \frac{1}{a - b'd^{-1}c} \begin{pmatrix} 1 & -b'd^{-1} \\ -d^{-1}c & ad^{-1} \end{pmatrix} \tag{62}$$

Lemma 3 *With $n \in \mathbb{N}_+^*$, $a \in \mathbb{R}, b \in \mathbb{R}^n$, and $d \in \mathbb{R}^{n\times n}$. Call 0_n the $n-$dimensional vector made of 0's, I_n the $n-$dimensional identity matrix, and suppose that $aI_n - d$ is invertible. Then, for $T \in \mathbb{N}^*$,*

$$\begin{pmatrix} a & b' \\ 0_n & d \end{pmatrix}^{T} = \begin{pmatrix} a^T & b'\left(a^T I_n - d^T\right)(aI_n - d)^{-1} \\ 0_n & d^T \end{pmatrix}$$

and, for $T \in \mathbb{R}$,

$$\exp\left[\begin{pmatrix} a & b' \\ 0_n & d \end{pmatrix} T\right] = \begin{pmatrix} e^{aT} & b'\left(e^{aT} I_n - e^{dT}\right)(aI_n - d)^{-1} \\ 0_n & e^{dT} \end{pmatrix}$$

Appendix B. Additional Derivations

7.1 Derivation of Theorems and Propositions

Proof of Theorem 1 Recall (14), $E_t[Y_{t+1}] = \Omega Y_t$. Iterating on T, it implies that for all $T \geq 0$, $E_t[Y_{t+T}] = \Omega^T Y_t$. Given $M_{t+T} = \nu' Y_{t+T}$,

$$\begin{aligned} Z_t(T) &= (M_t D_t)^{-1} E_t[M_{t+T} D_{t+T}] = (M_t D_t)^{-1} E_t[\nu' Y_{t+T}] = (M_t D_t)^{-1} \nu' E_t[Y_{t+T}] \\ &= (M_t D_t)^{-1} \nu' \Omega^T Y_t = \nu' \Omega^T \left((M_t D_t)^{-1} Y_t\right) = \nu' \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0_n \end{pmatrix} \Omega^T \begin{pmatrix} 1 \\ X_t \end{pmatrix} \end{aligned}$$

i.e. (15). The formula for $\gamma = 0$ comes from Lemma 3 in Appendix A.

Proof of Theorem 2 We use (15), which gives the perpetuity price:

$$P_t/D_t = \sum_{T=0}^{\infty} Z_t(T) = \nu' \left(\sum_{T=0}^{\infty} \Omega^T \right) \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \nu' (I_{n+1} - \Omega)^{-1} \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

$\sum_{T=0}^{\infty} \Omega^T$ is summable because all eigenvalues of Ω have a modulus less than 1. We use Lemma 2 from Appendix A to calculate $(I_n - \Omega)^{-1}$, and conclude.[18]

Proof of Theorem 3 Recall the definition of ω in (35), and $E_t [d(Y_t)] = -\omega Y_t dt$. It is well-known that this implies[19]: $\forall T \geq 0, E_t [Y_{t+T}] = e^{-\omega T} Y_t$. Given $M_{t+T} = \nu' Y_{t+T}$,

$$\begin{aligned} Z_t(T) &= (M_t D_t)^{-1} E_t [M_{t+T} D_{t+T}] = (M_t D_t)^{-1} E_t [\nu' Y_{t+T}] = (M_t D_t)^{-1} \nu' E_t [Y_{t+T}] \\ &= (M_t D_t)^{-1} \nu' e^{-\omega T} Y_t = \nu' e^{-\omega T} \left((M_t D_t)^{-1} Y_t \right) = \nu' e^{-\omega T} \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \begin{pmatrix} 1 & 0_n \end{pmatrix} e^{-\omega T} \begin{pmatrix} 1 \\ X_t \end{pmatrix}. \end{aligned}$$

i.e. Eq. 37. The formula for $b = 0$ comes from Lemma 3 in Appendix A.

Proof of Theorem 4 We use (37). The perpetuity price is:

$$P_t/D_t = \int_0^{\infty} Z_t(T) \, dT = \nu' \left(\int_0^{\infty} e^{-\omega T} dT \right) \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix} = \nu' \omega^{-1} \cdot \begin{pmatrix} 1 \\ X_t \end{pmatrix}$$

We use the Lemma 2 to calculate ω^{-1}, and conclude.[20]

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[18] There is a more elementary heuristic proof. We seek a solution of the type $P_t/D_t \equiv V_t = c - 1 + h' X_t$, which we know exists, by summation of (15). The arbitrage equation is: $V_t = 1 + E\left[\frac{M_{t+1} D_{t+1}}{M_t D_t} V_{t+1}\right]$, i.e.

$$c + h' X_t = 1 + E\left[\frac{M_{t+1} D_{t+1}}{M_t D_t} (c + h' X_t)\right] = 1 + c\left(\alpha + \delta' X_t\right) + h' (\gamma + \Gamma Y_t) = [1 + c\alpha + h'\gamma] + \left[c\delta' + h'\Gamma\right] X_t$$

i.e. (i) $c = 1 + c\alpha + h'\gamma$ and (ii) $h' = c\delta' + h'\Gamma$. (ii) gives $h' = c\delta' (1 - \Gamma)^{-1}$, and plugging in (i) yields $c\left[1 - \alpha - \delta' (1 - \Gamma)^{-1} \gamma\right] = 1$, hence c and (17).

[19] In the case $t = 0$ (which is enough), the proof is thus: define $f(T) = E_0 [Y_T]$. Then, $df(T) = E_0 [dY_T] = E_0 [-\omega Y_T dT] = -\omega E_0 [Y_T] dT = -\omega f(T) dT$, which integrates to $f(T) = e^{-\omega T} f(0)$, i.e. $E_0 [Y_T] = e^{-\omega T} Y_0$.

[20] The following elementary heuristic proof is useful to know. We seek a solution of the type $P_t/D_t \equiv V_t = c + h' X_t$, which we know exists, by integration of (37). The arbitrage equation is: $1 - r_t V_t + E[dV_t]/dt = 0$,

Proof of Proposition 5 Write $Y_t = (Y_{0t}, ..., Y_{nt})$, with $Y_{0t} = M_t D_t$, and define $H_t = Y_{0t} + \sum_{i=1}^n \frac{\min(\delta_i Y_{it}, 0)}{\alpha - \Gamma_i}$. Start with the case of where there is no noise, i.e. $\forall t, Y_{t+1} = \Omega Y_t$. Given Ω, this means $Y_{0,t+1} = \alpha Y_{0t} + \sum_i \delta_i Y_{it}$, and for $i \geq 1$, $Y_{i,t+1} = \Gamma_i Y_{i,t}$. So:

$$\begin{aligned} H_{t+1} &= \alpha Y_{0t} + \sum_i \delta_i Y_{it} + \sum_i \frac{\min(\delta_i \Gamma_i Y_{it}, 0)}{\alpha - \Gamma_i} \\ &= \alpha Y_{0t} + \sum_{i \text{ s.t. } \delta_i Y_{it} > 0} \delta_i Y_{it} + \sum_{i \text{ s.t. } \delta_i Y_{it} \leq 0} \left(1 + \frac{\Gamma_i}{\alpha - \Gamma_i}\right) \delta_i Y_{it} \\ &\geq \alpha Y_{0t} + \sum_{i \text{ s.t. } \delta_i Y_{it} \leq 0} \frac{\alpha}{\alpha - \Gamma_i} \delta_i Y_{it} = \alpha H_t. \end{aligned}$$

Hence, $H_{t+1} \geq \alpha H_t$. Hence, if $H_0 > 0$, then $\forall t \geq 0$, $H_t > 0$, and so that $M_t D_t = Y_{0t} \geq H_t > 0$.

In the case with noise, say that $Y_{t+1} = \Omega Y_t + u_{t+1}$, for some mean 0 noise u_{t+1}, and suppose that u_{t+1} is bounded. By continuity, if $H_t > 0$, H_{t+1} is positive with probability 1, if u_{t+1} is small enough. And again, $M_t D_t = Y_{0t} \geq H_t > 0$.

Proof of Proposition 6 Call $Y_t = M_t (1, X_t)'$, $\gamma_T = (\alpha_T, \beta_T)'$, so that $E_t [M_{t+T}] = \gamma_T' Y_t$. That implies $\gamma_{T+1}' Y_t = E_t [M_{t+T+1}] = E_t [E_{t+1} [M_{t+T+1}]] = E_t [\gamma_T Y_{t+1}]$, hence: $\gamma_{T+1}' Y_t = E_t [\gamma_T Y_{t+1}]$.

Call $e_k \in \mathbb{R}^{n+1}$, the vector with $k-$th coordinate equal to 1, and other coordinates equal to 0. As$\{\gamma_T, T = 1, 2, ..\}$ spans $\mathbb{R}^{n+1}$, there are reals λ_{kT} (with at most $n+1$ non-zero values) such that: $e_k = \sum_T \lambda_{kT} \gamma_T$. Define $\Omega = \sum_{k,T} e_k \lambda_{kT} \gamma_{T+1}'$. Given $I_{n+1} = \sum_{k=1}^{n+1} e_k e_k'$, we have:

$$\begin{aligned} E_t [Y_{t+1}] &= \left(\sum_k e_k e_k'\right) E_t [Y_{t+1}] = \left(\sum_k e_k \left(\sum_T \lambda_{kT} \gamma_T'\right)\right) E_t [Y_{t+1}] \\ &= \sum_{k,T} e_k \lambda_{kT} E_t [\gamma_T' Y_{t+1}] = \sum_{k,T} e_k \lambda_{kT} \gamma_{T+1}' Y_t = \Omega Y_t. \end{aligned}$$

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i.e.

$$1 - \left(r_* + \beta' X_t\right) \left(c + h' X_t\right) + h' \left[b - \Phi X_t + \left(\beta' X_t\right) X_t\right] = 0$$

This holds if and only if the constant and the term in X_t are zero, i.e. $r_* h' + \beta' c + h' \Phi = 0$ and $1 - r_* c + h' b = 0$. Hence $h' = -\beta' c \left(r_* + \Phi\right)^{-1}$ and $1 - c r_* + \beta' \left(r_* + \Phi\right)^{-1} b = 0$, which gives $c = 1 / \left[r_* + \beta' \left(r_* + \Phi\right)^{-1} b\right]$, and yields (39).

7.2 Derivations of Examples

Example 3 Define $\widehat{\pi}_t = \pi_t - \pi_*, \widehat{g}_t = g_t - g_*$, so that:

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right] = \frac{1}{1+r}(1+g_t-\pi_t) = \alpha + \frac{\widehat{g}_t - \widehat{\pi}_t}{1+r}$$

$$E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\widehat{g}_{t+1}\right] = E_t\left[\frac{M_{t+1}}{M_t}\frac{D_{t+1}}{D_t}\right]E_t\left[\widehat{g}_{t+1}\right] = \frac{1}{1+r}(1+g_t-\pi_t)\cdot\frac{1+g_*-\pi_*}{1+g_t-\pi_t}\rho_g\widehat{g}_t = \alpha\rho_g\widehat{g}_t$$

The analogue expression holds for $\widehat{\pi}_t$. Hence The process $Y_t = M_tD_t\left(1, \widehat{\pi}_t, \widehat{g}_t\right)'$ is LG, with generator $\begin{pmatrix} \alpha & 1/(1+r) & -1/(1+r) \\ 0 & \alpha\rho_g & 0 \\ 0 & 0 & \alpha\rho_\pi \end{pmatrix}$. Applying (17) yields the result.

Example 4 We have $E_t\left[\frac{M_{t+1}}{M_t}r_{i,t+1}\right] = \frac{1}{1+r_*}\rho_i r_{i,t}$. So process $M_t\left(1, r_{1,t}, ..., r_{n,t}\right)$ has generator: $\frac{1}{1+r_*}\begin{pmatrix} 1 & -1 & \cdots & -1 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \rho_n \end{pmatrix}$. By (11) and (16), the bond price obtains.

Example 5 $E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\right] = \sum_i G_iX_{it}$, and

$$E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}X_{i,t+1}\right] = E_t\left[\frac{M_{t+1}D_{t+1}}{M_tD_t}\right]E_t\left[X_{i,t+1}\right] = \left(\sum_k G_kX_{kt}\right)\left(\sum_j p_{ij}X_{jt}\right) = \sum_j p_{ij}G_jX_{jt}$$

as $X_{kt}X_{jt} = 0$ if $j \neq k$, and otherwise is equal to $X_{kt}X_{jt} = X_{jt}$, as exactly one of the X_{jt} is $\neq 0$.

Example 7 $E_t\left[d\left(M_tD_t\right)/\left(M_tD_t\right)\right] = \left(-r+g_t\right)dt$ and

$$E_td\left(M_tg_t\right) = g_t\cdot E_t\left[dM_t\right] + M_tE_tdg_t = g_t\cdot\left(-r+g_t\right)M_tdt + M_t\left(-\phi g_t - g_t^2\right)dt = -\left(r+\phi\right)g_tM_tdt$$

We note that the g_t^2 terms cancel out, which is their raison d'etre in (47). So $M_t(1, g_t)$ is a LG process with generator $\omega = \begin{pmatrix} r & -1 \\ 0 & r+\phi \end{pmatrix}$.

Example 8 We calculate the LG moments:

$$\begin{aligned}
E_t\left[\frac{d(M_tD_t)}{M_tD_t}\right]/dt &= -(r-g_*) + \sum_{i=1}^{n} X_{jt} \\
E_t\left[\frac{d(M_tD_tX_{it})}{M_tD_t}\right]/dt &= \left[-(r-g_*) + \sum_{i=1}^{n} X_{jt}\right] X_{it} + \left(-\phi_i X_{it} - \left(\sum_{i=1}^{n} X_{it}\right) X_{it}\right) \\
&= -(r-g_*+\phi_i) X_{it},
\end{aligned}$$

so $M_tD_t(1, X_{1t}, ..., X_{nt})$ is a LG process, with generator: $\omega = \begin{pmatrix} r-g_* & -1 & \cdots & -1 \\ 0 & r-g_*+\phi_1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & r-g_*+\phi_n \end{pmatrix}$.

We apply the Theorem 3, with $a = r-g_*, \beta' = (-1, ..., -1), \Phi = Diag(r-g_*+\phi_1, ..., r-g_*+\phi_n)$.

Example 9 We calculate the LG moments:

$$\begin{aligned}
E_t\left[\frac{d(M_tD_t)}{M_tD_t}\right]/dt &= -r-\pi_t+g_t = -R-\widehat{\pi}_t+\widehat{g}_t \\
E_t\left[\frac{d(M_tD_t\widehat{g}_t)}{M_tD_t}\right]/dt &= E_t\left[\frac{d(M_tD_t)/dt}{M_tD_t}\right]\widehat{g}_t + E_t[\widehat{g}_t]/dt \\
&= (-R-\widehat{\pi}_t+\widehat{g}_t)\widehat{g}_t + -\left(\phi_g-\widehat{\pi}_t+\widehat{g}_t\right)\widehat{g}_t = -\left(R+\phi_g\right)\widehat{g}_t
\end{aligned}$$

and likewise for $E_t\left[\frac{d(M_tD_t\widehat{\pi}_t)}{M_tD_t}\right]/dt = -(R+\phi_\pi)\widehat{\pi}_t$. So $M_tD_t(1, \widehat{g}, \widehat{\pi}_t)$ is LG with generator $\begin{pmatrix} R & 1 & 1 \\ 0 & R+\phi_g & 0 \\ 0 & 0 & R+\phi_\pi \end{pmatrix}$.

Example 11 Calculation shows that $Y_t = e^{-rT}\left(D_t, D_t g, D_t g_t^2\right)$ is a LG process, with generator $\omega = \begin{pmatrix} r & -1 & 0 \\ 0 & r+\phi & -1/2 \\ -kG^2 & -b & r+k+2\phi \end{pmatrix}$.

Example 12 We calculate the LG moments: $dM_t/M_t = -r_t dt = -\left(r_* + \widehat{r}_t\right)dt$, and:

$$\begin{aligned} \frac{d\left(M_t\widehat{r}_t\right)}{M_t} &= \widehat{r}_t\frac{dM_t}{M_t} + dX_t = -\widehat{r}_t\left(r_* + \widehat{r}_t\right)dt + -\left(\phi - \widehat{r}_t\right)\widehat{r}_t dt + \sigma_t dN_t \\ &= -\left(r_* + \phi\right)\widehat{r}_t dt + \sigma_t dN_t \end{aligned}$$

Importantly, the $\widehat{r}_t^2$ terms cancel out. So, using $E_t\left[dN_t\right] = 0$, we have the LG moments:

$$E_t\left[dM_t/M_t\right]/dt = -r_* - \widehat{r}_t \text{ and } E_t\left[d\left(M_t\widehat{r}_t\right)/M_t\right]/dt = -\left(r_* + \phi\right)\widehat{r}_t$$

So $Y_t = M_t\left(1, \widehat{r}_t\right)$ is LG with generator $\begin{pmatrix} r_* & 1 \\ 0 & r_* + \phi \end{pmatrix}$.

Example 14 $E_t\frac{dC_t^\alpha}{C_t^\alpha} = \alpha g_t + \alpha\left(\alpha - 1\right)\frac{\sigma^2}{2}$ and

$$\begin{aligned} \frac{E_t\left[d\left(C_t^\alpha g_t\right)\right]}{C_t^\alpha dt} &= \left(\alpha g_t + \alpha\left(\alpha-1\right)\frac{\sigma^2}{2}\right)g_t - \phi g_t + \alpha\left\langle dg_t, \frac{dC_t}{C_t}\right\rangle \\ &= \left(\alpha g_t + \alpha\left(\alpha-1\right)\frac{\sigma^2}{2}\right)g_t - \phi g_t - \alpha g_t\left(g_t - A\right) = \left(-\phi + \alpha A + \alpha\left(\alpha-1\right)\frac{\sigma^2}{2}\right)g_t dt \end{aligned}$$

so, $C_t^\alpha\left(1, g_t\right)$ is a LG process with generator $\omega = \begin{pmatrix} \alpha\left(\alpha-1\right)\frac{\sigma^2}{2} & \alpha \\ 0 & \alpha\left(\alpha-1\right)\frac{\sigma^2}{2} - \phi + \alpha A \end{pmatrix}$. The statement follows. The process is well-defined if $\alpha\left(g_t - A\right) > -\phi$, which is ensured if the volatility of g_t goes to 0 fast enough at $g_t = A$, so that always $g_t \leq A$.

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Online Appendix for "Linearity-Generating Processes: A Modelling Tool Yielding Closed Forms for Asset Prices"

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September 10, 2007

This online Appendix discusses a few additional issues related to LG processes. The first section discusses how to approximate non-LG processes with LG processes. The second section illustrates the difference between non-LG and LG processes, to see to what extent the quadratic terms makes a difference. The discussion uses heuristic arguments, and simulations.

This online Appendix contains ideas that might be profitably developed in further papers, notably that theme that LG processes might offer a way to arbitrarily approximate non-LG processes.

8 Approximating non-LG processes with LG processes

LG processes offer a way to approximate the price of stocks and bonds with non-LG processes, often to an arbitrary degree of precision. This section illustrates this in a particular example. The general properties of approximation with LG processes would require a full paper, but the present appendix simply illustrates that a preliminary investigation justifies being optimistic.

We suppose that the dividend growth (de-trended) follows an Ornstein-Uhlenbeck process. The price with dividend $D_t = D_0 \exp\left(\int_0^t g_s ds\right)$, $dg_t = -\phi g_t dt + \sigma dz_t$. The price $P_t = E_t\left[\int_0^\infty e^{-Rs} D_{t+s} ds\right]$ is:

$$P_t/D_t = \int_0^\infty \exp\left[-RT - \frac{1-e^{-\phi T}}{\phi}\widehat{\pi}_t + \frac{\sigma^2}{2\phi^3}\left(\phi T + 2e^{-\phi T} - \frac{e^{-2\phi T}+3}{2}\right)\right] dT \qquad (63)$$

which has no known closed-form solution. We consider how to approximate with a LG process or arbitrary precision.

8.1 First-order approximation

We define $Y_t^1 = e^{-Rt} D_t$, and $Y_t^2 = e^{-Rt} D_t g_t$. We have: $E_t [dY_{1,t}] / dt = (-R + g_t) Y_{1,t} = -R Y_{1,t} + Y_{2,t}$ and $dY_{2,t}/dt = Y_{1,t} \left(-(\phi + R) g_t + g_t^2\right)$.

To approximate g_t^2, we replace it by its steady state mean. To find it, we observe that $E_t [dg_t^2] / dt = -2\phi g_t^2 + \sigma^2$, so that taking the expectation at time 0, we obtain $\lim_{t\to\infty} E_0 [g_t^2] = \sigma^2 / (2\phi)$. Hence we approximate $dY_{2,t} \simeq Y_{1,t} \left(-(\phi + R) g_t + \sigma^2 / (2\phi)\right)$. Hence we approximate Y_t by Y_t^*, where

$$E_t [dY_t^*] / dt = - \begin{pmatrix} R & -1 \\ -\sigma^2 / (2\phi) & R + \phi \end{pmatrix} Y_t^*$$

Applying Theorem 4, we obtain:

$$V_t^{(1)} = P_t^* / D_t = \frac{1 + \frac{x_t}{R+\phi}}{R - \frac{\sigma^2}{(R+\phi) 2\phi}} \tag{64}$$

Figure 1 plots the LG approximation, and the exact expression. We find only a small discrepancy (less than 2%) between the two expressions. We conclude that the first order approximation of the Ornstein-Uhlenbeck process by a LG process will be rather good, and useful for theoretical purposes. If the goal is high-level numerical accuracy, we turn to an approximation of arbitrary order.

8.2 Approximation of arbitrary order

8.2.1 A simple truncation

In some examples, and perhaps virtually always (at least, when the processes defining the functions are analytic), it is possible to make LG processes approximate the prices of non-LG processes to an arbitrary degree of precision. We provide a simple illustration of this. Define

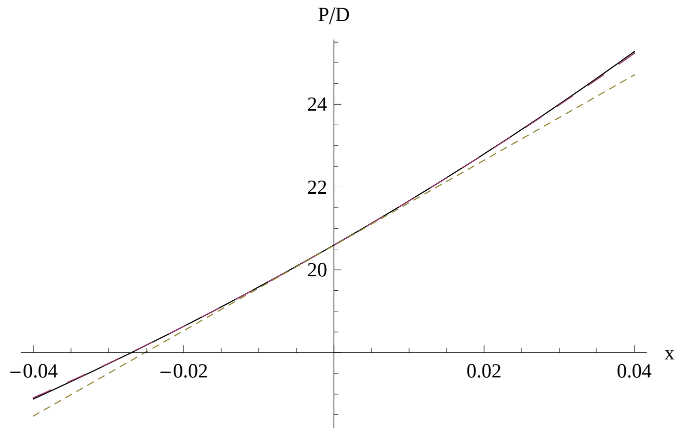

Figure 1: The Figure plots the true value of the P/D ratio of a stock with an Ornstein-Uhlenbeck process (solid line, Eq. 63), and the approximation by a LG process with $n = 1$ factor (short dashed line, Eq. 64), and $n = 2$ factors (longer dashed line, Eq. 71). The annualized values are: $R = 5\%$, $\sigma = 1\%$, $\phi = 15\%$. In the range of the Figure, the 1-factor approximation is within 2% of the true value, while the 2-factor approximation is withinn 0.15% of the true value. The 2-factor approximation is so close to true value that it is hard to distinguish visually.

$Y_{it} = e^{-rt} D_t g_t^{i-1}$ for $i = 1, 2, \ldots$. Hence, the vector of factors is $X_t = (g_t, g_t^2, g_t^3, \ldots)$. We have:

$$\begin{aligned} E_t \left[dY_{i,t} \right] / dt &= e^{-rt} D_t \left(g_t^{i+1} + (i-1)(-\phi) g_t^i + (i-1)(i-2) \frac{\sigma^2}{2} g_t^{i-3} \right) - r Y_{i,t} \\ &= (i-1)(i-2) \frac{\sigma^2}{2} Y_{i-2,t} - \left[r + (i-1) \phi \right] Y_{i,t} + Y_{i+1,t} \end{aligned}$$

so that $E_t [dY_t] = -\omega Y_t dt$, with $\omega_{i,i-2} = -(i-1)(i-2)\sigma^2/2$,$\omega_{i,i} = r + (i-1)\phi$, $\omega_{i,i+1} = -1$ and $\omega_{ij} = 0$ otherwise, i.e.

$$\omega = \begin{pmatrix} r & -1 & 0 & 0 & 0 & \cdots \\ 0 & r+\phi & -1 & 0 & 0 & \cdots \\ -\sigma^2 & 0 & r+2\phi & -1 & 0 & \cdots \\ 0 & -3\sigma^2 & 0 & r+3\phi & -1 & \cdots \\ 0 & 0 & -6\sigma^2 & 0 & r+4\phi & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

So the price is:

$$P_t/D_t = (1,0,...,0,..)\,\omega^{-1}\left(1, g_t, g_t^2, ..., g_t^n, ...\right)' \tag{65}$$

The sum can be truncated up to step n, i.e. be take to be the restriction of the vector to the first n dimensions. We compare the LG (65) to the exact expression (113). Numerical results show that the approximation is very good, even for $n = 5$.

8.2.2 An improved truncation

There is a better way to truncate the expansion (65). This section is heuristic, not rigorous. To study the "pure" behavior of the process, with the effect of discounting, we look at the process when $r = 0$. Then:

$$E_t\left[dY_{2,t}\right]/dt = -\phi Y_{2t} + Y_{3t} \tag{66}$$

$$E_t\left[dY_{3t}\right]/dt = \sigma^2 Y_{1t} - 2\phi Y_{3t} + Y_{4t} \tag{67}$$

Let us study the truncation to 2 terms. We replace Y_3 by the value that ensures the terms in (67) are equal to 0 (that's the "slaving principle"), so $Y_3 \simeq \sigma^2 Y_{1t}/\left(2\phi\right)$, and injecting this in (66), we get: $E_t\left[dY_{2,t}\right]/dt \simeq \frac{\sigma^2 Y_{1t}}{2\phi} - \left(\phi + R\right) Y_{2t}$. Hence, we approximate the OU process by the LG process

$$\omega^*_{(1)} = \begin{pmatrix} r & 1 \\ -\frac{\sigma^2}{2\phi} & r+\phi \end{pmatrix}$$

and the corresponding approximation is exactly the one we've previously derived, Eq. 64.

We can do the same for higher K's.For instance, with $r = 0$, $E_t\left[dY_{4t}\right]/dt = 3\sigma^2 Y_{2t} - 3\phi Y_{4t} - Y_{5t}$, so we replace $Y_{4t} \simeq Y_{2t}\sigma^2/\phi$. Injecting this in (67) we get:

$$\omega^*_{(2)} = \begin{pmatrix} r & -1 & 0 \\ 0 & r+\phi & -1 \\ -\sigma^2 & -\frac{\sigma^2}{\phi} & r+2\phi \end{pmatrix} \tag{68}$$

The corresponding approximation is (71) below.

Conclusion: The successive approximations, with 0, 1, and 2 factors respectively, are:

$$V_t^{(0)} = \frac{1}{R} \tag{69}$$

$$V_t^{(1)} = \frac{1+\frac{x_t}{R+\phi}}{R-\frac{\sigma^2}{(R+\phi)2\phi}} \tag{70}$$

$$V_t^{(2)} = \frac{1}{R}+\frac{g_t}{R(R+\phi)}+\frac{g_t^2+\frac{\sigma^2}{R\phi}(x_t+\phi)}{R(R+\phi)\left(R+2\phi-\frac{\sigma^2}{R\phi}\right)} \tag{71}$$

The approximation is very good. For instance, Figure 1 plots the LG approximation $V_t^{(2)}$, and it is extremely close, and very hard to distinguish visually from the true value.

8.3 Conclusion

It would be good to generalize the above procedure, probably in a future paper. It suggests that LG processes allow the evaluation of the price of many non-LG processes (e.g., those with analytic expansions), to an arbitrary degree of precision.

9 Twisted vs Non-twisted process

The "twist" term in LG processes may at first glance be strange. On the other hand, mathematically it is close to a simple AR(1). It may be useful to illustrate, in the same simulations, how an AR(1) and a LG process behave.

Consider the processes

$$x_{t+\Delta t}^* = \frac{(1-\phi\Delta t)\,x_t^*+v_x\Delta t}{1+x_t^*\Delta t}+\sigma(x_t^*)(\Delta t)^{1/2}\varepsilon_{t+\Delta t} \text{ with} \tag{72}$$

$$x_{t+\Delta t} = (1-\phi\Delta t)\,x_t+\sigma(x_t)(\Delta t)^{1/2}\varepsilon_{t+\Delta t} \tag{73}$$

where $\varepsilon_{t+\Delta t}$ is a uniformly distributed, with mean 0 distribution with variance 1. The ν_x term is an adjustment convexity, that will be determined very soon, and is analogous to the "$\sigma^2/(2\phi)$" in section 8. The processes mean-revert with a speed ϕ. The starred variables relate to the LG process, while the non-starred variables relate to the AR(1). The dividends start at $D_0^* = D_0 = 1$, and follow:

$$D_{t+\Delta t} = D_t \left(1 + \eta_{t+\Delta t}\right) \left(1 + x_t \Delta t\right) e^{g_D \Delta t}$$
$$D_{t+\Delta t}^* = D_t^* \left(1 + \eta_{t+\Delta t}\right) e^{x_t^* \Delta t + g_D \Delta t}$$

where η has mean 0 and standard deviation σ_η. In the above expressions, x_t is a growth rate. In another interpretation, useful when purely dealing with a price-dividend ratio, is that x_t is minus a risk premium. Stock prices are the net present value of dividend, discounted with a risk-neutral rate r.

The variance process is:

$$\sigma^2(X) = 2K\left(1 - X/X_{\min}\right)^2 \left(1 - X/X_{\max}\right)^2$$

with $K > 0$. It goes to 0 fast enough at $X_{\min}$ and $X_{\max}$, ensuring that x_t is within $[X_{\min}, X_{\max}]$. The average volatility of X is fairly well approximated by: $v_x = K^{1/2}\xi$, with $\xi = 1.35$, as proposed in the Appendix to Gabaix (2007), "A Simple, Unified, Exactly Solved Framework for Ten Puzzles in Macro-Finance".

For the simulations, each period lasts a month, $\Delta t = 1/12$. I use annual units, and I take $R = r - g_D = 0.05$, for a central P/D ratio of 20. Also, $(X_{\min}, X_{\max}) = (-8\%, 8\%)$, and $\phi = 0.15$ (which means that x_t mean-reverts with a half-life of mean-reversion of 5 years). Finally, $K = 0.1\phi \left|X_{\min}\right| X_{\max}$, which corresponds to a annual volatility of the log P/D ratio of 7% ($v_x / (R + \phi)$), based solely on the effect mentioned here. It is easy to increase this volatility, for instance by introducing a positive correlation between innovations to dividends, and innovations to dividend growth.

I simulate 100 years of data. Figures 2-4 illustrate a typical run. They show x_t and x_t^*, the corresponding price/dividend ratio P_t/D_t and P_t^*/D_t^*, and the (log) dividends D_t and D_t^*. (For the LG process, the PD ratio is in closed form; for the AR(1) process, I solve for it numerically). Given that the processes for x_t and x_t^* are similar up to second order terms, they are close, an expectation confirmed by Figure 2. The standard deviation of $x_t - x_{t*}$ is about 0.1 standard deviation of x_t.

Figure 3 shows the price-dividend ratio of the two processes. They are also reasonably

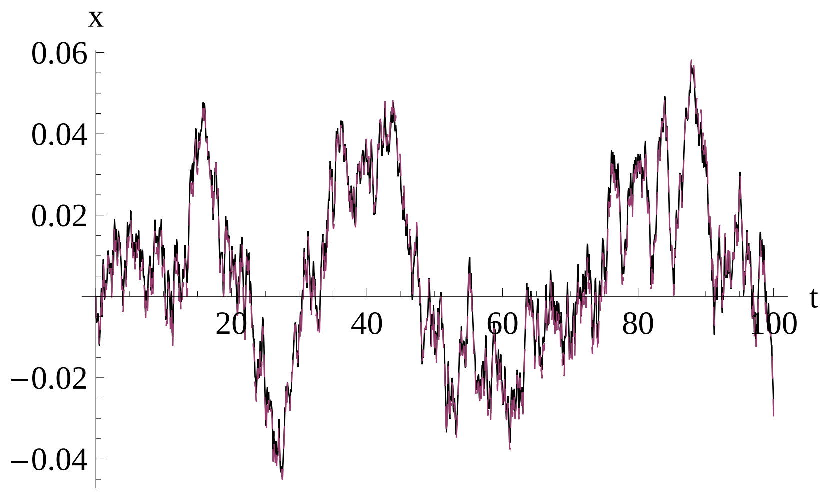

Figure 2: Process for x_t, simulated over 100 years. The solid black line is the LG process, and dashed purple line is the AR(1). The curves are very close, which was expected as the two processes are identical up to second order terms.

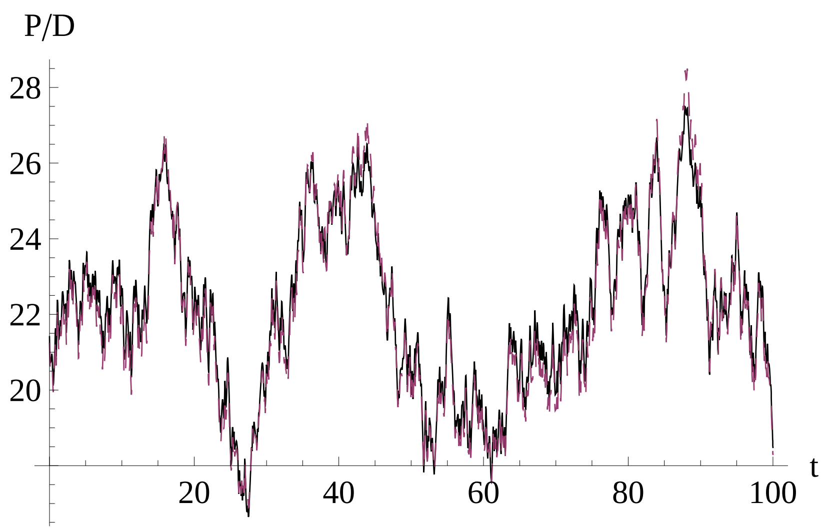

Figure 3: The price-dividend ratios, simulated over 100 years. The solid black line represents the P/D associated with the LG process, and dashed purple line the P/D ratio associated with the AR(1) process. The curves are quite close, which makes sense given that the two processes are identical up to second order terms.

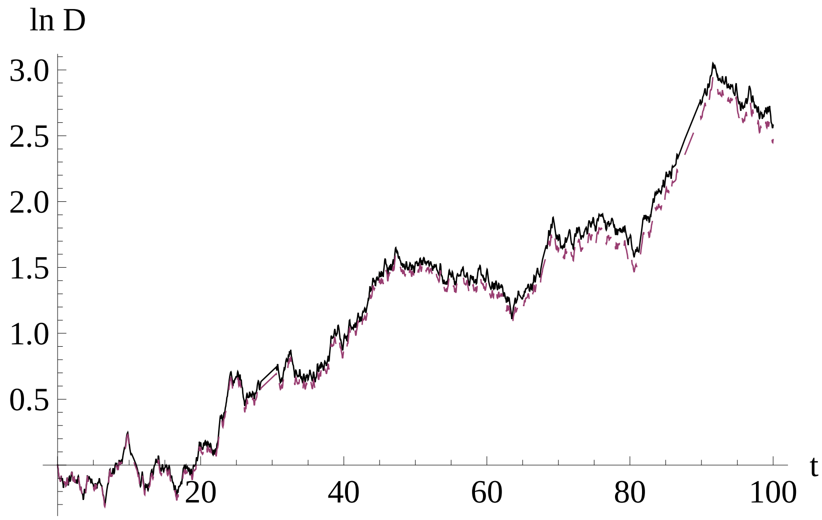

Figure 4: The dividend of the stock, simulated over 100 years. The solid black line represents the dividend associated with the LG process, and dashed purple line the dividend associated with the AR(1) process. The curves are rather close, in spite of the compounding in growth rates.

close. Finally, Figure 4 shows the dividend. Despite the compounding, the dividends are reasonably close.

The conclusion is that the processes are indeed quite close.

Of course, even if they had been quite different, this would not have been a problem for LG processes. We do not want to say that the true model is an AR(1), that a LG process approximates. It could as well be that the true model is a LG process, than an AR(1) model approximates. Or rather, as models are just approximation of a complex economic reality, the respective advantage of LG vs affine models depends on the specific task at hand. The modeler should be able to pick whichever approximation is most convenient. It is simply reassuring that the modelling choice does not make a large difference in terms of the economic processes.