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RESEARCH ARTICLE

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Consistent house allocation

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Abstract In practice we often face the problem of assigning indivisible objects (e.g., schools, housing, jobs, offices) to agents (e.g., students, homeless, workers, professors) when monetary compensations are not possible. We show that a rule that satisfies *consistency*, *strategy-proofness*, and *efficiency* must be an efficient generalized priority rule; i.e., it must adapt to an acyclic priority structure, except – maybe – for up to three agents in each object's priority ordering.

Keywords Indivisible objects · Priority structure · Consistency · Strategy-proofness

JEL Classification Numbers D63 · D70

1 Introduction

In real life we often face the problem of allocating heterogeneous indivisible objects (for instance, schools, housing, jobs, or offices) among a group of agents (for instance, students, homeless, workers, or professors) when monetary compensations are not possible. Agents have strict preferences over objects and remaining

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unassigned. An assignment is an allocation of the objects to the agents such that every agent receives at most one object. A rule is a systematic way of solving these assignment problems that are classically called house allocation problems (the implicit assumption being that each agent only needs at most one house to live in). The search for "good" house allocation rules, i.e., rules with desirable properties, is the subject of many recent papers (for instance, Abdulkadiroğlu and Sönmez 1998, 1999; Bogomolnaia and Moulin 2001; Chambers 2004; Ehlers et al. 2002; Ehlers and Klaus 2003, 2005; Ehlers et al. 2002; Ergin 2000; Kesten 2004a,b; Pápai 2000; Svensson 1999). Recent articles that pursue a similar research agenda in closely related models with monetary compensations are Miyagawa (2001), Ohseto (2005), Schummer (2000), Svensson (2004), Svensson and Larsson (2002), and Thomson (2003).[1]

In most real life problems "priorities" naturally arise. For example, in school choice students who live closer to a school and/or have siblings attending a school have higher priority to be admitted at that school (Abdulkadiroğlu and Sönmez 2003). When apartments are allocated, the homeless who have been waiting longer have higher priority to be assigned an apartment (similarly for the placement of students at colleges).

Balinski and Sönmez (1999) were the first to formulate the assignment problem based on priorities. The agents' priorities for a certain object are captured by an ordering of the agents. A priority structure is a profile specifying for each object a priority ordering. Given the agents' priorities, it is natural to require that the assignment does not violate any priorities. This means that there should be no agent who – conditional on higher priority – envies another agent (for receiving a better object). A rule adapts to a priority structure if it always chooses an assignment that does not violate any priorities.

It is well known, that for any profile of agents' preferences the assignment obtained from applying Gale and Shapley's 1962 deferred acceptance algorithm Pareto dominates any other assignment which does not violate any priorities. This algorithm is called the "best" rule among the rules adapting to a priority structure. Unfortunately the best rule may not be *efficient*. Ergin (2002) shows that "acyclicity" of the priority structure is equivalent to various properties (*efficiency*, *group strategy-proofness*,[2] and *consistency*) of the induced best rules.

Consistency, our main property, is a condition of stability when the set of agents and resources may change. To understand this property, suppose that after objects are allocated according to a rule, some agents leave the economy with their allotments, and the remaining agents "reassign" among themselves the remaining objects. What if the same rule is applied to their "reassignment problem"? A rule is considered "unstable" or "inconsistent" if its reassignment differs from its original assignment to the remaining agents.[3]

[1] This list is not exhaustive.

[2] By group strategy-proofness no group of agents can profit by joint misrepresentation of their preferences such that all members of the group weakly gain and at least one member of the group strictly gains.

[3] Ergin (2000) studies *consistency* for the house allocation problem in various combinations with *efficiency*, converse consistency, neutrality, and anonymity. Here, converse consistency pertains to the opposite operation of *consistency* (see Thomson 2004). By neutrality, the names of the objects do not matter. By anonymity, the names of the agents do not matter.

Our main result shows that a rule that satisfies *consistency*, *strategy-proofness*,[4] and *efficiency* must be an efficient generalized priority rule; i.e., it is *efficient* and adapts to an acyclic priority structure, except – maybe – for up to three agents in each object's priority ordering. Therefore, our properties imply that the assignment of objects must be based on an acyclic generalized priority structure and the rule chooses the same allocations as the associated deferred acceptance algorithm.

We proceed as follows. In Section 2 we introduce the model and the main properties. Section 3 we devote to (generalized) priority rules, two examples, and the main result. We conclude in Section 4 with a brief discussion of our results and a comparison to Ehlers and Klaus (2005). The proof of the main result is given in the Appendix.

2 House allocation

Let P denote the set of potential agents. We assume that P is finite and contains at least four agents.[5] Let $\mathcal{P}$ denote the set of all subsets of P containing at least two agents. Let K denote the set of potential real objects. The set K can be finite or infinite. Not receiving any real object is called "receiving the null object." The null object, denoted by 0, does not belong to K and is available in any economy. Let $\mathcal{H}$ denote the set of all finite subsets of K.

Each agent $i \in P$ is equipped with a strict preference relation R_i over all objects $K \cup \{0\}$. In other words, R_i is a linear order over $K \cup \{0\}$. Given $x, y \in K \cup \{0\}$, $x \; P_i \; y$ means that agent i strictly prefers x to y under R_i. Let $\mathcal{R}$ denote the set of all linear orders over $K \cup \{0\}$. Given $N \subseteq P$, let $\mathcal{R}^N$ denote the set of all *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$. Given $N \subseteq P$, $i \in N$, $R \in \mathcal{R}^N$, and $K' \subseteq K \cup \{0\}$, let $R_{i|K'}$ denote the restriction of R_i to K' and $R_{|K'} = (R_{i|K'})_{i \in N}$. An economy (or house allocation problem) consists of a set of agents, their preferences, and a finite set of real objects which have to be allocated among them. Formally, an *economy* is a triple (N, R, H) where $N \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $H \in \mathcal{H}$. We suppress the set of agents and write (R, H) instead of (N, R, H). Let $\mathcal{E}^N$ denote the set of all economies with the set of agents equal to N.

When allocating objects each agent receives one object. The null object is the only object which can be assigned to several agents. Formally, given a set of agents N, an *allocation* is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in K \cup \{0\}$, and none of the real objects in K is assigned to more than one agent. Note that not all real objects in K have to be assigned. An *(allocation) rule* is a function that assigns an allocation to every economy. Formally, a rule φ chooses for all $N \in \mathcal{P}$ and all economies $(R, H) \in \mathcal{E}^N$ an allocation $\varphi(R, H)$ such that for all $i \in N$, $\varphi_i(R, H) \in H \cup \{0\}$. Given $i \in N$, we call $\varphi_i(R, H)$ the *allotment* of agent i at $\varphi(R, H)$.

Next, we introduce our main properties for rules.

First, the rule chooses only (Pareto) efficient allocations.

[4] No agent can manipulate the allocation to his/her advantage by lying about his/her preferences.

[5] Example 2 considers $|P| = 3$ and clarifies why our main result has "no bite" in this case. Our results remain unchanged if P is infinite.

Efficiency: For all $N \in \mathcal{P}$ and all $(R, H) \in \mathcal{E}^N$, there is no allocation $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in H \cup \{0\}$ and $a_i \; R_i \; \varphi_i(R)$, and for some $j \in N$, $a_j \; P_j \; \varphi_j(R, H)$.

Given $N \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $M \subseteq N$, let R_M denote the profile $(R_i)_{i \in M}$. It is the restriction of profile R to the subset of agents M. We also use the notation $R_{-i} = R_{N \setminus \{i\}}$. For example, $(\bar{R}_i, R_{-i})$ denotes the profile obtained from R by replacing R_i by $\bar{R}_i$.

Second, no agent ever benefits from misrepresenting his/her preference relation.

Strategy-proofness: For all $N \in \mathcal{P}$, all $(R, H) \in \mathcal{E}^N$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}$, $\varphi_i(R, H) \; R_i \; \varphi_i((\bar{R}_i, R_{-i}), H)$.

Our last property is a stability condition. Consistency[6] requires that if some agents leave an economy with their allotments, then the rule should allocate the remaining objects among the agents who did not leave in the same way as in the original economy.

Consistency: For all $N, M \in \mathcal{P}$ such that $M \subseteq N$, all $(R, H) \in \mathcal{E}^N$, and all $i \in M$,

$$\varphi_i(R, H) = \varphi_i\left(R_M, H \setminus (\cup_{j \in N \setminus M} \{\varphi_j(R, H)\})\right).$$

Remark 1 Strategy-proofness and *consistency* imply that only preferences over objects that are present in an economy matter for the final allocation. Formally, for any *strategy-proof* and *consistent* rule φ, if two economies (N, R, H) and (N, R', H) are such that $R_{|H \cup \{0\}} = R'_{|H \cup \{0\}}$, then $\varphi(R, H) = \varphi(R', H)$.[7]

3 Generalized priority rules

We now introduce the idea that rules may adapt to priorities. In many situations "priorities" naturally arise. For example, when offices are assigned to the members of a department, seniority may matter; when apartments are assigned to graduate students, students who have been waiting longer should come first; and for jobs, a candidate with higher qualification may be ranked above lower qualified applicants.

We follow the notation and terminology introduced by Ergin (2002), who studies efficient house allocation on the basis of priorities.

Given $x \in K$, let $\succ_x$ denote a linear order over P. We call $\succ_x$ a priority ordering for object x. A priority structure is a profile $\succ = (\succ_x)_{x \in K}$ specifying for each object a priority ordering. Given $N \in \mathcal{P}$, $i \in N$, $R \in \mathcal{R}^N$, $x \in K$, and a priority structure $\succ$, an allocation *a violates the priority of i for x* if there exists $j \in N$ such that $a_j = x$, $i \succ_x j$, and $x \; P_i \; a_i$ (i.e., i has higher priority for object x than j

[6] For a recent overview see Thomson (2004).

[7] Without loss of generality, let $R' = (R'_i, R_{-i})$ for some $i \in N$. By *strategy-proofness* and $R_{i|H \cup \{0\}} = R'_{i|H \cup \{0\}}$, we obtain $\varphi_i(R, H) = \varphi_i(R', H)$. Thus, by *consistency* and $(R_{N \setminus \{i\}}, H \setminus \{\varphi_i(R, H)\}) = (R'_{N \setminus \{i\}}, H \setminus \{\varphi_i(R', H)\})$, we have for all $j \in N \setminus \{i\}$, $\varphi_j(R, H) = \varphi_j(R_{N \setminus \{i\}}, H \setminus \{\varphi_i(R, H)\}) = \varphi_j(R', H)$. Hence, $\varphi(R, H) = \varphi(R', H)$.

but j receives x and i envies j). A rule φ *adapts to a priority structure* $\succ$ if for all $N \in \mathcal{P}$ and all $(R, H) \in \mathcal{E}^N$, $\varphi(R, H)$ does not violate the priority of any agent for any object.

We say that φ is a priority rule if there exists a priority structure $\succ$ such that φ adapts to $\succ$.

Using a result from Balinski and Sönmez (1999, Theorem 2) it follows that an *efficient* priority rule φ must be a so-called best rule; i.e., for each economy $(R, H) \in \mathcal{E}^N$, $\varphi(R, H)$ is obtained from applying the agents-proposing deferred acceptance algorithm (Gale and Shapley 1962) to the two-sided matching problem where the agents' preferences are given by R and the "objects' preferences" are given by the priority structure $\succ$ (see also Ergin's 2002, Proposition 1). Ergin's (2002, Theorem 1) main result essentially states that for best rules, *efficiency*, *group strategy-proofness*, *consistency*, and the *acyclicity* of the priority structure are all equivalent.

While Ergin (2002) focuses on the class of rules that adapt to an exogenously given priority structure, we consider the general class of all rules. We show that if a rule satisfies *consistency*, *strategy-proofness*, and *efficiency*, then there must exist a priority structure that it "almost" adapts to. In order to formalize this "almost" adaptation, we introduce generalized priority rules next.

Let $x \in K$. We call a binary relation $\succ_x$ a *generalized priority ordering* if

(i) $\succ_x$ is transitive and antisymmetric and
(ii) there exists a set $Q^x \subseteq P$ such that $|Q^x| \leq 3$ and

(a) the restriction of $\succ_x$ to the agents in $P \backslash Q^x$, denoted $\succ_{x|P\backslash Q^x}$, is complete (all agents in P except for the agents in Q^x can be completely ranked according to priority order $\succ_x$) and
(b) for all $i \in P\backslash Q^x$ and all $j \in Q^x$, $i \succ_x j$ (agents in $P\backslash Q^x$ are ranked above agents in Q^x).

Note that any priority ordering is a generalized priority ordering since it satisfies the above definition for $Q^x = \emptyset$. Also, a generalized priority ordering that satisfies the above definition with $|Q^x| = 1$ is in fact a priority ordering.

A generalized *priority structure* is a profile $\succ = (\succ_x)_{x \in K}$ specifying for each object a generalized priority ordering. We say that φ is a *generalized priority rule* if there exists a generalized priority structure $\succ$ such that φ adapts to $\succ$; i.e., it adapts to all priorities that are specified in the generalized priority ordering, except possibly for priorities concerning an object x and agents in Q^x.

Theorem 1 *If a rule satisfies consistency, strategy-proofness, and efficiency, then it is an efficient generalized priority rule.*

In order to be more specific about the structure of efficient generalized priority rules, we use Ergin's (2002, Theorem 1) result that a priority rule is efficient if and only if the underlying priority structure is acyclic.

Let $\succ$ be a generalized priority structure. We say that $\succ$ has a cycle if there exist $x, y \in K$ and $i, j, k \in P$ such that $i \succ_x j \succ_x k \succ_y i$. A generalized priority structure is *acyclic* if it has no cycles. We can now rephrase Theorem 1.

Corollary 1 *If a rule satisfies consistency, strategy-proofness, and efficiency, then it adapts to an acyclic generalized priority structure* $\succ$.

Efficient priority rules satisfy all the axioms in Theorem 1. In the following example, we demonstrate that the class of efficient generalized priority rules is strictly larger than the class of efficient priority rules. In order to do so, we need some notation.

Let $\succ$ be a priority structure and φ be a priority rule adapting to $\succ$. Then we call φ a *serial dictatorship* if there exists an ordering $\succ_*$ over P such that for all $x \in K$, $\succ_x = \succ_*$. We denote by $\varphi^{\succ_*}$ the serial dictatorship with ordering $\succ_*$. It is easily checked that a serial dictatorship satisfies *consistency*, *strategy-proofness*, and *efficiency*.

Example 1 Let $\succ_*^1$ and $\succ_*^2$ be the following orderings over P:

$$|P| \succ_*^1 |P| - 1 \succ_*^1 \cdots \succ_*^1 3 \succ_*^1 2 \succ_*^1 1;$$

$$|P| \succ_*^2 |P| - 1 \succ_*^2 \cdots \succ_*^2 3 \succ_*^2 1 \succ_*^2 2.$$

The orderings $\succ_*^1$ and $\succ_*^2$ only differ in their last two entries. We define the following, "almost serial dictatorship:" Agents start choosing their objects according to their names, the agent with the highest number starts and chooses his/her favorite object, then the agents with the next higher number chooses his/her favorite object among the remaining objects, etc., until only agents 1 and 2 are left over. Now, the number of remaining objects determines who is allowed to choose next. For all $N \in \mathcal{P}$ and all $(R, H) \in \mathcal{E}^N$,

(i) if $|H \backslash (\cup_{i \in N \backslash \{1,2\}} \{\varphi_i^{\succ_*^1}(R, H)\})| \leq 1$, then $\varphi(R, H) \equiv \varphi^{\succ_*^1}(R, H)$ (if one or none object are left for agents 1 and 2, then agent 2 may choose among the remaining objects before agent 1); and

(ii) if $|H \backslash (\cup_{i \in N \backslash \{1,2\}} \{\varphi_i^{\succ_*^1}(R, H)\})| > 1$, then $\varphi(R, H) \equiv \varphi^{\succ_*^2}(R, H)$ (if more than one object is left for agents 1 and 2, then agent 1 may choose among the remaining objects before agent 2).

Because serial dictatorships are *efficient*, the rule φ is *efficient*. Since agents 1 and 2 cannot change the ordering by stating other preferences and $\succ_*^1$ and $\succ_*^2$ only differ in their last two entries, the rule φ is *strategy-proof*. To show *consistency* let $N \in \mathcal{P}$, $(R, H) \in \mathcal{E}^N$, and $j \in N$. It suffices to show that for all $i \in N \backslash \{j\}$,

$$\varphi_i(R_{-j}, H \backslash \{\varphi_j(R, H)\}) = \varphi_i(R, H). \tag{1}$$

If $i \in N \backslash \{1, 2\}$, then $\varphi_i^{\succ_*^1}(R, H) = \varphi_i^{\succ_*^2}(R, H)$ and $\varphi_i^{\succ_*^1}(R_{-j}, H \backslash \{\varphi_j(R, H)\}) = \varphi_i^{\succ_*^2}(R_{-j}, H \backslash \{\varphi_j(R, H)\})$. So, the fact that φ is a serial dictatorship for all agents except agents 1 and 2 implies (1).

Consider $i \in \{1, 2\}$ and denote $\{1, 2\} = \{i, k\}$. If either $j = k$ or $k \notin N$, then $\varphi_i^{\succ_*^1}(R_{-j}, H \backslash \{\varphi_j(R, H)\}) = \varphi_i^{\succ_*^2}(R_{-j}, H \backslash \{\varphi_j(R, H)\})$. Since either $\varphi(R, H) = \varphi^{\succ_*^1}(R, H)$ or $\varphi(R, H) = \varphi^{\succ_*^2}(R, H)$, the consistency property of $\varphi^{\succ_*^1}$ or $\varphi^{\succ_*^2}$ implies (1).

Finally, consider $j \in N \backslash \{1, 2\}$. For all $i \in N \backslash \{1, 2\}$, $\varphi_i^{\succ_*^1}(R_{-j}, H \backslash \{\varphi_j^{\succ_*^1}(R, H)\})$ $= \varphi_i^{\succ_*^1}(R, H)$ and $\varphi_j(R, H) = \varphi_j^{\succ_*^1}(R, H)$. Thus, $\big(H \backslash \{\varphi_j(R, H)\}\big) \backslash \Big(\cup_{i \in N \backslash \{1,2\}}$

$$\{\varphi_i^{\succ^1_*}(R_{-j}, H\backslash\{\varphi_j^{\succ^1_*}(R, H)\})\}\Big) = \Big(H\backslash\{\varphi_j^{\succ^1_*}(R, H)\}\Big) \backslash \Big(\cup_{i\in N\backslash\{1,2\}}\{\varphi_i^{\succ^1_*}(R, H)\}\Big)$$
$$= H\backslash \Big(\cup_{i\in N\backslash\{1,2\}}\{\varphi_i^{\succ^1_*}(R, H)\}\Big).$$

Then either (i) holds for (R, H) and $(R_{-j}, H\backslash\{\varphi_j^{\succ^1_*}(R, H)\})$ ($\varphi^{\succ^1_*}$ is used for both problems) or (ii) holds for (R, H) and $(R_{-j}, H\backslash\{\varphi_j^{\succ^1_*}(R, H)\})$ ($\varphi^{\succ^2_*}$ is used for both problems). Hence, the consistency property of $\varphi^{\succ^1_*}$ or $\varphi^{\succ^2_*}$ implies (1). Finally note that φ is a generalized priority rule where for all $x \in K$, $\succ_x = \succ^1_* \cap \succ^2_*$ and for all $x \in K$, $Q^x = \{1, 2\}$. □

Remark 2 In Example 1 the choice of $\succ^1_*$ or $\succ^2_*$ could be defined in a more complex manner without violating the axioms of Theorem 1. We could choose $\succ^1_*$ whenever a certain object x is available after the other agents (except agents 1 and 2) have received their allotments and when x is not available, then we pick $\succ^2_*$. On the other hand, we can easily define efficient generalized priority rules that do not satisfy the axioms of Theorem 1; for instance if the choice between $\succ^1_*$ and $\succ^2_*$ depends on the preferences of agent 1 (possible violations of *strategy-proofness*) or if the choice between $\succ^1_*$ and $\succ^2_*$ depends on the presence of certain agents (possible violations of *consistency*). Because it is intuitively clear, what the degrees of freedom in difference to the class of priority rules is, but since, at the same time, it is very tedious and technical to give a full characterization, we did not try to formulate Theorem 1 as a full characterization. The important point is that any rule satisfying the axioms of Theorem 1 is "almost" a priority rule. □

Remark 3 Theorem 1 and Example 1 show that *consistency*, *strategy-proofness* and *efficiency* characterize "almost" efficient priority rules. These axioms only allow more flexibility at the bottom of the priority orderings – up to three agents for each object's generalized priority ordering. A similar feature has been observed in the paper by Bogomolnaia et al. (2005). They show on the domain of weak preference relations that *non-bossiness*,[8] *strategy-proofness*, and *efficiency* "almost" characterize serial dictatorships. Those axioms only allow more flexibility for the first two agents of the serial order. □

In Example 1 the set of agents for which priorities are unspecified is equal to $\{1, 2\} = \cup_{x\in K} Q^x$. Next, we construct an *efficient*, *strategy-proof*, and *consistent* rule for $P = \{1, 2, 3\}$ and $K = \{a, b\}$ that does not adapt to any priority structure. Extending the example to $|P| > 3$, as explained after the example, yields an example where the set of agents for which priorities are unspecified is equal to $\{1, 2, 3\} = \cup_{x\in K} Q^x$.

Example 2 Let $P = \{1, 2, 3\}$ and $K = \{a, b\}$. The rule $\bar{\varphi}$ is defined as follows:

For $N = \{i, j, k\}$ such that $|N| = 3$ define φ^{ijk} as the serial dictatorship $\varphi^{\succ}$ where $i \succ_a j \succ_a k$ and $i \succ_b j \succ_b k$.

- If $|N| = 3$, $H = \{a, b\}$, and $R \in \mathcal{R}^N$ such that
  - $\varphi_1^{132}(R, H) = a$ and $\varphi_3^{132}(R, H) = b$, then $\bar{\varphi}(R, H) = \varphi^{132}(R, H)$,

[8] No agent can influence another agent's final allotment without changing his/her final consumption.

- $\varphi_1^{132}(R,H) = a$ and $\varphi_3^{132}(R,H) = 0$, then $\bar{\varphi}(R,H) = \varphi^{213}(R,H)$,
- $\varphi_1^{123}(R,H) = b$ and $\varphi_2^{123}(R,H) = a$, then $\bar{\varphi}(R,H) = \varphi^{123}(R,H)$,
- $\varphi_1^{123}(R,H) = b$ and $\varphi_2^{123}(R,H) = 0$, then $\bar{\varphi}(R,H) = \varphi^{312}(R,H)$.

- If $H = \{a\}$ and $R \in \mathcal{R}^N$, then $\bar{\varphi}(R,H) = \varphi^{123}(R,H)$.
- If $H = \{b\}$, and $R \in \mathcal{R}^N$, then $\bar{\varphi}(R,H) = \varphi^{132}(R,H)$.

For $N = \{1,2\}$ let $\succ^{[12]}$ denote the priority structure $2 \succ_a^{[12]} 1$ and $1 \succ_b^{[12]} 2$.

- If $N = \{1,2\}$, $H = \{a,b\}$, and $R \in \mathcal{R}^N$, then $\bar{\varphi}(R,H) = \varphi^{\succ^{[12]}}(R,H)$.

For $N = \{1,3\}$ let $\succ^{[13]}$ denote the priority structure $1 \succ_a^{[13]} 3$ and $3 \succ_b^{[13]} 1$.

- If $N = \{1,3\}$, $H = \{a,b\}$, and $R \in \mathcal{R}^N$, then $\bar{\varphi}(R,H) = \varphi^{\succ^{[13]}}(R,H)$.

For $N = \{2,3\}$ let $\succ^{[23]}$ denote the priority structure $2 \succ_a^{[23]} 3$ and $3 \succ_b^{[23]} 2$.

- If $N = \{2,3\}$, $H = \{a,b\}$, and $R \in \mathcal{R}^N$, then $\bar{\varphi}(R,H) = \varphi^{\succ^{[23]}}(R,H)$.
- If $|N| = 3$, $H = \{a,b\}$, and $R \in \mathcal{R}^N$ is such that the null object is the most preferred object under R_1, then $\bar{\varphi}(R,H) = (0, \varphi_2^{\succ^{[23]}}(R_{\{2,3\}},H), \varphi_3^{\succ^{[23]}}(R_{\{2,3\}},H))$.

Since all rules used to define $\bar{\varphi}$ are priority rules, $\bar{\varphi}$ is *efficient*. The proof that $\bar{\varphi}$ satisfies *strategy-proofness* and *consistency* is straightforward, but tedious and is available from the authors upon request. □

Example 2 demonstrates that at the bottom of each generalized priority ordering there may be a set of three agents which are not related to each other, i.e., $Q^a = Q^b = \{1,2,3\}$. Thus, the upper bound 3 for the cardinality of Q^x cannot be lowered in the definition of a generalized priority ordering $\succ_x$. Furthermore, Example 2 can be easily adapted to any number of agents. Just let all other agents choose according to a serial dictatorship before agents $\{1,2,3\}$ and use then the rule of Example 2 to allocate objects to $\{1,2,3\}$, depending on which objects are left behind by the agents $N \backslash \{1,2,3\}$.[9]

4 Discussion

Ehlers and Klaus (2005) consider an alternative consistency condition, called *reallocation-consistency*. In difference to *consistency*, this notion assumes that if some agents leave an economy with their allotments, then the rule should assign the objects in the same way to these agents for the economy where only the agents that left and their previous allotments are present.

[9] In Example 2 for any two agents the priority ordering of an object depends on which objects are available. Let $i \succ_x^H j$ mean that i has higher priority for object x than j when $H \cup \{0\}$ is the set of available objects. Then we have $1 \succ_a^{\{a\}} 2 \succ_a^{\{a\}} 3$ and $2 \succ_a^{\{a,b\}} 1 \succ_a^{\{a,b\}} 3$; and $1 \succ_b^{\{b\}} 3 \succ_b^{\{b\}} 2$ and $3 \succ_b^{\{a,b\}} 1 \succ_b^{\{a,b\}} 2$. These priorities contain some "acyclicities" depending on which objects are left in the economy. In other words, certain "acyclicities" in the definition of priorities are compatible with *consistency*, *strategy-proofness*, and *efficiency*. Unfortunately, we do not know more about the precise structure of these acyclicities and rules based on such priorities (apart from the rules based on Example 2).

Reallocation-consistency: For all $N, M \in \mathcal{P}$ such that $M \subseteq N$, all $(R, H) \in \mathcal{E}^N$, and all $i \in M$,

$$\varphi_i(R, H) = \varphi_i\left(R_M, \cup_{j \in M}\{\varphi_j(R, H)\}\right).$$

The difference between the two consistency properties is as follows. When defining *consistency*, we define the so-called *reduced economy for consistent rules* to equal the set of agents that were left behind and all objects not consumed by the agents that were leaving. Hence, in such a reduced economy there may be some unassigned objects in addition to the remaining agents' allotments – an incidence that cannot occur in a *reduced economy for reallocation-consistent rules* where agents can only reallocate their allotments among themselves. A priori, no logical relation exists between *consistency* and *reallocation-consistency*. Ehlers and Klaus (2005, Theorem 1) show that a rule satisfies *reallocation-consistency*, *strategy-proofness*, and *efficiency* if and only if it is an efficient priority rule. Since the set of efficient priority rules is a strict subset of the set of efficient generalized priority rules, *reallocation-consistency* – in combination with *strategy-proofness* and *efficiency* – is a more demanding property than *consistency*. Hence, using the weaker (and more standard) condition of *consistency* here, yields a larger class of rules, but not a fully characterized set of rules.

This is also the reason why our Theorem 1 requires arguments that are different from Ehlers and Klaus (2005). In seeing why, fix a *strategy-proof* and *efficient* rule φ and let $x \in K$ be an arbitrary object. An obvious way to derive a priority ordering for x is the following: for all $i \in N$, let $R_i^x \in \mathcal{R}$ be such that for all $y \in K \backslash \{x\}$, $x \; P_i^x \; 0 \; P_i^x \; y$ (object x is the only object which is preferred to the null object); then define

$$i \succ_x j \Leftrightarrow \varphi_i((R_i^x, R_j^x), \{x\}) = x.$$

If φ is *reallocation-consistent*, then it is easy to see that $\succ_x$ is a linear order and for all $H \in \mathcal{H}$ such that $x \in H$,

$$i \succ_x j \Leftrightarrow \varphi_i((R_i^x, R_j^x), H) = x. \qquad (*)$$

Hence, we could have also used $(*)$ to define $\succ_x$. If φ is *consistent*, then it is still true that $\succ_x$ is a linear order but $(*)$ may not be true. In Example 1, we have $2 \succ_x 1$ but $\varphi_1((R_1^x, R_2^x), H) = x$ for all $H \in \mathcal{H}$ such that $\{x\} \subsetneq H$. The main difficulty of our proof is to show that for any set consisting of four agents, say $N = \{1, 2, 3, 4\}$, $(*)$ holds for the highest $\succ_x$-ranked agent in this set and any other agent in N, i.e., if agent 1 is the highest $\succ_x$-ranked agent in N, then $(*)$ holds for agents 1 and 2, for agents 1 and 3, and for agents 1 and 4. Once this is shown, the rest of our proof uses arguments similar to Ehlers and Klaus (2005).

In real life agents may be indifferent between several objects because, for instance, they do not have enough information to distinguish any two of them. Unfortunately, our result (and Theorem 1 of Ergin 2002) is not robust when allowing for indifferences. The reason is as follows. It is easy to see that any *strategy-proof* and *consistent* rule satisfies *non-bossiness*. Then, on the domain of weak preferences any rule of Theorem 1 satisfies *strategy-proofness*, *non-bossiness*, and *efficiency*. By Bogomolnaia et al. (2005, Theorem 4) such a rule is "almost" a serial dictatorship. Therefore, on the domain of weak preferences, any rule that satisfies *consistency*, *strategy-proofness*, and *efficiency* must be "almost" a serial dictatorship.

Ehlers and Klaus (2005) consider the richer model of house allocation with quotas, i.e., possibly multiple copies of objects may be available. We could adjust our model accordingly without affecting the results, but chose for the clarity and notational simplicity of the classical house allocation model.

Appendix

Proof of Theorem 1

To prove Theorem 1 let φ be a rule satisfying *consistency*, *strategy-proofness*, and *efficiency*.

For each object we construct a generalized priority ordering. Given $x \in K$, let $R_i^x \in \mathcal{R}$ denote a preference relation such that for all $y \in K\backslash\{x\}$, $x \; P_i^x \; 0 \; P_i^x \; y$, i.e., x is the unique object that is preferred to the null object at R_i^x. Then

$$i \succ_x j \Leftrightarrow \text{for all } H \in \mathcal{H} \text{ such that } x \in H,\; \varphi_i((R_i^x, R_j^x), H) = x.$$

The difficulty of the proof is to show that $\succ_x$ is complete whenever necessary in the definition of a generalized priority ordering. Example 1 proves that there may exist agents i, j such that neither $i \succ_x j$ nor $j \succ_x i$.

Lemma 1 *Let* $x \in K$ *and* $N = \{1, 2, 3, 4\} \subseteq P$. *If* $\varphi_1(R_N^x, \{x\}) = x$, *then* $1 \succ_x 2$, $1 \succ_x 3$, *and* $1 \succ_x 4$.

Proof Let

$$\varphi_1(R_N^x, \{x\}) = x. \tag{2}$$

In order to prove $1 \succ_x 2$, $1 \succ_x 3$, and $1 \succ_x 4$, we show that for all $H \in \mathcal{H}$ such that $x \in H$, $\varphi_1(R_N^x, H) = x$.

Let $y \in K\backslash\{x\}$. We first show that

$$\varphi_1(R_N^x, \{x, y\}) = x. \tag{3}$$

The proof involves many different profiles, which we depict as explained after the two first profiles we introduce now. Recall that by Remark 1, for any economy it suffices to specify the preferences over the objects that are present in that economy. Without loss of generality, we assume

$$(R^4, \{x, y\}) : \begin{array}{cccc} R_1^x & R_2 & R_3 & R_4 \\ \hline \boxed{x} & x & x & x \\ & \boxed{y} & y & y \end{array}, \tag{4}$$

and

$$(R^5, \{x, y\}) : \begin{array}{cccc} R_1^x & R_2^x & R_3 & R_4 \\ \hline \boxed{x} & x & x & x \\ & & \boxed{y} & y \end{array}. \tag{5}$$

We only specify the ranking of the objects which are preferred to the null object. In (4), the set of objects is $\{x, y\}$ and the rule φ allocates x to agent 1 and y to agent 2. Furthermore, we denote the profile in (4) by R^4. In (4) agent 1 has to receive x because otherwise we obtain a contradiction from (2) and *consistency* [the agent who receives y in (4) can leave the economy in (2) and (4)]. Then by *efficiency*, one of the agents 2, 3, or 4 receives y. Without loss of generality, we suppose that agent 2 receives y. Similar arguments yield $\varphi_1(R^5, \{x, y\}) = x$.

From now on we assume that (3) is not true, i.e., $\varphi_1(R^x_N, \{x, y\}) \neq x$.

Step 1: If $\varphi_1(R^x_N, \{x, y\}) \neq x$, then

$$(R^6, \{x, y\}) :\ \begin{array}{cccc} R_1 & R_2^x & R_3^x & R_4^x \\ \hline x & x & x & \boxed{x} \\ \boxed{y} & & & \end{array} . \tag{6}$$

Because $\varphi_1(R^x_N, \{x, y\}) \neq x$, *strategy-proofness* implies that in the above economy agent 1 cannot receive x. Then by *efficiency*, $\varphi_1(R^6, \{x, y\}) = y$. We have to show that neither agent 2 nor agent 3 receives x at $(R^6, \{x, y\})$. Suppose

$$(R^7, \{x, y\}) :\ \begin{array}{cccc} R_1 & R_2^x & R_3^x & R_4^x \\ \hline x & \boxed{x} & x & x \\ \boxed{y} & & & \end{array} . \tag{7}$$

Then *consistency* and *strategy-proofness* imply

$$(R^8, \{x, y\}) :\ \begin{array}{cc} R_1 & R_2 \\ \hline x & \boxed{x} \\ \boxed{y} & y \end{array} . \tag{8}$$

Using *consistency* and *strategy-proofness* in (4) yields that agent 1 receives x in the economy (8), a contradiction. Hence, $\varphi_2(R^6, \{x, y\}) \neq x$.

Suppose

$$(R^9, \{x, y\}) :\ \begin{array}{cccc} R_1 & R_2^x & R_3^x & R_4^x \\ \hline x & x & \boxed{x} & x \\ \boxed{y} & & & \end{array} . \tag{9}$$

Then *consistency* and *strategy-proofness* imply

$$(R^{10}, \{x, y\}) :\ \begin{array}{cc} R_1 & R_3 \\ \hline x & \boxed{x} \\ \boxed{y} & y \end{array} . \tag{10}$$

Using *consistency* and *strategy-proofness* in (5) yields that agent 1 receives x in the economy (8), a contradiction. Hence, $\varphi_3(R^6, \{x, y\}) \neq x$. Since agents 1, 2, and 3 do not receive x in the economy (6), *efficiency* implies that 4 receives x in the economy (6).

Step 2: We finish the proof by showing that in the economy $(R^{11}, \{x, y\}) \in \mathcal{E}^{\{2,3,4\}}$ any efficient allocation yields a contradiction to our previous deductions. Let

$$(R^{11}, \{x, y\}) : \begin{array}{ccc} R_2 & R_3 & R_4 \\ \hline x & x & y \\ y & y & x \end{array} . \tag{11}$$

We distinguish three cases.

Case 1 $\varphi_4(R^{11}, \{x, y\}) = y$.

Then $\varphi_2(R^{11}, \{x, y\}) = 0$ or $\varphi_3(R^{11}, \{x, y\}) = 0$. If $\varphi_2(R^{11}, \{x, y\}) = 0$, then $\varphi_3(R^{11}, \{x, y\}) = x$ and by *consistency* and *strategy-proofness*, $\varphi_2(R^y_{\{2,4\}}, \{y\}) = 0$ and $\varphi_4(R^y_{\{2,4\}}, \{y\}) = y$.

Applying *consistency* to (4) yields $\varphi_2(R^4_{\{2,4\}}, \{y\}) = y$ and $\varphi_4(R^4_{\{2,4\}}, \{y\}) = 0$. Thus, by *strategy-proofness*, $\varphi_2(R^y_{\{2,4\}}, \{y\}) = y$ and $\varphi_4(R^y_{\{2,4\}}, \{y\}) = 0$, a contradiction.

If $\varphi_3(R^{11}, \{x, y\}) = 0$, then we obtain a contradiction by applying *consistency* for agents 3 and 4 to the economies $(R^{11}, \{x, y\})$ and $(R^5, \{x, y\})$ [in (5)]. Hence, Case 1 cannot occur.

Case 2 $\varphi_4(R^{11}, \{x, y\}) = x$.

Then $\varphi_2(R^{11}, \{x, y\}) = y$ or $\varphi_3(R^{11}, \{x, y\}) = y$. If $\varphi_2(R^{11}, \{x, y\}) = y$, then agents 2 and 4 gain from exchanging their objects, a contradiction to *efficiency*. If $\varphi_3(R^{11}, \{x, y\}) = y$, then agents 3 and 4 gain from exchanging their objects, a contradiction to *efficiency*. Hence, Case 2 cannot occur.

Case 3 $\varphi_4(R^{11}, \{x, y\}) = 0$.

Then, by *efficiency*, $\varphi_2(R^{11}, \{x, y\}) = x$ or $\varphi_3(R^{11}, \{x, y\}) = x$. If $\varphi_2(R^{11}, \{x, y\}) = x$, then by *consistency* and *strategy-proofness*, $\varphi_2(R^x_{\{2,4\}}, \{x\}) = x$ and $\varphi_4(R^x_{\{2,4\}}, \{x\}) = 0$.

Applying *consistency* to (6) yields $\varphi_2(R^x_{\{2,4\}}, \{x\}) = 0$ and $\varphi_4(R^x_{\{2,4\}}, \{x\}) = x$, a contradiction.

If $\varphi_3(R^{11}, \{x, y\}) = x$, then we obtain a contradiction by applying *consistency* for agents 3 and 4 to the economies $(R^{11}, \{x, y\})$ and $(R^6, \{x, y\})$ (in (6)). Hence, Case 3 cannot occur.

We have shown that $\varphi_1(R^x_N, \{x, y\}) = x$. Finally, in order to prove that for all $H \in \mathcal{H}$ such that $x \in H$, $\varphi_1(R^x_N, H) = x$, we repeat the previous arguments by adding another object $z \in H \backslash \{x, y\}$ and considering similar preference profiles where y is an undesirable object, *etc.* $\square$

Next, we show that $\succ_x$ is a generalized priority ordering.

Lemma 2 *For all $x \in K$, $\succ_x$ is transitive and antisymmetric; i.e., $\succ_x$ satisfies condition (i) in the definition of a generalized priority ordering.*

Proof Antisymmetry follows from the definition of $\succ_x$.

Let $1, 2, 3 \in P$. Suppose $1 \succ_x 2$ and $2 \succ_x 3$. Let $H \in \mathcal{H}$ be such that $x \in H$. In order to prove $1 \succ_x 3$, we need to show that $\varphi_1((R^x_1, R^x_3), H) = x$. Consider $R^x = (R^x_1, R^x_2, R^x_3)$. By *efficiency*, for some $l \in \{1, 2, 3\}$, $\varphi_l(R^x, H) = x$.

If $l = 2$, then by *consistency*, $\varphi_1((R_1^x, R_2^x), H) = 0$ and $\varphi_2((R_1^x, R_2^x), H) = x$, which contradicts $1 \succ_x 2$ and the definition of $\succ_x$.

If $l = 3$, then by *consistency*, $\varphi_2((R_2^x, R_3^x), H) = 0$ and $\varphi_3((R_2^x, R_3^x), H) = x$, which contradicts $2 \succ_x 3$ and the definition of $\succ_x$.

Thus, $l = 1$. By *consistency*, $\varphi_1((R_1^x, R_3^x), H) = x$ and $\varphi_3((R_1^x, R_3^x), H) = 0$. Hence, by the definition of $\succ_x$, $1 \succ_x 3$. □

In order to determine the set of agents Q^x for $\succ_x$, we define the following ordering:

$$i \succ'_x j \Leftrightarrow \varphi_i(R^x_{\{i,j\}}, \{x\}) = x.$$

Note that by *efficiency* we either have $i \succ'_x j$ or $j \succ'_x i$, but not both. Thus, $\succ'_x$ is complete. The proof that $\succ'_x$ is transitive is similar to the proof of Lemma 2.

Let $Q^x \subseteq P$ denote the set of the three bottom ranked agents under $\succ'_x$, i.e., for all $i \in P \backslash Q^x$ and all $j \in Q^x$, $i \succ'_x j$ and $|Q^x| = 3$.

Lemma 3 *For all $x \in K$, $\succ_x$ satisfies condition (ii) in the definition of a generalized priority ordering.*

Proof Let $Q^x = \{l, m, q\}$.

Let $i, j \in P \backslash Q^x$. Then by completeness of $\succ'_x$, $i \succ'_x j$, or $j \succ'_x i$. Suppose $i \succ'_x j$. By the definition of Q^x, $i \succ'_x l$, and $i \succ'_x m$. Thus, by *consistency*, $\varphi_i(R^x_{\{i,j,l,m\}}, \{x\}) = x$. By Lemma 1 (with $i = 1$ and $\{i, j, l, m\} = \{1, 2, 3, 4\}$), $i \succ_x j$. Hence, $\succ_{x|P\backslash Q^x}$ is complete and $\succ_x$ satisfies (a) in (ii).

Let $i \in P \backslash Q^x$. Then by definition of $\succ'_x$ and *consistency*, $\varphi_i(R^x_{\{i,l,m,q\}}, \{x\}) = x$. By Lemma 1 (with $i = 1$ and $\{i, j, l, m\} = \{1, 2, 3, 4\}$), $i \succ_x l$, $i \succ_x m$, and $i \succ_x q$. Hence, $\succ_x$ satisfies (b) in (ii). □

Let $\succ \equiv (\succ_x)_{x \in K}$. Then, $\succ$ is a generalized priority structure.

Lemma 4 *The rule φ adapts to the generalized priority structure $\succ$.*

Proof Let $N \in \mathcal{P}$ and $(R, H) \in \mathcal{E}^N$. Suppose that there exist $x \in H$ and $i \in N$ such that $\varphi(R, H)$ violates the priority of i for x. Then for some $j \in N$, $\varphi_j(R, H) = x$, $i \succ_x j$, and $x \; P_i \; \varphi_i(R, H)$. Let $\bar{R} \in \mathcal{R}^N$ be such that (i) for all $l \in N \backslash \{i\}$ and all $y \in K \backslash \{\varphi_l(R, H)\}$, $\varphi_l(R, H) \; \bar{R}_l \; 0 \; \bar{P}_l \; y$ and (ii) for all $y \in K \backslash \{x\}$, $x \; \bar{P}_i \; 0 \; \bar{P}_i \; y$.

Let $l \in N \backslash \{i\}$. By *strategy-proofness*, $\varphi_l((\bar{R}_l, R_{-l}), H) = \varphi_l(R, H)$. Thus, by *consistency*, $\varphi((\bar{R}_l, R_{-l}), H) = \varphi(R, H)$. Applying the same arguments repeatedly we obtain that $\varphi((R_i, \bar{R}_{-i}), H) = \varphi(R, H)$. Let $\bar{H} \equiv H \backslash (\cup_{l \in N \backslash \{i,j\}} \{\varphi_l((R_i, \bar{R}_{-i}), H)\})$.

By *consistency*, $\varphi_i((R_i, \bar{R}_j), \bar{H}) = \varphi_i(R, H)$ and $\varphi_j((R_i, \bar{R}_j), \bar{H}) = x$. By *strategy-proofness* and $x \; P_i \; \varphi_i(R, H)$, $\varphi_i(\bar{R}_{\{i,j\}}, \bar{H}) = 0$, and $\varphi_j(\bar{R}_{\{i,j\}}, \bar{H}) = x$. Thus, by *strategy-proofness*, $\varphi_i(R^x_{\{i,j\}}, \bar{H}) = 0$ and $\varphi_j(R^x_{\{i,j\}}, \bar{H}) = x$. By $i \succ_x j$ and the definition of $\succ_x$, we have for all $x \in H' \in \mathcal{H}$, $\varphi_i(\bar{R}_{\{i,j\}}, H') = x$. Since $x \in \bar{H}$, the previous two facts constitute a contradiction. □

By Lemma 4, φ is an efficient generalized priority rule. The proof of Theorem 1 is now complete.

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