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CORE IMPLEMENTATION IN MODFIED STRONG AND COALITION PROOF NASH EQUILIBRIA

P.Borm*, G.-J. Otten*, H. Peters**
**Department of Econometrics, Tilburg University - Tilburg - The Netherlands*
***Department of Mathematics - University of Limburg - Maastricht - The Netherlands*

ABSTRACT

In Borm and Tijs (1992) it was shown that the strong core of a superadditive NTU-game can be implemented in strong Nash equilibria. In this note it is demonstrated that it can be implemented in modified strong and coalition proof Nash equilibria.

1. Introduction

In Borm and Tijs (1992) it was shown that the strong core elements of a superadditive nontransferable-utility game are implemented by the strong Nash equilibria of a corresponding so-called strategic claim game. This result confirms the apparent analogy between the definition of the (strong) core and the definition of strong Nash equilibrium: both reflect the unprofitability of coalitional deviations.

Ray (1989) and Greenberg (1987) independently proposed a modification of the core in the following sense. A coalitional deviation is allowed as long as it is not credible, which means that there exists a subcoalition which, at its turn, can deviate credibly and profitably from the original deviation. They showed that the (strong) core already satisfies this internal stability requirement, so that imposing this seemingly weaker condition instead of the usual (strong) core constraints never leads to a larger set. This brings one to the conjecture that the - analogous - weakening of the notion of strong Nash equilibrium to so-called modified strong Nash equilibrium in the strategic claim games of Borm and Tijs (1992) might still entail implementation of the strong core. This note will show that, indeed, this conjecture is true.

The above notion of modified strong Nash equilibrium seems related to the more familiar weakening of strong Nash equilibrium to coalition proof Nash equilibrium as introduced by Bernheim, Peleg and Whinston (1987). However, we provide an example showing that for general strategic games these equilibrium concepts are logically independent. Then, interestingly, we will prove that for the special class of strategic claim games the sets of modified strong and coalition proof Nash equilibria coincide. In particular, this implies that the strong core of a superadditive NTU-game can be implemented by coalition proof Nash equilibria.

Finally, it should be noted that this last result establishes an asymmetry with the results of Otten, Borm, Storcken and Tijs (1992) in the context of effectivity functions. There, similar to the NTU-case, it is shown that the core of an (upper cycle free) effectivity function (at a given preference profile) can be implemented by strong Nash equilibria of an associated claim game correspondence (at that profile). However, using the (equivalent) reformulation of Abdou and Keiding (1991) for coalition proofness, it was shown that this could not be achieved by coalition proof Nash equilibria.

2. Strategic claim games

Let $N = \{1, \ldots, n\}$ denote the set of players. An n-person strategic game is represented by

$$\Gamma = ((X_i)_{i \in N}, (K_i)_{i \in N})$$

where X_i is player i's strategy set, and $K_i : \prod_{i \in N} X_i \rightarrow \mathbb{R}$ is player i's payoff function. For every $S \in 2^N \setminus \{\emptyset\}$ let $X_S := \prod_{i \in S} X_i$, and for $x \in X_N$ let $x_S := (x_i)_{i \in S}$. Elements of 2^N are called *coalitions.*

Definition. A strategy combination $x \in X_N$ is a *Nash equilibrium* if there do not exist a player $i \in N$ and a strategy $y_i \in X_i$ such that

$$K_i(x_{N \setminus \{i\}}, y_i) > K_i(x).$$

Further, x is a *strong Nash equilibrium* if there do not exist a coalition S and a strategy vector $y_S \in X_S$ such that

$$K_S(x_{N \setminus S}, y_S) \gneq K_S(x).$$

The concept of strong Nash equilibrium was introduced by Aumann (1959).

Let V be a map assigning to every non-empty coalition S a subset $V(S)$ of $\mathbf{R}^S$ such that:

(i) $V(S)$ is comprehensive (i.e. $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$)

(ii) For each $x \in V(S)$ there is a $y \in V(S)$ with $y \geq x$ and $[z \geq y \Rightarrow z = y]$ for all $z \in V(S)$.

Condition (ii) is a weak boundedness condition. Combined, conditions (i) and (ii) imply that each $V(\{i\})$ is of the form $(-\infty, v(i)]$. The pair (N, V) or just V is called an *NTU-game*. The *strong core SC(V)* consists of those payoff vectors which are attainable by the grand coalition N and which are "stable" with respect to domination. More specifically,

$$SC(V) := \{a \in V(N) \mid \neg\exists S \in 2^N \setminus \{\emptyset\}\ \exists b \in V(S) : b \gneqq a_S\}.$$

An NTU-game V is *superadditive* if for all $S, T \in 2^N \setminus \{\emptyset\}, S \cap T = \emptyset$ implies $V(S) \times V(T) \subset V(S \cup T)$.

Given an NTU-game V the *claim game* $\Gamma(V) = ((X_i)_{i \in N}, (K_i)_{i \in N})$ is defined as follows. For $i \in N$

$$X_i := \{S \in 2^N \mid i \in S\} \times \mathbf{R}$$

and, for $x = (S_j, t_j)_{j \in N} \in X_N$,

$$K_i(x) := \begin{cases} t_i & \text{if } S_j = S_i\ \forall j \in S_i \text{ and } (t_j)_{j \in S_i} \in V(S_i) \\ \min\{t_i, v(i)\} & \text{otherwise.} \end{cases}$$

In the claim game, each player mentions a coalition and a payoff, and exactly those coalitions are formed that are mentioned by all their members, provided that the payoffs are feasible.

Theorem 1. *(Borm and Tijs (1992)) If the NTU-game V is superadditive, then $a \in SC(V)$ if and only if there is a strong Nash equilibrium x of the corresponding claim game $\Gamma(V) = ((X_i)_{i\in N}, (K_i)_{i\in N})$ with $K_N(x) = a$.*

In section 3 and 4 will be shown that in this theorem "strong" can be replaced by "modified strong" or "coalition proof", respectively.

3. Modified strong Nash equilibria

We now introduce a weakening of strong Nash equilibrium to modified strong Nash equilibrium based on Ray (1989) and Greenberg (1987).

Let $\Gamma = ((X_i)_{i\in N}, (K_i)_{i\in N})$ be an n-person strategic game.
For $S \in 2^N \setminus \{\emptyset\}$, $x \in X_N$, $y_S \in X_S$, we say that y_S is *blocked by $T \subset S$ given x* if there is a vector $z_T \in X_T$ such that

$$K_T(z_T, y_{S\setminus T}, x_{N\setminus S}) \gneq K_T(y_S, x_{N\setminus S}).$$

S is *credible given x* if there is a $y_S \in X_S$, $y_S \neq x_S$, that is not blocked by any credible $T \subsetneq S$ given x.

Note that a strategy vector cannot be blocked by the empty coalition. Consequently (if $|X_i| > 1$ for all $i \in N$) all 1-person coalitions are credible, and the notion of credibility is well-defined.

Definition. A strategy combination $x \in X_N$ is a *modified strong (m-strong) Nash equilibrium* if it is not blocked by any credible coalition (given x).

Clearly, since x is a strong Nash equilibrium if it is not blocked by any coalition (given x), and a Nash equilibrium if it is not blocked by any 1-person coalition (given x), strong implies m-strong and m-strong implies Nash.

The following simple example shows that an m-strong Nash equilibrium need not be strong.

Example 1. Consider the prisoner's dilemma game as represented in figure 1. Player 1 chooses between the rows A_1 and A_2, player 2 between the columns B_1 and B_2. In each cell the first number corresponds to the payoff to player 1, the second to the payoff to player 2.

Figure 1. Prisoner's dilemma

	B_1	B_2
A_1	3,3	1,4
A_2	4,1	2,2

This game has a unique Nash equilibrium $x = (A_2, B_2)$. This equilibrium is not strong since it can be blocked by the coalition $\{1,2\}$ using (A_1, B_1). However, x is an m-strong equilibrium because $\{1,2\}$ is the only coalition that can block x while $\{1,2\}$ is not credible given x: every strategy combination $y \neq x$ is blocked by a (credible) 1-person coalition.

Remark. Example 1 shows that the condition "$y_S \neq x_S$" in the definition of credibility can not be dispensed with. In fact, without this condition it is not hard to show that the concepts of strong and m-strong equilibrium coincide.

For the special case of strategic claim games m-strong and strong Nash equilibria do coincide.

Theorem 2. *Let V be an NTU-game and $\Gamma = ((X_i)_{i \in N}, (K_i)_{i \in N})$ the corresponding claim game. Then $x \in X_N$ is a strong Nash equilibrium if and only if it is a modified strong Nash equilibrium.*

In the proof of this theorem we will use the following notation.
For $x = (S_k, t_k)_{k\in N}$, $S \in F_x$ if and only if $S_j = S$ for all $j \in S$ and $(t_j)_{j\in S} \in V(S)$.
Proof. Since any strong Nash equilibrium is m-strong only the converse implication needs to be proved. Let $x \in X_N$ be an m-strong Nash equilibrium, and suppose x is not strong. Then there exists a coalition S and a strategy vector $y_S \in X_S$ with

$$K_S(y_S, x_{N\setminus S}) \underset{\neq}{\geq} K_S(x) \geq (v(j))_{j\in S}$$

where the second inequality follows since x is a Nash equilibrium. With $j \in S$ such that $K_j(y_S, x_{N\setminus S}) > K_j(x)$ and letting $y_S = (S_k, t_k)_{k\in S}$, we have that $t_j > v(j)$ and $S_j \in F_{(y_S, x_{N\setminus S})}$, hence

$$K_{S_j}(y_{S_j}, x_{N\setminus S_j}) \underset{\neq}{\geq} K_{S_j}(x). \tag{1}$$

In view of the boundedness condition on $V(S_j)$ we may assume that

$$\forall z \in V(S_j)[z \geq K_{S_j}(y_{S_j}, x_{N\setminus S_j}) \Rightarrow z = K_{S_j}(y_{S_j}, x_{N\setminus S_j})]. \tag{2}$$

By (1), since x is m-strong, S_j can not be credible given x. So there exists a credible coalition $T \underset{\neq}{\subseteq} S_j$ and a strategy vector $z_T \in X_T$ such that

$$K_T(z_T, y_{S_j\setminus T}, x_{N\setminus S_j}) \underset{\neq}{\geq} K_T(y_{S_j}, x_{N\setminus S_j}) \geq K_T(x). \tag{3}$$

Let $z_T = (A_k, b_k)_{k\in T}$ and $i \in T$. If $A_i \notin F_{(z_T, y_{S_j\setminus T}, x_{N\setminus S_j})}$, (3) implies

$$K_i(z_T, x_{N\setminus T}) \geq \min\{b_i, v(i)\} = K_i(z_T, y_{S_j\setminus T}, x_{N\setminus S_j}) \geq K_i(x). \tag{4}$$

If $A_i \in F_{(z_T, y_{S_j\setminus T}, x_{N\setminus S_j})}$, then $A_i \cap (S_j \setminus T) \neq \emptyset$ would imply $A_i = S_j$ (since $S_j \in F_{(y_S, x_{N\setminus S})}$) and, consequently, $K_{S_j\setminus T}(z_T, y_{S_j\setminus T}, x_{N\setminus S_j}) = K_{S_j\setminus T}(y_{S_j}, x_{N\setminus S_j})$. However, combining this with (3), we would find

$$K_{S_j}(z_T, y_{S_j\setminus T}, x_{N\setminus S_j}) \underset{\neq}{\geq} K_{S_j}(y_{S_j}, x_{N\setminus S_j}),$$

violating (2). Therefore, $A_i \subset T \cup (N \setminus S_j)$. Hence, with (3):

$$K_i(z_T, x_{N\setminus T}) = K_i(z_T, y_{S_j\setminus T}, x_{N\setminus S_j}) \geq K_i(x). \tag{5}$$

Combining (4) and (5) gives:

$$K_T(z_T, x_{N\setminus T}) \geq K_i(x). \tag{6}$$

Now let $i \in T$ with $K_i(z_T, y_{S_j\setminus T}, x_{N\setminus S_j}) > K_i(y_{S_j}, x_{N\setminus S_j})$.
Then $A_i \in F_{(z_T, y_{S_j\setminus T}, x_{N\setminus S_j})}$ since otherwise, by (3), $K_i(x) < v(i)$, violating the fact that x is a Nash equilibrium. So, as before, $A_i \subset T \cup (N \setminus S_j)$, and

$$K_i(z_T, x_{N\setminus T}) = K_i(z_T, y_{S_j\setminus T}, x_{N\setminus S_j}) > K_i(y_{S_j}, x_{N\setminus S_j}) \geq K_i(x).$$

Hence, by (6), $K_T(z_T, x_{N\setminus T}) \gneq K_T(x)$, whereas T is a credible coalition. This contradicts the fact that x is m-strong. □

Theorem 1 and theorem 2 imply

Corollary 3. *If the NTU-game V is superadditive, then b is a strong core element of V if and only if there is a modified strong Nash equilibrium of the corresponding claim game $\Gamma(V)$ having payoff vector b.*

4. Coalition proof Nash equilibria

Bernheim, Peleg and Whinston (1987) introduced coalition proof Nash equilibria in the following recursive way.

Definition. Let $\Gamma = ((X_i)_{i\in N}, (K_i)_{i\in N})$ be an n-person strategic game and let $x \in X_N$. Then x is a *coalition proof Nash equilibrium of* Γ if either

(i) $n = 1$ and $K_1(x) = \max_{y\in X_1} K_1(y)$

or

(ii) $n \geq 2$ and

(a) x is *self-enforcing*, i.e. for all $S \subsetneq N$, x_S is a coalition proof Nash equilibrium of the game $\Gamma_S^x := ((X_i)_{i\in S}, (\widetilde{K}_i)_{i\in S})$ where

$$\widetilde{K}_i(y_S) := K_i(y_S, x_{N \setminus S})$$

for all $y_S \in X_S$ and $i \in S$

and

(b) there is no other self-enforcing strategy combination $y \in X_N$ such that $K_N(y) \gneq K_N(x)$.

Bernheim, Peleg and Whinston (1987) note that each strong Nash equilibrium is coalition proof and that for 2-person games the set of coalition proof Nash equilibria coincides with the set of Nash equilibria that are not Pareto dominated by any other Nash equilibrium. In fact, it is not difficult to see that the same is true for m-strong Nash equilibria. So, for 2-person games, each coalition proof Nash equilibrium is m-strong and conversely. However, for general n-person games ($n \geq 3$) the two equilibrium concepts are logically independent. This is seen in the following example, which in essence is based on the example of Bernheim, Peleg and Whinston (1987) showing that for $n \geq 3$, coalition proof Nash equilibria can be Pareto dominated within the set of Nash equilibria (although this phenomenon does not occur here).

Example 2. We consider the 3-person game Γ as represented in figure 2 in which player 1 chooses between the rows A_1, A_2 and A_3, player 2 between the columns B_1, B_2 and B_3, and player 3 between the boxes C_1 and C_2. In each cell the first number corresponds to the payoff to player 1, the second to the payoff to 2 and the third to the payoff to 3.

Figure 2. The 3-person game Γ

C_1		B_1	B_2	B_3
	A_1	$-2,-2,-10$	$-10,-10,-10$	$-10,-10,-10$
	A_2	$-10,-10,-10$	$1,1,-5$	$-10,2,-10$
	A_3	$-10,-10,-10$	$2,-10,-10$	$0,0,10$

C_2		B_1	B_2	B_3
	A_1	$-1,-1,5$	$-5,-5,0$	$-10,-10,-10$
	A_2	$-5,-5,0$	$-2,-2,-10$	$-10,-10,-10$
	A_3	$-10,-10,-10$	$-10,-10,-10$	$-15,-15,-15$

The game Γ has two Nash equilibria $x^1 := (A_3, B_3, C_1)$ and $x^2 := (A_1, B_1, C_2)$. We show that x^1 is coalition proof but not m-strong and that x^2 is m-strong but not coalition proof.

(i) x^1 is coalition proof: x^1 is self-enforcing since for each S with $|S| = 2$ the restriction x^1_S is a Nash equilibrium of $\Gamma^{x^1}_S$ which is not Pareto dominated by any other Nash equilibrium of $\Gamma^{x^1}_S$. Moreover, there is no strategy combination $y \neq x$ such that y Pareto dominates x.
(ii) x^2 is not coalition proof: x^2 is Pareto dominated by the self-enforcing strategy combination x^1.
(iii) x^2 is m-strong: note that the coalition $\{1,2\}$ is credible given x^2 since (A_2, B_2) can not be blocked by the coalitions $\{1\}$ and $\{2\}$ given x^2 (i.e. C_2 is fixed). Further, the only coalition that can block x^2 is $\{1,2,3\}$ (using x^1). However, $\{1,2,3\}$ is not credible given x^2 since each strategy combination $y \neq x^2$ is blocked either by a 1-person coalition or (in case $y = x^1$) by the (credible) coalition $\{1,2\}$.
(iv) x^1 is not m-strong: the coalition $\{1,2\}$ is credible given x^1 since (A_1, B_1) can not be blocked by $\{1\}$ and $\{2\}$ given x^1. Further, x^1 is blocked by $\{1,2\}$ using (A_2, B_2).

For strategic claim games corresponding to an NTU-game coalition proof and m-strong Nash equilibria do coincide. This is an immediate consequence of theorem 2 and

Theorem 4. *Let V be an NTU-game and $\Gamma = ((X_i)_{i\in N}, (K_i)_{i\in N})$ the corresponding claim game. Then $x \in X_N$ is a strong Nash equilibrium if and only if it is a coalition proof Nash equilibrium.*

Proof. It suffices to show that each coalition proof Nash equilibrium of $\Gamma(V)$ is strong. This is obvious if $n = 1$. So, let $n \geq 2$ and let x be a coalition proof Nash equilibrium of $\Gamma(V)$. Suppose x is not strong. Then, using similar arguments as in the proof of theorem 2, there exists a coalition S and a strategy vector $y_S := (S, t_i)_{i\in S} \in X_S$ such that

$$t_S = K_S(y_S, x_{N\setminus S}) \gneqq K_S(x) \tag{7}$$

and t_S is Pareto undominated within $V(S)$.
Moreover, without loss of generality we may assume that S is minimal in the sense that there exist no $U \subsetneqq S$ and $z_U \in X_U$ such that $K_U(z_U, x_{N\setminus U}) \gneqq K_U(x)$.
Since x is coalition proof, it follows that y_S is not self-enforcing in $(\Gamma(V))^{x}_{S}$. Hence, there is a $T \subsetneqq S$ such that y_T is not a coalition proof Nash equilibrium in the restricted game $((X_i)_{i\in T}, (L_i)_{i\in T})$ where

$$L_i(c_T) := K_i(c_T, y_{S\setminus T}, x_{N\setminus S})$$

for all $c_T \in X_T$ and $i \in T$. In particular, it can be seen that there exists a coalition $U \subset T$ and a strategy vector $z_U \in X_U$ such that $L_U(z_U, y_{T\setminus U}) \gneqq L_U(y_T)$, i.e.,

$$K_U(z_U, y_{S\setminus U}, x_{N\setminus S}) \gneqq K_U(y_S, x_{N\setminus S}). \tag{8}$$

Since t_S is Pareto undominated within $V(S)$, (8) can only be true if S is not formed, which implies

$$K_U(z_U, y_{S\setminus U}, x_{N\setminus S}) = K_U(z_U, x_{N\setminus U}). \tag{9}$$

However, combining (7)-(9), we obtain

$$K_U(z_U, x_{N\setminus U}) \mathrel{\substack{\geq \\ \neq}} K_U(y_S, x_{N\setminus S}) \geq K_U(x)$$

contradicting the minimality of S. □

Corollary 5. *If the NTU-game V is superadditive, then b is a strong core element of V if and only if there is a coalition proof Nash equilibrium of the corresponding claim game $\Gamma(V)$ having payoff vector b.*

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