Int. J. Game Theory (2006) 34: 1–11
DOI 10.1007/s00182-006-0009-6

ORIGINAL ARTICLE

Bettina Klaus · Flip Klijn

Median stable matching for college admissions

Published online: 28 February 2006

Abstract We give a simple and concise proof that so-called generalized median stable matchings are well-defined for college admissions problems. Furthermore, we discuss the fairness properties of median stable matchings and conclude with two illustrative examples of college admissions markets, the lattices of stable matchings, and the corresponding generalized median stable matchings.

Keywords Matching · College admissions · Stability · Fairness

JEL Classification C78 · D63

1 Introduction

We study a specific class of two-sided matching problems, so-called college admissions problems, in which students have to be matched to colleges (Gale and Shapley 1962) based on the students' and the colleges' preferences over the other side of the market and colleges' capacity constraints. An outcome for such a college admissions market, a matching, is an assignment of students to colleges such that each student is matched to at most one college and no college is matched to more students than its capacity allows for. A key property for college admissions markets is stability: a matching is stable if it satisfies individual rationality and no coalition of agents can improve by rematching among themselves (no blocking).

B. Klaus (✉)
Department of Economics, Maastricht University,
P.O. Box 616, 6200 MD Maastricht, The Netherlands
E-mail: B.Klaus@algec.unimaas.nl

F. Klijn
Institut d'Anàlisi Econòmica (CSIC), Campus UAB,
08193 Bellaterra (Barcelona), Spain
E-mail: Flip.Klijn@uab.es

For college admissions markets with responsive preferences,[1] the set of stable matchings is nonempty (Roth 1985) and has a specific lattice structure (Roth and Sotomayor 1990). A direct consequence of this lattice structure is the polarization of stable matchings in the sense that there is a best stable matching for the colleges (students) which is at the same time the worst stable matching for the students (colleges). Thus, both extreme stable matchings clearly favor one side of the market over the other. Masarani and Gokturk (1989) showed several impossibilities to obtain a fair deterministic matching mechanism within the context of Rawlsian justice based on cardinal preference information. One way to recover fairness is to use probabilistic (stable) matching mechanisms that are ex ante fair and/or 'procedurally fair;' see for instance Aldershof et al. (1999), Klaus and Klijn (2006), and Ma (1996).

Using another approach, Teo and Sethuraman (1998) and Sethuraman et al. (2004) established the existence of natural deterministic 'compromising mechanisms' for marriage and college admissions models, respectively. Specifically, they showed that if all agents order their (possibly non-distinct) matches at the, say, k stable matchings from best to worst, then the map that assigns to each agent of one side of the market its lth best match and to each agent of the other side its $(k-l+1)$st best match constitutes a stable matching. Teo and Sethuraman (1998) and Sethuraman et al. (2004) used linear programming tools to prove that these '(generalized) median stable matchings' are indeed well-defined and stable. We use the term '(generalized) median' to emphasize not only the formal equivalence of this solution concept to (generalized) medians in voting theory, but also to its similar spirit of compromise (Moulin 1980; Barberà et al. 1993).

In this note, we provide a very short and direct proof that all (generalized) median stable matchings are well-defined and stable. Our proof is based on the lattice structure of the set of stable matchings (Fleiner 2002, Theorem 5.5, independently obtained the same result for a more abstract two-sided matching model). Given that for responsive preferences the lattice structure reflects the polarization and trade-offs that occur between the two sides of the market, any median stable matching combines stability with some degree of 'endstate' fairness. Hence, median stable matchings are compromise solutions that can be applied to conflict situations that resemble college admissions problems.

The rest of the note is organized as follows. In section 2, we introduce college admissions markets and recall some results concerning stable matchings that we need for our proof. In section 3, we present our proof of the existence of generalized median stable matchings, define the subset of median stable matchings and discuss their fairness properties. We conclude with two examples of college admissions markets for which we illustrate the associated lattices of stable matchings and the corresponding generalized median stable matchings.

2 College admissions markets

There are two finite and disjoint sets of agents: a set $S = \{s_1, \ldots, s_m\}$ of students and a set $\mathcal{C} = \{C_1, \ldots, C_n\}$ of colleges. We denote a generic student by s and a

[1] By responsiveness (Roth 1985), a college's preference relation over sets of students is related to its ranking of single students in the following way: the college always prefers to add an acceptable student to any set of students (provided this does not violate the capacity constraint) and it prefers to replace any student by a better student.

generic college by C. For each college C, there is a fixed quota q_C that represents the number of positions it offers.[2]

Each student s has a complete, transitive, and strict preference relation $\succeq_s$ over the colleges and the prospect of being *unmatched*. Hence, student s's preferences can be represented by a strict ordering $P(s)$ of the elements in $\mathcal{C} \cup \{s\}$. If $C \in \mathcal{C}$ such that $C \succ_s s$, then we call C an *acceptable college for student* s. Let $P^S = \{P(s)\}_{s \in S}$.

A set of students $S' \subseteq S$ *is feasible for college* C if $|S'| \leq q_C$. Each college C has a complete and transitive preference relation $\succeq_C$ over feasible sets of students, which can be represented by a weak ordering $P(C)$ of the elements in $\mathcal{P}(S, q_C) \equiv \{S' \subseteq S : |S'| \leq q_C\}$. We make two assumptions on the preferences of a college C.[3]

First, C's preferences over singleton sets of students, or equivalently over individual students, are strict. For notational convenience we denote a singleton set $\{s\}$ by s. The second assumption describes comparisons of feasible sets of students when a single student is added or replaced. If $s \in S$ is such that $s \succ_C \emptyset$, then we call s an *acceptable student for college* C. If $s, s' \in S$ are such that $s \succ_C s'$, then we call student s a *better student than student* s' *for college* C. We assume that each college C's preferences over feasible sets of students are based on preferences over individual students such that C always prefers to add an acceptable student and it also prefers to replace any student by a better student. More formally, we assume that C's preferences are *responsive*, i.e., for all $S' \in \mathcal{P}(S, q_C)$,

(r1) if $s \notin S'$ and $|S'| < q_C$, then $(S' \cup s) \succ_C S'$ if and only if $s \succ_C \emptyset$ and
(r2) if $s \notin S'$ and $t \in S'$, then $\big((S' \backslash t) \cup s\big) \succ_C S'$ if and only if $s \succ_C t$.

Let $P^{\mathcal{C}} = \{P(C)\}_{C \in \mathcal{C}}$.

A *college admissions market* is a triple $(S, \mathcal{C}, P)$, where $P = (P^S, P^{\mathcal{C}})$. A *matching* for college admissions market $(S, \mathcal{C}, P)$ is a function μ on the set $S \cup \mathcal{C}$ such that

(m1) each student is either matched to exactly one college or unmatched, i.e., for all $s \in S$, either $\mu(s) \in \mathcal{C}$ or $\mu(s) = s$,
(m2) each college is matched to a feasible set of students, i.e., for all $C \in \mathcal{C}$, $\mu(C) \in \mathcal{P}(S, q_C)$, and
(m3) a student is matched to a college if and only if the college is matched to the student, i.e., for all $s \in S$ and $C \in \mathcal{C}$, $\mu(s) = C$ if and only if $s \in \mu(C)$.

Given matching μ, we call $\mu(s)$ *student* s*'s match* and $\mu(C)$ *college* C*'s match*.

A key property of matchings is *stability*. First, we impose a voluntary participation condition. A matching μ is *individually rational* if neither a student nor a college would be better off by breaking a current match, i.e., if $\mu(s) = C$, then $C \succ_s s$ and $\mu(C) \succ_C (\mu(C) \backslash s)$. By responsiveness of $\succeq_C$, the latter requirement can be replaced by $s \succ_C \emptyset$. Thus alternatively, a matching μ is *individually rational* if any student and any college that are matched to one another are mutually acceptable. Second, if a student s and a college C are not matched to one another at a matching μ but the student would prefer to be matched to the college and

[2] The marriage model is the special case of one-to-one (two-sided) matching where for all $C \in \mathcal{C}$, $q_C = 1$.

[3] See Roth and Sotomayor (1989) for a discussion of these assumptions.

the college would prefer to either add the student or replace another student by student s, then we would expect this mutually beneficial adjustment to be carried out. Formally, a pair (s, C), $s \notin \mu(C)$, is a *blocking pair* if $C \succ_s \mu(s)$ and (a) $[\,|\mu(C)| < q_C$ and $s \succ_C \emptyset\,]$ or (b) [there exists $t \in \mu(C)$ such that $s \succ_C t$].[4] A matching is *stable* if it is individually rational and there are no blocking pairs. With a slight abuse of notation, we denote the set of stable matchings for college admissions market $(S, \mathcal{C}, P)$ by $\Sigma(P) = \{\mu_1, \ldots, \mu_k\}$.

Gale and Shapley (1962) proved that $\Sigma(P) \neq \emptyset$. Roth and Sotomayor (1989) proved that each college has *strict* preferences over different sets of students that they are matched to at different stable matchings (even though they may be indifferent between other feasible sets of students). Note that a similar result for the students holds trivially because of the strictness of the students' preferences.

Theorem 2.1 [Roth and Sotomayor (1989), Theorem 3] *Let $\mu, \mu' \in \Sigma(P)$ and $C \in \mathcal{C}$. Then, either $\mu(C) \succ_C \mu'(C)$, $\mu'(C) \succ_C \mu(C)$, or $\mu(C) = \mu'(C)$.*

In fact, the set of stable matchings has several other appealing features, all of which are due to its specific lattice structure, which we explain next.

For any two stable matchings μ and μ' we define the function $\mu \vee_S \mu'$ that assigns to each student his/her more preferred match from μ and μ' and to each college its less preferred match from μ and μ'.[5] Formally, we define the function $\lambda := \mu \vee_S \mu'$ on the set $S \cup \mathcal{C}$ as follows. For all $s \in S$, let $\lambda(s) := \mu(s)$ if $\mu(s) \succ_s \mu'(s)$ and $\lambda(s) := \mu'(s)$ otherwise. For all $C \in \mathcal{C}$, let $\lambda(C) := \mu'(C)$ if $\mu(C) \succ_C \mu'(C)$ and $\lambda(C) := \mu(C)$ otherwise. In a similar way we define the function $\mu \wedge_S \mu'$ that assigns to each student his/her less preferred match and to each college its more preferred match.

Theorem 2.2 [Roth and Sotomayor (1990), Theorem 5.31] *Let $\mu, \mu' \in \Sigma(P)$. Then, $\mu \vee_S \mu' \in \Sigma(P)$ and $\mu \wedge_S \mu' \in \Sigma(P)$.*

Let μ, μ' be two stable matchings. We write $\mu \succ_S \mu'$ if for all $s \in S$, $\mu(s) \succeq_s \mu'(s)$, and for some $s' \in S$, $\mu(s') \succ_{s'} \mu'(s')$. Similarly, we write $\mu \succ_{\mathcal{C}} \mu'$ if for all $C \in \mathcal{C}$, $\mu(C) \succeq_C \mu'(C)$, and for some $C' \in \mathcal{C}$, $\mu(C') \succ_{C'} \mu'(C')$. Note that $\succ_S$ and $\succ_{\mathcal{C}}$ are partial orders on the set of stable matchings $\Sigma(P)$.

Theorem 2.3 [Roth and Sotomayor (1990), Theorem 5.29] *The partial orders $\succ_S$ and $\succ_{\mathcal{C}}$ are dual partial orders on the set of stable matchings $\Sigma(P)$, i.e., for any $\mu, \mu' \in \Sigma(P)$, $\mu \succ_S \mu'$ if and only if $\mu' \succ_{\mathcal{C}} \mu$.*

Summarizing Theorems 2.2 and 2.3 in algebraic terms, we obtain the following characterization of the set of stable matchings.

Corollary 2.4 [Roth and Sotomayor (1990), Corollary 5.32] *The set $\Sigma(P)$ forms a lattice under the partial orders $\succ_{\mathcal{C}}$ or $\succ_S$ with the lattice under the first partial order being the dual to the lattice under the second partial order.*

[4] Recall that by responsiveness (a) implies $(\mu(C) \cup s) \succ_C \mu(C)$ and (b) implies $((\mu(C) \backslash t) \cup s) \succ_C \mu(C)$.

[5] Note that by Theorem 2.1 each college's less (or more) preferred match from μ and μ' is well-defined.

3 Generalized median stable matchings

We first introduce generalized median stable matchings. The main result of this section is a very simple proof that for any college admissions market, generalized median stable matchings are well-defined and stable (Theorem 3.2). In contrast to Sethuraman et al. (2004), who used a linear programming approach, our proof is based on the lattice structure of the set of stable matchings. Proceeding from the existence of generalized median stable matchings, we define the subset of median stable matchings and discuss their fairness properties (Definition 3.5, Remark 3.6, Example 3.7). Finally, we give an illustrative example of generalized median stable matchings in a college admissions market (Example 3.8).

Consider a college admissions market $(S, \mathcal{C}, P)$. Then, the set of stable matchings $\Sigma(P) = \{\mu_1, \ldots, \mu_k\}$ is nonempty. Each student can order the matchings in $\Sigma(P)$ according to his/her preferences over the corresponding colleges. Formally, for each $s \in S$ there is a sequence of matchings $(\mu_1^s, \ldots, \mu_k^s)$ such that $\{\mu_1^s, \ldots, \mu_k^s\} = \{\mu_1, \ldots, \mu_k\}$ and for any $l \in \{1, \ldots, k-1\}$, either $\mu_l^s(s) \succ_s \mu_{l+1}^s(s)$ or $\mu_l^s(s) = \mu_{l+1}^s(s)$. Thus, for any $l \in \{1, \ldots, k\}$, at μ_l^s student s is assigned to his/her lth '(weakly) best' match among all k stable matchings. For any $l \in \{1, \ldots, k\}$, define the function α_l^S on the set S such that for all $s \in S$, $\alpha_l^S(s) := \mu_l^s(s)$.

It follows from Theorem 2.1 that each college can proceed similarly. Formally, for each $C \in \mathcal{C}$ there is a sequence of matchings $(\mu_1^C, \ldots, \mu_k^C)$ such that $\{\mu_1^C, \ldots, \mu_k^C\} = \{\mu_1, \ldots, \mu_k\}$ and for any $l \in \{1, \ldots, k-1\}$, either $\mu_l^C(C) \succ_C \mu_{l+1}^C(C)$ or $\mu_l^C(C) = \mu_{l+1}^C(C)$. Thus, for any $l \in \{1, \ldots, k\}$, at μ_l^C college C is assigned to its lth '(weakly) best' match among all k stable matchings. For any $l \in \{1, \ldots, k\}$, define the function $\alpha_l^{\mathcal{C}}$ on the set $\mathcal{C}$ such that for all $C \in \mathcal{C}$, $\alpha_l^{\mathcal{C}}(C) := \mu_l^C(C)$.

In Theorem 3.2 we state that for any $l \in \{1, \ldots, k\}$, functions α_l^S and $\alpha_{k-l+1}^{\mathcal{C}}$ together constitute a well-defined and stable matching.

Definition 3.1 (Generalized median stable matchings) *Let $l \in \{1, \ldots, k\}$. Then,* the lth student optimal generalized median stable matching *is defined by function α_l^S that assigns all students to their lth (weakly) best match among all k stable matchings. Similarly, the* lth college optimal generalized median stable matching *is defined by function $\alpha_l^{\mathcal{C}}$ that assigns all colleges to their lth (weakly) best match among all k stable matchings.*[6]

[6] We would like to make two short remarks on the definition of generalized median stable matchings:
(1) We use the term 'generalized median stable matching' because for any $l \in \{1, \ldots, k\}$ the lth (weakly) best match can be represented as the median of all k matches with $k-1$ extra weights on the lth (weakly) best match; i.e., $l = \text{med}\{1, \ldots, \underbrace{l, \ldots, l}_{k \text{ times}}, \ldots, k\}$ (for a similar use of the term 'generalized median' see for instance Moulin's (1980) generalized median voter rules).
(2) Determining the set of generalized median stable matchings is only possible if the set of stable matchings is known. Gusfield and Irving (1989, section 3.5, pp. 121) provided a time- and space-optimal algorithm for enumerating all stable matchings for one-to-one matching markets $(S, \mathcal{C}, P)$ (i.e., for all $C \in \mathcal{C}$, $q_C = 1$) where $|S| = |\mathcal{C}| = m$; their algorithm needs $O(m^2 + m|\sum(P)|)$ total time. Using responsiveness and Roth and Sotomayor (1989), Lemma 5.6, Gusfield and Irving's (1989) time- and space-optimal algorithm can be straightforwardly adjusted to college admissions markets with a complexity of time of $O(m \sum_{C \in \mathcal{C}} q_C + \min\{m, \sum_{C \in \mathcal{C}} q_C\} |\sum(P)|)$.

Sethuraman et al. (2004) used linear programming tools to prove the following theorem. We give a simple proof of this result by exploiting the lattice structure of the set of stable matchings.

Theorem 3.2 (Generalized median stable matchings are well-defined and stable) *All student optimal and all college optimal generalized median stable matchings are well-defined and stable matchings. Furthermore, for any $l \in \{1, \ldots, k\}$, the lth student optimal generalized median stable matching equals the $(k-l+1)$st college optimal generalized median stable matching. Formally, for all $l \in \{1, \ldots, k\}$ there exists a stable matching $\gamma \in \sum(P)$ such that for all $s \in S$, $\gamma(s) = \alpha_l^S(s)$, and for all $C \in \mathcal{C}$, $\gamma(C) = \alpha_{k-l+1}^{\mathcal{C}}(C)$.*

Proof Let $l \in \{1, \ldots, k\}$. For all choices of l matchings $\nu_1, \ldots, \nu_l$ out of the k stable matchings it follows from Theorem 2.2 that $\nu_1 \wedge_S \cdots \wedge_S \nu_l$, the matching where all students are assigned to their least preferred match of all matchings in $\{\nu_1, \ldots, \nu_l\}$, is well-defined and stable. Denote the $N = \binom{k}{l}$ (possibly non-distinct) stable matchings obtained in this way by $\beta_1, \ldots, \beta_N$. By Theorem 2.2, $\gamma = \beta_1 \vee_S \cdots \vee_S \beta_N$, the matching where all students are assigned to their most preferred match of all matchings in $\{\beta_1, \ldots, \beta_N\}$, is well-defined and stable, i.e., $\gamma \in \Sigma(P)$.

Consider a student $s \in S$. Note that for all $r \in \{1, \ldots, N\}$, $\alpha_l^S(s) \succeq_s \beta_r(s)$, which implies $\alpha_l^S(s) \succeq_s \gamma(s)$. Also, for some $r' \in \{1, \ldots, N\}$, $\beta_{r'}(s) = \mu_1^s \wedge_S \cdots \wedge_S \mu_l^s = \alpha_l^S(s)$, which implies $\gamma(s) \succeq_s \alpha_l^S(s)$. Hence, for all $s \in S$, $\gamma(s) = \alpha_l^S(s)$. Finally, from the definition of $\vee_S$ and $\wedge_S$ and Corollary 2.4 it follows that for all $C \in \mathcal{C}$, $\gamma(C) = \alpha_{k-l+1}^{\mathcal{C}}(C)$. □

Given Theorem 3.2, from now on and with some abuse of notation, the functions α_l^S and $\alpha_{k-l+1}^{\mathcal{C}}[l \in \{1, \ldots, k\}]$ denote the same stable matching, i.e., $\alpha_l^S = \alpha_{k-l+1}^{\mathcal{C}} \in \sum(P)$.

Next, we comment on the relation between our result and Fleiner (2002, Theorem 5.5) and describe an alternative proof of Theorem 3.2.

Remark 3.3 (Alternative proofs) After finishing the first draft of this article, thanks to Jay Sethuraman, we became aware of a similar proof for the marriage model due to Fleiner (2002, Theorem 5.5). In fact, Fleiner (2002) noted that his Theorem 5.5 can be generalized to a more abstract setting. However, for more general models than the college admissions market (with responsive preferences!) the set of stable matchings can only be endowed with a lattice structure if $\vee_S$ and $\wedge_S$ are replaced by binary operations that do not necessarily reflect agents' preferences over stable matchings (cf. Blair 1988; Martínez et al. 2001). Consequently, since then the lattice structure does not reflect any polarization between agents according to their preferences over stable matchings, for more general models it is no longer clear how far the 'generalized median stable matchings' are natural compromises. In fact, apart from giving a simple proof of Theorem 3.2 for the college admissions market, we would like to argue that Fleiner's (2002, Theorem 5.5) and our (Theorem 3.2) result does not only describe '(mathematical) operations' that induce stable matchings, but that these 'operations' reflect fairness trade-offs between stable matchings. In Remark 3.6 we comment on these fairness aspects of median stable matchings.

Finally, there is another simple proof of Theorem 3.2 based on Teo and Sethuraman's (1998, Theorem 2) result for marriage markets. The proof works as follows. First, one uses the well-known technique of transforming a college admissions market (with responsive preferences) into a related marriage market by replicating all colleges according to their quota such that all $q_C - 1$ copies of a college have the same preferences as the original college C and such that each student replaces a college by a fixed order over the college and its replicas (see Roth and Sotomayor 1990; section 5.3.1, pp. 131). Then, Teo and Sethuraman's (1998, Theorem 2) result applies to the related marriage market. Since by Roth and Sotomayor (1990, Lemma 5.6) stable matchings in the related marriage market correspond to stable matchings in the original college admissions market, any generalized median stable matching for the related marriage market is also a generalized median stable matching for the original college admissions market and vice versa.[7]

The next remark clarifies two implications of Theorem 3.2.

Remark 3.4 (i) Note that Definition 3.1 and Theorem 3.2 can be straightforwardly generalized to any subset $\Sigma' \subseteq \Sigma(P)$ of stable matchings. In this case, however, a generalized median stable matching may be a stable matching that is no longer in Σ'.
(ii) It is also interesting to note that by construction of the generalized median stable matchings each side of the market is unanimous on the set of generalized median stable matchings. More precisely, for all $l \in \{1, \ldots, k-1\}$, either $[\alpha_l^S = \alpha_{l+1}^S$ and $\alpha_{k-l+1}^{\mathcal{C}} = \alpha_{k-l}^{\mathcal{C}}]$ or $[\alpha_l^S \succ_S \alpha_{l+1}^S$ and $\alpha_{k-l+1}^{\mathcal{C}} \prec_C \alpha_{k-l}^{\mathcal{C}}]$.

When the number of stable matching is odd, Teo and Sethuraman (1998) and Sethuraman et al. (2004) proved the existence of the so-called median stable matching for the marriage model and the college admissions model, respectively, i.e., giving each student and each college the 'median' match in the set of stable matches yields again a stable matching. Next, independently of the number of stable matching being odd or even, we define 'median stable matchings.'

Definition 3.5 (Median stable matchings) *If k is odd, then the* set of median stable matchings $\mathcal{M}(P)$ *is a singleton that consists of the matching where each student and each college is assigned the* $(\frac{k+1}{2})$*th (weakly) best match, i.e.,* $\mathcal{M}(P) = \{\alpha^S_{\frac{k+1}{2}}\} = \{\alpha^{\mathcal{C}}_{\frac{k+1}{2}}\}$.
If k is even, then we call matching $\alpha^S_{\frac{k}{2}}$ $(\alpha^{\mathcal{C}}_{\frac{k+2}{2}})$ *the* lower student (upper college) median stable matching *and matching* $\alpha^S_{\frac{k+2}{2}}$ $(\alpha^{\mathcal{C}}_{\frac{k}{2}})$ *the* upper student (lower college) median stable matching. *The* set of median stable matchings $\mathcal{M}(P)$ *consists of the (possibly distinct) upper and lower median stable matchings, i.e.,* $\mathcal{M}(P) = \{\alpha^S_{\frac{k}{2}}, \alpha^S_{\frac{k+2}{2}}\} = \{\alpha^{\mathcal{C}}_{\frac{k}{2}}, \alpha^{\mathcal{C}}_{\frac{k+2}{2}}\}$.

Theorem 3.2 implies that for any college admissions market the set of median stable matchings $\mathcal{M}(P)$ is well-defined and $\mathcal{M}(P) \subseteq \Sigma(P)$.

[7] This alternative and elegant proof of Theorem 3.2 was suggested by one of the referees.

Remark 3.6 (Fairness aspects) Even when using cardinal preference information (either based on cardinal utility functions or using the rankings of matches in the preference orders as cardinal measurement) a matching that combines stability and certain (endstate) fairness criteria may not exist (Masarani and Gokturk 1989). In the absence of a clear criterion for what constitutes a fair outcome, Klaus and Klijn (2006) therefore applied Rawls's (1971) principle of 'pure procedural justice' and identified two procedurally fair and stable matching mechanisms. Given Masarani and Gokturk's (1989) negative and Klaus and Klijn's (2006) positive results, it would seem that we could only expect procedural fairness, but not endstate fairness, in combination with stability. However, median stable matchings satisfy various aspects of endstate fairness different from those of Masarani and Gokturk (1989). First, the mere fact that based on the ordinal preferences, each agent is assigned to a median stable match should be considered an endstate fairness result given the stability constraints. In addition to this endstate fairness aspect that is induced by using medians, it is interesting to note that two further fairness properties are satisfied by median stable matchings for marriage markets (where all colleges have quota one). First of all, both sides of the market are treated *symmetrically*; i.e., exchanging the roles of students and colleges will not change the median stable matching(s). Second, an agent who is assigned to the same match at all stable matchings, called a dummy agent, does not influence the matches of other agents. Thus, median stable matchings are *independent of dummy agents*.[8]

Our next example illustrates how the median stable matching coincides with what for this example may be called the endstate compromise matching. Klaus and Klijn (2006) demonstrated for the same example that none of the procedurally fair and stable matching mechanisms they analyzed ever chooses the endstate compromise matching. For notational convenience, in this and the next example we only list acceptable colleges (students) in students' (colleges') preferences.

Example 3.7 (The median stable matching equals the endstate compromise) Let $(S, \mathcal{C}, P)$ with $S = \{s_1, s_2, s_3\}$, $\mathcal{C} = \{C_1, C_2, C_3\}$, $q_{C_1} = q_{C_2} = q_{C_3} = 1$, and P listed below. The three stable matchings for this market are listed below as well (for example, μ_1 matches s_1 to C_1, s_2 to C_3, and s_3 to C_2).

Preferences				Stable matchings			
$P(s_1) =$	C_1	C_2	C_3	$\mu_1 =$	C_1	C_3	C_2
$P(s_2) =$	C_3	C_1	C_2	$\mu_2 =$	C_2	C_1	C_3
$P(s_3) =$	C_2	C_3	C_1	$\mu_3 =$	C_3	C_2	C_1
$P(C_1) =$	s_3	s_2	s_1				
$P(C_2) =$	s_2	s_1	s_3				
$P(C_3) =$	s_1	s_3	s_2				

At matching μ_1 all students (colleges) are assigned to their most (least) preferred match. Matching μ_3 establishes the other extreme: all colleges (students) are assigned to their most (least) preferred match. At matching μ_2 all agents are matched to their second choice, which is why we consider μ_2 to be an endstate compromise in this situation. We depict the corresponding lattice in Fig. 1. The nodes denote the stable matchings. The solid arcs denote comparability or unanimity on each

[8] The exact formulation of this property is given in Klaus and Klijn (2006) and can easily be adjusted to college admissions markets.

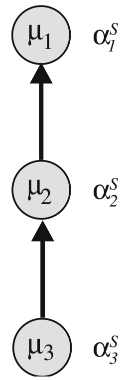

Fig. 1 Example 3.7 – lattice of stable matchings

side of the market. For instance $\mu_2 \to \mu_1$ in Fig. 1 means that all students weakly prefer their matches at μ_1 to their matches at μ_2 and all colleges weakly prefer their matches at μ_2 to their matches at μ_1. The generalized median stable matchings coincide with the stable matchings: $\alpha_1^S = \mu_1, \alpha_2^S = \mu_2$, and $\alpha_3^S = \mu_3$. In particular, the median stable matching α_2^S equals the endstate compromise matching μ_2. □

We conclude with an example of a more general college admissions market and its generalized median stable matchings.

Example 3.8 Let $(S, \mathcal{C}, P)$ with $S = \{s_1, \ldots, s_{11}\}$, $\mathcal{C} = \{C_1, \ldots, C_5\}$, and P be given by Tables 1 and 2 (by responsiveness it suffices to present colleges'

Table 1 Example 3.8 – students' preferences

Students' preferences					
$P(s_1)$	$= C_3$	C_1	C_5	C_4	
$P(s_2)$	$= C_1$	C_3	C_4	C_2	C_5
$P(s_3)$	$= C_4$	C_5	C_3	C_1	C_2
$P(s_4)$	$= C_3$	C_4	C_1	C_5	
$P(s_5)$	$= C_1$	C_4	C_2		
$P(s_6)$	$= C_4$	C_3	C_2	C_1	C_5
$P(s_7)$	$= C_2$	C_5	C_1	C_3	
$P(s_8)$	$= C_1$	C_3	C_2	C_5	C_4
$P(s_9)$	$= C_4$	C_1	C_5		
$P(s_{10})$	$= C_3$	C_1	C_5	C_2	C_4
$P(s_{11})$	$= C_5$	C_4	C_1	C_3	C_2

Table 2 Example 3.8 – quota and colleges' preferences

Quota	Colleges' preferences											
4	$P(C_1) =$	s_3	s_7	s_9	s_{11}	s_5	s_4	s_{10}	s_8	s_6	s_1	s_2
3	$P(C_2) =$	s_5	s_7	s_{10}	s_6	s_8	s_2	s_3	s_{11}			
3	$P(C_3) =$	s_{11}	s_6	s_8	s_3	s_2	s_4	s_7	s_1	s_{10}		
2	$P(C_4) =$	s_{10}	s_1	s_2	s_{11}	s_4	s_9	s_5	s_3	s_6	s_8	
1	$P(C_5) =$	s_2	s_4	s_{10}	s_7	s_6	s_1	s_8	s_3	s_{11}	s_9	

Table 3 Example 3.8 – stable matchings

Matching	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}
μ_1	C_3	C_1	C_4	C_3	C_1	C_3	C_2	C_1	C_4	C_1	C_5
μ_2	C_1	C_3	C_4	C_3	C_1	C_3	C_2	C_1	C_4	C_1	C_5
μ_3	C_3	C_1	C_5	C_3	C_1	C_3	C_2	C_1	C_4	C_1	C_4
μ_4	C_1	C_3	C_5	C_3	C_1	C_3	C_2	C_1	C_4	C_1	C_4
μ_5	C_5	C_3	C_3	C_4	C_1	C_3	C_2	C_1	C_1	C_1	C_4
μ_6	C_5	C_4	C_3	C_1	C_1	C_3	C_2	C_3	C_1	C_1	C_4
μ_7	C_4	C_4	C_3	C_1	C_1	C_3	C_2	C_3	C_1	C_5	C_1

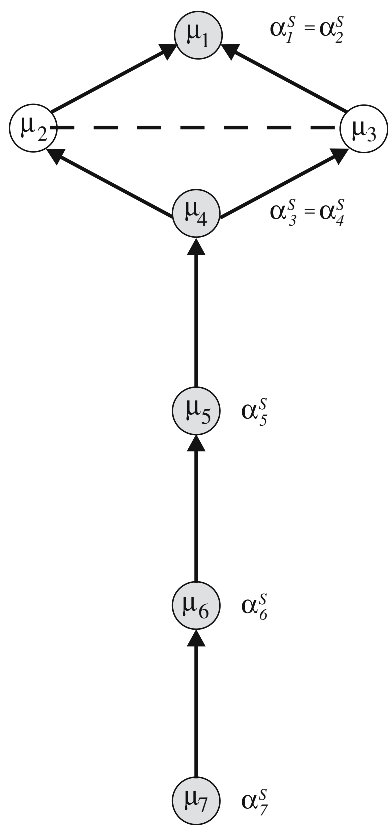

Fig. 2 Example 3.8 – lattice of stable matchings

preferences by strict orderings of individual students).[9] We list all seven stable matchings in Table 3. We depict the corresponding lattice in Fig. 2. Again, the nodes denote the stable matchings and the solid arcs denote comparability or unanimity on each side of the market. The dotted edge $\mu_2 \cdots \mu_3$ denotes incomparability on each side of the market. In other words, there is disagreement among the students (colleges) about which matching is better (for instance, $\mu_3(s_1) \succ_{s_1} \mu_2(s_1)$, but $\mu_2(s_3) \succ_{s_2} \mu_3(s_3)$). The generalized median stable matchings are depicted by the

[9] This college admissions market is taken from Gusfield and Irving (1989). It is also used by Sethuraman et al. (2004).

gray-filled nodes: $\alpha_1^S = \alpha_2^S = \mu_1$, $\alpha_3^S = \alpha_4^S = \mu_4$, $\alpha_5^S = \mu_5$, $\alpha_6^S = \mu_6$, and $\alpha_7^S = \mu_7$. Since the number of stable matchings is odd, the set of median stable matchings is a singleton given by $\mathcal{M}(P) = \{\mu_4\}$. □

Acknowledgements We thank Tamás Fleiner, Rob Irving, Howard Petith, Jay Sethuraman, William Thomson, and three anonymous referees for their helpful comments. The authors' research was supported by Ramón y Cajal contracts of the Spanish *Ministerio de Ciencia y Tecnología*, Research Grant BEC2002-02130 from the Spanish *Ministerio de Ciencia y Tecnología*, and the Barcelona Economics Program of CREA.

References

Aldershof B, Carducci OM, Lorenc DC (1999) Refined inequalities for stable marriage. Constraints 4:281–292

Barberà S, Gul F, Stacchetti E (1993) Generalized median voter schemes and committees. J Econ Theory 61:262–289

Blair C (1988) The lattice structure of the set of stable matchings with multiple partners. Math Oper Res 13:619–628

Fleiner T (2002) Some results on stable matchings and fixed points. Technical Report, EGRES Report TR-2002-8, Egerváry Research Group, Budapest, Hungary [http://www.cs.elte.hu/egres/tr/egres-02-08.pdf]

Gale D, Shapley LS (1962) College admissions and the stability of marriage. Am Math Mon 69:9–15

Gusfield D, Irving RW (1989) The stable marriage problem: structure and algorithms. MIT Press, Cambridge

Klaus B, Klijn F (2006) Procedurally fair and stable matching. Econ Theory 27:431–447

Ma J (1996) On randomized matching mechanisms. Econ Theory 8:377–381

Martínez R, Massó J, Neme A, Oviedo J (2001) On the lattice structure of the set of stable matchings for a many-to-one model. Optimization 50:439–457

Masarani F, Gokturk SS (1989) On the existence of fair matching algorithms. Theory Decis 26:305–322

Moulin H (1980) On strategy-proofness and single peakedness. Public Choice 35:437–455

Rawls J (1971) A theory of justice. Harvard University Press, Cambridge, MA

Roth AE (1985) The college admissions problem is not equivalent to the marriage problem. J Econ Theory 36:277–288

Roth AE, Sotomayor MAO (1989) The college admissions problem revisited. Econometrica 57:559–570

Roth AE, Sotomayor MAO (1990) Two-sided matching: a study in game-theoretic modeling and analysis. Econometric Society Monograph Series. Cambridge University Press, New York

Sethuraman J, Teo C-P, Qian L (2004) Many-to-one stable matching: geometry and fairness. Working Paper TR-2004-02, Computational Optimization Research Center, Columbia University [http://www.corc.ieor.columbia.edu/reports/techreports.html]

Teo C-P, Sethuraman J (1998) The geometry of fractional stable matchings and its applications. Math Oper Res 23:874–891