Target Rules

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Abstract

We consider a probabilistic approach to collective choice problems where a group of agents with single-peaked preferences have to decide on the level or location of a public good. We show that every probabilistic rule that satisfies Pareto efficiency and "solidarity" (population-monotonicity or replacement-domination) must equal a so-called target rule.

1. Introduction

We consider the problem of choosing the provision level or the location of a public good along a one-dimensional continuum. Each agent has a "single-peaked" preference relation over the continuum. An agent's preference relation is "single-peaked" if up to a certain point, his "peak level," his welfare is strictly increasing, and it is strictly decreasing beyond that point. An economy is given by a population of agents and a profile of single-peaked preferences. Examples are voting situations where candidates, or alternatives, can be ordered on a left-to-right spectrum. Another example is the choice of the quality of a public service under budgetary constraints as in public education. Monetary transactions between the social planner and the agents are not allowed.

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1.1 Solidarity

We consider changes in the economic environment; for instance, new agents may enter the economy, or the availability of resources may vary. The decisions made by the social planner, say, the government, are typically different before and after such changes. Such situations include the allocation of resources before and after an earthquake or a growth in population and the compulsory duration of schooling before and after the change of the interests of some members of the society. Solidarity means that in such situations the welfares of the agents whose characteristics did not change will be affected in the same direction: as a result of the change either they all (weakly) benefit or they all (weakly) lose. Solidarity ensures that changes of the decisions made by the social planner are "fair" in the following sense: there cannot be a pair of agents whose characteristics did not change such that one of them strictly benefits whereas the other strictly loses.

We apply the idea of solidarity to situations when the population of agents varies or the preference relations of some agents change. If some agents leave, then as a result either all remaining agents (weakly) gain or they all (weakly) lose. This requirement is called *population-monotonicity* (Thomson 1983). Alternatively, if the preference relations of some agent change, then as a result either all the agents whose preference relations did not change (weakly) gain or they all (weakly) lose. This requirement is called *replacement-domination* (Moulin 1987).[1] The idea of solidarity has been applied to various contexts. Thomson (1983) studied the idea in bargaining; Chichilnisky and Thomson (1987), Kim (1999), and Sprumont and Zhou (1999) in classical exchange economies; Thomson (1997), Chun (1999), and Klaus (2000) in private good economies with single-peaked preferences; Sprumont (1990) and Hokari (1999) in cooperative games with transferable utility; and Miyagawa (1998a, 1998b) for the provision of multiple public goods.[2]

The idea of solidarity has also been applied to the problem of choosing the level of a public good along a one-dimensional continuum. A rule assigns to every economy a level. Thomson (1993) and Ching and Thomson (1996) showed that if a rule satisfies Pareto efficiency and either of the two solidarity properties described above, then it must be a so-called target rule. Such a rule is determined by its target (level). If the target is Pareto efficient, then the target is chosen by the rule. If the target is not Pareto efficient, then the Pareto efficient level that is closest to the target is chosen by the rule. Note that the target that essentially determines any target rule might be the status quo of the present level of the public good,

[1] Moulin (1987) calls it "agreement."

[2] This list is not exhaustive.

or alternatively it may be a target level for the public good that is imposed by the social planner.

1.2 Probabilistic Rules

We consider a probabilistic approach. Alternatives are distributions from which the final level of the public good is drawn. A deterministic rule assigns to every economy a distribution that places probability one on exactly one level. Agents with peak levels that are far away from that uniquely chosen level may regard the outcome to be unfair because more favorable levels of the public good were excluded a priori. By using a probabilistic approach, it is possible to assign positive probabilities to several public good levels that might improve “a priori fairness.” For example, consider the probabilistic rule that assigns equal probability to every agent’s peak level of the public good. Since now, at least a priori, each agent has an equal chance that his peak level is the outcome assigned by the probabilistic rule, this rule would be “a priori fair.” A probabilistic rule assigns to every economy a distribution. Recently, probabilistic studies have received renewed attention: Abdulkadiroğlu and Sönmez (1998) and Bogomolnaia and Moulin (1999) for house allocation problems, Crès and Moulin (1998) for the tragedy of the commons, and Ehlers (1998) for private good economies with single-dipped preferences. We use the ordinal extension of preferences from levels to distributions to define gains and losses (Gibbard 1977).

1.3 Results

It turns out that Pareto efficiency is equivalent to ex-post efficiency: the probabilistic rule assigns to every preference profile a distribution that places probability 1 on the interval having as boundary points the smallest peak level and the greatest peak level. Our main results are the following. First, we show that Pareto efficiency and either of the two solidarity properties imply anonymity (the names of the agents do not matter) and strategy-proofness (truthtelling is a weakly dominant strategy). Those properties are very desirable. Second, we show that deterministic target rules are the only probabilistic rules satisfying Pareto efficiency and replacement-domination. Surprisingly, even if we allow rules to be probabilistic, then still only deterministic rules satisfy these two requirements. The class we characterize is the same that Thomson (1993) characterized in the deterministic model. Furthermore, we extend deterministic target rules to the probabilistic setting. Given a (fixed) target distribution, a probabilistic target rule assigns to every economy the following distribution: in the interior of the interval having as endpoints the smallest peak level and the greatest peak level the target probability distribution is applied; all probability outside that interval is “projected” to the smallest peak level and the greatest peak level, respectively. Third, we show that the probabilistic target rules are the only probabilistic rules satisfying Pareto efficiency and population-monotonicity. Therefore, a probabilistic rule that satisfies Pareto efficiency and solidarity must be of the "target type," using a target probability distribution or a target level.

1.4 Target Rules

Samuelson and Zeckhauser (1988) prove that in many situations individuals disproportionally stick to the status quo. In other words, a "target bias" with the target equal to the status quo is present in many decisions. Our main results imply that in public good economies Pareto efficiency and solidarity imply such a target bias. Target rules with the target equal to the status quo are useful in economic situations when agents have veto power over changes in the status quo, as this might be the case for risky undertakings. A practical advantage of target rules is that they are simple and can be implemented easily and quickly. Furthermore, they are strategy-proof and to some extent fair if they use "fair" target probability distributions for example, the uniform probability distribution. On the other hand, since target rules are almost constant (apart from ruling out probabilities on inefficient levels) one can interpret our (and previous) results concerning target rules as impossibility results.

The paper is organized as follows. Section 2 introduces the model and some basic properties. Section 3 defines the solidarity properties and presents relations and implications of those properties. Section 4 contains the characterizations. The proofs are relegated to the Appendix.

2. The Model and Basic Properties

We consider the problem of choosing a level of a public good on the real line $\mathbb{R}$.[3] Let $\mathbb{N}$ denote the set of natural numbers. There is a population $\mathbb{P} \subseteq \mathbb{N}$ of "potential" agents. The population $\mathbb{P}$ can be either finite or infinite. Let $\mathbb{P}$ contain at least four agents. Let $\mathcal{P}$ denote the class of nonempty and finite subsets of $\mathbb{P}$ with cardinality greater than or equal to three. Each agent $i \in \mathbb{P}$ is equipped with a continuous and "single-peaked" preference relation R_i defined over $\mathbb{R}$. *Single-peakedness* of R_i means that there exists a level $p(R_i) \in \mathbb{R}$, called the *peak level of* R_i, with the following property: for all $x, y \in \mathbb{R}$, if $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, then $yP_i x$. As usual, $xR_i y$ means "x is weakly preferred to y," and $xP_i y$ means "x is strictly preferred to y." Let $\mathcal{R}$ denote the class of all continuous, single-peaked preference relations over $\mathbb{R}$. Given $N \in \mathcal{P}$, let $\mathcal{R}^N$ denote the set of all (preference) profiles $R = (R_i)_{i \in N}$

[3] All results remain true if we restrict $\mathbb{R}$ to a closed interval $[a, b]$ or an open interval $]a, b[$, $a, b \in \mathbb{R}$.

such that for all $i \in N$, $R_i \in \mathcal{R}$. We also call R an *economy*. Given $R \in \mathcal{R}^N$, let $\underline{p}(R) := \min_{i \in N} p(R_i)$, $\bar{p}(R) := \max_{i \in N} p(R_i)$, and $E(R) := [\underline{p}(R), \bar{p}(R)]$.[4]

A *deterministic (decision) rule* Φ is a function that selects for every $N \in \mathcal{P}$ and every $R \in \mathcal{R}^N$ a level in $\mathbb{R}$, denoted by $\Phi(R)$. We extend the original analysis of deterministic rules by considering "probabilistic" rules. A *probabilistic (decision) rule* φ is a function that selects for every $N \in \mathcal{P}$ and every $R \in \mathcal{R}^N$ a (probability) distribution over $\mathbb{R}$, denoted by $\varphi(R)$. For all (measurable) subsets X of $\mathbb{R}$, the number $\varphi(R)(X)$, which the distribution $\varphi(R)$ assigns to X, is the probability that the final level belongs to X. We consider distributions defined on the Borel σ-algebra $\mathcal{L}$. Elements of $\mathcal{L}$ are called *Borel sets*.[5]

A deterministic rule is a probabilistic rule that selects for every $N \in \mathcal{P}$ and every $R \in \mathcal{R}^N$ a distribution placing probability 1 on a single level in $\mathbb{R}$. We extend preferences over levels in $\mathbb{R}$ to preferences over distributions ordinally (see Gibbard 1977; or Ehlers, Peters, and Storcken 1999). This ordinal extension is based on the concept of upper contour sets.

Given $x \in \mathbb{R}$ and $R_i \in \mathcal{R}$, the *weak upper contour set of x at R_i* is defined by $B(x, R_i) := \{y \in \mathbb{R} | yR_i x\}$, and the *strict upper contour set of x at R_i* is defined by $B^\circ(x, R_i) := \{y \in \mathbb{R} | yP_i x\}$. Because of single-peakedness, all upper contour sets are measurable.

An agent i prefers distributions that assign larger probabilities to all his upper contour sets. Given a preference relation $R_i \in \mathcal{R}$ and two distributions Q, Q' over $\mathbb{R}$, agent i weakly prefers Q to Q', if Q assigns to each weak upper contour set at least the probability that is assigned by Q' to this set. Abusing notation we use the same symbols to denote preferences over distributions and preferences over levels.

Ordinal extension of preferences: *For all $R_i \in \mathcal{R}$ and all distributions Q, Q' over $\mathbb{R}$, QR_iQ' if and only if*

$$\text{for all } x \in \mathbb{R},\ Q(B(x, R_i)) \geq Q'(B(x, R_i)). \tag{1}$$

Furthermore, QP_iQ' if and only if

$$QR_iQ' \quad \text{and for some } y \in \mathbb{R},\ Q(B(y, R_i)) > Q'(B(y, R_i)). \tag{2}$$

Inequality (1) is a first order stochastic dominance condition; in particular it requires that the distributions Q and Q' are comparable in that respect. Therefore, our extension is not complete over the set of all distributions over $\mathbb{R}$. Note that for preferences over distributions completeness is a demanding requirement.

Our extension of preferences is equivalent to the following. Assume that each agent is a von Neumann–Morgenstern (vNM) expected utility

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[4]As explained later (Remark 1), $E(R)$ is the set of Pareto efficient levels at profile R.

[5]All basic results of measure theory that are used in the paper can be found in Halmos (1970).

maximizer. That means that each agent has a vNM-utility function and compares two distributions via the expected values relative to this function. Then (1) is equivalent to the fact that the expected value relative to *any* vNM-utility function representing R_i is at Q greater or equal than at Q'. Thus, regardless what vNM-utility function the agent has, he will weakly prefer Q to Q' although the rule depends only on the ordinal information of the preference profile.

Ehlers et al. (1999, Lemma 2.1) show that we can equivalently extend preferences with respect to strict upper contour sets.

LEMMA 1: *Let $R_i \in \mathcal{R}$ and Q, Q' be distributions over $\mathbb{R}$. Then QR_iQ' if and only if*

$$\textit{for all } x \in \mathbb{R},\ Q(B^\circ(x, R_i)) \geq Q'(B^\circ(x, R_i)). \tag{3}$$

Furthermore, QP_iQ' if and only if

$$QR_iQ' \quad \textit{and for some } y \in \mathbb{R},\ Q(B^\circ(y, R_i)) > Q'(B^\circ(y, R_i)). \tag{4}$$

From now on, we use [(1) and (2)] or [(3) and (4)] for our ordinal extension of preferences. We are interested in the following properties.

First we consider Pareto efficiency: for each profile, there does not exist a distribution that all agents weakly prefer to the distribution assigned by the probabilistic rule to this profile with strict preference for some agent.

Let $N \in \mathcal{P}$ and Q, Q' be distributions over $\mathbb{R}$. If for all $i \in N$, QR_iQ' and for some $j \in N$, QP_jQ', then we call Q a *Pareto improvement* of Q'.

PARETO EFFICIENCY: *For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, there exists no Pareto improvement of $\varphi(R)$.*

Remark 1: A deterministic rule satisfies Pareto efficiency if and only if for all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, $\Phi(R) \in E(R)$. Therefore, we call $E(R)$ the *Pareto set* of R.

Ehlers et al. (1999, Lemma 2.2) show that a probabilistic rule satisfies Pareto efficiency if and only if it only selects distributions that place for every profile probability 1 on its Pareto set. Therefore, in our model, ex-post efficiency is equivalent to Pareto efficiency.

LEMMA 2: *Let φ be a probabilistic rule. Then φ satisfies Pareto efficiency if and only if for all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, $\varphi(R)(E(R)) = 1$.*

The next requirement ensures that no agent can ever benefit by misrepresenting his true preference relation.

Let $N, M \in \mathcal{P}$, $N \subseteq M$, and $R \in \mathcal{R}^M$. Then $M \backslash N := \{i \in M | i \notin N\}$. Let R_N denote the restriction $(R_i)_{i \in N} \in \mathcal{R}^N$ of R to N.

STRATEGY-PROOFNESS: *For all* $N \in \mathcal{P}$, *all* $j \in N$, *and all* $R, \bar{R} \in \mathcal{R}^N$ *such that* $R_{N \setminus \{j\}} = \bar{R}_{N \setminus \{j\}}$, $\varphi(R) R_j \varphi(\bar{R})$.

Note that our notion of strategy-proofness also contains the requirement that the distributions that are assigned by the probabilistic rule before and after any unilateral deviation are comparable. For a more detailed discussion of strategy-proofness we refer to Ehlers et al. (1999).

The next requirement says that the probabilistic rule is symmetric in its arguments; in other words, the names of the agents do not matter.

Let $N \in \mathcal{P}$ and Π^N be the class of all permutations on N. For all $R \in \mathcal{R}^N$ and all $\pi \in \Pi^N$, let R_π denote the profile $(R_{\pi(i)})_{i \in N}$.

ANONYMITY: *For all* $N \in \mathcal{P}$, *all* $R \in \mathcal{R}^N$, *and all* $\pi \in \Pi^N$, $\varphi(R) = \varphi(R_\pi)$.

Ehlers et al. (1999, Corollary 5.4) give a full characterization of all probabilistic rules satisfying Pareto efficiency, strategy-proofness, and anonymity. Given $N \in \mathcal{P}$, a probabilistic rule on $\mathcal{R}^N$ satisfying these properties is determined by $(|N| - 1)$ fixed distributions. These fixed distributions play the same role as the fixed $(|N| - 1)$ "phantom voters," or calibration points, that characterize any Pareto efficient, strategy-proof, and anonymous deterministic rule on $\mathcal{R}^N$ (Moulin 1980, Ching 1997).

Let $Q^{+\infty}$ and $Q^{-\infty}$ denote the probability distributions placing probability 1 on $+\infty$ and $-\infty$, respectively. For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, let $\hat{p}_1 < \cdots < \hat{p}_t$ be such that $\{\hat{p}_1, \ldots, \hat{p}_t\} = \{p(R_i) | i \in N\}$. Given a Borel set X, let 1_X denote the *indicator function of* X: for all $x \in \mathbb{R}$, if $x \in X$, then $1_X(x) = 1$, and if $x \notin X$, then $1_X(x) = 0$.

THEOREM 1: *Let* φ *be a probabilistic rule. Then* φ *satisfies Pareto efficiency, strategy-proofness, and anonymity if and only if for all* $N \in \mathcal{P}$, *there exist* $|N| - 1$ *distributions over* $\mathbb{R} \cup \{-\infty, +\infty\}$, *denoted by* $D_1^N, \ldots, D_{|N|-1}^N$, *such that*

(i) *for all* $l \in \{2, \ldots, |N| - 1\}$ *and all* $x \in \mathbb{R}$, $D_l^N([-\infty, x]) \geq D_{l-1}^N([-\infty, x[)$ *and*

(ii) *for all* $R \in \mathcal{R}^N$, *all* $X \in \mathcal{L}$, $D_0^N := Q^{+\infty}$, *and* $D_{|N|}^N := Q^{-\infty}$,

$$\varphi(R)(X) = \sum_{l=1}^{t-1} D_l^N(X \cap]\hat{p}_l, \hat{p}_{l+1}[) + \sum_{l=1}^{t} 1_X(\hat{p}_l)(D_l^N([-\infty, \hat{p}_l]) - D_{l-1}^N([-\infty, \hat{p}_l[)). \qquad (5)$$

Moreover, all D_l^N *are uniquely determined.*

3. Solidarity Properties and Their Logical Relations

In this section we introduce two solidarity properties and show that Pareto efficiency and either of the two solidarity properties imply anonymity and strategy-proofness.

First, we apply the idea of solidarity to population changes. We require that if an agent leaves, then all agents that are present after weakly gain as a result of the departure.

POPULATION-MONOTONICITY:[6] *For all* $N \in \mathcal{P}$, *all* $i \in N$, *all* $j \in \mathbb{P}\backslash N$, *and all* $R \in \mathcal{R}^{N \cup \{j\}}$, $\varphi(R_N) R_i \varphi(R)$.

Thomson (1983) introduced population-monotonicity in the context of bargaining. For a survey on population-monotonicity we refer to Thomson (1995). We prove the following logical relationship of properties in the Appendix.

LEMMA 3: *Pareto efficiency and population-monotonicity imply anonymity and strategy-proofness.*

Second, we apply the idea of solidarity to changes in preference relations. When the preference relation of an agent changes, then the welfares of all agents whose preference relations remained the same should be affected in the same direction. Either they all weakly gain or they all weakly lose as a result of the change of an agent's preference relation.

REPLACEMENT-DOMINATION: *For all* $N \in \mathcal{P}$, *all* $j \in N$, *and all* $R, \bar{R} \in \mathcal{R}^N$ *such that* $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$, *either [for all* $i \in N\backslash\{j\}$, $\varphi(R) R_i \varphi(\bar{R})$*] or [for all* $i \in N\backslash\{j\}$, $\varphi(\bar{R}) R_i \varphi(R)$*].*

Moulin (1987) introduced replacement-domination in the context of binary choice with quasi-linear preferences. Thomson (1993,1997) called it *welfare-domination under preference-replacement* and studied its implications in public and private good economies with single-peaked preferences. For a survey, we refer to Thomson (1999). We prove the following logical relationship of properties in the Appendix.

LEMMA 4: *Pareto efficiency and replacement-domination imply anonymity and stratgy-proofness.*

4. Target Rules

The following classes of "target rules" play the central role in the sequel. A deterministic target rule is determined by its target level. If the target level is Pareto efficient, then the rule chooses the target level. If the target level is not Pareto efficient, then the rule chooses the Pareto efficient level that is closest to the target level.

[6] The "standard population-monotonicity property" would require that if a group of agents leave or come in, then the welfares of all agents who are present before and after the change in population are affected in the same direction. Instead of considering all possible changes in population we only consider the arrival or departure of one agent. Note that Pareto efficiency together with this weaker population-monotonicity property implies our population-monotonicity property.

Deterministic target rules, Φ^a: *Given $a \in \mathbb{R} \cup \{-\infty,+\infty\}$, the deterministic target rule with target level a, denoted by Φ^a, is defined as follows. For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, $\Phi^a(R) = \operatorname{med}(\underline{p}(R), a, \bar{p}(R))$.*[7]

Given $N \in \mathcal{P}$, we say that a probabilistic rule φ is a deterministic target rule on the domain $\mathcal{R}^N$, if there exists $a^N \in \mathbb{R} \cup \{-\infty,+\infty\}$ such that for all $R \in \mathcal{R}^N$, $\varphi(R)(\{\Phi^{a^N}(R)\}) = 1$.

In many public choice decision processes, target oriented decisions where the target equals the status quo are prevailing, for instance, if all participants have veto power over changes in the status quo and unanimity is required to change it.

A probabilistic target rule is determined by its target probability distribution. In the interior of the Pareto set, the target probability distribution is applied. Probability outside the Pareto set is projected to the closest level belonging to the Pareto set.

Probabilistic target rules, φ^Q: *Given distribution Q over $\mathbb{R} \cup \{-\infty,+\infty\}$, the probabilistic target rule with target distribution Q, denoted by φ^Q, is defined as follows. For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, if $\underline{p}(R) \neq \bar{p}(R)$, then for all $X \in \mathcal{L}$,*

$$\varphi^Q(R)(X) = 1_X(\underline{p}(R))\,Q([-\infty, \underline{p}(R)]) + Q(X \cap\,]\underline{p}(R), \bar{p}(R)[)$$

$$+\, 1_X(\bar{p}(R))\,Q([\bar{p}(R), +\infty]),$$

and if $\underline{p}(R) = \bar{p}(R)$, then for all $X \in \mathcal{L}$,

$$\varphi^Q(R)(X) = 1_X(\underline{p}(R)).$$

It is easy to check that all probabilistic target rules satisfy Pareto efficiency and population-monotonicity. If the target distribution assigns probability 1 to a single point $a \in \mathbb{R} \cup \{-\infty,+\infty\}$, then the probabilistic target rule φ^Q is identical to the deterministic target rule Φ^a.

Remark 2: All probabilistic target rules satisfy Pareto efficiency, strategy-proofness, and anonymity. Probabilistic target rules are a special subclass of the class of rules as characterized in Theorem 1: a probabilistic rule as described in Theorem 1 is a probabilistic target rule if and only if all fixed probability distributions are the same; that is, for all $N, M \in \mathcal{P}$, $D_1^N = \cdots = D_{|N|-1}^N = D_1^M = \cdots = D_{|M|-1}^M$.

Thomson (1993) shows that all deterministic target rules satisfy replacement-domination. However, the following example shows that not all probabilistic target rules satisfy replacement-domination.

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[7] By med we denote the median operator; that is, if $x \leq y \leq z$, then $\operatorname{med}(x, y, z) = y$. For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, $\underline{p}(R), \bar{p}(R) \in \mathbb{R}$. Hence, for all $a \in \mathbb{R} \cup \{-\infty,+\infty\}$, $\operatorname{med}(\underline{p}(R), a, \bar{p}(R)) \in \mathbb{R}$.

Example 1: Let Q be the distribution over $\mathbb{R} \cup \{-\infty, +\infty\}$ assigning probability $\frac{1}{2}$ to $-\infty$ and probability $\frac{1}{2}$ to $+\infty$. The resulting probabilistic target rule φ^Q is determined as follows. For all $N \in \mathcal{P}$, all $R \in \mathcal{R}^N$, and all $X \in \mathcal{L}$,

$$\varphi^Q(R)(X) = \tfrac{1}{2}1_X(\underline{p}(R)) + \tfrac{1}{2}1_X(\bar{p}(R)).$$

We show that φ^Q violates replacement-domination. Let $N = \{1,2,3\}$ and $R \in \mathcal{R}^N$ be such that $(p(R_1), p(R_2), p(R_3)) = (0,1,2)$. Let $\bar{R}_3 \in \mathcal{R}$ be such that $p(\bar{R}_3) = -1$ and $\bar{R} = (R_1, R_2, \bar{R}_3)$. Then, $\bar{R}_{N\setminus\{3\}} = R_{N\setminus\{3\}}$, and by definition of φ^Q,

$$\varphi^Q(R)(\{0\}) = \tfrac{1}{2} > 0 = \varphi^Q(\bar{R})(\{0\})$$

and

$$\varphi^Q(R)(\{1\}) = 0 < \tfrac{1}{2} = \varphi^Q(\bar{R})(\{1\}).$$

Since $p(R_1) = 0$ and $p(R_2) = 1$, this contradicts replacement-domination.

The following result states that Pareto efficiency and replacement-domination essentially characterize the class of deterministic target rules.

THEOREM 2: *A probabilistic rule satisfies Pareto efficiency and replacement-domination if and only if, for all $N \in \mathcal{P}$, it is a deterministic target rule on the domain $\mathcal{R}^N$.*

Pareto efficiency and population-monotonicity characterize the class of probabilistic target rules.[8]

THEOREM 3: *A probabilistic rule satisfies Pareto efficiency and population-monotonicity if and only if it is a probabilistic target rule.*

Remark 3: If in Theorem 3 we replace population-monotonicity by a weaker replacement-domination property, then for each $N \in \mathcal{P}$, a probabilistic rule satisfying these two properties must be a probabilistic target rule on $\mathcal{R}^N$ (Appendix, Theorem 4).

The Appendix contains the proofs of Theorems 2 and 3 and Remark 3 (Appendix, Theorem 4). Our proofs use the fact that Pareto efficiency and either one of the solidarity properties imply anonymity and strategy-proofness. Then we apply Theorem 1 and Remark 2 to deduce that a rule satisfying these requirements must be a probabilistic target rule. Moreover, we show that if a probabilistic target rule satisfies replacement-domination, then it must be a deterministic rule. Interestingly, our proofs provide an alternative way to show the results of Thomson (1993) and Ching and Thomson (1996).

[8]Theorem 3 remains valid if $|\mathbb{P}| \geq 3$ and $\mathcal{P}$ contains all nonempty subsets of $\mathbb{P}$.

The last requirement we consider says that if all agents have the same preference relation, then the probabilistic rule places probability 1 on the unanimous peak level.

UNANIMITY: *For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$, if for all $i, j \in N$, $R_i = R_j$, then* $\varphi(R)(\{p(R_i)\}) = 1$.

Ehlers et al. (1999) show that when in Theorem 1 Pareto efficiency is replaced by unanimity, their characterization still holds. The following example shows that we cannot weaken Pareto efficiency to unanimity in Theorems 2 and 3.

Example 2: Let $a \in \mathbb{R}$. We define the deterministic rule Ψ^a as follows. For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$,

$$\Psi^a(R) = \begin{cases} \underline{p}(R) & \text{if for all } i, j \in N,\ R_i = R_j, \\ a & \text{otherwise.} \end{cases}$$

The deterministic rule Ψ^a satisfies unanimity, population-monotonicity, and replacement-domination but not Pareto efficiency.

Examples 2 and 3 establish the independence of Theorems 2 and 3.

Example 3: We define the deterministic rule Θ as follows. For all $N \in \mathcal{P}$ and all $R \in \mathcal{R}^N$,

$$\Theta(R) = p(R_{\min\{i \mid i \in N\}}).$$

The deterministic rule Θ satisfies Pareto efficiency but neither population-monotonicity nor replacement-domination.

Appendix

Proofs of Lemmas 3 and 4

The following lemma will be a useful tool.

LEMMA 5: *For all $N \in \mathcal{P}$, all $R \in \mathcal{R}^N$, all $i, j \in N$ such that $p(R_i) \leq p(R_j)$, and all distributions Q, Q' over $\mathbb{R}$, if QR_iQ', QR_jQ', and $Q([p(R_i), p(R_j)]) = 1 = Q'([p(R_i), p(R_j)])$, then $Q = Q'$.*

Proof: Let $x \in [p(R_i), p(R_j)]$. Then, $Q([p(R_i), p(R_j)]) = 1 = Q'([p(R_i), p(R_j)])$ and QR_iQ' imply $Q([p(R_i), x]) = Q(B(x, R_i)) \geq Q'(B(x, R_i)) = Q'([p(R_i), x])$. Similarly, QR_jQ' implies $Q(]x, p(R_j)]) \geq Q'(]x, p(R_j)]$. Because $Q([p(R_i), p(R_j)]) = 1 = Q'([p(R_i), p(R_j)])$, all inequalities are equalities. Therefore, for all intervals $]a, b] \subset \mathbb{R}$, $Q(]a, b]) = Q'(]a, b])$. Because the σ-algebra $\mathcal{L}$ is generated by those intervals, it follows that $Q = Q'$. ■

Applying Pareto efficiency, population-monotonicity, and Lemma 5 yields the following.

LEMMA 6: *Let φ be a probabilistic rule satisfying Pareto efficiency and population-monotonicity. For all $N \in \mathcal{P}$, all $j \in \mathbb{P} \backslash N$, and all $R \in \mathcal{R}^{N \cup \{j\}}$, if $E(R_N) = E(R)$, then $\varphi(R_N) = \varphi(R)$.*

Successive application of Lemma 6 yields the following.

LEMMA 7: *Pareto efficiency and population-monotonicity imply anonymity.*

In order to prove Lemmas 3 and 4, we need to introduce a weaker replacement-domination property. This condition only requires solidarity when the Pareto set before the change of an agent's preference relation is either a subset or a superset of the Pareto set after the change. We call these changes *one-sided* because we restrict changes in the Pareto set to changes of "one of its sides."

ONE-SIDED REPLACEMENT-DOMINATION: *For all $N \in \mathcal{P}$, all $j \in N$, and all $R, \bar{R} \in R^N$ such that $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$ and $E(R) \subseteq E(\bar{R})$ or $E(R) \supseteq E(\bar{R})$, either [for all $i \in N\backslash\{j\}$, $\varphi(R) R_i \varphi(\bar{R})$] or [for all $i \in N\backslash\{j\}$, $\varphi(\bar{R}) R_i \varphi(R)$].*

LEMMA 8: *Pareto efficiency and population-monotonicity imply one-sided replacement-domination.*

Proof: Let φ be a probabilistic rule satisfying Pareto efficiency and population-monotonicity. By Lemma 7, φ satisfies anonymity. Let $N \in \mathcal{P}$, $j \in N$, and $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$ and $E(R) \subseteq E(\bar{R})$. Without loss of generality, we suppose that $\underline{p}(R) = \underline{p}(\bar{R})$. We show that for all $i \in N\backslash\{j\}$,

$$\varphi(R) R_i \varphi(\bar{R}), \tag{6}$$

which proves one-sided replacement-domination. We distinguish two cases.

Case 1: $\mathbb{P}\backslash N \neq \emptyset$.

Let $k \in \mathbb{P}\backslash N$ and $\tilde{R} \in \mathcal{R}^{N \cup \{k\}}$ be such that $\tilde{R}_N = R$ and $\tilde{R}_k = \bar{R}_j$. By population-monotonicity, for all $i \in N$, $\varphi(R) R_i \varphi(\tilde{R})$. By Lemma 6, $\varphi(\tilde{R}_{(N\cup\{k\})\backslash\{j\}}) = \varphi(\tilde{R})$. By anonymity, $\varphi(\tilde{R}_{(N\cup\{k\})\backslash\{j\}}) = \varphi(\bar{R})$. The previous three facts imply (6).

Case 2: $N = \mathbb{P}$.

If $E(\bar{R}) = E(R)$, then Lemma 6 and Case 1 imply (6). Let $E(R) \subsetneq E(\bar{R})$ and suppose that $p(R_j) < \bar{p}(R)$. By Lemma 6, $\varphi(R_{N\backslash\{j\}}) = \varphi(R)$. Let $R' \in \mathcal{R}^N$ be such that $R'_{N\backslash\{j\}} = R_{N\backslash\{j\}}$ and $p(R'_j) = \bar{p}(R_{N\backslash\{j\}})$. By Lemma 6 and the previous fact, $\varphi(R') = \varphi(R_{N\backslash\{j\}}) = \varphi(R)$. Therefore, without loss of generality, we may assume that $p(R_j) = \bar{p}(R)$.

Because $|\mathbb{P}| \geq 4$, $\underline{p}(R) = \underline{p}(\bar{R})$, $p(R_j) = \bar{p}(R)$, and $p(\bar{R}_j) = \bar{p}(\bar{R})$, there exist $l, k \in N$ such that $E(R_{N\backslash\{k\}}) = E(R) = E(R_{N\backslash\{l\}})$ and

$E(\bar{R}_{N\setminus\{k\}}) = E(\bar{R}) = E(\bar{R}_{N\setminus\{l\}})$. By Lemma 6, $\varphi(R_{N\setminus\{k\}}) = \varphi(R)$ and $\varphi(\bar{R}_{N\setminus\{k\}}) = \varphi(\bar{R})$. Thus, by Case 1, for all $i \in N\setminus\{j, k\}$,

$$\varphi(R) R_i \varphi(\bar{R}). \tag{7}$$

Similarly, by considering $N\setminus\{l\}$, it follows that for all $i \in N\setminus\{j, l\}$, $\varphi(R) R_i \varphi(\bar{R})$. The previous fact and (7) imply (6). ■

Applying Pareto efficiency, one-sided replacement-domination, and Lemma 5 yields the following.

LEMMA 9: *Let φ be a probabilistic rule satisfying Pareto efficiency and one-sided replacement-domination. For all $N \in \mathcal{P}$, all $j \in N$, and all $R, \bar{R} \in \mathcal{R}^N$ such that $R_{N\setminus\{j\}} = \bar{R}_{N\setminus\{j\}}$, if $E(R) = E(\bar{R})$, then $\varphi(R) = \varphi(\bar{R})$.*

Successive application of Lemma 9 yields the following.

LEMMA 10: *Pareto efficiency and one-sided replacement-domination imply anonymity.*

The following implication of Pareto efficiency and one-sided replacement-domination will be useful to finish the proofs of Lemmas 3 and 4.

LEMMA 11: *Let φ be a probabilistic rule satisfying Pareto efficiency and one-sided replacement-domination. For all $N \in \mathcal{P}$, all $j \in N$, and all $R, \bar{R} \in \mathcal{R}^N$ such that $R_{N\setminus\{j\}} = \bar{R}_{N\setminus\{j\}}$, if $E(R) \subsetneq E(\bar{R})$, then for all $i \in N\setminus\{j\}$, $\varphi(R) R_i \varphi(\bar{R})$.*

Proof: Without loss of generality, we suppose that $\underline{p}(R) = \underline{p}(\bar{R})$. Assume, by contradiction, that for all $i \in N\setminus\{j\}$, $\varphi(\bar{R}) R_i \varphi(R)$, and for some $l \in N\setminus\{j\}$, $\varphi(\bar{R}) P_l \varphi(R)$. Thus, for some $x \in \mathbb{R}$, $\varphi(\bar{R})(B(x, R_l)) > \varphi(R)(B(x, R_l))$. Without loss of generality, we suppose that $B(x, R_l) = [y, x]$. Hence,

$$\varphi(\bar{R})([y, x]) > \varphi(R)([y, x]). \tag{8}$$

Let $k \in N\setminus\{j\}$ be such that $p(R_k) = \underline{p}(R)$. We show that

$$\varphi(\bar{R}) P_k \varphi(R). \tag{9}$$

If $y \leq \underline{p}(R)$, then by Pareto efficiency, $p(R_k) = \underline{p}(R) = \underline{p}(\bar{R})$, and (8), $\varphi(\bar{R})(B(x, R_k)) > \varphi(R)(B(x, R_k))$. The previous fact and $\varphi(\bar{R}) R_k \varphi(R)$ imply (9).

If $\underline{p}(R) < y$, then, $\varphi(\bar{R})([\underline{p}(R), y[) = \varphi(\bar{R})(B^o(y, R_k)) \geq \varphi(R)(B^o(y, R_k)) = \varphi(R)([\underline{p}(R), y[)$, where the equalities follow from Pareto efficiency, and the inequality from $\varphi(\bar{R}) R_k \varphi(R)$. Therefore, $\varphi(\bar{R})([\underline{p}(R), y[) \geq \varphi(R)([\underline{p}(R), y[)$. This and (8) yields $\varphi(\bar{R})([\underline{p}(R), x]) > \varphi(R)([\underline{p}(R), x])$. So, by Pareto efficiency and $p(R_k) = \underline{p}(R) = \underline{p}(\bar{R})$, $\varphi(\bar{R})(B(x, R_k)) > \varphi(R)(B(x, R_k))$. The previous fact and $\varphi(\bar{R}) R_k \varphi(R)$ imply (9).

Let $h \in N\setminus\{k, j\}$ and $R' \in \mathcal{R}^N$ be such that $R'_{N\setminus\{h\}} = R_{N\setminus\{h\}}$ and $p(R'_h) = \bar{p}(R)$. Let $\bar{R}' \in \mathcal{R}^N$ be such that $\bar{R}'_{N\setminus\{h\}} = \bar{R}_{N\setminus\{h\}}$ and $\bar{R}'_h = R'_h$.

By Lemma 9, $\varphi(R') = \varphi(R)$ and $\varphi(\bar{R}') = \varphi(\bar{R})$. Since $R'_k = R_k = \bar{R}'_k$, then (9) and the previous two equalities imply

$$\varphi(\bar{R}')P_k\varphi(R'). \tag{10}$$

Then, $\varphi(\bar{R}')(B(\bar{p}(R'), R_k)) \geq \varphi(R')(B(\bar{p}(R'), R_k)) = \varphi(R')([\underline{p}(R'), \bar{p}(R')]) = 1$, where the inequality follows from (10) and the equalities from Pareto efficiency. Hence, $\varphi(\bar{R}')([\underline{p}(R'), \bar{p}(R')]) = 1$. Thus, by one-sided replacement-domination and (10), $\varphi(\bar{R}')P_k\varphi(R')$ and $\varphi(\bar{R}')R_h\varphi(R')$. Because $[p(R_k), p(R_h)] = [\underline{p}(R'), \bar{p}(R')]$, the three previous facts and Lemma 5 imply $\varphi(\bar{R}') = \varphi(R')$. Hence, $\varphi(R')R_k\varphi(\bar{R}')$, which contradicts (10). Therefore, assumption (8) was wrong, and it follows that for all $i \in N\backslash\{j\}$, $\varphi(R)R_i\varphi(\bar{R})$. ■

LEMMA 12: *Pareto efficiency and one-sided replacement-domination imply strategy-proofness.*

Proof: By Lemma 10, φ satisfies anonymity. Let $N \in \mathcal{P}$, $j \in N$, and $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$. We have to show that $\varphi(R)R_j\varphi(\bar{R})$. We consider four cases.

Case 1: $E(R) = E(\bar{R})$.

By Lemma 9, $\varphi(R) = \varphi(\bar{R})$. Thus, $\varphi(R)R_j\varphi(\bar{R})$, the desired conclusion.

Case 2: $E(R) \subsetneq E(\bar{R})$.

Without loss of generality, we suppose that $\underline{p}(R) = \underline{p}(\bar{R})$. Let $k \in N\backslash\{j\}$ be such that $p(R_k) = \underline{p}(R)$, and $l \in N\backslash\{k, j\}$. Let $R' \in \mathcal{R}^N$ be such that $R'_{N\backslash\{l\}} = R_{N\backslash\{l\}}$ and $R'_l = \bar{R}_j$. Thus, $E(R') = E(\bar{R})$ and $E(R) \subsetneq E(R')$. By Lemma 11, for all $i \in N\backslash\{l\}$, $\varphi(R)R_i\varphi(R')$. In particular, $\varphi(R)R_j\varphi(R')$. By anonymity and Lemma 9, $\varphi(R') = \varphi(\bar{R})$. Thus, $\varphi(R)R_j\varphi(\bar{R})$, the desired conclusion.

Case 3: $E(R) \supsetneq E(\bar{R})$.

Without loss of generality, we suppose that $\underline{p}(R) = \underline{p}(\bar{R})$. Let $k \in N\backslash\{j\}$ be such that $p(R_k) = \underline{p}(R)$. By Lemma 11, $\varphi(\bar{R})R_k\varphi(R)$. Thus, for all $x \in [\underline{p}(\bar{R}), \bar{p}(\bar{R})]$,

$$\varphi(\bar{R})([\underline{p}(\bar{R}), x]) \geq \varphi(R)([\underline{p}(\bar{R}), x]). \tag{11}$$

By Pareto efficiency, $\varphi(R)([\underline{p}(R), \bar{p}(R)]) = 1$. Because $\underline{p}(R) = \underline{p}(\bar{R})$ and $\bar{p}(\bar{R}) < \bar{p}(R)$, (11) implies that for all $x \in [\underline{p}(R), \bar{p}(R)]$, $\varphi(R)(]x, \bar{p}(R)]) \geq \varphi(\bar{R})(]x, \bar{p}(R)])$. Hence, by Pareto efficiency and $p(R_j) = \bar{p}(R)$, for all $x \in [\underline{p}(R), \bar{p}(R)]$, $\varphi(R)(B^o(x, R_j)) \geq \varphi(\bar{R})(B^o(x, R_j))$. Thus, $\varphi(R)R_j\varphi(\bar{R})$, the desired conclusion.

Case 4: $E(R)\backslash E(\bar{R}) \neq \emptyset$ and $E(\bar{R})\backslash E(R) \neq \emptyset$.

Without loss of generality, we suppose that $p(R_j) > \bar{p}(R_{N\backslash\{j\}})$. Thus, $p(\bar{R}_j) < \underline{p}(R_{N\backslash\{j\}})$. Let $R' \in \mathcal{R}^N$ be such that $R'_{N\backslash\{j\}} = R_{N\backslash\{j\}}$ and $p(R'_j) = \bar{p}(R_{N\backslash\{j\}})$. By Case 3, $\varphi(R)R_j\varphi(R')$. By Case 2,

$\varphi(R')R'_j\varphi(\bar{R})$. By Pareto efficiency, for all $x \in [\underline{p}(R'), \bar{p}(R')]$, $\varphi(R')(B(x, R'_j)) = \varphi(R')([x, p(R'_j)]) = \varphi(R')(B(x, R_j))$. Therefore, it follows that for all $x \in [\underline{p}(R), \bar{p}(R)]$, $\varphi(R)(B(x, R_j)) \geq \varphi(\bar{R})(B(x, R_j))$. Hence, $\varphi(R)R_j\varphi(\bar{R})$, the desired conclusion. ■

Thus, Lemmas 7, 8, and 12 prove Lemma 3 and Lemmas 10 and 12 prove Lemma 4.

Proofs of Theorems 2 and 3

Given $N \in \mathcal{P}$, we show that the class of probabilistic target rules that is restricted to the domain $\mathcal{R}^N$ is characterized by Pareto efficiency and one-sided replacement-domination on $\mathcal{R}^N$. Theorem 4 also proves Remark 3.

THEOREM 4: *Let φ be a probabilistic rule and $N \in \mathcal{P}$. Then φ satisfies Pareto efficiency and one-sided replacement-domination on $\mathcal{R}^N$ if and only if there exists a distribution Q^N over $\mathbb{R} \cup \{-\infty, +\infty\}$ such that for all $R \in \mathcal{R}^N$, $\varphi(R) = \varphi^{Q^N}(R)$.*

Proof: Let φ be a probabilistic rule satisfying Pareto efficiency and one-sided replacement-domination. Let $N \in \mathcal{P}$. By Lemmas 10 and 12, φ satisfies strategy-proofness and anonymity on $\mathcal{R}^N$. Without loss of generality, we assume that $N = \{1, \ldots, n\}$. Therefore, by Theorem 1, φ is characterized on $\mathcal{R}^N$ by $(n-1)$ fixed distributions $D_1^N, \ldots, D_{n-1}^N$ over $\mathbb{R} \cup \{-\infty, +\infty\}$. By Remark 2 it is sufficient to show that $D_1^N = \cdots = D_{n-1}^N =: Q^N$.

Suppose that there exist some $l, t \in \{1, \ldots, n-1\}$ such that $D_l^N \neq D_t^N$. Since the Borel σ-algebra on $\mathbb{R} \cup \{-\infty, +\infty\}$ is generated by all intervals $[x, +\infty]$, $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $D_l^N([y, +\infty]) \neq D_t^N([y, +\infty])$. Without loss of generality, we assume that $l < t$. Let $R_1, R_n \in \mathcal{R}$ be such that $p(R_1) = y - 1$ and $p(R_n) = y$. By $R \in \mathcal{R}^N$ we denote the profile such that for all $i \in \{1, \ldots, l\}$, $R_i = R_1$, and for all $i \in \{l+1, \ldots, n\}$, $R_i = R_n$. By $\bar{R} \in \mathcal{R}^N$ we denote the profile such that for all $i \in \{1, \ldots, t\}$, $\bar{R}_i = R_1$, and for all $i \in \{t+1, \ldots, n\}$, $\bar{R}_i = R_n$. By Theorem 1, equality (5),

$$\varphi(R)(\{y\}) = D_l^N([y, +\infty]) \neq D_t^N([y, +\infty]) = \varphi(\bar{R})(\{y\}). \tag{12}$$

Notice that $\bar{R}$ can be obtained from changing successively $(t - l)$ agents' preference relations from R_1 to R_n. Since at these unilateral deviations the Pareto set does not change, Lemma 10 implies $\varphi(R) = \varphi(\bar{R})$, which contradicts (12). ■

Theorem 4 and the following proposition yield the proof of Theorem 2.

PROPOSITION 1: *Let $N \in \mathcal{P}$ and Q be a distribution over $\mathbb{R} \cup \{-\infty, +\infty\}$. If φ^Q satisfies replacement-domination on $\mathcal{R}^N$, then Q places probability 1 on a single point in $\mathbb{R} \cup \{-\infty, +\infty\}$.*

Proof: Suppose that Q does not place probability 1 on a single level in $\mathbb{R} \cup \{-\infty, +\infty\}$. Then there exist $y, x \in \mathbb{R}$ such that

$$y < x,\ Q([-\infty, y]) > 0 \text{ and } Q([x, +\infty]) > 0. \tag{13}$$

We show that φ^Q violates replacement-domination on $\mathcal{R}^N$.

Without loss of generality, we assume that $\{1,2,3\} \subseteq N$. Let $R \in \mathcal{R}^N$ be such that $p(R_1) = \frac{1}{2}(y + x)$, $p(R_2) = \frac{1}{4}y + \frac{3}{4}x$, $p(R_3) = x$, and for all $i \in N\backslash\{1,2,3\}$, $R_i = R_1$. Let $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{N\backslash\{3\}} = R_{N\backslash\{3\}}$ and $p(\bar{R}_3) = y$. By definition of φ^Q and (13),

$$\begin{aligned}\varphi^Q(R)(\{\tfrac{1}{4}y + \tfrac{3}{4}x\}) &= Q(\{\tfrac{1}{4}y + \tfrac{3}{4}x\}) < Q([\tfrac{1}{4}y + \tfrac{3}{4}x, +\infty]) \\ &= \varphi^Q(\bar{R})(\{\tfrac{1}{4}y + \tfrac{3}{4}x\})\end{aligned}$$

and

$$\begin{aligned}\varphi^Q(R)(\{\tfrac{1}{2}(y + x)\}) &= Q([-\infty, \tfrac{1}{2}(y + x)]) > Q(\{\tfrac{1}{2}(y + x)\}) \\ &= \varphi^Q(\bar{R})(\{\tfrac{1}{2}(y + x)\}).\end{aligned}$$

Since $p(R_1) = \frac{1}{2}(y + x)$ and $p(R_2) = \frac{1}{4}y + \frac{3}{4}x$, the previous two inequalities contradict replacement-domination. ■

Finally, we prove Theorem 3 by using Theorem 4.

Proof of Theorem 3: Let φ be a probabilistic rule satisfying Pareto efficiency and population-monotonicity. By Lemma 8, φ satisfies one-sided replacement-domination. Hence, by Theorem 4, for each $N \in \mathcal{P}$ there exists a distribution Q^N such that for all $R \in \mathcal{R}^N$, $\varphi(R) = \varphi^{Q^N}(R)$.

We show that for all $N, M \in \mathcal{P}$, $Q^N = Q^M$. Let $x \in \mathbb{R}$, $R^1 \in \mathcal{R}^N$, $R^2 \in \mathcal{R}^M$, and $R^3 \in \mathcal{R}^{N \cup M}$ be such that $R^3_N = R^1$, $R^3_M = R^2$, and $E(R^1) = E(R^2) = E(R^3) = [x - 1, x]$.

Successive application of Lemma 6 yields $\varphi(R^3) = \varphi(R^3_N) = \varphi(R^1)$ and $\varphi(R^3) = \varphi(R^3_M) = \varphi(R^2)$. Thus, $\varphi(R^1) = \varphi(R^2)$. Therefore,

$$\begin{aligned}Q^N([x, +\infty]) &= \varphi^{Q^N}(R^1)(\{x\}) = \varphi(R^1)(\{x\}) = \varphi(R^2)(\{x\}) \\ &= \varphi^{Q^M}(R^2)(\{x\}) = Q^M([x, +\infty]),\end{aligned}$$

where the first and the last equality follow from (5). Hence, $Q^N([x, +\infty]) = Q^M([x, +\infty])$. Since x was arbitrary, it follows that for all $x \in \mathbb{R}$, $Q^N([x, +\infty]) = Q^M([x, +\infty])$. Because the Borel σ-algebra on $\mathbb{R} \cup \{-\infty, +\infty\}$ is generated by all intervals $[x, +\infty]$, $x \in \mathbb{R}$, it follows that $Q^N = Q^M$. Hence, $\varphi = \varphi^Q$ where $Q := Q^N = Q^M$. ■

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