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Lifting valid inequalities for the precedence constrained knapsack problem

Received July 10, 1997 / Revised version received January 9, 1999
Published online May 12, 1999

Abstract. This paper considers the precedence constrained knapsack problem. More specifically, we are interested in classes of valid inequalities which are facet-defining for the precedence constrained knapsack polytope. We study the complexity of obtaining these facets using the standard sequential lifting procedure. Applying this procedure requires solving a combinatorial problem. For valid inequalities arising from minimal induced covers, we identify a class of lifting coefficients for which this problem can be solved in polynomial time, by using a supermodular function, and for which the values of the lifting coefficients have a combinatorial interpretation. For the remaining lifting coefficients it is shown that this optimization problem is strongly NP-hard. The same lifting procedure can be applied to (1,k)-configurations, although in this case, the same combinatorial interpretation no longer applies. We also consider K-covers, to which the same procedure need not apply in general. We show that facets of the polytope can still be generated using a similar lifting technique. For tree knapsack problems, we observe that all lifting coefficients can be obtained in polynomial time. Computational experiments indicate that these facets significantly strengthen the LP-relaxation.

1. Introduction

The knapsack problem is a classical problem in combinatorial optimization. In this problem, there is a set V of items, $V = \{1, 2, \dots, n\}$ and a knapsack with capacity $b \in \mathbb{Z}^+$. Each item $i \in V$ has a value $c_i \in \mathbb{Z}$, and a weight $w_i \in \mathbb{Z}^+$. The problem is to find a maximum value subset of the set of items whose total weight does not exceed the knapsack capacity. This is modeled in the following integer linear programming formulation:

$$\max \quad \sum_{i=1}^n c_i x_i \quad (1)$$

$$\text{s.t.} \quad \sum_{i=1}^n a_i x_i \leq b \quad (2)$$

$$x_i \in \{0, 1\} \quad i = 1, \dots, n \quad (3)$$

The knapsack problem has received considerable attention, not only because it has several important applications in itself, but also because it arises as a substructure in many combinatorial problems.

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Mathematics Subject Classification (1991): 90C10

This paper studies the precedence constrained knapsack problem, which generalizes the knapsack problem by including a partial order on the items. We say that there is a precedence constraint *from* item i *to* item j if item j can be included in the knapsack only if item i is included. Thus, $x_i = 0 \Rightarrow x_j = 0$. The set of precedence constraints can be represented by a directed graph $D(V, A)$, where the node set V is the set of items, and each precedence constraint is represented by a directed arc in A . The precedence constraints are given by

$$x_i \geq x_j \quad (i, j) \in A \quad (4)$$

The precedence constrained knapsack problem (PCKP) is now formulated by (1)-(4). In this paper, we are interested in facet-defining inequalities for the precedence constrained knapsack polytope, and more specifically, the complexity of obtaining these facets using lifting techniques. The remainder of this section discusses PCKP, the literature on its polyhedral structure, and the contributions of the present paper.

As is the case for the ordinary knapsack problem, PCKP is an interesting problem in itself, which also arises naturally as a substructure in several other combinatorial problems. Consider for instance a tool management problem that arises in automated manufacturing, where each part requires a specific set of tools in order to be processed. Hence, a part can only be processed on a machine if the required tools are loaded. In our model this would correspond to a precedence constraint from tool i to job j if tool i is required to process job j . The knapsack constraint stems from the limited capacity of the tool magazine. Crama [8] and Stecke and Kim [19] discuss several problems containing both knapsack and precedence constraints in the context of tool management, and provide pointers to literature on related combinatorial problems, of which we mention only a few here. Mamer and Shogan [14] and Hwan and Shogan [12] consider capital constrained repair kit selection problems, which also have both knapsack and precedence constraints. A similar formulation arises in strip mining where digging in lower layers is impossible without digging in higher layers (see Johnson and Niemi [13]). Shaw [17] and Cho and Shaw [7] employ a decomposition technique to tackle network design problems, for which the precedence constrained knapsack problem arises as a subproblem (see Cho and Shaw [6] and Shaw and Cho [18]).

Garey and Johnson [10] prove that the decision version of PCKP is NP-complete in the strong sense, but solvable in pseudopolynomial time if the underlying precedence graph is a tree (see Johnson and Niemi [13] for a dynamic programming algorithm). Hence, in order to solve the general PCKP to optimality, a further understanding of the structure of the precedence constrained knapsack polytope can be expected to accelerate general integer programming schemes, as it has led to more powerful exact solution methods for ordinary knapsack problems (see Crowder, Johnson and Padberg [9]). For polyhedral results on the ordinary knapsack problem we refer to Balas [3], Balas and Zemel [4] and Zemel [21]. Hartvigsen and Zemel [11] discuss the complexity of the recognition of (lifted) valid knapsack inequalities.

As is observed by Boyd [5], problems which are defined entirely by precedence constraints can be solved using standard LP-techniques, since a set of precedence constraints itself defines a totally unimodular matrix and hence, a polyhedron with integral vertices. For the PCKP, Boyd [5] analyzes two classes of valid inequalities

arising from K -covers and $(1,k)$ -configurations. He identifies conditions under which these inequalities define facets of a lower dimensional polytope, in which case lifting may lead to a facet of the precedence constrained knapsack polytope itself. Park and Park [16] consider a special case of K -covers which they refer to as minimal induced covers. In general, inequalities arising from minimal induced covers will not define facets of the precedence constrained knapsack polytope. Park and Park [16] consider a lifting technique to obtain valid inequalities. The reader may observe that all of the aforementioned classes of valid inequalities are natural extensions of classes of inequalities for the ordinary knapsack problem.

In this paper, we present various new results on facets of the PCKP-polytope. In Section 3 we state a class of lifting orders which guarantees that valid inequalities of the PCKP-polytope can be lifted to obtain facet-defining inequalities for the polytope, using a standard sequential lifting procedure. For valid inequalities arising from minimal induced covers and $(1,k)$ -configurations we identify the lower dimensional polytope for which valid inequalities arising from minimal induced covers and $(1,k)$ -configurations are facet-defining. We specifically consider valid inequalities arising from minimal induced covers; the variables for which lifting coefficients have yet to be determined are partitioned into two classes. For one of these classes, we establish a relation between the lifting coefficients and the number of components in two related subgraphs of D . Based on this characterization, these lifting coefficients can be seen to be computable in polynomial time. For the second class of lifting coefficients however, we prove that their computation is strongly NP-hard. A special case of the PCKP which has received considerable attention is the tree knapsack problem, in which the underlying precedence graph is a tree. Aghezzaf, Magnanti and Wolsey [2] for instance, study the polyhedral structure of the problem. Our results allow for more general graph structures and extend their findings. Moreover, our results easily imply that, for the tree knapsack problem, all lifting coefficients can be obtained in polynomial time. In Section 4 we consider valid inequalities arising from K -covers. For these valid inequalities standard sequential lifting techniques cannot always be applied. We show that by applying a related lifting procedure facets of the PCKP-polytope can still be obtained. To illustrate the effect of lifted inequalities and their applicability in integer solution procedures, we report our computational results in Section 5.

2. Notation

Throughout this paper, the following definitions and notation will turn out to be convenient. For $(i, j) \in A$, item i is called a *predecessor* of item j and item j is called a *successor* of item i . For all $W \subseteq V$, we denote by $F(W) = \{j \in V \setminus W \mid \exists i_1 \in W : (i_1, j) \in A, \exists i_2 \in W : (j, i_2) \in A\}$ the set of elements in $V \setminus W$ which are both a successor of an element in W and a predecessor of an element in W , by $P(W) = \{j \in V \setminus W \mid \exists i \in W : (j, i) \in A\} \setminus F(W)$ the set of predecessors of a set W excluding items in W and $F(W)$, by $T(W) = W \cup P(W) \cup F(W)$ the set of predecessors of set W including W , and by $R(W) = V \setminus T(W)$ the set of remaining items (variables). For ease of exposition, $P(\{i\})$, $T(\{i\})$ and $R(\{i\})$ will be denoted $P(i)$, $T(i)$ and $R(i)$, respectively. Furthermore, for all $W \subseteq V$ define $a(W) = \sum_{i \in W} a_i$. Note that if $(i, j) \in A$

and $(j, k) \in A$, then, by transitivity of the precedence relations, (i, k) can be assumed to be an element of A . Arcs in A induced by transitivity will be omitted in the figures. Moreover, arcs (i, j) are depicted downward.

The following two assumptions can be made without loss of generality.

Assumption 1. *The directed graph D is acyclic.*

If D contains a cycle, nodes (variables) in this cycle must either all be included in, or all be excluded from the knapsack. Hence, the cycle can be contracted into a single node, with cumulative value and weight coefficient.

Assumption 2. *$a(T(i)) \leq b$, for all $i = 1, \dots, n$.*

This simply implies that for every item i there exists a feasible solution in which it is included in the knapsack. Items violating this assumption can be deleted from the problem instance.

3. Minimal induced covers and (1,k)-configurations

In the literature on the polyhedral structure of the knapsack problem, minimal cover inequalities have been investigated (see for instance Balas [3], Balas and Zemel [4]). In order to generalize these concepts to PCKP, we must take into account that if an item i is included in the knapsack, so must all the items in $T(i)$. In Subsection 3.1 we therefore consider a straightforward generalization of minimal covers, the so-called minimal *induced* cover (see Park and Park [16]). We show how valid inequalities for the precedence constrained knapsack polytope can be strengthened using standard sequential lifting techniques. Different lifting orders are discussed, and we derive sufficient conditions for classes of valid inequalities under which lifting leads to facet-defining inequalities for the PCKP-polytope.

Given a minimal induced cover $C \subseteq V$, we give a combinatorial characterization of the value of the lifting coefficients for the variables in $P(C)$ in Subsection 3.2. This leads to the conclusion that these values can be computed in polynomial time. Subsection 3.3 shows that the computation of lifting coefficients for variables in $R(C)$ is, in general, strongly NP-hard, but solvable in polynomial time in the special case where the underlying precedence graph is a tree. Finally, Subsection 3.4 concludes this section with an illustrative example.

3.1. Generic sequential lifting

Item $i \in V$ and $j \in V$ are called *incomparable* if both $(i, j) \notin A$ and $(j, i) \notin A$. A set $W \subseteq V$ is called *incomparable* if the elements in W are pairwise incomparable. Note that if W is incomparable then $F(W) = \emptyset$.

Definition 1. $C \subseteq V$ is a minimal induced cover (MIC) if

- C is incomparable
- $a(T(C)) > b$
- $a(T(C) \setminus \{i\}) \leq b$, for all $i \in C$

In words, a minimal induced cover is a set of incomparable items, which together do not fit in the knapsack, whereas all but one of them do fit in the knapsack together. The above definition follows the work of Boyd [5]. An alternative definition would be to replace $a(T(C) \setminus \{i\}) \leq b$, for all $i \in C$ by $a(T(C \setminus \{i\})) \leq b$, for all $i \in C$. In fact, the latter inequality appears in the definition of minimal induced covers of Park and Park [16]. However, on close inspection it can be verified that the results of Park and Park [16] are derived under conditions for which the two definitions coincide. Since our results are only applicable under the current definition, we follow the original definition of Boyd [5].

For $C \subseteq V$ a MIC, the following inequality is valid:

$$\sum_{i \in C} x_i \leq |C| - 1 \quad (5)$$

We refer to this inequality as the *MIC-inequality*.

We define X to be the set of feasible solutions of the PCKP, and $\text{conv}(X)$ to be the convex hull of the set X . Furthermore, for any $W_1, W_2 \subseteq V$ such that $W_2 \cap T(W_1) = \emptyset$, we define the subset $X_{W_1|W_2} = X \cap \{x \in \{0, 1\}^n \mid x_i = 1, \text{ for } i \in W_1, \text{ and } x_i = 0, \text{ for } i \in W_2\}$. For $W \subseteq V$, we denote by x^W the characteristic vector of W , that is, $x_i^W = 1$, if $i \in W$, and $x_i^W = 0$, otherwise.

Proposition 1. (see Boyd [5]) *The dimension of $\text{conv}(X)$ is $|V|$.*

Proof. The vectors $x^{T(i)}, i = 1, \dots, |V|$ together with the zero vector give $|V| + 1$ affinely independent vectors in $\text{conv}(X)$ (note that we use Assumption 2 here). □

Proposition 2. *Let $C \subseteq V$ be a MIC. Then (5) is facet-defining for $\text{conv}(X_{P(C)|R(C)})$.*

Proof. By Proposition 1, the dimension of $\text{conv}(X_{P(C)|R(C)})$ is $|C|$. We specify $|C|$ affinely independent vectors in $\text{conv}(X_{P(C)|R(C)})$ satisfying (5) at equality. Let $\theta^j = x^{T(C) \setminus \{j\}}$, for all $j \in C$. It can easily be checked that the vectors $\theta^j, j \in C$ satisfy the inequality at equality and are affinely independent. □

Proposition 2 enables us to lift the variables in $P(C)$ and $R(C)$ into the MIC-inequality using the following technique (see Nemhauser and Wolsey [15]). Let $B^n = \{0, 1\}^n$. For some j , suppose $Y \subseteq B^n$, $Y^0 = Y \cap \{x \in B^n \mid x_j = 0\}$, and $Y^1 = Y \cap \{x \in B^n \mid x_j = 1\}$. If the inequality

$$\sum_{i \neq j} \alpha_i x_i \leq \alpha_0 \quad (6)$$

is facet-defining for $\text{conv}(Y^1)$ and $Y^0 \neq \emptyset$, then

$$\sum_{i \neq j} \alpha_i x_i + \alpha_j (1 - x_j) \leq \alpha_0 \quad (7)$$

is facet-defining for $\text{conv}(Y)$ if

$$\alpha_j = \alpha_0 - \max_{x \in Y^0} \left\{ \sum_{i \neq j} \alpha_i x_i \right\}. \quad (8)$$

Similarly, if (6) defines a facet for $\text{conv}(Y^0)$ and $Y^1 \neq \emptyset$, then

$$\sum_{i \neq j} \alpha_i x_i + \alpha_j x_j \leq \alpha_0 \quad (9)$$

defines a facet of $\text{conv}(Y)$ if

$$\alpha_j = \alpha_0 - \max_{x \in Y^1} \left\{ \sum_{i \neq j} \alpha_i x_i \right\}. \quad (10)$$

In order to apply this lifting technique repeatedly to MIC-inequalities, we have to be careful with the order in which the variables are lifted. Otherwise we might, at some point, violate the condition $Y^0 \neq \emptyset$ when applying (8) or the condition $Y^1 \neq \emptyset$ when applying (10).

Definition 2. For $W \subseteq V$, π is called a PFRS-order (predecessors first, remaining variables second) for W if π is a one-to-one mapping $\pi : P(W) \cup R(W) \rightarrow \{1, \dots, |P(W) \cup R(W)|\}$ satisfying the following conditions:

- (i) $\pi(i) < \pi(j)$ if $i \in P(W)$, $j \in R(W)$
- (ii) $\pi(i) < \pi(j)$ if $i, j \in P(W)$ and $j \in P(i)$ (reversed topological ordering on $P(W)$)
- (iii) $\pi(i) < \pi(j)$ if $i, j \in R(W)$ and $i \in P(j)$ (topological ordering on $R(W)$)

Note that under Assumption 1 such an order always exists. Given a MIC $C \subseteq V$ and a PFRS-order π for C , for all elements $j \in P(C) \cup R(C)$ we define

$$\begin{aligned} p^\pi(j) &= \{i \in P(C) \cup R(C) \mid \pi(i) < \pi(j)\} && \text{(predecessors of } j \text{ in order } \pi) \\ s^\pi(j) &= \{i \in P(C) \cup R(C) \mid \pi(i) > \pi(j)\} && \text{(successors of } j \text{ in order } \pi) \end{aligned}$$

During the lifting process, variables in $P(C) \cup R(C)$ are lifted sequentially, and hence, the lifting problem for a variable $j \in P(C)$ is defined as follows:

Given a MIC $C \subseteq V$ and its associated MIC inequality

$$\sum_{i \in C} x_i + \sum_{i \in P(C) \cap p^\pi(j)} \alpha_i (1 - x_i) \leq |C| - 1 \quad (11)$$

which is valid for $\text{conv}(X)$, determine

$$\alpha_j = |C| - 1 - \max_{\substack{x \in X_{P(C) \cap s^\pi(j) | R(C)} \\ x_j=0}} \left\{ \sum_{i \in C} x_i + \sum_{i \in P(C) \cap p^\pi(j)} \alpha_i (1 - x_i) \right\}. \quad (12)$$

Likewise, for a variable $j \in R(C)$ the lifting problem is defined as follows:
Given a MIC $C \subseteq V$ and an inequality

$$\sum_{i \in C} x_i + \sum_{i \in P(C)} \alpha_i (1 - x_i) + \sum_{i \in R(C) \cap p^\pi(j)} \alpha_i x_i \leq |C| - 1 \quad (13)$$

which is valid for X , determine

$$\alpha_j = |C| - 1 - \max_{\substack{x \in X_{\emptyset | R(C) \cap s^\pi(j)} \\ x_j=1}} \left\{ \sum_{i \in C} x_i + \sum_{i \in P(C)} \alpha_i (1 - x_i) + \sum_{i \in R(C) \cap p^\pi(j)} \alpha_i x_i \right\}. \quad (14)$$

Theorem 1. *Let*

- $C \subseteq V$ be a MIC with its corresponding valid inequality (5)
- π be a PFRS-order for C .
- lifting coefficients for variables in $P(C)$ be determined according to (12)
- lifting coefficients for variables in $R(C)$ be determined according to (14)

then the resulting inequality

$$\sum_{i \in C} x_i + \sum_{i \in P(C)} \alpha_i (1 - x_i) + \sum_{i \in R(C)} \alpha_i x_i \leq |C| - 1 \quad (15)$$

defines a facet of the PCKP-polytope $\text{conv}(X)$.

Proof. We construct $|V|$ affinely independent vectors in $\text{conv}(X)$, satisfying the inequality at equality. For $j \in C$, let θ^j be defined as in the proof of Proposition 2. For all $j \in P(C)$, let θ^j be the vector for which the maximum in (12) is attained. W.l.o.g. assume that $\theta_i^j = 1$ for all $i \in P(C) \cap s^\pi(j)$, and $\theta_i^j = 0$, for all $i \in R(C)$. Likewise, for all $j \in R(C)$, let θ^j be the vector for which the maximum in (14) is attained. W.l.o.g., assume that $\theta_i^j = 0$, for all $i \in R(C) \cap s^\pi(j)$. Then it is easy to verify that the vectors θ^j , $j \in V$, satisfy the inequality at equality, and moreover, are affinely independent. \square

In order for the lifting procedure that consists of repeatedly solving (12) and (14) to be applicable, the maximum that is taken in (12) and (14) has to be well defined. This is not the case, if the subset over which the maximum is taken is empty. We conclude that the procedure is only valid if, at each iteration, the subset is nonempty. In the current framework, of lifting MIC inequalities, this property is ascertained by the

ordering conditions stated in Definition 2. This result, of course, can be generalized to wider classes of valid inequalities, which may even be defined on subsets which need not necessarily be incomparable.

Theorem 2. *Let $W \subseteq V$, and let $\alpha^T x \leq \alpha_0$, where $\alpha_j = 0$ for $j \notin W$, be a facet-defining inequality for $\text{conv}(X_{P(W)|R(W)})$. Then, lifting the variables in $P(W)$ and $R(W)$ as specified in (8) and (10) in PFRS-order yields a facet of the PCKP-polytope $\text{conv}(X)$.*

Proof. Let $W \subseteq V$. If $W = \emptyset$, then $X_{P(W)|R(W)} = X$, no lifting has to be done, and hence the theorem obviously holds. So assume $W \neq \emptyset$. As we are given a facet-defining inequality, $X_{P(W)|R(W)} \neq \emptyset$, which implies that the vector $x^{P(W)} \in X$ (i.e. $a(P(W)) \leq b$). Now it only remains to prove that at every step of the lifting process the subset on which the maximum in (8) and (10) is defined is nonempty. Under a PFRS-order π , when lifting $j \in P(W)$, since $a(P(W) \cap s^\pi(j)) \leq a(P(W)) \leq b$, the vector $x^{P(W) \cap s^\pi(j)}$ is in the corresponding subset. When lifting $j \in R(W)$, the vector $x^{T(j)}$ is in the subset at hand. □

The conditions on the PFRS-order in Definition 2 are such that at each step of the lifting process the variables which are fixed do not violate the precedence constraints, and the variables which are fixed to one do not violate the knapsack constraint. Instead of considering a PFRS-order in which all elements in $P(W)$ are lifted before elements in $R(W)$, we might also allow for more general lifting orders, in which an element in $R(W)$ can be lifted before all predecessors in $P(W)$ are lifted. The existence of such an order is again guaranteed by Assumption 1. Next, we derive necessary and sufficient conditions for which this class of more general orders yields facet-defining inequalities.

Definition 3. *For $W \subseteq V$, π is called a valid order for W if π is a one-to-one mapping $\pi : P(W) \cup R(W) \rightarrow \{1, \dots, |P(W) \cup R(W)|\}$ satisfying the following conditions:*

- (i) $\pi(i) < \pi(j)$ if $i, j \in P(W) \setminus W$ and $j \in P(i)$ (reversed topological ordering on $P(W)$)
- (ii) $\pi(i) < \pi(j)$ if $i, j \in R(W) \setminus W$ and $i \in P(j)$ (topological ordering on $R(W)$)

Theorem 3. *Let $W \subseteq V$, and let $\alpha^T x \leq \alpha_0$, where $\alpha_j = 0$ for $j \notin W$, be a facet-defining inequality for $\text{conv}(X_{P(W)|R(W)})$. Let π be a valid order for W . Let the lifting coefficients of the variables in $P(W)$ and $R(W)$ be determined as in (8) and (10). Then the resulting inequality is facet-defining for $\text{conv}(X)$ if and only if $a((P(W) \cap s_j^\pi) \cup T(j)) \leq b$, for each $j \in R(W)$.*

Proof. Let $W \subseteq V$. If $W = \emptyset$, then $X_{P(W)|R(W)} = X$, no lifting has to be done, and hence the theorem obviously holds. So assume $W \neq \emptyset$. Using inductive arguments, when lifting a variable x_j we are given a facet-defining inequality for the polytope $\text{conv}(X_{P(W) \cap s_j^\pi | R(W) \cap s_j^\pi})$. If $j \in P(W)$ then by nonemptiness of the above polytope and the definition of a valid order, the subset on which the maximum as in (8) is defined is nonempty since the vector $x^{P(W) \cap s^\pi(j)}$ is in the corresponding polytope. If $j \in R(W)$, then the condition on the weights as mentioned in the theorem guarantees that the subset on which the maximum is defined as in (10) is nonempty since the vector

$x^{(P(W) \cap s^\pi(j)) \cup T(j)}$ is in the corresponding polytope. Conversely, assume the condition is not satisfied for a $j \in R(W)$. When lifting the corresponding variable, the subset on which the maximum is defined is empty.

□

It is easy to see that conditions (i) and (ii) in Definition 3 cannot be removed since then immediately the subset on which the maximum is defined in the lifting procedure will become empty. Hence the class of valid orders in Definition 3 is the most general class of orders which can be considered for the standard lifting procedure as defined by (8) and (10). Note that the above result paves the way for lifting other classes of valid inequalities. Here, we mention (1,k)-configurations and state the polytope for which the corresponding valid inequality is facet-defining.

Definition 4. Let

- $C \cup \{t\} \subseteq V$ be incomparable, with $t \notin C$
- $C \cup \{t\}$ be a cover and $a(T(C \cup \{t\}) \setminus \{t\}) \leq b$
- $Q \cup \{t\}$ be a minimal (induced) cover, $\forall Q \subseteq C$ with $|Q| = k$ where $2 \leq k \leq |C|$

then $C \cup \{t\}$ is called a (1,k)-configuration.

For a (1,k)-configuration the following inequalities are valid:

$$(r - k + 1)x_t + \sum_{i \in Z(r)} x_i \leq r \quad (16)$$

where r is such that $k \leq r \leq |C|$ and $Z(r)$ is any subset of C , with $|Z(r)| = r$.

Proposition 3. Let $C \cup \{t\} \subseteq V$ be a (1,k)-configuration, and let $Z(r)$ be any subset of C , with $|Z(r)| = r$. Then, (16) is facet-defining for $\text{conv}(X_{P(C \cup \{t\})|R(C \cup \{t\}) \cup (C \setminus Z(r))})$.

Proof. By Proposition 1, $\text{conv}(X_{P(C \cup \{t\})|R(C \cup \{t\}) \cup (C \setminus Z(r))})$ has dimension $r + 1$. Hence, to prove the proposition, we construct $r + 1$ affinely independent vectors in the polytope satisfying the inequality at equality. We assume without loss of generality that the elements are numbered such that $1, \dots, r$ denote the elements in $Z(r)$ and $r + 1$ denotes element t .

For $i = 1, \dots, r - k + 2$, let y^i be defined by

$$y_j^i = \begin{cases} 1 & j \in \{i, \dots, i + k - 2\} \\ 0 & j \in \{1, \dots, r\} \setminus \{i, \dots, i + k - 2\} \\ 1 & j = r + 1 \\ 1 & j \in P(C \cup \{t\}) \\ 0 & j \in R(C \cup \{t\}) \cup (C \setminus Z(r)) \end{cases}$$

For $i = r - k + 3, \dots, r$, let y^i be defined by

$$y_j^i = \begin{cases} 1 & j \in \{i, \dots, r\} \cup \{1, \dots, i + k - r - 2\} \\ 0 & j \in \{i + k - r - 1, \dots, i - 1\} \\ 1 & j = r + 1 \\ 1 & j \in P(C \cup \{t\}) \\ 0 & j \in R(C \cup \{t\}) \cup (C \setminus Z(r)) \end{cases}$$

and define y^{r+1} as

$$y^{r+1} = \begin{cases} 0 & j = r + 1 \\ 1 & j \in Z(r) \\ 1 & j \in P(C \cup \{t\}) \\ 0 & j \in R(C \cup \{t\}) \cup (C \setminus Z(r)) \end{cases}$$

Then it can easily be verified that the vectors $y^i, i = 1, \dots, r + 1$ are affinely independent.

□

Applying Theorem 2 or 3 now yields that the elements of $V \setminus (Z(r) \cup \{t\})$ can now be lifted in PFRS-order or valid order so as to obtain a facet of $\text{conv}(X)$.

3.2. Lifting predecessors of a minimal induced cover using a PFRS-order

In general, calculating α_j by (8) or (10) requires solving a difficult maximization problem. In fact, for PCKP, the optimization problems in (12) and (14) are in turn PCKP problems. In this Subsection we show that the lifting problem of predecessors under a PFRS-order has a combinatorial interpretation that leads to an algorithm that solves the lifting problem in polynomial time. In contrast, in Subsection 3.3, it is shown that the lifting problem for the remaining variables is strongly NP-hard.

Definition 5. Let $C \subseteq V$ be a MIC. For $W \subseteq P(C)$ let $f(W)$ be the number of (weak) components in the subgraph of G induced by $W \cup C$.

Lemma 1. f is supermodular: for all $W_1 \subseteq W_2 \subseteq P(C)$ and $i \in P(C) \setminus W_2$ it holds that

$$f(W_1 \cup \{i\}) - f(W_1) \leq f(W_2 \cup \{i\}) - f(W_2)$$

Proof. Choose $W_2 \subseteq P(C)$ and $W_1 \subseteq W_2$ arbitrarily. Notice first that, since we only consider $W_1 \subseteq P(C)$ and $W_2 \subseteq P(C)$, each component of a subgraph induced by $W_1 \cup C$ or $W_2 \cup C$ contains at least one element $c \in C$. Further, since $W_1 \subseteq W_2$, it must hold that if $c_1 \in C$ and $c_2 \in C$ are in different components of the subgraph induced by $W_2 \cup C$, then they also are in different components of the subgraph induced by $W_1 \cup C$.

Now, consider a component of $W_2 \cup C \cup \{i\}$, containing vertices Q , and let $K = Q \setminus C$. Since this component contains at least one element of C , the subgraph induced by $Q \setminus \{i\}$, consists of a strictly positive number of say k components. Let $K_j, j = 1, \dots, k$ be the nodes in the intersection of W_2 and the j -th of these components (the components may be numbered arbitrarily), and similarly, let $C_j, j = 1, \dots, k$ be the nodes in the intersection of C and the j -th of these components. The subgraphs induced by $K_j \cup C_j, j = 1, \dots, k$ form the distinct components of $W_2 \cup C$.

We first consider the case where $i \in Q$. Since for all $j = 1, \dots, k$, the subgraphs induced by $K_j \cup C_j$ contain at least one element of C , and since they are distinct components of the subgraph induced by $W_2 \cup C$, there must be distinct $c_j \in C_j, j =$

$1, \dots, k$ such that $i \in P(c_j)$, $j = 1, \dots, k$. Using that $W_1 \subseteq W_2$, we establish that all c_j , $j = 1, \dots, k$ are in distinct components of the subgraph induced by $W_1 \cup C$. Let K'_j be the set of elements of $P(C)$ that are in the component containing c_j in the subgraph induced by $W_1 \cup C$. Then, $K'_j \subseteq K_j$, since $W_1 \subseteq W_2$. Further, since $i \in P(c_j)$, $j = 1, \dots, k$, these k components K'_j are in a single component of the subgraph induced by $W_1 \cup C \cup \{i\}$. Let K' be the intersection of the set of nodes in this component and the nodes in $P(C)$, and let C' be the other nodes in the component. Hence, the component Q of the subgraph induced by $(W_2 \cup C \cup \{i\})$ containing i , which consists of k distinct components $K_1 \cup C_1, \dots, K_k \cup C_k$ of the subgraph induced by $W_2 \cup C$, contains as a subgraph a component $K' \cup C'$ of the subgraph induced by $W_1 \cup C \cup \{i\}$ that contains in turn at least k components $K'_1 \cup C'_1, \dots, K'_k \cup C'_k$ of the subgraph induced by $W_1 \cup C$, such that $K'_j \subseteq K_j$ for $j = 1, \dots, k$. Thus it holds that

$$f(W_2 \cap Q) - f((W_2 \cap Q) \cup \{i\}) \geq f(W_1 \cap Q) - f((W_1 \cap Q) \cup \{i\})$$

On the other hand, if $i \notin Q$, $Q \setminus \{i\} = Q$, and hence the subgraph induced by $Q \setminus \{i\}$ consists of one component of the subgraph induced by $W_2 \cup C$, namely the component induced by the vertices in Q . Now consider the set $Q \cap C$, and observe that i is not a predecessor of any of the elements in $Q \cap C$. Consider a component Q' in $W_1 \cup C$ for which it holds that $Q' \subseteq Q$. Since i is not a predecessor of any vertex in Q , it is not a predecessor of any element of Q' , and thus, Q' is a component of $W_1 \cup C \cup \{i\}$. Hence, in this case we have that

$$0 = f(W_2 \cap Q) - f((W_2 \cap Q) \cup \{i\}) = f(W_1 \cap Q) - f((W_1 \cap Q) \cup \{i\})$$

Since each component of $W_1 \cup \{i\}$ is contained in a component of $W_2 \cup \{i\}$, and since the components of $W_2 \cup \{i\}$ are, by definition, disjoint, this implies that

$$f(W_2) - f(W_2 \cup \{i\}) \geq f(W_1) - f(W_1 \cup \{i\})$$

as required. □

Definition 6. Let $C \subseteq V$ be a MIC and let $P(C) = \{1, \dots, m\}$. Then, for $i \in P(C)$, let γ_i be defined by

$$\gamma_i = f(\{1, \dots, i-1\}) - f(\{1, \dots, i\}). \quad (17)$$

Thus, γ_i represents the reduction in the number of components by adding node i and the arcs constituting the precedence relations in which i is involved to the subgraph of G induced by $C \cup \{1, \dots, i-1\}$. As node i is in $P(C)$, this automatically implies that $\gamma_i \geq 0$.

Proposition 4. Let $C \subseteq V$ be a MIC and $W \subseteq P(C)$ with $W = \{1, \dots, m\}$. Consider the subgraph of G induced by $W \cup C$. For each component K of this subgraph it holds that

$$\sum_{i \in K \cap W} \gamma_i = |C \cap K| - 1$$

Proof. We use induction on the elements in W . For $i = 1$, consider any component K of the subgraph induced by $C \cup \{i\}$. If component K does not contain node i , the result follows immediately, as both the summation of the coefficients γ_i in K and $|C \cap K| - 1$ equal zero. If component K does contain node i , this node is connected with $|C \cap K|$ nodes in the component K . Hence, $|C \cap K|$ components have been merged into one component, such that the reduction in the number of components γ_i equals $|C \cap K| - 1$.

Next assume that the result holds for the graph induced by $C \cup \{1, \dots, i-1\}$. Let K be any component of the graph induced by $C \cup \{1, \dots, i\}$. If K does not contain node i , the result follows directly from the induction hypothesis. If K does contain node i , then node i merges a number of components of the graph induced by $C \cup \{1, \dots, i-1\}$ together, say components K_1, \dots, K_k . Note that this implies $\gamma_i = k - 1$. Consequently,

$$\sum_{i \in K \cap \{1, \dots, i\}} \gamma_i = |C \cap K_1| - 1 + \dots + |C \cap K_k| - 1 + k - 1 = |C \cap K| - 1$$

□

In the sequel we will use a special ordering on the elements from $P(C)$, namely, a reversed topological ordering, i.e. a one-to-one mapping $\pi : P(C) \rightarrow \{1, \dots, |P(C)|\}$ satisfying $\pi(i) < \pi(j)$ for $i, j \in P(C)$ and $j \in P(i)$. The following theorem shows that under a reversed topological ordering γ_j is exactly the lifting coefficient α_j , as defined in (12), for $j \in P(C)$.

Theorem 4. *Let $C \subseteq V$ be a MIC and let π be a reversed topological ordering on $P(C)$. If the lifting coefficients γ_i are determined according to (17) under the order π , then for each $j = \pi^{-1}(1), \dots, \pi^{-1}(|P(C)|)$ the inequality*

$$\sum_{i \in C} x_i + \sum_{i=\pi^{-1}(1)}^{\pi^{-1}(j)} \gamma_i (1 - x_i) \leq |C| - 1 \quad (18)$$

is valid and facet-defining for the polytope $\text{conv}(X_{A|R(C)})$, where $A = \{\pi^{-1}(j+1), \dots, \pi^{-1}(|P(C)|)\}$.

Proof. To show validity of (18) it suffices to show this for $j = \pi^{-1}(|P(C)|)$. For other values of j validity then follows from the fact that we restrict the set of feasible solutions by setting the variables in $\{\pi^{-1}(j+1), \dots, \pi^{-1}(|P(C)|)\}$ to 1. Let x be an arbitrary feasible solution with $x_i = 0$ for all $i \in R(C)$. Define $C' = \{i \in C | x_i = 1\}$, and $W = \{i \in P(C) | x_i = 0\}$. Then obviously $C' \neq C$ since C is a MIC: $|C'| \leq |C| - 1 < |C|$. Hence,

$$\begin{aligned} \sum_{i \in C} x_i + \sum_{i=\pi^{-1}(1)}^{\pi^{-1}(|P(C)|)} \gamma_i (1 - x_i) &= |C'| + \sum_{i \in W} \gamma_i \leq |C'| + \sum_{i \in W} \gamma'_i \\ &\leq |C'| + |C| - f(W) \leq |C| - 1 \end{aligned}$$

The first inequality follows from the supermodularity of f , where the γ_i are obtained using the sequence of $P(C) = \pi^{-1}(1), \dots, \pi^{-1}(|P(C)|)$ and the γ'_i are obtained using

the subsequence of $\pi^{-1}(1), \dots, \pi^{-1}(|P(C)|)$ defined by W . Using (17), observe that $\sum_{i \in W} \gamma_i'$ equals $f(\emptyset) - f(W) = |C| - f(W)$. Finally, $f(W)$ equals the number of components in the graph induced by $C \cup W$. If $W = \emptyset$, then $f(W) = |C|$, and the result follows from $C' \subset C$ since C is a MIC. If $W \neq \emptyset$, it follows from solution vector x that nodes in C' are not successors of nodes in W . Hence, in the graph induced by $C \cup W$ the nodes in C' are $|C'|$ individual components. As there is at least one component containing elements in W , it follows that $f(W) \geq |C'| + 1$, which completes the proof of the validity of (18).

To show that (18) is facet-defining, we use induction on the elements $j = \pi^{-1}(1), \dots, \pi^{-1}(|P(C)|)$. It suffices to display a feasible solution θ^j with $\theta_j^j = 0$, $\theta_i^j = 1$, $i \in \{\pi^{-1}(j+1), \dots, \pi^{-1}(|P(C)|)\}$, and $\theta_i^j = 0$ for $i \in R(C)$, satisfying (18) at equality (see Nemhauser and Wolsey [15] Proposition 1.1., page 261). To construct this vector consider the graph induced by $C \cup \{\pi^{-1}(1), \dots, \pi^{-1}(j)\}$. Let K be the component containing node j . Define the vector θ^j as follows:

$$\theta_i^j = \begin{cases} 1 & i \in T(C) \setminus K \\ 0 & \text{otherwise} \end{cases}$$

To show that θ^j satisfies (18) at equality, note that

$$\begin{aligned} \sum_{i \in C} \theta_i^j + \sum_{i=\pi^{-1}(1)}^{\pi^{-1}(j)} \gamma_i(1 - \theta_i^j) &= |C \setminus K| + \sum_{i \in P(C) \cap K} \gamma_i \\ &= |C \setminus K| + |C \cap K| - 1 \\ &= |C| - 1 \end{aligned}$$

where the second equality follows from Proposition 4.

Hence, for $j = \pi^{-1}(1)$ the vector θ^j satisfies the requirements. Next assume that the result holds for all $i \in \{\pi^{-1}(1), \dots, \pi^{-1}(j-1)\}$. Hence, we are given $C + j - 1$ affinely independent vectors which all have their i^{th} component equal to 1 for $i \in P(C) \setminus \{\pi^{-1}(1), \dots, \pi^{-1}(j-1)\}$. By defining θ^j as in the above and following the same reasoning, θ^j satisfies (18) at equality and is affinely independent from the foregoing vectors, as $\theta_j^j = 0$.

□

Theorem 4 shows that f can be used in a lifting process to obtain facet-defining inequalities. The coefficients γ_i represent the reduction in the number of components by adding node i and the arcs constituting the precedence relations in which i is involved to the subgraph of G induced by $C \cup \{1, \dots, i-1\}$. This reduction number can be determined using a set union algorithm such as developed by Tarjan [20]. If n is the number of nodes in the graph, and m is the number of arcs, the algorithm runs in $\mathcal{O}((n+m)\alpha((n+m), n))$, where $\alpha((n+m), n)$ is a functional inverse of Ackerman's function [1].

3.3. Lifting of non-predecessor variables of a minimal induced cover

Although the maximization problem in (8) is NP-hard in general, for PCKP the resulting problem in (12) can be solved in polynomial time when lifting predecessors of a MIC C . For variables in $R(C)$, the maximization problem in equality (14) is also essentially a PCKP, but will turn out to be NP-hard in general. To give a formal proof of this statement, we introduce the following problem definitions:

Clique (see Garey and Johnson [10])

INSTANCE: Graph $G = (U, E)$, and a positive integer K , with $3 \leq K \leq |U|$.

QUESTION: Does G contain a clique of size K or more?

Note that the assumption $K \geq 3$ does not change the complexity of the problem.

PCKP-MIC-R(C)-lifting

INSTANCE:

- Instance I of PCKP, consisting of a directed acyclic graph $D^I = (V^I, A^I)$, a knapsack capacity $B^I \in \mathbb{Z}^+$, and for all $i \in V^I$ a value $c_i^I \in \mathbb{Z}$ and a weight $a_i^I \in \mathbb{Z}^+$.
- A MIC $C^I \subseteq V^I$
- A PFRS-order π^I on $V^I \setminus C^I$
- A facet-defining inequality

$$\sum_{i \in C^I} x_i + \sum_{i \in P(C^I)} \alpha_i (1 - x_i) \leq |C^I| - 1$$

obtained by applying the lifting procedure as defined in (12) to the variables in $P(C^I)$ under PFRS-order π^I .

- $\alpha^I \in \mathbb{Z}$

QUESTION: Is the lifting coefficient for the first variable in $R(C^I)$ under order π^I , as defined in (14), less than or equal to α^I ?

Theorem 5. *PCKP-MIC-R(C)-lifting is NP-complete in the strong sense.*

Proof. It can easily be checked that PCKP-MIC-R(C)-lifting is in NP. Hence it suffices to show that Clique reduces to PCKP-MIC-R(C)-lifting. To this purpose, the graph $G = (U, E)$ will be transformed into an instance $D^I = (V^I, A^I)$, in which there is a node for each $u \in U$ and for each $e \in E$. The nodes corresponding to U will function as predecessors of nodes corresponding to E . As shown in Subsection 3.2, a lifting coefficient of a predecessor is equal to the reduction in the number of components. Under the assumption that, for all $u \in U$, the degree $|\delta(u)| \geq 2$, this reduction, and hence the lifting coefficients, are enforced to be 1. Since we are looking for cliques of size at least 3, this assumption causes no loss of generality.

Let $G = (U, E)$ be an instance of Clique, let K be any integer satisfying $3 \leq K \leq |U|$, and let π be any order on the nodes in U , i.e., $U = \{\pi^{-1}(1), \dots, \pi^{-1}(n)\}$. We define an index set J , consisting of nodes $u \in U$ which are currently not adjacent to a node with higher index. Hence,

$$J = \{u \in U \mid \nexists w \in U : \pi(w) > \pi(u), \{u, w\} \in E\}$$

Extend the graph G to $\tilde{G} = (\tilde{U}, \tilde{E})$, where

$$\tilde{U} = U \cup \bigcup_{u \in J} \{\tilde{u}\}; \quad \tilde{E} = E \cup \bigcup_{u \in J} \{u, \tilde{u}\}$$

Furthermore, let $\tilde{K} = K$, and $\tilde{\pi}$ an order on \tilde{U} such that $\pi(u) = \tilde{\pi}(u)$, for $u \in U$. Then, since nodes in \tilde{U} have degree 1, G contains a clique of size $K \geq 3$ if and only if \tilde{G} contains a clique of size \tilde{K} .

Next, we determine a subset $L \subseteq E$ whose elements share a common predecessor in the directed graph to be introduced shortly. Let $\delta(u)$ be the set of edges incident to node u in graph \tilde{G} . Then, the set L is determined by the following algorithm, which is to be explained shortly:

```

L = ∅; W =  $\tilde{E}$ ; /* initialisation */
for u =  $\tilde{\pi}^{-1}(1)$  to  $\tilde{\pi}^{-1}(|U|)$  do
  begin
    let e ∈ W ∩ δ(u);
    L = L ∪ (W ∩ δ(u) \ {e});
    W = W \ δ(u);
  end;
endfor;

```

This algorithm processes the vertices $u \in \tilde{U}$ in increasing order of their indices, and considers the intersection of $\delta(u)$ and W , where W initially consist of all edges in \tilde{E} . In each iteration, the algorithm selects an arbitrary edge e in the intersection of $\delta(u)$ and W , and eliminates all edges in $\delta(u)$ from W . Except for e , the thus eliminated edges are added to a set L which is initially empty. Notice that such an e always exists since each vertex $u \in U$ is adjacent to a higher index vertex in \tilde{U} . Notice also that all edges in $W \cap \delta(u) \setminus \{e\}$ are in E since vertices in U only contain edges to \tilde{U} , and hence in $\tilde{E} \setminus E$, if they are not adjacent to a higher index vertex in U , and should this be the case, then they contain only one such edge (which per force is chosen to be e).

Now, we are able to define the instance I with directed graph $D^I = (V^I, A^I)$ where

$$V^I = \tilde{U} \cup \tilde{E} \cup \{Q, q\}; \quad A^I = \{(u, e) | u \in \tilde{U}, e \in \tilde{E}, e \in \delta(u)\} \cup (Q, q) \cup \bigcup_{e \in L} (Q, e)$$

To complete the instance I of PCKP-MIC-R(C)-lifting, let $c_v \in \mathbb{Z}$ for all $v \in V^I$, and $a(Q) \in \mathbb{Z}^+$. Further, let

$$B^I = \frac{|U|}{\tilde{K}} + |\tilde{U} \setminus U| \cdot \left[1 + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} \right] + \frac{|\tilde{E}| - 1}{\tilde{K}^3} + a(Q)$$

and define weights for nodes in V^I as follows:

$$a_u^I = \begin{cases} \frac{1}{\tilde{K}} & u \in U \\ 1 + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} & u \in \tilde{U} \setminus U \\ \frac{1}{\tilde{K}^3} & u \in \tilde{E} \\ a(Q) & u = Q \\ B^I - \left[1 + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} + a(Q) \right] & u = q \end{cases}$$

Now consider \tilde{E} . In order to include all $e \in \tilde{E}$ in the knapsack, we must also include all items in $P(\tilde{E})$, which means in this case that all $u \in \tilde{U}$ and Q must be in the knapsack. Since the nodes in $u \in \tilde{U}$ and Q already account for a weight of

$$\frac{|U|}{\tilde{K}} + |\tilde{U} \setminus U| \cdot \left[1 + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} \right] + a(Q)$$

only $|\tilde{E}| - 1$ elements from \tilde{E} can be included in the knapsack. This yields that \tilde{E} is a MIC. Now, let π^I be a PFRS-order on $\tilde{U} \cup \{Q, q\}$ defined by:

$$\pi^I(i) = \begin{cases} 1 & i = Q \\ \tilde{\pi}(i) + 1 & i \in \tilde{U} \\ |\tilde{U}| + 2 & i = q \end{cases}$$

and the corresponding facet-defining inequality after lifting Q and \tilde{U} be given by

$$\sum_{i \in \tilde{E}} x_i + (|L| - 1)(1 - x_Q) + \sum_{i \in U} (1 - x_i) \leq |\tilde{E}| - 1 \quad (19)$$

The lifting coefficients can be explained as follows. Notice that by the construction of A^I , Q connects $|L|$ elements of \tilde{E} , and thus adding Q yields a reduction in the number of components of $|L| - 1$. Further, by the construction of D^I , there are $|U|$ edges not connected to Q . Now, using our assumption all vertices $u \in U$ have $|\delta(u)| \geq 2$, each vertex $u \in U$ connects the edge e selected in the $\tilde{\pi}(u)$ -th iteration of the previously described algorithm to the component which became connected when x_Q was lifted. Hence $\alpha_u = 1$ for all $u \in U$. Finally, the vertices $\tilde{u} \in \tilde{U}$ can be seen not to cause a reduction in the number of components at all, and hence have coefficient $\alpha_{\tilde{u}} = 0$.

Finally, let

$$\alpha^I = |\tilde{E}| - 1 - \left[\binom{\tilde{K}}{2} + |U| - \tilde{K} \right]$$

We leave it to the reader to verify that the above transformation from G , via \tilde{G} , to instance I is polynomial.

Next let us consider the problem (14) that arises when lifting x_q , the only variable in $R(C^I)$. When lifting variable x_q according to (14), nodes q and Q must be included in the knapsack.

We are now going to show that \tilde{G} contains a clique of size $\tilde{K} \geq 3$ if and only if the maximal value of the lifting coefficient for variable x_q is less than or equal to α^I .

(\Rightarrow) If G contains a clique of size \tilde{K} , then by including the nodes u in $\tilde{U} \cup \tilde{E}$ corresponding to the vertices and edges in the clique in the knapsack together with Q and q , we have a cumulative weight $\frac{\tilde{K}}{\tilde{K}} + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} + a(Q) + B^I - \left[1 + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} + a(Q) \right] = B^I$. Thus this set of items is feasible. Furthermore, together these items yield a value for α^I of

$$|\tilde{E}| - 1 - \left[\binom{\tilde{K}}{2} + |U| - \tilde{K} \right]$$

as required.

(\Leftarrow) Let the value of the lifting coefficient for x_q (being $\alpha_q \leq \alpha^I$) be obtained by a solution in which i items (say) corresponding to nodes $u \in U$ in the graph \tilde{G} are included in the knapsack. Let us first assume $i > \tilde{K}$. Then the weight in the knapsack amounts to at least

$$1 + \frac{1}{\tilde{K}} + a(Q) + B^I - \left[1 + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} + a(Q) \right] > B^I$$

which yields a contradiction. Hence $i \leq \tilde{K}$. Assume $i < \tilde{K}$. Then,

$$\alpha_q \geq |\tilde{E}| - 1 - \left[\binom{i}{2} + |U| - i \right] > |\tilde{E}| - 1 - \left[\binom{\tilde{K}}{2} + |U| - \tilde{K} \right] = \alpha^I$$

for $\tilde{K} \geq 3$. Again a contradiction. Let us finally consider the case where $i = \tilde{K}$. Then by the above reasoning we find $\alpha_q \geq \alpha^I$ which together with $\alpha_q \leq \alpha^I$ implies that $\alpha_q = \alpha^I$ and hence the remaining knapsack capacity

$$B^I - \frac{\tilde{K}}{\tilde{K}} - a(Q) - B^I + \left[1 + \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3} + a(Q) \right] = \frac{\binom{\tilde{K}}{2}}{\tilde{K}^3}$$

must account for an increase in the solution of the maximization problem in (14) of at least $\left(\frac{\tilde{K}}{2}\right)$. Hence, at least $\left(\frac{\tilde{K}}{2}\right)$ vertices from \tilde{E} must be included in the knapsack, and moreover, these vertices must have their predecessors in the graph in the knapsack. This can only be achieved if these vertices are in E , and hence we have found a clique of size K in the graph G . □

Although lifting variables for a PCKP is strongly NP-hard in general, in the special case where the precedence graph is a tree and the size of the coefficients in the given valid inequality is polynomially bounded, then lifting coefficients can be obtained in polynomial time.

Theorem 6. *Given a PCKP for which the precedence graph is a tree, and a valid inequality with coefficients whose size is bounded by a polynomial in the size of the tree, then all lifting coefficients can be determined in polynomial time.*

Proof. Lifting a variable requires solving a tree knapsack problem on a subtree of the original tree. Tree knapsack problems with possibly negative objective coefficients can be solved in pseudo-polynomial time $\mathcal{O}(nQ^2)$ by an extension of a standard dynamic programming algorithm for tree knapsacks (see for instance Jonson and Niemi ([13]), where Q is an upper bound on the maximum value that can be achieved in the optimization problem. If all coefficients of the inequality are polynomially bounded in the size of the original tree, then Q is polynomially bounded, and therefore the tree knapsack problem can be solved in polynomial time. \square

3.4. An example

Consider the example in Fig. 1. Table 1 gives an example of both a minimal induced

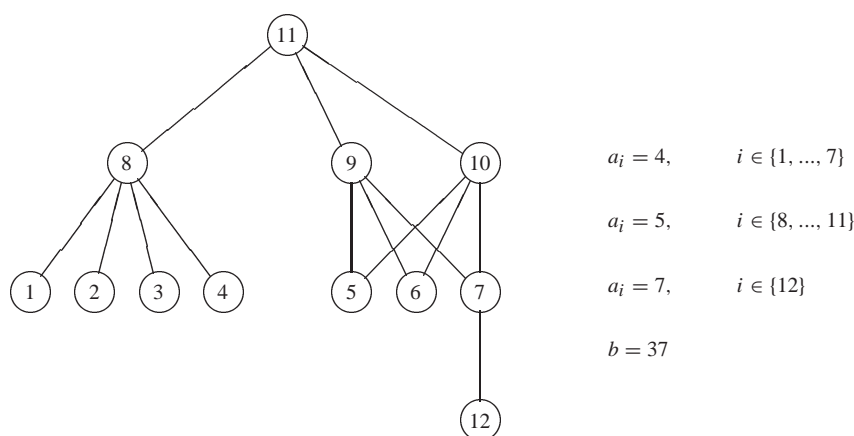


Fig. 1. Example

cover for which two different PFRS-orders are stated and a (1,k)-configuration for which only two valid inequalities out of the total set as defined by (16) are listed. The resulting facet-defining inequalities are listed below. The combinatorial interpretation of the predecessors of a minimal induced cover can be seen from this table, which also illustrates that different PFRS-orders can lead to different facets. The reader may note that many more covers and (1,k)-configurations are present in the problem instance (see also Section 5).

Table 1. A minimal induced cover and (1,k)-configuration for problem instance in Fig. 1

C	t	type	$Z(r)$	lifting order
$\{1,2,5,6,7\}$	–	MIC	–	8,9,10,11,3,4,12
$\{1,2,5,6,7\}$	–	MIC	–	8,10,9,11,3,4,12
$\{1,2,5\}$	12	(1,2)-conf	$\{1,2\}$	7,8,9,10,11,3,4,6
$\{1,2,5\}$	12	(1,2)-conf	$\{1,2,5\}$	7,8,9,10,11,3,4,6

Resulting facet-defining inequalities for Table 1:

$$\begin{array}{rcl}
x_1 + x_2 + x_5 + x_6 + x_7 + (1 - x_8) + 2(1 - x_9) & & + (1 - x_{11}) + x_{12} \leq 4 \\
x_1 + x_2 + x_5 + x_6 + x_7 + (1 - x_8) & + 2(1 - x_{10}) & + (1 - x_{11}) + x_{12} \leq 4 \\
x_1 + x_2 & + (1 - x_8) & + (1 - x_{11}) + x_{12} \leq 2 \\
x_1 + x_2 + x_5 & + (1 - x_8) + (1 - x_9) & + (1 - x_{11}) + 2x_{12} \leq 3
\end{array}$$

4. K-covers

In this section, we discuss valid inequalities arising from K-covers. Although this class of inequalities is a direct generalization of minimal induced covers, it is not always immediately clear for which subset of the polytope the corresponding valid-inequalities are facet-defining. One way to obtain facets for the PCKP-polytope would be to follow two steps: first, the exact polytope for which the valid inequality is facet-defining could be determined and next, the same lifting procedure as mentioned in Section 3 could be applied. In this section, we show that if the first step is skipped, and a different lifting procedure is applied, again facets for the PCKP-polytope are obtained.

Definition 7. $C \subseteq V$ is a K-cover if

- C is incomparable
- $\forall S \subseteq C$, with $|S| = K$ it holds that $a(T(S)) > b$, but $a(T(S) \setminus \{i\}) \leq b$, $\forall i \in S$.

Let $C \subseteq V$ be a K-cover, then

$$\sum_{i \in C} x_i \leq K - 1 \quad (20)$$

is a valid inequality for the PCKP-polytope.

Figure 2 shows that if $C \subseteq V$ is a K-cover the subset $X_{P(C)|R(C)}$ can be empty, in which case the aforementioned lifting procedure cannot directly be applied to the variables in $P(C) \cup R(C)$. In this example $C = \{1, 2, 3, 4\}$ is a 3-cover, but $X_{P(C)|R(C)} = \emptyset$. In fact, the corresponding valid inequality is facet-defining for $\text{conv}(X_{\{9,10,11\}|R(C)})$.

Below we define a different lifting procedure which is still defined on $P(C) \cup R(C)$ and generates facet-defining inequalities. Note that the only difference between the two lifting procedures consists of the polytope over which the maximization problem is defined.

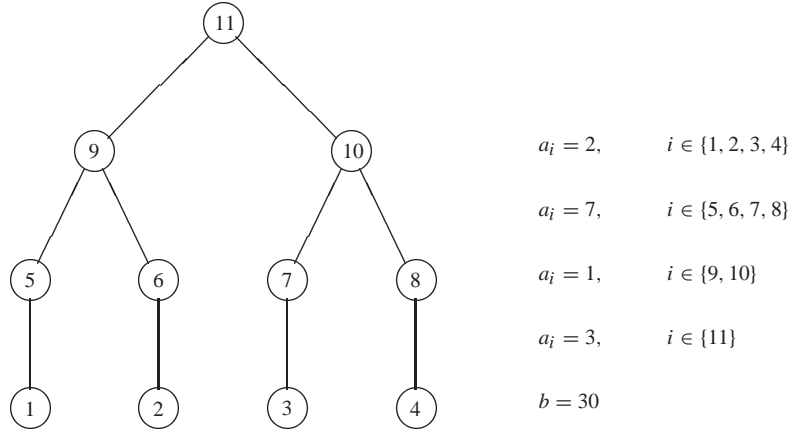


Fig. 2. Example k-cover

Definition 8. Let C be a K -cover and π be a PFRS-order for C which represents the order of lifting variables. Let the lifting coefficient for a variable $j \in P(C)$ be determined by

$$\alpha_j = K - 1 - \max_{\substack{x \in X: \\ x_j = 0}} \left[\sum_{i \in C} x_i + \sum_{i \in P(C) \cap p^\pi(j)} \alpha_i (1 - x_i) \right] \quad (21)$$

and for a variable $j \in R(C)$ by

$$\alpha_j = K - 1 - \max_{\substack{x \in X: \\ x_j = 1}} \left[\sum_{i \in C} x_i + \sum_{i \in P(C)} \alpha_i (1 - x_i) + \sum_{i \in R(C) \cap p^\pi(j)} \alpha_i x_i \right] \quad (22)$$

Theorem 7. Let

- $C \subseteq V$ be a K -cover
- π be a PFRS-order for C
- the lifting coefficients for variables in $P(C)$ be determined according to (21)
- the lifting coefficients for variables in $R(C)$ be determined according to (22)

then the resulting inequality

$$\sum_{i \in C} x_i + \sum_{i \in P(C)} \alpha_i (1 - x_i) + \sum_{i \in R(C)} \alpha_i x_i \leq K - 1 \quad (23)$$

a) is valid, and $\alpha_i \geq 0$, for all $i \in P(C) \cup R(C)$

b) defines a facet of the PCKP-polytope $\text{conv}(X)$.

Proof. Both validity and nonnegativity of the lifting coefficients can be proved easily using inductive arguments. We will prove the remainder of our claim by constructing $|V| - 1$ linearly independent vectors (directions) in the face described by the inequality. These vectors are constructed as the difference of two vectors, both with the following properties:

- (i) the vector satisfies the knapsack constraint;
- (ii) the vector satisfies the precedence constraints;
- (iii) the vector satisfies the lifted inequality at equality.

Properties (i) and (ii) imply that the vector is in $\text{conv}(X)$, whereas property (iii) guarantees that the vector is in the face described by the inequality. For ease of notation, let $P^0(C) = \{j \in P(C) | \alpha_j = 0\}$ and $P^>(C) = \{j \in P(C) | \alpha_j > 0\}$. Note that for each $i \in P^>(C)$ the number of items in C which are *not* successors of i is less than or equal to $K - 2$. If there were more than $K - 2$ items in C which are *not* successors of i , say set $S \subseteq C$ consisting of $K - 1$ items, then the maximization problem in (21) for variable x_i would have value at least $K - 1$ since the items in $T(S)$ could be set to one. In other words, if $S \subset C$ contains $K - 1$ elements, then i is a predecessor of at least one element from S . This property is used at several occasions in the remainder of the proof.

Let $C = \{1, \dots, |C|\}$. For $j = 1, \dots, |C| - 1$ let $C^j \subseteq C \setminus \{j, j + 1\}$ with $|C^j| = K - 2$. Next, define

$$\begin{aligned}\phi^j &= x^{T(C^j \cup \{j, j+1\}) \setminus \{j+1\}} \\ \psi^j &= x^{T(C^j \cup \{j, j+1\}) \setminus \{j\}} \\ y^j &= \phi^j - \psi^j\end{aligned}$$

Then the vectors y^j , $j = 1, \dots, |C| - 1$ are clearly linearly independent. Moreover, ϕ^j satisfies properties (i) and (ii) by definition: we take K items from C and all their predecessors; after that we remove one of the items from the K items chosen. Clearly, the vector ϕ^j also satisfies property (iii): we have $K - 1$ elements from C , and thus the first term of the left-hand side of (23) equals $K - 1$. Since the other terms are nonnegative and the equation is valid, we must have that equality holds. For vector ψ^j , (i), (ii), and (iii) can be shown similarly.

For $j \in P^0(C)$, let θ^j be the vector for which the maximum in (21) is attained. Let $C^j = \{i \in C | \theta_i^j = 1\}$. Thus, $\forall_{i \in T(C^j)} \theta_i^j = 1$. Moreover, we may assume that $\forall_{i \notin T(C^j)} \theta_i^j = 0$: Clearly, this maintains feasibility with regard to both the knapsack constraint, and the precedence constraints. It remains to show that the maximum in (21) is not decreased. If $i \in C \setminus C^j$ $\theta_i^j = 0$ by definition of C^j . If $i \in P(C) \setminus T(C^j)$, setting $\theta_i^j = 0$ can not have a decreasing effect on the maximum in (21), since $(1 - x_i)$ becomes positive, and the objective coefficient of $(1 - x_i)$ in (21) is nonnegative. If $i \in R(C)$, the objective coefficient of x_i is zero in (21).

We define

$$\phi_i^j = \begin{cases} 1 & i \in T(C^j) \cup T(j) \cup T(P^>(C) \cap s^\pi(j)) \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_i^j = \begin{cases} 1 & i \in T(C^j) \cup P(j) \cup T(P^>(C) \cap s^\pi(j)) \\ 0 & \text{otherwise} \end{cases}$$

Note that $\phi^j - \psi^j$ is the j -th unit vector, since $j \in T(j)$ but $j \notin T(C^j) \cup P(j) \cup T(P^>(C) \cap s^\pi(j))$ (Note that $\theta_j^j = 0$, see (21)). It remains to show that ϕ and ψ satisfy (i), (ii), and (iii). To show that the knapsack constraint is satisfied, we construct an extension of ϕ that does so. Let \bar{C}^j be an extension of C^j with $K - 1$ elements from C . If $j \in T(\bar{C}^j)$ take an arbitrary $i \in C \setminus \bar{C}^j$. Otherwise choose $i \in C \setminus \bar{C}^j$ such that j is a predecessor of i . By definition of K -covers, the set $T(\bar{C}^j \cup \{i\}) \setminus \{i\}$ satisfies the knapsack constraint. Clearly, $T(C^j)$ and $T(j)$ are subsets of $T(\bar{C}^j \cup \{i\}) \setminus \{i\}$. Furthermore, $P^>(C) \subset T(\bar{C}^j)$, since $T(\bar{C}^j)$ contains $K - 1$ elements from C . Thus, $P^>(C) \cap s^\pi(j) \subset T(\bar{C}^j)$, and thus $T(P^>(C) \cap s^\pi(j)) \subset T(\bar{C}^j)$, and $T(\bar{C}^j) \subseteq T(\bar{C}^j \cup \{i\}) \setminus \{i\}$.

The precedence constraints hold by construction of ϕ^j .

To show that (iii) is satisfied by ϕ^j , we show that ϕ^j obtains the same value in the maximum of (21) as θ^j , i.e., $K - 1$. This is true because ϕ^j is an extension of θ^j with elements from $s^\pi(j) \cup \{j\}$. This trivially holds for $P^>(j) \cap s^\pi(j)$ and $\{j\}$. It also holds for the predecessors of both sets, i.e., $T(P^>(j) \cap s^\pi(j))$ and $T(j)$ by the definition of π . Since elements from $s^\pi(j)$ have no contribution to the maximum of (21) ϕ^j and θ^j have the same value. Next, the terms in the maximum of (21) are a subset of the terms in (23) with a value of $K - 1$. The remaining terms are nonnegative, and hence, by validity of the inequality, therefore zero. Thus, (23) is satisfied at equality. Similar arguments show that (i), (ii), and (iii) hold for ψ .

Next, let $j \in P^>(C)$, and let θ^j be the vector for which the maximum in (21) is attained. W.l.o.g. assume that $\theta_i^j = 0$ for $i \in R(C)$ and for $i \in P(C)$ such that i is not a predecessor of an element in C which is set to one. As indicated in the above, we can again extend the solution θ^j to a solution ψ^j , in which all variables in $T(P^>(C) \cap s^\pi(j))$ are included in the knapsack. Next, let ϕ^j be any vector with $K - 1$ elements in C equal to one, and $\phi_i^j = 0$, for $i \in R(C)$. Analogously as in the above, one can verify that ψ^j and ϕ^j satisfy properties (i)-(iii). Define $y^j = \phi^j - \psi^j$, then $y_j^j = 1$ and $y_i^j = 0$, for $i \in P^>(C) \cap s^\pi(j)$ and $i \in R(C)$.

Finally, for $j \in R(C)$, let θ^j be the vector for which the maximum in (22) is attained. W.l.o.g., assume that $\theta_i^j = 0$, for $i \in R(C) \cap s^\pi(j)$. Let $C^j \subset C$ with $|C^j| = K - 1$. Next, define

$$\begin{aligned} \phi^j &= \theta^j \\ \psi^j &= x^{T(C^j)} \\ y^j &= \phi^j - \psi^j \end{aligned}$$

Then both ϕ^j and ψ^j satisfy properties (i)-(iii). Finally, $y_j^j = 1$ and $y_i^j = 0$ for $i \in s^\pi(j)$.

We leave it to the reader to verify that the $|V| - 1$ vectors as defined in the above are linearly independent. □

The following example shows that Theorem 7 does not necessarily hold if we allow for more general lifting orders.

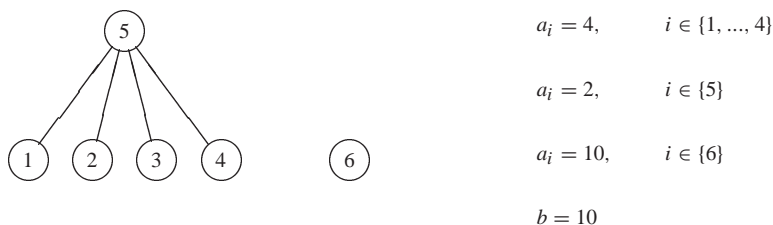


Fig. 3. Example 2

Now the set $\{1, 2, 3, 4\}$ is a 3-cover and hence

$$\sum_{i=1}^4 x_i \leq 2$$

is a valid inequality. If we first lift variable x_6 according to (22) and then x_5 according to (21) we obtain the inequality

$$\sum_{i=1}^4 x_i + 2x_6 \leq 2$$

which is valid but not facet-defining for $\text{conv}(X)$.

5. Computational results and concluding remarks

To gain insight in the effectiveness of the proposed facets in this paper, reconsider the example of Subsection 3.4. In this small example 36 covers were found, which by applying different PFRS-orders led to a total of 63 different facet-defining inequalities. Furthermore seven K -covers were found (11 different facets), and 24 (1,k)-configurations (leading to 100 different facets).

In the five problem instances in Table 2 nodes in the same "layer" of the graph are given the same, but randomly chosen, objective function coefficient. For each of these 5 problem instances, the value of the LP-relaxation, the IP-value and the value of the LP-relaxation after adding all 174 facets to the description were computed.

These results indicate that the effect of the valid inequalities may be significant. Problem instance obj1b represents a situation where nodes in layer three (i.e. nodes 1 through 7) have a low objective coefficient compared to the other nodes.

Next, 9 objective functions in which each node is given a random objective coefficient were generated. Different ranges of objective coefficients were tested. Table 3 reports on the computational results for these problem instances.

Table 2. Computational results for problem instance in Fig. 1

problem	LP-value	LP + facets	IP-value	%gap closed
obj1	49.3	47.8	46	45%
obj1a	57.0	51.8	48	58%
obj1b	104.0	104.0	100	0%
obj1c	242.0	235.9	227	41%
obj1d	270.6	244.6	225	57%

Table 3. Computational results for problem instance in Fig. 1

problem	LP-value	LP + facets	IP-value	%gap closed
obj2	124.0	119.1	119	97%
obj3	116.0	110.3	110	95%
obj4	104.6	97.0	97	100%
obj5	179.8	176.0	176	100%
obj6	231.7	228.3	226	60%
obj7	233.7	223.7	222	85%
obj8	274.0	251.4	248	87%
obj9	53.5	51.5	51	80%
obj10	43.2	41.0	41	100%

The results show that a large proportion of the gap *can* be closed by including the facets proposed in this paper. In fact, only PFRS-orders were considered in the tests, hence more valid inequalities can be included by allowing for more general lifting orders as indicated in this paper. Finally we state some remarks on possible future research directions.

Firstly, note that problem instance obj1c shows that for certain problem instances the facet-defining inequalities discussed in this paper are not very useful. Direct generalizations of other well-known classes of valid inequalities for regular knapsack problems could of course form a fruitful area for future research. Next, to incorporate such inequalities into a branch-and-cut procedure, the separation problem must be addressed. Again, generalizations of separation heuristics for ordinary cover inequalities can be investigated. Thirdly, the difference in the definition of a minimal induced cover used in this paper and by Boyd ([5]), and the definition employed by Park and Park ([16]) deserves more research. Finally, the lifting procedure as proposed in Section 4 is only proven to be a valid procedure for valid inequalities arising from K-covers. In fact, in the proof detailed information from the definition of a K-cover is used. The question arises whether these lifting ideas can also be used for different and/or more general classes of valid inequalities.

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