Economics Letters 11 (1983) 217–221
North-Holland Publishing Company

TRANSFORMED BETA-CAPACITY DISTRIBUTIONS OF PRODUCTION UNITS *

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Received 29 April 1982

In this paper the capacity distributions corresponding to several well-known aggregate production functions (Cobb–Douglas, CES and VES) are derived. It is shown that these distributions can be transformed to beta-distributions in a straightforward way.

1. The distribution approach

According to the distribution approach to the aggregation of production functions, each different form of the aggregate short-run production function of an industry corresponds to a different form of the capacity distribution of production units in that industry [Johansen (1972), Sato (1975)]. In this paper we shall derive the capacity distributions corresponding to several well-known aggregate production functions and investigate their properties. It will be shown that all derived capacity distributions can be transformed to beta-distributions in a straightforward way. The beta-distribution therefore appears to be a rather general form for the capacity distribution. This is also interesting since the beta-distribution allows for a bell-shaped form, a property of the capacity distribution which is frequently observed.

A central concept in the distribution approach is the capacity distribution, $\phi(\xi)$, the distribution of capacity output in terms of the labour

* This paper was written with financial support of the Netherlands Organization for the Advancement of Pure Research (ZWO). I would like to thank Professor L. Johansen and Dr. F.R. Førsund for their comments.

input coefficient, ξ.[1] Its normalised form is

$$\hat{\phi}(\xi)=\phi(\xi)/J, \qquad \xi \geqslant \xi^0 > 0 \quad \text{where} \quad J=\int_{\xi^0}^{\infty}\phi(\xi)\mathrm{d}\xi. \tag{1}$$

J is the capacity output of the industry. Also normalising aggregate output and labour of the industry, Y and L, respectively, yields $y = Y/J$ and $l = L/J$. Assuming efficient behaviour, the short-run production function of the industry is uniquely determined by the capacity distribution and vice-versa. This function, $y=f(l)$, has the following properties [Muysken (1979, pp. 77–79)]:

$$\begin{array}{llll} f(0)=0, & f(l)>0 & \text{for} \quad 0<l<\bar{l}, & f(\bar{l})=1, \\ f'(0)=1/\xi^0, & f'(l)=q & \text{for} \quad 0<l<\bar{l}, & f'(\bar{l})=0, \\ & f''(l<0) & \text{for} \quad 0<l<\bar{l}, & \end{array} \tag{2}$$

where $\bar{l}=\bar{L}/J$, $\bar{L}$ is the amount employed when the industry is operating at full capacity, and q stands for the wage rate in terms of output price.

Finally the capacity distribution underlying a given short-run production function is given by

$$\hat{\phi}(\xi)=\frac{-1}{\xi^3}\cdot d'\left(\frac{1}{\xi}\right), \tag{3}$$

where the function d is given by $f'(l)=q \Leftrightarrow l=d(q)$.

2. Transformed beta-capacity distributions [2]

The capacity distribution can always be expressed as a function of ξ^0/ξ: $\phi(\xi)=h(\xi^0/\xi)$. Then holds

$$\phi(\xi)\mathrm{d}\xi=\xi^0\cdot\left(\frac{\xi^0}{\xi}\right)^{-2}\cdot h(\xi^0/\xi)\mathrm{d}(\xi^0/\xi)=w(\xi^0/\xi)\mathrm{d}(\xi^0/\xi), \tag{4}$$

[1] Cf. Johansen (1972) and Sato (1975). We assume a putty-clay model in which production units operate with only two inputs: one variable input, labour, and one fixed input, capital. For the putty-putty case see Sato (1975); for the extension to more variable inputs see Johansen (1972) and Sato (1975, ch. IV).

[2] The idea to transform capacity distributions to beta functions is based on Sato (1975, chs. XIV and XVII).

Table 1
Maximum and mean of the (transformed) $B(\lambda, \mu)$ – capacity distributions, $\hat{w}(x)$ and $\hat{\phi}(\xi)$.

	$\hat{w}(x)$	$\hat{\phi}(\xi)$
Maximum	$x^* = \dfrac{\lambda - 1}{\lambda + \mu - 2}$	$\xi^* = \dfrac{\lambda + \mu}{\lambda + 1} \cdot \xi^0$
Mean	$\bar{x} = \dfrac{\lambda}{\lambda + \mu}$	$\bar{\xi} = \dfrac{\lambda + \mu - 1}{\lambda - 1} \cdot \xi^0$

where $w(\xi^0/\xi)$ is called the transformed capacity distribution. The normalized form of this distribution is

$$\hat{w}(x) = \frac{w(x)}{J}, \qquad 0 \leqslant x = \xi^0/\xi \leqslant 1, \qquad J = \int_0^1 w(x)\mathrm{d}x. \tag{5}$$

Now we are interested in those cases where $\hat{w}(x)$ is a beta function, i.e.,

$$\hat{w}(x) = \frac{1}{B(\lambda, \mu)} \cdot x^{\lambda - 1} \cdot (1 - x)^{\mu - 1}, \qquad 0 \leqslant x \leqslant 1, \qquad \lambda, \mu > 0. \tag{6}$$

The capacity distribution corresponding to $\hat{w}(x)$ is

$$\hat{\phi}(\xi) = \frac{1}{B(\lambda, \mu)} \cdot \frac{1}{\xi^0} \left(\frac{\xi^0}{\xi} \right)^{\lambda + 1} \cdot \left(1 - \frac{\xi^0}{\xi} \right)^{\mu - 1}. \tag{7}$$

Table 1 summarises the mean and the maximum for both distributions; $\hat{w}(x)$ and $\hat{\phi}(\xi)$ have a maximum, and hence are bell-shaped, when $\lambda > 1$, $\mu > 1$ and $\lambda > 0$, $\mu > 1$, respectively.

The aggregate short-run production function corresponding to the capacity distribution (7) has the additional property to eq. (2) that it can be written as an explicit function of $l/\bar{l}$.[3] It is impossible, however, to derive its explicit form without further assumptions. In fact, its derivation only is analytically manageable when we have either $\mu = 1$ or $\lambda = 1$ or 2 [Muysken (1981b, p. 5)]. As we shall show below, these are the cases of the Cobb–Douglas function, the CES function and a VES function, respectively.

[3] Muysken (1981b, p. 5). This property allows one to derive the long-run production function as an envelope of the short-run production functions with respect to $\bar{l}$, Muysken (1981a).

3. The Cobb–Douglas, the CES and a VES production function [4]

$B(\beta, 1)$. Houthakker (1956) already proved that the capacity distribution underlying the Cobb–Douglas function is a Pareto distribution, which is a special form of the beta-distribution. However, a well-known problem is that one has to assume $\xi^0 = 0$ in order to derive the Cobb–Douglas production function. In the more realistic case of $\xi^0 > 0$, the Pareto capacity distribution:

$$\hat{\phi}(\xi) = (\beta/\xi^0)(\xi^0/\xi)^{\beta+1}, \qquad \beta > 1 \tag{8}$$

which is a transformed $B(\beta, 1)$-distribution, yields

$$y = 1 - \{1 - l/\bar{l}\}(\beta/(\beta - 1)). \tag{9}$$

This short-run production function has been studied in Kuipers (1970) for the case of $\beta = 2$.

$B(1, \sigma/(1 - \sigma))$. Levhari (1968) derived the capacity distribution corresponding to the CES function below:

$$y = \left[\alpha \cdot l^P + \beta\right]^{1/P}, \qquad P < 0, \quad 0 < \sigma = \frac{1}{1 - P} < 1. \tag{10}$$

The distribution is a transformed $B(1, \sigma/(1 - \sigma))$-distribution and has the following form: [5]

$$\phi(\xi) = h(z) = \sigma \cdot \beta^{\sigma/(\sigma-1)} \cdot \frac{1}{\xi^0} \cdot z^{(2-\sigma)/(1-\sigma)} \cdot (1 - z)^{(1-2\sigma)/(\sigma-1)},$$

$$0 < z = (\xi^0/\xi)^{1-\sigma} < 1. \tag{11}$$

This distribution is bell-shaped for $\sigma > \frac{1}{2}$; the greater σ, the flatter the distribution is and the further stretched out away from ξ^0. A drawback is that the distribution does not converge: $\bar{l}$ is infinite. Hence the amount of

[4] The trans-log production function is inconsistent with the distribution approach, Muysken (1981b, sect. 6).

[5] It is obvious that it is necessary to transform the distribution to $z = (\xi^0/\xi)^{1-\sigma}$ instead of $x = \xi^0/\xi$.

labour employed in an industry operating at full capacity is infinite.

$B(2, 1/(\alpha - 1))$. The production function:

$$y = a \cdot l - b \cdot l^{\alpha}, \qquad \alpha > 1 \tag{12}$$

is a function with a variable elasticity of substitution. The case of $\alpha = 2$ has been elaborated in Kuipers (1970), the case of $\alpha = 3/2$ in Muysken (1979). The corresponding capacity distribution is

$$\phi(\xi) = \frac{1}{\xi^0} \cdot \frac{1}{B\left(2, \frac{1}{\alpha - 1}\right)} \cdot \left(\frac{\xi^0}{\xi}\right)^3 \cdot \left(1 - \frac{\xi^0}{\xi}\right)^{1/(\alpha - 1) - 1} \tag{13}$$

which is a transformed $B(2, 1/(\alpha - 1))$-distribution. The distribution is bell-shaped for $1 < \alpha < 2$; the lower α, the flatter the distribution is and the further stretched out away from ξ^0.

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