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Independence of inadmissible strategies and best reply stability: a direct proof

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Abstract. Hillas (1990) introduced a definition of strategic stability based on perturbations of the best reply correspondence that satisfies all of the requirements given by Kohlberg and Mertens (1986). Hillas et al. (2001) point out though that the proofs of the iterated dominance and forward induction properties were not correct. They also provide a proof of the IIS property, a stronger version of both iterated dominance and forward induction, using the results of that paper. In this note we provide a direct proof of the IIS property.

Key words: Game Theory, Nash equilibrium, Stable sets, Admissibility of strategies

1. Introduction

The theory of strategic stability originates from the theory of refinements of Nash equilibrium. Refinements were usually designed to eliminate Nash equilibria that (for whatever reason) did not look like the "right" solution for the game under consideration. The problem with this approach was that this way refinements got designed on a rather ad hoc basis. They tended to cure one specific problem of Nash equilibrium, but usually still, like the notion of Nash equilibrium itself, did not solve various other problems.

The theory of strategic stability is geared towards a more systematic approach of equilibrium selection. First a number of desirable properties of the equilibrium to be selected is chosen. The next step is to try to find a notion of strategic stability that satisfies all these properties. The selection criterion is usually defined in terms of an appropriate way to perturb the game.

One such attempt, based on the requirements for strategic stability as they were originally formulated in Kohlberg and Mertens (1986), is given in Hillas (1991). He uses perturbations of the best reply correspondence of a game as a means to test for strategic stability. Hillas (1991) claims that the resulting type of strategic stability satisfies all requirements. Nevertheless, Hillas et. al (2001) point out that the proofs of iterated dominance and forward induction, both requirements involving the deletion of a certain type of pure strategy from the game, were not correct. They proceed to give an alternative proof of a property called *independence of inadmissible strategies*. This property implies both iterated dominance and forward induction.

Since the proof of the IIS property of best reply stability in Hillas et. al (2001) is based on results obtained earlier in that paper, the proof of this property is fairly elaborate though. In this note we present a direct, and much shorter, proof of the IIS property of best reply stability.

Notation: For $k \in \mathbb{N}$, $K = \{1, 2, \dots, k\}$ and $\mathbb{R}^k$ is the vector space of k-tuples of real numbers. For a finite set T, $\Delta(T)$ denotes the set of probability distributions on T. For $x \in \mathbb{R}^k$, $\|x\| = \max\{|x_i| | 1 \leq i \leq k\}$. For a set $C \subset \mathbb{R}^k$, $x \in C$ and $\varepsilon > 0$, $B_\varepsilon(x) = \{y \in C | \|x - y\| < \varepsilon\}$. For a set $S \subset C$ and $\varepsilon > 0$, $B_\varepsilon(S) = \bigcup_{x \in S} B_\varepsilon(x)$. For a subset C of $\mathbb{R}^k$, the convex hull of C is denoted by $\text{ch}(C)$. For two nonempty, compact sets C and D in $\mathbb{R}^k$, $d_H(C, D) = \max\{\|x - y\| \,|\, x \in C, y \in D\}$ is the Hausdorff distance between C and D.

2. Preliminaries

A (*finite n-person*) *game* (in normal form) is a pair

$$\Gamma = \langle A, u \rangle.$$

The finite set A is the Cartesian product $\prod_{i \in N} A_i$ where A_i is the set of pure strategies of player i. The vector $u = (u_i)_{i \in N}$ lists the payoff functions $u_i : A \to \mathbb{R}$ for the players i in N.

The set of mixed strategies of player i is $\Delta_i = \Delta(A_i)$. We will abuse notation and simply write a_i for the mixed strategy that puts all weight on the pure strategy a_i.

The expected payoff of a strategy profile $x \in \Delta = \prod_{j \in N} \Delta_j$ to player i is defined by

$$u_i(x) = \sum_{(a_1, \dots, a_n)} \prod_{j \in N} x_{j a_j} u_j(a_1, \dots, a_n).$$

The strategy profile where player i uses strategy $y_i \in \Delta_i$ and his opponents play $x_{-i} = (x_j)_{j \neq i}$ is denoted by $(x_{-i} | y_i)$. For player i and a strategy profile x in Δ a strategy $y_i \in \Delta_i$ is called a *best reply* to x if

$$u_i(x_{-i} | y_i) \geq u_i(x_{-i} | z_i)$$

holds for all $z_i \in \Delta_i$. A strategy profile $x \in \Delta$ is called a *Nash equilibrium* of Γ if x_i is a best reply to x for each player i. The set of best replies for player i to x is denoted by $\text{BR}_i(x)$. The correspondence $\text{BR}\colon \Delta \twoheadrightarrow \Delta$ defined by

$$\mathrm{BR}(x) = \prod_{j \in N} \mathrm{BR}_j(x) \qquad (x \in \Delta)$$

is called the *best reply correspondence* of Γ.

Selten (1975) introduced perfect equilibria as a refinement of the set of Nash equilibria of Γ. Take a positive number η and a completely mixed strategy profile x (i.e. all coordinates are positive). The profile x is called *η-perfect* if for each player $j \in N$ and pure strategy a_j in A_j,

$$x_{ja_j} < \eta \quad \text{whenever } u_j(x_{-j}|a_j) < u_j(x_{-j}|b_j) \quad \text{for some } b_j \in A_j.$$

A strategy profile x is called *perfect* if there is a sequence $(\eta_k)_{k \in \mathbb{N}}$ of positive real numbers converging to zero and a sequence $(x^k)_{k \in \mathbb{R}}$ of completely mixed strategy profiles converging to x such that x^k is η_k-perfect.

3. BR-stable sets of equilibria

Kohlberg and Mertens (1986) initiated a search for solution concepts that satisfy a number of stability requirements. They argued that these requirements were needed for a solution in order to be "well behaved". Such a "well behaved" solution concept is called a stability concept. However, none of the three candidate stability concepts introduced in their paper satisfied all the minimum requirements. This triggered a further search, in particular by Mertens (1989, 1991) and Hillas (1990). In his paper Hillas presented an alternative definition of stability, what we will call BR-stability. His definition is repeated here, together with some basic facts on BR-stable sets that will be useful in subsequent sections.

Let Γ be a game. The *perturbation space* $\mathscr{H}$ of Γ is the set of all (nonempty) closed- and convex-valued upper hemicontinuous correspondences $\varphi: \Delta \twoheadrightarrow \Delta$ equipped with the metric d, defined by

$$d(\varphi, \psi) = \max\{d_H(\varphi(x), \psi(x)) | x \in \Delta\} \qquad (\varphi, \psi \in \mathscr{H}).$$

A strategy profile $x \in \Delta$ is called a *fixed point* of $\varphi \in \mathscr{H}$ if $x \in \varphi(x)$. The set of fixed points of φ is denoted by $\mathrm{fix}(\varphi)$.

It is well known that the set of fixed points of an element of $\mathscr{H}$ is not empty (cf. Kakutani (1941)). Furthermore, the best reply correspondence BR of the game Γ is an element of $\mathscr{H}$ and its set of fixed points equals the set of Nash equilibria of the game Γ.

Definition 1. *A nonempty closed set $S \subset \Delta$ is called a* BR-*set of* Γ *if for every neighborhood V of S there exists an $\varepsilon > 0$ such that* $\mathrm{fix}(\varphi) \cap V$ *is not empty whenever* $d(\mathrm{BR}, \varphi) < \varepsilon$. A BR-*set is called* BR-*stable if it is a connected set of perfect equilibria.*

Along with a number of properties of BR-stable sets, their existence is established in Hillas (1990). In what is coming in this paper we need a characterization of BR-stability in terms of completely mixed perturbations.

A perturbation φ in $\mathscr{H}$ is called *completely mixed* if $\varphi(x)$ only contains completely mixed strategy profiles for every strategy profile $x \in \Delta$. We will need the following characterization of BR-sets in terms of completely mixed perturbations.

Lemma 1. *For a closed set $S \subset \Delta$, the following two statements are equivalent:*

(1) *S is a* BR-*set*
(2) *for every neighborhood V of S there exists an $\varepsilon > 0$ such that* $\operatorname{fix}(\varphi) \cap V$ *is not empty for any completely mixed element φ of $\mathscr{H}$ with $d(\mathrm{BR}, \varphi) < \varepsilon$.*

4. An extension theorem

The proof of the IIS property of BR-stable sets presented in this note is based on the following extension theorem for nonempty- compact-valued upper hemicontinuous correspondences.

Theorem 1. *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be nonempty and compact. Let X' be a closed subset of X. Let $\beta : X \twoheadrightarrow Y$ and $\varphi' : X' \twoheadrightarrow Y$ be nonempty-compact-valued upper hemicontinuous correspondences. Suppose further that $d_H(\beta(x), \varphi'(x)) < \delta$ for all x in X'. Then there exists a nonempty-compact-valued upper hemicontinuous correspondence*

$$\varphi : X \twoheadrightarrow Y$$

whose restriction to X' equals φ' and $d(\beta, \varphi) \leq 2\delta$.

Proof: For $z \in X'$ and $k \in \mathbb{N}$, define the open neighborhood $O_k(z) \subset X$ of z by

$$O_k(z) = B_{k^{-1}}(z) \cap \left\{x \in X | \beta(x) \subset B_{k^{-1}\delta}(\beta(z))\right\}.$$

Write $O_k = \bigcup_{z \in X'} O_k(z)$. Now define $A_0 = X \setminus O_1$ and for $k \in \mathbb{N}$, $A_k = O_k \setminus O_{k+1}$. Then, since X' is closed in X, it is easy to see that the sets

$$X', A_0, A_1, A_2, \ldots\ldots$$

form a partition of X. Define the correspondence $\varphi^* : X \twoheadrightarrow Y$ by

$$\varphi^*(x) = \begin{cases} B_{2\delta}(\beta(x)) & \text{if } x \in A_0 \\ B_{2\delta}(\beta(x)) \cap \bigcup_{\{z \in X' | x \in O_k(z)\}} \varphi'(z) & \text{if } x \in A_k \text{ for some } k \in \mathbb{N} \\ \varphi'(x) & \text{if } x \in X'. \end{cases}$$

Using the fact that $d(\beta, \varphi') < \delta$, it is straightforward to show that the values of φ^* are not empty. Let $\varphi : X \twoheadrightarrow Y$ be the correspondence whose graph is the closure of the graph of φ^* in $X \times Y$. Clearly, φ is upper hemicontinuous and nonempty- compact-valued, since its graph is closed in $X \times Y$. Claim: its restriction to X' equals φ' and $d(\beta, \varphi) \leq 2\delta$.

(a) First we will show that $\varphi(x) = \varphi'(x)$ for all $x \in X'$. To this end, take an x in X'. Note that $\varphi^*(x) = \varphi'(x)$. Now take a sequence $(x_m, y_m)_{m \in \mathbb{N}}$ in $X \times Y$ converging to (x, y) for which y_m is an element of $\varphi^*(x_m)$. It suffices to show that y is an element of $\varphi^*(x)$.
To this end, take a closed neighborhood F of $\varphi^*(x)$. It is sufficient to show that y is an element of F, since $\varphi^*(x) = \varphi'(x)$ is compact.

Since φ' is upper hemicontinuous on X', we can take $k \in \mathbb{N}$ such that for all elements z of X' in $B_{k^{-1}}(x)$ it holds that $\varphi'(z)$ is a subset of F. Furthermore, $O_{2k}(x)$ is a (non-empty) neighborhood of x, so we can take $M \in \mathbb{N}$ such that $x_m \in O_{2k}(x)$ whenever $m \geq M$. Now take such an $m \geq M$. Because x_m is an element of $O_{2k}(x)$, it is an element of either A_l for some $l \geq 2k$ or of X'. However, in both cases there is an $l \geq 2k$ such that

$$\varphi^*(x_m) \subset \bigcup_{\{z \in X' \mid x_m \in O_l(z)\}} \varphi'(z).$$

Therefore, since y_m is an element of $\varphi^*(x_m)$, we can find an $l \geq 2k$ and a $z_m \in X'$ such that $x_m \in O_l(z_m)$ and $y_m \in \varphi'(z_m)$. For this z_m,

$$\|x - z_m\| \leq \|x - x_m\| + \|x_m - z_m\| < (2k)^{-1} + l^{-1} \leq k^{-1}.$$

Hence, by the choice of k, $\varphi^*(z_m) = \varphi'(z_m)$ is a subset of F. Thus we get that y_m is an element of F and, since $m \geq M$ was arbitrary, also y is an element of F.

(b) Secondly we will show that $d(\beta, \varphi) \leq 2\delta$. The proof is divided in two parts.
   (b1) First we will prove that for all $x \in X$, $\varphi(x)$ is a subset of the closure of $B_{2\delta}(\beta(x))$. To this end, notice that $\varphi^*(x)$ is a subset of $B_{2\delta}(\beta(x))$ for all $x \in X$. Therefore the graph of φ, being the closure of the graph of φ^*, is a subset of the closure of the graph of the correspondence that assigns $B_{2\delta}(\beta(x))$ to x. This however implies that $\varphi(x)$ is a subset of the closure of $B_{2\delta}(\beta(x))$ for each x in X.
   (b2) Next we will prove that $\beta(x)$ is a subset of the closure of $B_{2\delta}(\varphi(x))$ for all $x \in X$. If x is an element of A_0, then $\beta(x)$ is clearly a subset of

$$B_{2\delta}(\beta(x)) \subset B_{2\delta}(\varphi^*(x)) \subset B_{2\delta}(\varphi(x)).$$

If x is an element of X', then – since $d_H(\beta(x), \varphi'(x)) < \delta$ by assumption – $\beta(x)$ is a subset of

$$B_{2\delta}(\varphi'(x)) = B_{2\delta}(\varphi^*(x)) \subset B_{2\delta}(\varphi(x)).$$

So suppose that x is an element of A_k for some $k \in \mathbb{N}$ and take an element y of $\beta(x)$. We will show that there exists an element w of $\varphi^*(x)$ with $\|y - w\| \leq 2\delta$.

Since x is an element of $A_k \subset O_k$, there is an element z of X' with $x \in O_k(z)$. Therefore y is an element of $\beta(x) \subset B_{k^{-1}\delta}(\beta(z))$ by the definition of $O_k(z)$. This implies that we can find an element w of $\varphi'(z)$ with $\|y - w\|_\infty < 2\delta$ since $d_H(\beta(z), \varphi'(z)) < \delta$ by assumption. However, since y is an element of $\beta(x)$, $\|y - w\|_\infty < 2\delta$ implies that w is also an element of $B_{2\delta}(\beta(x))$. Hence, w is an element of $\varphi^*(x)$ which is what we wanted to show. ■

Independence of inadmissible strategies

Originally Kohlberg and Mertens required that a stability concept should satisfy, among other conditions, iterated dominance and forward induction. However, both these conditions are implied by independence of inadmissible strategies and the extension theorem of the previous section enables us to prove the latter requirement. Therefore we will work with the independence of inadmissible strategies in this note instead of using the original requirements.

A strategy y_i of player i is an *admissible best reply* against an element $x \in \Delta$ if there is a sequence $(x^k)_{k \in \mathbb{N}}$ of completely mixed strategy profiles in Δ converging to x such that $y_i \in \mathrm{BR}_i(x^k)$ for all k. For a subset S of Δ, a pure strategy b_j of player j is called an *inadmissible reply* against S if b_j is not an admissible best reply against any strategy profile x in S.

Loosely speaking, independence of inadmissible strategies means that a stable set S of a game Γ remains stable when a pure inadmissible reply against S is deleted from Γ.

In order to get a more formal treatment, we need to introduce some terminology. Let $\Gamma = \langle A, u \rangle$ be a game and let b_j be a pure strategy of player j. The restricted game Γ' of Γ is the game that results when the pure strategy b_j of player j is removed from player j's set of pure strategies.

Notice that we can simply identify the game Γ' with the game that results if we allow player j only to use those strategies x_j in Δ_j for which $x_{jb_j} = 0$. Therefore, we will denote the set of strategy profiles in Δ in which the pure strategy b_j of player j is played with zero probability by Δ' and view Δ' as being the strategy space of Γ'.

Theorem 2 (IIS). *Let S be a* BR-*stable set of* Γ *and suppose that the pure strategy b_j of player j in* Γ *is an inadmissible reply against S. Then S is a subset of Δ' and a* BR-*set of the restricted game* Γ'.

Proof: Take a BR-stable set S of Γ. Since S is a collection of perfect equilibria and b_j is an inadmissible reply against S, it is clear that S is a subset of Δ'. Furthermore, let BR and BR$'$ denote the respective best reply correspondences of Γ and Γ' and similarly distinguish between $\mathscr{H}$ and $\mathscr{H}'$.

In order to show that S is even a BR-set of Γ', take a neighborhood V' of S in Δ'. Take a neighborhood V of S in Δ whose restriction to Δ' equals V'. Furthermore, since b_j is an inadmissible reply against S, we can w.l.o.g. assume V to be sufficiently small to guarantee that b_j is not a best reply against any completely mixed strategy profile in V. Let W denote the closure of V.

Now, since S is a BR-set of the game Γ, there exists an $\varepsilon > 0$ such that $\mathrm{fix}(\varphi) \cap V$ is not empty whenever $d(\mathrm{BR}, \varphi) < \varepsilon$.

Take such an ε and take a completely mixed element φ' in $\mathscr{H}'$ with $d(\mathrm{BR}', \varphi') < \frac{1}{2}\varepsilon$. By lemma 1 it suffices to show that φ' has a fixed point in V'.

First notice that the correspondence $\beta : W \twoheadrightarrow \Delta'$ defined by

$$\beta(x) := \mathrm{BR}(x) \cap \Delta' \qquad (x \in W)$$

is compact- and convex-valued upper hemicontinuous. Moreover, its values are not empty because b_j is not a best reply against any completely mixed strategy profile in V. Next, let W' be the restriction of W to Δ'. It is straightforward to check that $\beta(x)$ is a subset of $\mathrm{BR}'(x)$ for any element x of W'. However, again because b_j is not a best reply against any completely mixed strategy profile in V, the converse also holds. Hence, β equals BR$'$ on W' and we get that $d_H(\beta(x), \varphi'(x)) < \frac{1}{2}\varepsilon$ for all x in W'. Thus, taking $X = W$, $X' = W'$ and $Y = \Delta'$ theorem 1 tells us that there exists a non-empty- compact-valued and hemicontinuous extension

$$\varphi^* : W \twoheadrightarrow \Delta'$$

of φ' such that $d(\beta, \varphi^*) < \varepsilon$ on W. Of course we can in addition assume w.l.o.g. that φ^* has convex values. Now let

$$G := \{x \in \Delta \mid b_j \in \mathrm{BR}_j(x)\}$$

be the collection of strategy profiles against which b_j is a best reply. Furthermore, take an arbitrary completely mixed element β^* in $\mathscr{H}$ with $d(\mathrm{BR}, \beta^*) < \varepsilon$. Define a new correspondence φ from Δ to Δ by

$$\varphi(x) := \begin{cases} \beta^*(x) & \text{if } x \notin W \\ \varphi^*(x) & \text{if } x \in V \text{ and } x \notin G \\ \mathrm{ch}\Big(\varphi^*(x) \cup \beta^*(x)\Big) & \text{else.} \end{cases}$$

This is clearly an upper hemicontinuous correspondence whose values are not empty, convex and compact. Moreover, it is straightforward to check that $d(\mathrm{BR}, \varphi) < \varepsilon$. So, by the choice of ε it has a fixed point, say x^*, in V.

We will first show that x^* must be an element of Δ'. Suppose it is not. We will derive a contradiction.

So, we will first show that x^* is not an element of G. In order to do that, suppose that x^* is an element of G. In that case x^* is an element of the convex hull of $\varphi^*(x^*) \cup \beta^*(x^*)$. Thus, since all values of φ^* are subsets of Δ' and x^* is not by assumption, x^* can be written as

$$x^* = \lambda y + (1 - \lambda) z$$

for certain profiles y in $\varphi^*(x^*)$ and z in $\beta^*(x^*)$ and $0 \leq \lambda < 1$. In particular, since β^* is completely mixed and $\lambda < 1$, x^* must be completely mixed. However, since x^* is an element of V, this means that b_j cannot be a best reply against x^* by the choice of V. This contradicts the assumption that x^* is an element of G.

So, x^* is not an element of G. Thus, since it is an element of V, the definition of φ tells us that x^* must be an element of $\varphi^*(x^*)$. However, φ^* only takes values in Δ'. Hence, x^* must be an element of Δ'.

Finally, notice that this implies that x^* is a fixed point of φ' by the construction of φ^*. Moreover, since x^* is an element of both Δ' and V, it is necessarily an element of V'. Thus, φ' has a fixed point in V', which is what we wanted to show. ■

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