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The selectope for games with partial cooperation ☆

J.M. Bilbao[a], N. Jiménez[a], E. Lebrón[a], H. Peters[b,*]

[a]Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain
[b]Department of Economics, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, Netherlands

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Abstract

In this paper we define and study some solution concepts for games in which there does not have to be total cooperation between the players. In particular, we show the relations of these solution concepts with the selectope. In this way, we extend the work of Derks, Haller and Peters, METEOR Research Memorandum, Maastricht, RM/97/016 (revised version, 1998).

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1. Introduction

In game theory it is often assumed that either all players will cooperate or that the game will be played noncooperatively. However, there are many intermediate possibilities between universal cooperation and no cooperation. In this paper, we discuss a class of partial cooperation structures and develop a model of cooperative games in which only certain coalitions are allowed to form. This interesting idea was studied already in [12] and this line of research was continued by, among others, Owen [14], van den Nouweland and Borm [13], and Borm, Owen and Tijs [3].

Faigle [8] and Faigle and Kern [9] proposed a model of partial cooperation by considering a family of feasible coalitions and defining a game on this family. In our model, we will define the feasible coalitions by using combinatorial theory. The model of Faigle and Kern, and the classical model of a cooperative game are particular cases.

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* Corresponding author.
E-mail addresses: mbilbao@matinc.us.es (J.M. Bilbao), h.peters@ke.unimaas.nl (H. Peters)

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The central concept of the present paper is the selectope, first introduced in [11]. Briefly, the selectope contains all ways of distributing the so-called dividends of the coalitions between their members. Game theoretically, can be seen as imposing reasonable bounds on such distributions. Indeed, for a classical cooperative game the selectope is equal to the core of a game that arises by imposing on every coalition additionally the negative dividends of all coalitions in which members of the coalition under consideration participate. See [5] for further details.

Let us briefly outline the contents. We begin by introducing games defined on a family of subsets of a finite set N and in this context, different solution concepts are considered. In the first section, we define imputations, core and selectope for games defined on a family $\mathscr{L} \subseteq 2^N$ and we establish the relations between these solution concepts. If $\mathscr{L} = 2^N$ we obtain a classical cooperative game. In this case, we know that the core of a game v is included in the selectope. However, this is not true when $\mathscr{L} \neq 2^N$.

In the third section, some conditions on the family $\mathscr{L}$ are imposed and we prove that under those conditions, the selectope contains the core of the game. To this end, closure spaces are introduced, and in particular, intersecting families. It turns out, roughly, that the latter structure of partial cooperation is necessary and sufficient for the core to be always included in the selectope. Moreover, the concept of a closure space generalizes those partial cooperation structures where the set of feasible coalitions is closed under taking intersections.

In the last section, we define convex geometries and describe some of their fundamental properties. The convex geometry structure generalizes many forms of partial cooperation studied in the literature, as we will show by means of examples. Moreover, it facilitates the extension of the definition of marginal worth vectors to games with partial cooperation because the role of permutations of the player set in this definition is played by maximal chains. We introduce the Weber set as the convex hull of the marginal worth vectors, and we show that for games on convex geometries the Weber set is contained in the selectope. In classical cooperative games the Weber set plays a role as a core catcher. Moreover, for the important class of convex games it coincides with the core and, thus, implies a simple (greedy) algorithm to compute core elements. Here, as we will see, it singles out distributions according to consistent selectors from the selectope.

2. Solution concepts for games with partial cooperation

In this section we consider cooperative games in which there is a restriction on the formation of coalitions; the collection of feasible coalitions is denoted by $\mathscr{L}$. If we denote the set of players by $N = \{1, \dots, n\}$, then $\mathscr{L} \subseteq 2^N$ and we only assume that $\emptyset, N \in \mathscr{L}$ throughout the paper. When $\{i\} \in \mathscr{L}$ for all $i \in N$, the family $\mathscr{L}$ is called *atomic*. Sometimes, additional conditions are required on $\mathscr{L}$. The interpretation is that only the feasible coalitions are worthwhile in the game. A *game on* $\mathscr{L}$ is a function $v : \mathscr{L} \to \mathbb{R}$ such that $v(\emptyset) = 0$.

We denote by $\Gamma(\mathscr{L})$ the set of all games defined on the family $\mathscr{L}$. The set $\Gamma(\mathscr{L})$ is a vector space over $\mathbb{R}$, and there is a collection of games which deserves special consideration because it is a basis of $\Gamma(\mathscr{L})$. These *unanimity games* are defined for any nonempty $T \in \mathscr{L}$, and denoted by $\zeta_T : \mathscr{L} \to \{0,1\}$, where

$$\zeta_T(S) = \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise} \end{cases}$$

for every coalition $S \in \mathscr{L}$. To show linear independence of the collection of unanimity games is left as an easy exercise to the reader; the spanning property is then obvious, and the associated coefficients play a major role in this paper and are derived below.

One of the main topics dealt with in cooperative game theory, is to divide the amount $v(N)$ between the players if the grand coalition N is formed. The solution concepts that we study will prescribe distributions of the worth $v(N)$ among the players. Let $v : \mathscr{L} \to \mathbb{R}$ be a game on $\mathscr{L}$. For $x \in \mathbb{R}^n$ and $S \subseteq N$ define $x(S) = \sum_{i \in S} x_i$, with $x(\emptyset) = 0$.

We introduce the following solution concepts.

The set of *preimputations* of the game v: $\mathscr{L} \to \mathbb{R}$ is the set

$$I^*(\mathscr{L}, v) = \{x \in \mathbb{R}^n \colon x(N) = v(N)\},$$

and the *imputation set* is given by

$$I(\mathscr{L}, v) = \{x \in I^*(\mathscr{L}, v) \colon x_i \geqslant v(\{i\}) \text{ for all } i \in N \text{ with } \{i\} \in \mathscr{L}\}.$$

The *core* of the game v is the set

$$\mathrm{Core}(\mathscr{L}, v) = \{x \in \mathbb{R}^n \colon x(N) = v(N),\ x(S) \geqslant v(S) \text{ for all } S \in \mathscr{L}\}.$$

Note that the core of v is a polyhedron that is not necessarily bounded. However, if the family $\mathscr{L}$ is atomic then the core is a polytope.

Hammer et al. [11] introduced another solution concept which was called *selectope*. This solution concept was investigated by Derks et al. [5] in the case of cooperative games. We extend it for games defined on an atomic family of feasible coalitions $\mathscr{L}$.

First of all, we define the dividends for the game $v \in \Gamma(\mathscr{L})$, recursively, by

$$\triangle_v(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ v(S) - \sum_{\{T \in \mathscr{L} : T \subset S\}} \triangle_v(T) & \text{otherwise} \end{cases}$$

for every coalition $S \in \mathscr{L}$, where $\subset$ denotes strict inclusion. Obviously, for every $v \in \Gamma(\mathscr{L})$ and $S \in \mathscr{L}$,

$$v(S) = \sum_{\{T \in \mathscr{L} : T \subseteq S\}} \triangle_v(T)$$

and hence every $v \in \Gamma(\mathscr{L})$ can be uniquely represented by

$$v = \sum_{\{T \in \mathscr{L} : T \neq \emptyset\}} \triangle_v(T) \zeta_T,$$

where $\zeta_T \in \Gamma(\mathscr{L})$ is the unanimity game for the nonempty coalition $T \in \mathscr{L}$.

Table 1

α	$\{1,2\}$	$\{1,2,3\}$	$m_1^\alpha(v)$	$m_2^\alpha(v)$	$m_3^\alpha(v)$
1	1	1	3	0	0
2	1	2	2	1	0
3	1	3	2	0	1
4	2	1	1	2	0
5	2	2	0	3	0
6	2	3	0	2	1

Definition 1. Let $\mathscr{L} \subseteq 2^N$ be an atomic family such that $\emptyset, N \in \mathscr{L}$. A selector on $\mathscr{L}$ is a function $\alpha : \mathscr{L} \backslash \{\emptyset\} \to N$ with $\alpha(S) \in S$ for every nonempty coalition $S \in \mathscr{L}$.

We denote by $\mathscr{A}(\mathscr{L})$ the set of all selectors on $\mathscr{L}$. Note that the number of selectors on $\mathscr{L}$ is $\prod_{T \in \mathscr{L}, T \neq \emptyset} |T|$.

Definition 2. Let $\alpha \in \mathscr{A}(\mathscr{L})$ be a selector. The selection corresponding to α is the vector $m^\alpha(v) \in \mathbb{R}^n$ defined by

$$m_i^\alpha(v) = \sum_{\{S \in \mathscr{L} : \alpha(S) = i\}} \triangle_v(S)$$

for every $i \in N$ and $v \in \Gamma(\mathscr{L})$. The selectope for a game $v \in \Gamma(\mathscr{L})$ is given by

$$\mathrm{Sel}(\mathscr{L}, v) = \mathrm{conv}\{m^\alpha(v) \colon \alpha \in \mathscr{A}(\mathscr{L})\}.$$

Obviously, for every $v \in \Gamma(\mathscr{L})$, and $\alpha \in \mathscr{A}(\mathscr{L})$, we have $m^\alpha(v) \in I^*(\mathscr{L}, v)$ and hence, by convexity of $I^*(\mathscr{L}, v)$,

$$\mathrm{Sel}(\mathscr{L}, v) \subseteq I^*(\mathscr{L}, v).$$

Example. Let $N = \{1, 2, 3\}$ and consider the family

$$\mathscr{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}.$$

We define $v \in \Gamma(\mathscr{L})$ by $v(1) = v(2) = v(3) = 0$, $v(12) = 2$, and $v(123) = 3$. There are six selectors on $\mathscr{L}$ which are listed in Table 1 along with the corresponding selections.

The unitary coalitions are not in this table because all selectors satisfy $\alpha(\{i\}) = i$ for all $\{i\} \in \mathscr{L}$. For this game v, we have

$$\mathrm{Sel}(\mathscr{L}, v) = \mathrm{conv}\{(3,0,0), (2,1,0), (2,0,1), (1,2,0), (0,3,0), (0,2,1)\}.$$

Note that $\mathrm{Core}(\mathscr{L}, v) = \mathrm{Sel}(\mathscr{L}, v)$ as we illustrate in Fig. 1.

Example. Let $N = \{1, 2, 3, 4\}$ and consider the family

$$\mathscr{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{1,2,3,4\}\}$$

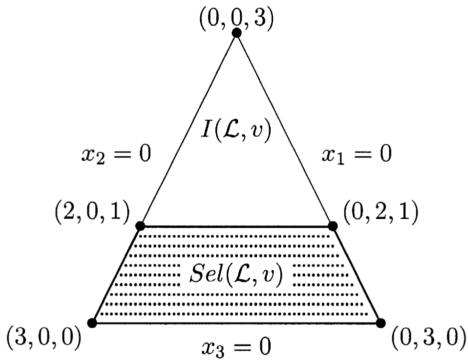

Fig. 1.

Table 2

α	$\{1,2\}$	$\{2,3\}$	$\{1,2,3,4\}$	$m_1^\alpha(v)$	$m_2^\alpha(v)$	$m_3^\alpha(v)$	$m_4^\alpha(v)$
1	1	2	1	1	3	−1	−1
2	1	2	2	3	1	−1	−1
3	1	2	3	3	3	−3	−1
4	1	2	4	3	3	−1	−3
5	1	3	1	1	−1	3	−1
6	1	3	2	3	−3	3	−1
7	1	3	3	3	−1	1	−1
8	1	3	4	3	−1	3	−3
9	2	2	1	−3	7	−1	−1
10	2	2	2	−1	5	−1	−1
11	2	2	3	−1	7	−3	−1
12	2	2	4	−1	7	−1	−3
13	2	3	1	−3	3	3	−1
14	2	3	2	−1	1	3	−1
15	2	3	3	−1	3	1	−1
16	2	3	4	−1	3	3	−3

and the game $v \in \Gamma(\mathscr{L})$ defined, for every $S \in \mathscr{L}\setminus\{\emptyset\}$, by

$$v(S) = \begin{cases} -1 & \text{if } |S| = 1, \\ 2 & \text{otherwise.} \end{cases}$$

If we calculate the dividends of the game, we get

$$\triangle_v(S) = \begin{cases} -1 & \text{if } |S| = 1, \\ 4 & \text{if } |S| = 2, \\ -2 & \text{if } S = N \end{cases}$$

and we show the selections in following Table 2.

On the other hand, Core($\mathscr{L}, v$) is the set of vectors $x \in \mathbb{R}^4$ such that

$$x_1 + x_2 + x_3 + x_4 = 2,$$
$$x_1 + x_2 \geqslant 2,$$
$$x_2 + x_3 \geqslant 2,$$
$$x_1, x_2, x_3, x_4 \geqslant -1.$$

Note that we have that Sel($\mathscr{L}, v$) $\nsubseteq$ Core($\mathscr{L}, v$) and Core($\mathscr{L}, v$) $\nsubseteq$ Sel($\mathscr{L}, v$). Specifically, $(3, 3, -3, -1) \in$ Sel($\mathscr{L}, v$)$\setminus$Core($\mathscr{L}, v$) since $x_3 \ngeqslant -1$ and $(0, 2, 0, 0) \in$ Core($\mathscr{L}, v$)$\setminus$ Sel($\mathscr{L}, v$) as it cannot be written as convex combination of $m^\alpha(v)$ with $\alpha \in \mathscr{A}(\mathscr{L})$, because $x_4 < 0$ for all $x \in$ Sel($\mathscr{L}, v$) (see Table 2).

The inclusion Sel($\mathscr{L}, v$) $\subseteq$ Core($\mathscr{L}, v$) characterizes (see below) the almost positive games. A game $v \in \Gamma(\mathscr{L})$ is called *almost positive* if the dividends of all non-unitary coalitions are nonnegative, that is

$$\triangle_v(S) \geqslant 0 \quad \text{for all } S \in \mathscr{L} \text{ with } |S| \geqslant 2.$$

Theorem 1. *Let $\mathscr{L} \subseteq 2^N$ be an atomic family with $\emptyset, N \in \mathscr{L}$ and let $v \in \Gamma(\mathscr{L})$. The following statements are equivalent*:

(a) Sel($\mathscr{L}, v$) $\subseteq$ Core($\mathscr{L}, v$).

(b) Sel($\mathscr{L}, v$) $\subseteq I(\mathscr{L}, v)$.

(c) *The game v is almost positive.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious, as by definition Core($\mathscr{L}, v$) $\subseteq I(\mathscr{L}, v)$.

For (b) $\Rightarrow$ (c), assume that there exists $S \in \mathscr{L}$ with $|S| \geqslant 2$ and $\triangle_v(S) < 0$ and consider a selector $\alpha \in \mathscr{A}(\mathscr{L})$ such that $\alpha(S) = i$ and $\alpha(T) \neq i$ for all $T \in \mathscr{L}$, $T \neq S$, $\{i\}$. Then $m_i^\alpha(v) = v(\{i\}) + \triangle_v(S) < v(\{i\})$, contradicting (b).

For (c) $\Rightarrow$ (a), let v be an almost positive game. We assert that $m^\alpha(v) \in$ Core($\mathscr{L}, v$) for every selector α. Indeed, for every nonempty coalition $S \in \mathscr{L}$, we have

$$\begin{aligned}\sum_{i \in S} m_i^\alpha(v) &= \sum_{i \in S} \sum_{\{T \in \mathscr{L} : \alpha(T) = i\}} \triangle_v(T) \\ &\geqslant \sum_{\{T \in \mathscr{L} : T \subseteq S\}} \triangle_v(T) \\ &= v(S),\end{aligned}$$

where the inequality follows because all dividends of the non-unitary coalitions are nonnegative. □

Although Core($\mathscr{L}, v$) $\subseteq$ Sel($\mathscr{L}, v$) has already been proved by Hammer et al. [11] in the case $\mathscr{L} = 2^N$, this result is not true when the family of the feasible coalitions $\mathscr{L} \neq 2^N$ as the above example shows. Even, Sel($\mathscr{L}, v$) $\subset$ Core($\mathscr{L}, v$) is possible. For instance, consider $\mathscr{L} \subset 2^N$ with $N = \{1, 2, 3, 4\}$ and $\emptyset, N, \{1, 2\}, \{2, 3\}, \{i\} \in \mathscr{L}$ for all $i \in N$, and the almost positive game v with $v(\{1, 2\}) = v(\{2, 3\}) = 1$, $v(N) = 4$, and

$v(S)=0$ otherwise. Then it is easily checked that $(0,1,0,3)\in \mathrm{Core}(\mathscr{L},v)$, whereas player 4 obtains at most $\triangle_v(N)=2$ in the selectope.

The inclusion $\mathrm{Core}(\mathscr{L},v)\subseteq \mathrm{Sel}(\mathscr{L},v)$ is true if we consider the family $\mathscr{L}$ with some additional conditions, as established in the next section.

3. Games on closure spaces

We now introduce a certain class of families of feasible coalitions. A family $\mathscr{L}$ of subsets of N with the properties

(1) $\emptyset\in\mathscr{L}$ and $N\in\mathscr{L}$,
(2) $A\in\mathscr{L}$ and $B\in\mathscr{L}$ implies that $A\cap B\in\mathscr{L}$,

is called a *closure space* on N. The elements of a closure space are called *closed sets*. When $\mathscr{L}$ is a closure space on N ordered by inclusion, it is a complete lattice.

Two special cases of closure spaces are *intersecting families* and *convex geometries*. Convex geometries will be defined in the next section.

The closure space $\mathscr{L}$ is called an intersecting family when

If $S,T\in\mathscr{L}$ with $S\cap T\neq\emptyset$, then $S\cup T\in\mathscr{L}$.

Note that all the closure spaces on $N=\{1,2,3\}$ are intersecting families.

In order to prove the next theorem, it is necessary to first define some concepts. Let $\mathscr{L}\subseteq 2^N$ be an atomic intersecting family and let $S\subseteq N$. We consider the set of all closed sets of $\mathscr{L}$ contained in S, i.e.,

$$\mathscr{L}_S=\{T\in\mathscr{L}:\ T\subseteq S\}.$$

Obviously, $\mathscr{L}_S\neq\emptyset$ since $\emptyset$, $\{i\}\in\mathscr{L}_S$ for all $i\in S$. Moreover, $\mathscr{L}_S$ is an atomic intersecting family on S if, and only if, $S\in\mathscr{L}$.

If we consider $\mathscr{L}_S$ ordered by inclusion, then $(\mathscr{L}_S,\subseteq)$ is a finite partially ordered set where the first element is $\emptyset$. Let $\Pi_{\mathscr{L}}(S)$ be the set of all maximal elements of $(\mathscr{L}_S,\subseteq)$. Every one of these elements is called a *maximal closed set* of S. Under these conditions we have the following property.

Proposition 2. *For every $S\subseteq N$, the coalitions of $\Pi_{\mathscr{L}}(S)$ form a partition of S.*

Proof. Note that if $S\in\mathscr{L}$, then $\Pi_{\mathscr{L}}(S)=\{S\}$ and therefore, the proposition is trivially true. If $S\notin\mathscr{L}$ and we write $\Pi_{\mathscr{L}}(S)=\{S_1,\ldots,S_k\}$, then $k\geqslant 2$ and $\bigcup_{i=1}^k S_i\subseteq S$. Also, $S\subseteq\bigcup_{i=1}^k S_i$ because $\mathscr{L}$ is an atomic family.

In order to prove that the coalitions in $\Pi_{\mathscr{L}}(S)$ form a partition of S, we claim that $S_i\cap S_j=\emptyset$ for all $1\leqslant i,\ j\leqslant k$, $i\neq j$. Indeed, if $S_i\cap S_j\neq\emptyset$, then $S_i\cup S_j\in\mathscr{L}$ since $\mathscr{L}$ is an intersecting family and also $S_i\cup S_j\subseteq S$, and hence $S_i\cup S_j\in\mathscr{L}_S$ but this is a contradiction because S_i and S_j are maximal closed sets of S. $\square$

Theorem 3. *Let $\mathscr{L} \subseteq 2^N$ be an atomic intersecting family and let $v \in \Gamma(\mathscr{L})$. Then* $\mathrm{Core}(\mathscr{L}, v) \subseteq \mathrm{Sel}(\mathscr{L}, v)$.

Proof. Assume that there exists $x \in \mathrm{Core}(\mathscr{L}, v)$ such that $x \notin \mathrm{Sel}(\mathscr{L}, v)$. By convexity and closedness of $\mathrm{Sel}(\mathscr{L}, v)$ and applying a separation theorem, there exists $y \in \mathbb{R}^n$ such that $z \cdot y > x \cdot y$ for every $z \in \mathrm{Sel}(\mathscr{L}, v)$. In particular, this holds for every $m^\alpha(v)$ with $\alpha \in \mathscr{A}(\mathscr{L})$. If the components of vector y are ordered in nonincreasing order

$$y_{i_1} \geqslant y_{i_2} \geqslant \cdots \geqslant y_{i_{n-1}} \geqslant y_{i_n},$$

then we have

$$\begin{aligned}
x \cdot y &= \sum_{j=1}^{n} x_{i_j} y_{i_j} = y_{i_n} \sum_{j=1}^{n} x_{i_j} + \sum_{k=1}^{n-1} (y_{i_k} - y_{i_{k+1}}) \sum_{j=1}^{k} x_{i_j} \\
&\geqslant y_{i_n} v(N) + \sum_{k=1}^{n-1} (y_{i_k} - y_{i_{k+1}}) \sum_{S \in \Pi_{\mathscr{L}}(\{i_1, i_2, \ldots, i_k\})} v(S) \\
&= y_{i_1} v(\{i_1\}) + \sum_{j=2}^{n} y_{i_j} \left(\sum_{S \in \Pi_{\mathscr{L}}(\{i_1, \ldots, i_j\})} v(S) - \sum_{S \in \Pi_{\mathscr{L}}(\{i_1, \ldots, i_{j-1}\})} v(S) \right) \\
&= y_{i_1} \triangle_v (\{i_1\}) + \sum_{j=2}^{n} y_{i_j} \left(\sum_{S \in \Pi_{\mathscr{L}}(\{i_1, \ldots, i_j\})} \sum_{\{T \in \mathscr{L} : T \subseteq S\}} \triangle_v(T) \right. \\
&\quad \left. - \sum_{S \in \Pi_{\mathscr{L}}(\{i_1, \ldots, i_{j-1}\})} \sum_{\{T \in \mathscr{L} : T \subseteq S\}} \triangle_v(T) \right) \\
&= y_{i_1} \triangle_v (\{i_1\}) + \sum_{j=2}^{n} y_{i_j} \left(\sum_{\{T \in \mathscr{L} : T \subseteq \{i_1, \ldots, i_j\}\}} \triangle_v(T) \right. \\
&\quad \left. - \sum_{\{T \in \mathscr{L} : T \subseteq \{i_1, \ldots, i_{j-1}\}\}} \triangle_v(T) \right) \\
&= \sum_{j=1}^{n} y_{i_j} \left(\sum_{\{T \in \mathscr{L} : T \subseteq \{i_1, i_2, \ldots, i_j\}, i_j \in T\}} \triangle_v(T) \right),
\end{aligned}$$

where the inequality follows because $x \in \mathrm{Core}(\mathscr{L}, v)$, the y_{i_j} are nonincreasing, and by Proposition 2, and the before last equality also follows by Proposition 2. Obviously, it is sufficient to take the selector $\alpha \in \mathscr{A}(\mathscr{L})$ such that

$$m^\alpha_{i_j}(v) = \sum_{\{T \in \mathscr{L} : T \subseteq \{i_1, i_2, \ldots, i_j\},\ i_j \in T\}} \triangle_v(T)$$

for all $1 \leqslant j \leqslant n$ in order to have a contradiction. This selector $\alpha \in \mathscr{A}(\mathscr{L})$ is defined by

$$\alpha(S) = i_k \quad \text{for all } S \in \mathscr{L},$$

where $k = \max\{p\colon i_p \in S\}$. $\square$

It should be noted that the proof of Theorem 3 is closely related to the proof in [4] of the inclusion of the core in the Weber set for games on 2^N. See also Section 4 below.

There is also a converse to Theorem 3.

Theorem 4. *Let* $\mathscr{L} \subseteq 2^N$ *be an atomic closure space such that* $\mathrm{Core}(\mathscr{L}, v) \subseteq \mathrm{Sel}(\mathscr{L}, v)$ *for all* $v \in \Gamma(\mathscr{L})$. *Then* $\mathscr{L}$ *is an intersecting family.*

Proof. Suppose $\mathscr{L}$ is not an intersecting family. Then there are $S, T \in \mathscr{L}$ with $S \cap T \neq \emptyset$ but $S \cup T \notin \mathscr{L}$. Let the number of players in $N \backslash (S \cup T)$ be equal to $k > 0$. Define v by $v(M) = 1$ if $S \subseteq M$ or $T \subseteq M$ but $S \cup T \nsubseteq M$, $v(M) = 2$ if $S \cup T \subseteq M$ but $M \neq N$, $v(N) = k + 1$, and $v(M) = 0$ otherwise. Then the dividends of S and T are equal to 1, the dividend of N is equal to $k - 1$, and all other dividends are equal to 0. Fix a player $i \in S \cap T$. Then it is easily seen that the vector x with $x_i = 1$, $x_j = 1$ for all $j \in N \backslash (S \cup T)$, and $x_j = 0$ otherwise, is a core element. In this core element the players outside $S \cup T$ receive k together. In the selectope, however, they receive at most the dividend $k - 1$ of N together. Hence, the core is not contained in the selectope. (Observe that in this example it is indeed essential that $S \cup T \notin \mathscr{L}$, because otherwise x as above would not be a core element.) $\square$

As has already been indicated, the game $v \in \Gamma(\mathscr{L})$ can be written as a linear combination of unanimity games, where the coefficients are the dividends of the coalitions in $\mathscr{L}$. So, if we consider

$$v^+ = \sum_{\{S \in \mathscr{L} : \triangle_v(S) > 0\}} \triangle_v(S)\zeta_S \quad \text{and} \quad v^- = \sum_{\{S \in \mathscr{L} : \triangle_v(S) < 0\}} -\triangle_v(S)\zeta_S,$$

then we have the decomposition $v = v^+ - v^-$.

Theorem 5. *Let* $\mathscr{L} \subseteq 2^N$ *be an atomic intersecting family and* $v \in \Gamma(\mathscr{L})$. *Then*

(a) $\mathrm{Sel}(\mathscr{L}, v^+) = \mathrm{Core}(\mathscr{L}, v^+)$ *and* $\mathrm{Sel}(\mathscr{L}, v^-) = \mathrm{Core}(\mathscr{L}, v^-)$.

(b) $\mathrm{Sel}(\mathscr{L}, v) = \mathrm{Core}(\mathscr{L}, v^+) - \mathrm{Core}(\mathscr{L}, v^-) = \{x \in \mathbb{R}^n\colon x = y - z$ *with* $y \in \mathrm{Core}(\mathscr{L}, v^+)$ *and* $z \in \mathrm{Core}(\mathscr{L}, v^-)\}$.

Proof. (a) This is a direct consequence of Theorems 1 and 3.

(b) For every $\alpha \in \mathscr{A}(\mathscr{L})$, we have $m\alpha(v) = m\alpha(v^+) - m\alpha(v^-)$, and by part (a), $m\alpha(v) \in \mathrm{Core}(\mathscr{L}, v^+) - \mathrm{Core}(\mathscr{L}, v^-)$. Therefore,

$$\mathrm{Sel}(\mathscr{L}, v) \subseteq \mathrm{Core}(\mathscr{L}, v^+) - \mathrm{Core}(\mathscr{L}, v^-).$$

In order to prove the converse inclusion it is, in view of part (a), sufficient to prove that for any two selectors $\alpha, \beta \in \mathscr{A}(\mathscr{L})$, we have $m^{\alpha}(v^{+}) - m^{\beta}(v^{-}) \in \mathrm{Sel}(\mathscr{L}, v)$. We define $\gamma \in \mathscr{A}(\mathscr{L})$ by

$$\gamma(S) = \begin{cases} \alpha(S) & \text{if } \triangle_v(S) \geqslant 0, \\ \beta(S) & \text{if } \triangle_v(S) < 0 \end{cases}$$

for every nonempty $S \in \mathscr{L}$, and thus $m^{\gamma}(v) = m^{\alpha}(v^{+}) - m^{\beta}(v^{-})$. Therefore, $m^{\alpha}(v^{+}) - m^{\beta}(v^{-}) \in \mathrm{Sel}(\mathscr{L}, v)$. □

Note that if family $\mathscr{L}$ is an atomic family such that $\emptyset$, $N \in \mathscr{L}$, but not necessarily an intersecting family, then the assertions in this theorem hold only with $\subseteq$ instead of $=$.

4. Games on convex geometries

In this section, we define the concept of convex geometry (see [6]) and we describe some of their fundamental properties. We introduce the Weber set for games on convex geometries and investigate its relation with the selectope. A closure space $\mathscr{L}$ is a *convex geometry* on N if it satisfies the one-player extension property, i.e., If $A \neq N$ is a closed set, then $A \cup \{i\}$ is closed for some $i \in N \backslash A$.

An element in a convex geometry $\mathscr{L}$ is called *convex set*. For $A \subseteq N$, an element $a \in A$ is an *extreme point* of A if $a \notin \overline{A \backslash a}$. (Here, $\bar{B}$ is the smallest element of $\mathscr{L}$ containing $B \subseteq N$. This is well defined because $\mathscr{L}$ is closed under taking intersections.) For a closed set $A \in \mathscr{L}$ this is equivalent to $A \backslash a \in \mathscr{L}$. Moreover, for $A, B \in \mathscr{L}$ with $B \subseteq A$, any extreme point of A belonging to B is also an extreme point of B. Let $\mathrm{ex}(A)$ be the set of all extreme points of A. The convex geometries are the abstract closure spaces satisfying the finite Minkowski–Krein–Milman property: *Every closed set is the closure of its extreme points* [6]. The following result shows that convex geometries have some properties of Euclidean convexity.

Theorem 6. *Let $-: 2^N \to 2^N$ be a closure operator on N and let $\mathscr{L}$ be the family of its closed sets. Then the following statements are equivalent*:

(a) *$\mathscr{L}$ is a convex geometry.*

(b) *$a, b \notin \bar{A}$ and $b \in \overline{A \cup a}$ imply $x \notin \overline{A \cup b}$, for every $A \subseteq N$ and all $a, b \in N$ with $a \neq b$.*

(c) *For every closed set $C \subseteq N$, $C = \overline{\mathrm{ex}(C)}$.*

Proof. See [6, Theorem 2.1]. □

We give some examples which exhibit how convex geometries have already arisen in other papers on games with partial cooperation.

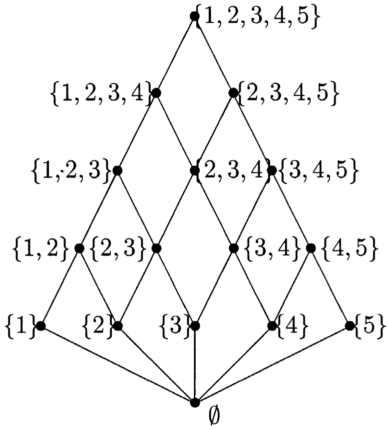

Fig. 2. A convex geometry Co({1, 2, 3, 4, 5}).

Example. A subset S of a poset (partially ordered set) $(P, \leqslant)$ is *convex* whenever $a \in S$, $b \in S$ and $a \leqslant b$ imply $[a, b] \subseteq S$. The convex subsets of any poset P form a closure system $\mathrm{Co}(P)$. If P (or, equivalently $\mathrm{Co}(P)$) is finite, then each element is between a maximal and a minimal one. If $C \in \mathrm{Co}(P)$ then $\mathrm{ex}(C)$ is the union of the maximal and minimal elements of C. Moreover, $\mathrm{Co}(P)$ is a convex geometry [2, Theorem 3]. Edelman [7] studies *voting games* such that the feasible coalitions are the convex sets in $\mathrm{Co}(P)$, where $(P, \leqslant)$ is the chain defined by the policy order (see Fig. 2).

Example. A graph $G = (N, E)$ is *connected* if any two vertices can be joined by a path. A maximal connected subgraph of G is a *component* of G. A *cutvertex* is one whose removal increases the number of components, and a *bridge* is an edge with the same property. A graph is 2-*connected* if it is connected, has at least 3 vertices and contains no cutvertex. A subgraph B of a graph G is a *block* of G if either B is a bridge or else it is a maximal 2-connected subgraph of G. A graph G is a *block graph* if every block is a complete graph. The block graphs are called *cycle-complete* graphs in [13].

A *communication situation* is a triple (N, G, v), where (N, v) is a game and $G = (N, E)$ is a graph. This concept was first introduced in [12], and investigated in [3]. If $G = (N, E)$ is a connected block-graph, then the family of all coalitions of N that induce connected subgraph

$$\mathscr{L} = \{S \subseteq N\colon\ (S, E(S)) \text{ is connected}\},$$

is a convex geometry [6, Theorem 3.7].

Example. Let $(P, \leqslant)$ be a poset (with, as usual in this paper, P finite). For any $X \subseteq P$,

$$X \mapsto \bar{X} := \{y \in P\colon\ y \leqslant x \text{ for some } x \in X\},$$

defines a closure operator on P. Its closed sets are the *order ideals* (down sets) of P, and we denote this lattice by $J(P)$. Since the union and intersection of order ideals is again an order ideal, it follows that $J(P)$ is a sublattice of 2^P. Then $J(P)$ is a distributive lattice and so, $J(P)$ is a convex geometry closed under set-union and $\mathrm{ex}(S)$ is the set of all maximal points $\mathrm{Max}(S)$ of the subposet $S \in J(P)$. There is a 1-1 correspondence between antichains of P and order ideals. The games $(\mathscr{C}, v)$ and $(\mathscr{A}, c)$ of Faigle and Kern [9,10], where $\mathscr{C}$ is the family of down sets of P and $\mathscr{A}$ is the set of antichains of P, are games on distributive lattices.

Edelman and Jamison defined a *compatible ordering* of a convex geometry $\mathscr{L} \subseteq 2^N$ as a total ordering of the elements of N, $i_1 < i_2 < \cdots < i_n$ such that

$$\{i_1, i_2, \ldots, i_k\} \in \mathscr{L} \quad \text{for all } 1 \leqslant k \leqslant n.$$

A compatible ordering of $\mathscr{L}$ corresponds to a *maximal chain* of $\mathscr{L}$. Here, a maximal chain C of $\mathscr{L}$ is an ordered collection of convex sets

$$C : (\emptyset =)\ C_0 \subset C_1 \subset \cdots \subset C_{n-1} \subset C_n\ (=N).$$

We denote by $\mathscr{C}(\mathscr{L})$ the set of all maximal chains of $\mathscr{L}$. Note that if $\mathscr{L} = 2^N$, then there are $n!$ maximal chains.

Thus, in every maximal chain of $\mathscr{L}$, there is a minimal convex set that contains a player i. Moreover, i is extreme in this convex set. Thus, for every $i \in N$ and $C \in \mathscr{C}(\mathscr{L})$, we denote by $C(i)$ the minimal convex set in chain C which contains player i, i.e.,

$$C(i) = \{j \in N\colon j \leqslant i \text{ in the chain } C\}.$$

In other words, $C(i)$ is the set of the predecessors of player i with respect to chain C together with player i. It is clear that $i \in \mathrm{ex}(C(i))$ since $C(i) \setminus i \in \mathscr{L}$.

Let $\mathscr{L}$ be a convex geometry on N, $C \in \mathscr{C}(\mathscr{L})$ and $v \in \Gamma(\mathscr{L})$. The *marginal worth vector* with respect to chain C in game v is the vector $a^C(v) \in \mathbb{R}^N$ given by

$$a_i^C(v) = v(C(i)) - v(C(i) \setminus i) \quad \text{for all } i \in N$$

and the Weber set is defined by

$$\mathrm{Weber}(\mathscr{L}, v) = \mathrm{conv}\{a^C(v)\colon C \in \mathscr{C}(\mathscr{L})\}.$$

For an atomic convex geometry $\mathscr{L}$ on N we will investigate the relation between the Weber set and the selectope. The following result establishes connections between selections corresponding to selectors on $\mathscr{L}$ and the marginal worth vectors associated with the maximal chains of $\mathscr{L}$.

Proposition 7. *Let $C \in \mathscr{C}(\mathscr{L})$ be a maximal chain and let $\alpha : \mathscr{L} \setminus \{\emptyset\} \to N$ be defined for every nonempty coalition $S \in \mathscr{L}$, by*

$$\alpha(S) = j \quad \text{where } j \in S \text{ and } S \subseteq C(j).$$

Then α is a selector and $m^\alpha(v) = a^C(v)$ for every $v \in \Gamma(\mathscr{L})$.

Proof. First of all, we prove that α is well defined. For every nonempty coalition $S \in \mathscr{L}$, it is clear that there exists a unique $j \in S$ such that $S \subseteq C(j)$. Here, j is the last element of S that is incorporated in the maximal chain C, i.e., this element $j \in S$ is such that $C(k) \subseteq C(j)$ for all $k \in S$. Moreover, $j \in \text{ex}(S)$ since

$$\left.\begin{array}{l} C(j)\backslash j \in \mathscr{L} \\ S \in \mathscr{L} \end{array}\right\} \Rightarrow (C(j)\backslash j) \cap S = S\backslash j \in \mathscr{L}.$$

Therefore, α is well defined. Now, for every $i \in N$ and $v \in \Gamma(\mathscr{L})$, we have

$$\begin{aligned} a_i^C(v) &= v(C(i)) - v(C(i)\backslash i) \\ &= \sum_{\{T \in \mathscr{L}: T \subseteq C(i)\}} \triangle_v(T) - \sum_{\{T \in \mathscr{L}: T \subseteq C(i)\backslash i\}} \triangle_v(T) \\ &= \sum_{\{T \in \mathscr{L}: T \subseteq C(i), i \in T\}} \triangle_v(T) \\ &= \sum_{\{T \in \mathscr{L}: \alpha(T) = i\}} \triangle_v(T) \\ &= m_i^\alpha(v). \qquad \square \end{aligned}$$

As the selectope for game $v \in \Gamma(\mathscr{L})$ is a convex set, one consequence of this Proposition is that $\text{Weber}(\mathscr{L}, v) \subseteq \text{Sel}(\mathscr{L}, v)$.

In order to prove a converse of Proposition 7, we have to impose a condition on the selector.

Definition 3. A selector $\alpha \in \mathscr{A}(\mathscr{L})$ is called consistent if it satisfies the following conditions:

1. $\alpha(S) \in \text{ex}(S)$ for all $S \in \mathscr{L}\backslash\{\emptyset\}$.
2. For all S, $T \in \mathscr{L}$, if $S \subset T$ and $\alpha(T) \in S$, then $\alpha(S) = \alpha(T)$.

Note that if we take a maximal chain $C \in \mathscr{C}(\mathscr{L})$ and we define the selector α as in the above proposition, then α is consistent. Moreover, different chains correspond to different selectors.

Theorem 8. *Let $\mathscr{L}$ be an atomic convex geometry and let $\alpha \in \mathscr{A}(\mathscr{L})$. Then α is consistent if and only if there exists a maximal chain $C \in \mathscr{C}(\mathscr{L})$ such that $m^\alpha(v) = a^C(v)$ for every $v \in \Gamma(\mathscr{L})$. In this case, the maximal chain*

$$C: \emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset \{i_1, \ldots, i_{n-1}\} \subset \{i_1, \ldots, i_n\} = N$$

is unique and it is recursively defined by

$$\begin{aligned} i_n &= \alpha(N), \\ i_k &= \alpha(N\backslash\{i_n, i_{n-1}, \ldots, i_{k+1}\}) \quad \textit{for all } 1 \leqslant k < n. \end{aligned}$$

Proof. First of all, note that for every unanimity game $\zeta_S \in \Gamma(\mathscr{L})$, we have

$$\triangle_{\zeta_S}(S) = 1 \quad \text{and} \quad \triangle_{\zeta_S}(R) = 0 \text{ for } R \in \mathscr{L},\ R \neq S,$$

because $\{\zeta_T\colon T \in \mathscr{L}, T \neq \emptyset\}$ is a basis of $\Gamma(\mathscr{L})$ and, for every $v \in \Gamma(\mathscr{L})$, we have

$$v = \sum_{\{T \in \mathscr{L}: T \neq \emptyset\}} \triangle_v(T)\zeta_T.$$

($\Leftarrow$) Let $\alpha \in \mathscr{A}(\mathscr{L})$ be a selector such that there exists a maximal chain $C \in \mathscr{C}(\mathscr{L})$ satisfying $m^\alpha(v) = a^C(v)$ for all $v \in \Gamma(\mathscr{L})$. We assert that α is consistent. In order to prove the first condition of consistency, we consider the unanimity game $\zeta_T \in \Gamma(\mathscr{L})$. If $\alpha(T) = i$, we have

$$m_i^\alpha(\zeta_T) = \sum_{\{R \in \mathscr{L}: \alpha(R) = i\}} \triangle_{\zeta_T}(R) = 1$$

and by hypothesis, $m_i^\alpha(\zeta_T) = a_i^C(\zeta_T)$. Therefore

$$\zeta_T(C(i)) - \zeta_T(C(i)\backslash i) = 1.$$

Hence, $T \subseteq C(i)$ but $T \nsubseteq C(i)\backslash i$, i.e., i is the last element of T in the order of C. So $\alpha(T) \in T$ and $T \subseteq C(\alpha(T))$, and thus $\alpha(T) \in \mathrm{ex}(T)$ because $\alpha(T) \in \mathrm{ex}(C(\alpha(T)))$.

Now, we prove the second condition of consistency. Let $S \subset T$ and $\alpha(T) = i$ with $i \in S$. Since $i \in \mathrm{ex}(T)$ and $S \subset T$ we have $i \in \mathrm{ex}(S)$. Then $S \subset C(i)$ since $T \subseteq C(i)$ and we have

$$a_i^C(\zeta_S) = \zeta_S(C(i)) - \zeta_S(C(i)\backslash i) = 1$$

and thus

$$m_i^\alpha(\zeta_S) = \sum_{\{R \in \mathscr{L}: \alpha(R) = i\}} \triangle_{\zeta_S}(R) = 1.$$

Hence, $i = \alpha(S)$ and so, α is consistent.

($\Rightarrow$) Let $\alpha \in \mathscr{A}(\mathscr{L})$ be a consistent selector. First, we prove that there exists a maximal chain $C \in \mathscr{C}(\mathscr{L})$ which satisfies $a^C(v) = m^\alpha(v)$ for every $v \in \Gamma(\mathscr{L})$. We consider the following maximal chain:

$$C : \emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset \{i_1, \ldots, i_{n-1}\} \subset \{i_1, \ldots, i_n\} = N$$

or equivalently

$$C : \emptyset \subset C(i_1) \subset C(i_2) \subset \cdots \subset C(i_{n-1}) \subset C(i_n) = N,$$

where

$$i_n = \alpha(N),$$
$$i_k = \alpha(N\backslash\{i_n, i_{n-1}, \ldots i_{k+1}\}) \quad \text{for all } 1 \leqslant k < n.$$

Note that $N, N\backslash\{i_n\}, \ldots, N\backslash\{i_n, i_{n-1}, \ldots i_{k+1}\} \in \mathscr{L}$ because α is consistent. Moreover, the selector α coincides with the selector defined in Proposition 7 for this chain, i.e., if $\beta \in \mathscr{A}(\mathscr{L})$ is such that for all $S \in \mathscr{L}$, $S \neq \emptyset$,

$$\beta(S) = j \quad \text{where } j \in S \text{ and } S \subseteq C(j),$$

then $\alpha = \beta$. Indeed, it is clear that $\alpha(S) = \beta(S)$ for every nonempty coalition $S \in C$; if $S \notin C$ then $\beta(S) = i \in \text{ex}(S)$ and $S \subset C(i)$ where $C(i)$ is the least convex set in the chain that contains i. Since $S \subset C(i)$ and $\alpha(C(i)) = i \in \text{ex}(S)$, the consistency of the selector α implies that $\alpha(S) = i$. Thus $\beta(S) = \alpha(S)$ for every nonempty coalition $S \in \mathscr{L}$.

Applying Proposition 7 for the selector α, we obtain $a^C(v) = m^\alpha(v)$ for every $v \in \Gamma(\mathscr{L})$.

In order to prove that chain C is the unique chain such that $a^C(v) = m^\alpha(v)$ for all $v \in \Gamma(\mathscr{L})$, we consider the unanimity game $\zeta_N \in \Gamma(\mathscr{L})$. For every $j \in N$, we have

$$m_j^\alpha(\zeta_N) = \sum_{\{R \in \mathscr{L} : \alpha(R) = j\}} \triangle_{\zeta_N}(R) = \begin{cases} 1 & \text{if } \alpha(N) = j, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$a_j^C(\zeta_N) = \zeta_N(C(j)) - \zeta_N(C(j) \backslash j) = \begin{cases} 1 & \text{if } C(j) = N, \\ 0 & \text{otherwise,} \end{cases}$$

so, as $\alpha(N) = i_n$, the vectors coincide if and only if $C(i_n) = N$.

Next, consider $\zeta_{N \backslash \{i_n\}} \in \Gamma(\mathscr{L})$. For every $j \in N$, we have

$$m_j^\alpha(\zeta_{N \backslash \{i_n\}}) = \sum_{\{R \in \mathscr{L} : \alpha(R) = j\}} \triangle_{\zeta_{N \backslash \{i_n\}}}(R) = \begin{cases} 1 & \text{if } \alpha(N \backslash \{i_n\}) = j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} a_j^C(\zeta_{N \backslash \{i_n\}}) &= \zeta_{N \backslash \{i_n\}}(C(j)) - \zeta_{N \backslash \{i_n\}}(C(j) \backslash j) \\ &= \begin{cases} 1 & \text{if } C(j) \supseteq N \backslash \{i_n\} \text{ and } C(j) \backslash j \not\supseteq N \backslash \{i_n\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that if $C(j) \supseteq N \backslash \{i_n\}$ and $C(j) \backslash j \not\supseteq N \backslash \{i_n\}$, then $C(j)$ is a convex set with n or $n-1$ elements, but in every maximal chain there is only one convex set with k elements, for all $1 \leqslant k \leqslant n$, and the convex set of n elements is excluded. Therefore,

$$a_j^C(\zeta_{N \backslash \{i_n\}}) = \begin{cases} 1 & \text{if } C(j) = N \backslash \{i_n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, since $\alpha(N \backslash \{i_n\}) = i_{n-1}$, we have $a^C(\zeta_{N \backslash \{i_n\}}) = m^\alpha(\zeta_{N \backslash \{i_n\}})$ if and only if $C(i_{n-1}) = N \backslash \{i_n\}$. By repeating this argument, the maximal chain C is obtained. □

Example. Let $N = \{1, 2, 3\}$ and consider the convex geometry

$$\mathscr{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$$

Table 3

α	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$	$m_1^\alpha(v)$	$m_2^\alpha(v)$	$m_3^\alpha(v)$
1	1	2	3	2	2	−1
2	2	2	3	0	4	−1
3	1	3	3	2	0	1
4	2	3	3	0	2	1
5	1	2	1	1	2	0
6	2	2	1	−1	4	0
7	1	3	1	1	0	2
8	2	3	1	−1	2	2
9	1	2	2	2	1	0
10	2	2	2	0	3	0
11	1	3	2	2	−1	2
12	2	3	2	0	1	2

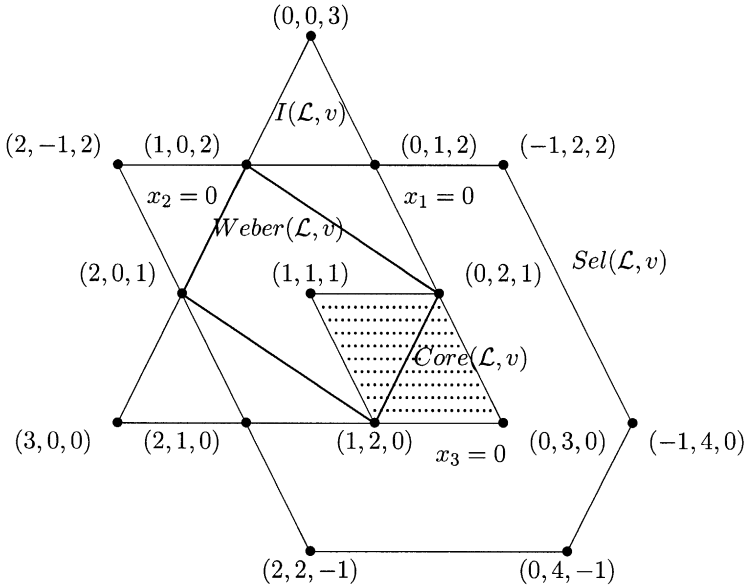

Fig. 3.

and the game $v \in \Gamma(\mathscr{L})$ given by $v(1)=v(2)=v(3)=0$, $v(12)=v(23)=2$, and $v(123)=3$. There are 12 selectors and are given in Table 3.

Note that in this example, selectors 3–5, and 7 are consistent and the selections corresponding to these selectors coincide with the marginal worth vectors with respect to the four maximal chains in $\mathscr{L}$. In Fig. 3, we show the position of the Weber set, the selectope and the core of the game v in the above example.

In this example, note that the Weber set is strictly contained in the selectope; in general, $\text{Weber}(\mathscr{L}, v) \neq \text{Sel}(\mathscr{L}, v)$. Also, $\text{Core}(\mathscr{L}, v) \subseteq \text{Sel}(\mathscr{L}, v)$.

It is worth noting that the inclusion of $\text{Core}(\mathscr{L}, v) \subseteq \text{Weber}(\mathscr{L}, v)$ (see [15,4]) is not true if (the convex geometry) $\mathscr{L} \neq 2^N$, as the above example shows (see also

[1]). In view of this fact, the proof of $\mathrm{Core}(\mathscr{L}, v) \subseteq \mathrm{Sel}(\mathscr{L}, v)$ cannot be supported by $\mathrm{Weber}(\mathscr{L}, v) \subseteq \mathrm{Sel}(\mathscr{L}, v)$.

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