Rev. Econ. Design 6, 85–98 (2001)

Review of Economic Design

Uniform allocation and reallocation revisited

Bettina Klaus

Department of Economics, University of Nebraska at Lincoln, Lincoln, NE 68588-0489, USA
(e-mail: bklaus1@unl.edu)

Received: 20 February 1999 / Accepted: 15 February 2000

Abstract. Thomson (1995a) proved that the uniform allocation rule is the only allocation rule for allocation economies with single-peaked preferences that satisfies *Pareto efficiency*, *no-envy*, *one-sided population-monotonicity*, and *replication-invariance* on a restricted domain of single-peaked preferences. We prove that this result also holds on the unrestricted domain of single-peaked preferences.

Next, replacing *one-sided population-monotonicity* by *one-sided replacement-domination* yields another characterization of the uniform allocation rule, Thomson (1997a). We show how this result can be extended to the more general framework of reallocation economies with individual endowments and single-peaked preferences.

Following Thomson (1995b) we present allocation and reallocation economies in a unified framework of open economies.

JEL classification: D63, D71

Key words: Fair allocation and reallocation, open economies, single-peaked preferences, population-monotonicity, replacement-domination

1 Introduction

We consider the allocation of some perfectly divisible commodity in economies with single-peaked preferences. Typical examples can be derived from rationing (Benassy 1982) or from simple resource allocation problems; for instance the

Part of this paper was written while the author was visiting the University of Rochester. I acknowledge the hospitality of the Department of Economics at the University of Rochester and the financial support of the Netherlands Organization for Scientific Research (NWO). I wish to thank William Thomson, Gabrielle Demange, and the referees for many helpful comments.

allocation of a task among the members of a team. In many real world situations one can assume that the members of the team have single-peaked preferences over their share in the completion of the task. Since Sprumont's (1991) fundamental article a wide literature has been concerned with the search for, and analysis of, rules with socially and economically appealing properties; for a survey of this literature we refer to Thomson (1997b). An important conclusion of this research is that the uniform allocation rule is now accepted as the most important rule for allocation economies with single-peaked preferences.

A natural extension of allocation economies is obtained by the introduction of individual endowments. A reallocation problem may occur when preferences change over time and previous allotments are converted into individual endowments. Barberà et al. (1997) study reallocation economies in the context of investment situations where natural claims or priorities may serve as individual endowments. The extension of the uniform allocation rule to these reallocation economies, the uniform reallocation rule, satisfies many desirable properties; for a survey of this literature we refer to Klaus (1998).

In this article we introduce allocation and reallocation economies in a unified framework of open economies as suggested by Thomson (1995b). In an open economy agents may have individual endowments as well as a joint obligation to, or from, the outside world. For instance, each member of a team might be already endowed with some share of the work-load, but in addition to these endowments the team might have to fulfill some additional tasks.

The purpose of this article is twofold. First we would like to contribute to a full understanding of allocation and reallocation economies with single-peaked preferences. Second, by helping to "complete the allocation and reallocation picture", we also hope to contribute to an understanding of open economies with single-peaked preferences and facilitate future research.

We proceed from two well-known characterizations of the uniform allocation rule by Thomson (1995a,1997a). Thomson (1995a) proved that the uniform allocation rule is the only allocation rule satisfying *Pareto efficiency*, *no-envy in allotments*, *one-sided population-monotonicity*, and *replication-invariance* on a restricted domain of single-peaked preferences. *No-envy in allotments* states that no agent prefers the allotment of another agent to his own allotment. The property *one-sided population-monotonicity* is a restricted version of *population-monotonicity*: if the change of the population is one-sided, *i.e.*, it does not change an economy where there is too less to distribute into an economy where there is too much to distribute or *vice versa*, then all agents initially present are affected in the same direction, *i.e.*, these agents all (weakly) gain or they all (weakly) lose. *Replication-invariance* requires the following: if an economy is replicated, then the replica of the allocation assigned by the rule for the initial economy equals the allocation assigned by the rule for the replicated economy. Our first result is that Thomson's (1995a) characterization also holds on the whole domain of single-peaked preferences (Theorem 1).

Replacing *one-sided population-monotonicity* by *one-sided replacement-domination* yields another characterization of the uniform allocation rule, Thomson (1997a). The property *one-sided replacement-domination* is a restricted version of *replacement-domination:* if the unilateral change of an agent's preference relation is one-sided, then the remaining agents are affected in the same direction. We show how this result can be extended to the reallocation model. Since in reallocation situations one can interpret the individual endowments of the agents as individual rights, we formulate the notion of *no-envy* for reallocation rules in terms of trade rather than in terms of net allotments. All other properties we mentioned before are essentially the same for reallocation rules. We prove that the uniform reallocation rule is the unique reallocation rule that satisfies *Pareto efficiency*, *no-envy in net trades*, *one-sided replacement-domination*, and *replication-invariance* (Theorem 3).

The paper is organized as follows. In Sect. 2 we introduce the allocation and the reallocation model with single-peaked preferences in a unified framework of open economies. In Sect. 3 we extend Thomson's (1995a) characterization of the uniform allocation rule to the domain of single-peaked preferences. In Sect. 4 we show how Thomson's (1997a) result can be extended to reallocation rules. In Sect. 5 we conclude.

2 Open economies, allocation economies, and reallocation economies

In this section we present allocation and reallocation economies when preferences are single-peaked in a unified framework of so-called open economies as introduced by Thomson (1995b). For more details on allocation or reallocation economies with single-peaked preferences we refer to the related literature as discussed in Klaus (1998) and Thomson (1997b). Apart from Thomson (1995b), Herrero (1998a,b) and Schummer and Thomson (1997) study open economies with single-peaked preferences.

2.1 Open economies

There is an infinite population of potential agents, indexed by the natural numbers $\mathbb{N}$. Each agent $i \in \mathbb{N}$ is equipped with a continuous and single-peaked preference relation R_i defined over the non-negative real numbers $\mathbb{R}_+$. Single-peakedness of R_i means that there exists a point $p(R_i) \in \mathbb{R}_+$, called *agent i's peak amount*, with the following property: for all $x, y \in \mathbb{R}_+$ with $x < y \leq p(R_i)$ or $x > y \geq p(R_i)$, we have $y \, P_i \, x$.[1] Each preference relation R_i can be described in terms of the *indifference function* $r_i : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ that is defined as follows. If $x \leq p(R_i)$, then $r_i(x) \geq p(R_i)$ and either $r_i(x) \, I_i \, x$ (if such a point exists) or $r_i(x) = \infty$. If $x \geq p(R_i)$, then $r_i(x) \leq p(R_i)$ and $r_i(x) \, I_i \, x$ (if such a point exists) or $r_i(x) = 0$.

By $\mathscr{R}$ we denote the class of all continuous, single-peaked preference relations over $\mathbb{R}_+$ and by $\mathscr{R}_b \subsetneq \mathscr{R}$ the subclass of preferences $R_i \in \mathscr{R}_b$ such that the corresponding indifference function r_i is *b*ounded, *i.e.*, $r_i(0) < \infty$. Throughout the paper we assume that any subset of agents $N \subset \mathbb{N}$ is non-empty and

[1] P_i denotes the strict preference relation associated with R_i and I_i the indifference relation.

finite. Then, $\mathscr{R}^N$ denotes the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathscr{R}$; $\mathscr{R}_b^N$ has a similar meaning.

Now, an open economy can be formalized as follows. Let $\omega = (\omega_i)_{i \in N} \in \mathbb{R}_+^N$ be a *vector of individual endowments* and $T \in \mathbb{R}$ be an *obligation to, or from, the outside world* such that $\sum_N \omega_i + T \geq 0$. For short, we call T the *outside obligation*. Then, we call a triple $e = (R, \omega, T) \in \mathscr{R}^N \times \mathbb{R}_+^N \times \mathbb{R}$ such that $\sum_N \omega_i + T \geq 0$ an *open economy*. Let $\mathscr{OE}^N$ be the class of all open economies for the set of agents N and $\mathscr{OE} = \bigcup_N \mathscr{OE}^N$ be the class of all open economies. Similarly, let $\mathscr{OE}_b^N \subsetneq \mathscr{OE}^N$ be the class of all open economies for N where $R \in \mathscr{R}_b^N$ and $\mathscr{OE}_b = \bigcup_N \mathscr{OE}_b^N$. A *feasible allocation* for $e = (R, \omega, T) \in \mathscr{OE}^N$ is a vector $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = \sum_N \omega_i + T$.[2] For any $e \in \mathscr{OE}^N$ the *set of feasible allocations* is denoted by $X(e)$. A *rule* is a function φ that assigns to every $e \in \mathscr{OE}$ a feasible allocation $\varphi(e) \in X(e)$. Given $i \in N$, we call $\varphi_i(e)$ the *allotment* of agent i and $\triangle\varphi_i(e) := \varphi_i(e) - \omega_i$ his *allotment change*. For $e = (R, \omega, T) \in \mathscr{OE}^N$, let $z(e) := \sum_N p(R_i) - \sum_N \omega_i - T$ denote *aggregate excess demand for the economy* e. If $z(e) = 0$, then e is *balanced*. If $z(e) > 0$, then e is an economy with *excess demand*. If $z(e) < 0$, then e is an economy with *excess supply*.

2.2 The allocation model and the uniform allocation rule

We call an open economy $e = (R, \omega, T)$ where $\omega = 0$ an *allocation economy*. The class of allocation economies for the set of agents N equals $\mathscr{AE}^N = \{e = (R, \omega, T) \in \mathscr{OE}^N \mid \omega = 0\} \subsetneq \mathscr{OE}^N$. With some abuse of notation, we set $\mathscr{AE}^N = \mathscr{R}^N \times \mathbb{R}_+$ and denote an allocation economy by $e = (R, \Omega) \in \mathscr{R}^N \times \mathbb{R}_+$ where $\Omega \in \mathbb{R}_+$ is the amount of an infinitely divisible commodity, the *social endowment*, that has to be distributed among a group of agents N with profile $R \in \mathscr{R}^N$. Let $\mathscr{AE} = \bigcup_N \mathscr{AE}^N$. Similarly, let $\mathscr{AE}_b^N = \mathscr{R}_b^N \times \mathbb{R}_+$ and $\mathscr{AE}_b = \bigcup_N \mathscr{AE}_b^N$. Note that $x \in X(e)$ implies that $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = \Omega$. An *allocation rule* is a function φ that assigns to every $e \in \mathscr{AE}$ a feasible allocation $\varphi(e) \in X(e)$. Obviously, for all $i \in N$, $\varphi_i(e) = \triangle\varphi_i(e)$.

The following allocation rule, known as the *uniform allocation rule*, has played a central role in the literature of fair allocation when preferences are single-peaked.

Uniform Allocation Rule U^a. For all $e = (R, \Omega) \in \mathscr{AE}^N$, and all $j \in N$,

$$U_j^a(e) := \begin{cases} \min\{p(R_j), \lambda\} & \text{if } z(e) \geq 0, \\ \max\{p(R_j), \lambda\} & \text{if } z(e) \leq 0, \end{cases}$$

where λ solves $\sum_N U_i^a(e) = \Omega$.

So, in case of excess demand, each agent either receives his peak amount or his allotment is greater than or equal to the allotment of each other agent.

[2] Note that free disposal of the commodity is not allowed.

Similarly, in case of excess supply, each agent either receives his peak amount or his allotment is smaller than or equal to the allotment of each other agent. If the economy is balanced, each agent receives his peak amount.

2.3 The reallocation model and the uniform reallocation rule

We call an open economy $e = (R, \omega, T)$ where $T = 0$ a *reallocation economy*. The class of reallocation economies for the set of agents N equals $\mathscr{RE}^N = \{e = (R, \omega, T) \in \mathscr{OE}^N \mid T = 0\} \subsetneq \mathscr{OE}^N$. With some abuse of notation, we set $\mathscr{RE}^N = \mathscr{R}^N \times \mathbb{R}^N_+$ and denote a reallocation economy by $e = (R, \omega) \in \mathscr{R}^N \times \mathbb{R}^N_+$ where $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_+$ is a *vector of individual endowments* that have to be reallocated among the agents in N with profile $R \in \mathscr{R}^N$. Let $\mathscr{RE} = \bigcup_N \mathscr{RE}^N$. Note that $x \in X(e)$ implies that $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \sum_N \omega_i$. A *reallocation rule* φ is a function that assigns to every $e \in \mathscr{RE}$ a feasible (re)allocation $\varphi(e) \in X(e)$.

Until now the "reallocation model" almost equals the "allocation model": the only difference is that we have a vector of individual endowments $\omega \in \mathbb{R}^N_+$ instead of a social endowment $\Omega \in \mathbb{R}_+$. Since for each vector of individual endowments $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_+$, we can interpret the sum $\sum_N \omega_i$ as a social endowment, each allocation rule can be easily applied as a reallocation rule. However, in contrast with the allocation model, not only the individual preferences, but also the individual endowments can be used to discriminate between the agents. Some of these "discriminations", *e.g.*, having a greater endowment than one's peak amount, turn out to be important in the sequel.

Let $e = (R, \omega) \in \mathscr{RE}^N$. We call agent $i \in N$ a *demander* if his endowment is strictly less than his peak amount: he "demands" $p(R_i) - \omega_i$ units of the commodity. We denote the set of demanders by $D(e)$. We call agent $i \in N$ a *supplier* if his endowment is strictly greater than his peak amount: he wants to "supply" $\omega_i - p(R_i)$ units of the commodity. We denote the set of suppliers by $S(e)$. We call agent $i \in N$ a *non-trader* if his endowment is equal to his peak amount: he favors no trade.

The following reallocation rule, known as the *uniform reallocation rule*, has played a central role in the axiomatic literature of fair reallocation when preferences are single-peaked.

Uniform Reallocation Rule U^r. For all $e = (R, \omega) \in \mathscr{RE}^N$, and all $j \in N$,

$$U^r_j(e) := \begin{cases} \min\{p(R_j), \omega_j + \lambda\} & \text{if } z(e) \geq 0, \\ \max\{p(R_j), \omega_j + \lambda\} & \text{if } z(e) \leq 0, \end{cases}$$

where $\lambda \in \mathbb{R}$ solves $\sum_N U^r_i(e) = \sum_N \omega_i$.

In case of excess demand, suppliers and non-traders receive their peak amounts. The amount they supply is divided among the demanders so that each of them receives in addition to his individual endowment an amount that is as equal as possible, with each demander's peak amount as upper bound. If the economy is balanced, each agent receives his peak amount. In case of excess supply, the reallocation is dual to the excess demand case: all demanders and non-traders receive their peak amounts. The amount they demand is subtracted from the suppliers so that each of them diminishes his individual endowment by an amount that is as equal as possible, with each supplier's peak amount as lower bounds.[3]

2.4 Properties of allocation and reallocation rules

In this subsection we introduce several properties for allocation and reallocation rules. It is without loss of generality that we introduce properties for rules that are defined on the whole class of open economies.

Our first (standard) requirement for rules is *Pareto efficiency*.

Pareto efficiency. For all N and all $e = (R, \omega, T) \in \mathscr{OE}$, there is no $x \in X(e)$ such that for all $i \in N$, $x_i \, R_i \, \varphi_i(e)$, and for some $j \in N$, $x_j \, P_j \, \varphi_j(e)$.

It follows easily that a rule is *Pareto efficient* if and only if it is *same-sided*, that is: for all N and all $e = (R, \omega, T) \in \mathscr{OE}^N$, either [for all $i \in N$, $\varphi_i(e) \leq p(R_i)$], or [for all $i \in N$, $\varphi_i(e) \geq p(R_i)$].

The next property we analyze is *no-envy*. *No-envy* can be traced back to Foley (1967) who considers it in the context of resource allocation problems. A rule satisfies *no-envy in allotments* if no agent strictly prefers the allotment of another agent to his own allotment.

No-envy in allotments. For all N, all $e = (R, \omega, T) \in \mathscr{OE}^N$, and all $i, j \in N$, $\varphi_i(e) \, R_i \, \varphi_j(e)$.

Because of individual endowments, in reallocation and open economies it might make less sense to compare final allotments only when performing a "no-envy test". In order to incorporate the individual endowments into the notion of *no-envy*, we formulate *no-envy* in terms of trade or allotment changes: no agent strictly prefers the allotment change of another agent or the part of another agent's allotment change that is feasible for him to his own allotment change.

Let $\alpha \in \mathbb{R}$. Then, $\alpha^+ := \max\{0, \alpha\}$.

No-envy in net trades. For all N, all $e = (R, \omega, T) \in \mathscr{OE}^N$, and all $i, j \in N$, $\varphi_j(e) \, R_j \, (\omega_j + \triangle\varphi_i(e))^+$.

So, agent j envies agent i if j prefers i's allotment change or the part of i's allotment change which is feasible for him, to his own allotment change. A concept of *no-envy* in terms of allotment changes—called "fair" net trade—as

[3] It is worth noting that the uniform allocation rule as well as the uniform reallocation rule can be interpreted as Walrasian solutions; see Thomson (1995b). For exchange economies with a single commodity, the uniform reallocation rule U^r is as a special case of Mas-Colell's (1992) "Walrasian solution with slack".

An extension of the uniform allocation and the uniform reallocation rule to open economies, the so-called generalized uniform rule, was introduced by Thomson (1995b).

introduced above was formulated by Schmeidler and Vind (1972) in the more general context of exchange economies.

Next, we introduce the slightly weaker notion of *weak no-envy in net trades*: in difference to *no-envy in net trades* an agent is not considered to envy another agent if he strictly prefers the part of the other agent's allotment change that is feasible for him to his own allotment change while in fact the whole allotment change of the other agent would not be feasible for him.

Weak no-envy in net trades. For all N, all $e = (R, \omega, T) \in \mathscr{OE}^N$, and all $i, j \in N$ with $\omega_j + \triangle\varphi_i(e) \geq 0$, $\varphi_j(e)\, R_j\, (\omega_j + \triangle\varphi_i(e))$.

So, agent j envies agent i if j prefers i's allotment change, added to his endowment, to his own allotment—provided the former is feasible.

Note that on the class of allocation economies $\mathscr{AE}$, both *no-envy* properties *in net trades* are reduced to *no-envy in allotments*. Allocation and reallocation rules satisfying *no-envy in net trades* always exist. However, Thomson (1995b) shows that on the domain of open economies no rule satisfies *weak no-envy in net trades*.[4] Thomson's (1995b) "*weak no-envy*" property is a modification of the *weak no-envy in net trades* property used here.

Next, we discuss two so-called "solidarity properties" that describe the effect of certain changes in a single parameter of an open economy while the other parameters are kept fixed. If after any arrival of new agents either all agents initially present (weakly) lose together or all (weakly) gain together, then the rule satisfies *population-monotonicity*.[5] If after any change of a single agent's preference relation either all remaining agents (weakly) lose together or all (weakly) gain together, then the rule satisfies *welfare-domination under preference-replacement*, or *replacement-domination* for short.[6]

Thomson (1995a,1997a) shows that for allocation rules *population-monotonicity* and *replacement-domination* are generally incompatible with *Pareto efficiency* and *no-envy in allotments*. A similar incompatibility holds for reallocation rules; see for instance Moreno (1996) and Thomson (1995b). However, these incompatibilities only occur when the change in the parameter is such that it turns an allocation or reallocation economy in which there is "too much" to divide into an economy in which there is "too little" to divide, or *vice versa*. We call a change where this does not occur *one-sided*. In the sequel, we consider the one-sided versions of *population-monotonicity* and *replacement-domination, i.e.*, solidarity among the agents is only required for one-sided changes in the initial allocation or reallocation economy.

---

[4] Thomson (1995b) demonstrates this result with the following example. Let $e = (R, \omega, T) \in \mathscr{OE}^N$ be such that $N = \{1, 2\}$, $p(R_1) = 1$, $p(R_2) = 2$, $\omega = (0, 1)$, and $T = -1$. Note that $X(e) = \{(0, 0)\}$ and that at $(0, 0)$ agent 2 envies agent 1 for his allotment change.

[5] Thomson (1983a,b) introduced *population-monotonicity* in the context of bargaining. For a survey on *population-monotonicity* we refer to Thomson (1995c).

[6] Moulin (1987) introduced *replacement-domination* in the context of binary choice with quasi-linear preferences. *Replacement-domination* has been studied in a variety of settings and we refer the interested reader to a recent review of the literature by Thomson (1999).

Let $N \subseteq M$, and $R \in \mathcal{R}^M$. Then, the *restriction* $(R_i)_{i \in N} \in \mathcal{R}^N$ *of* $R \in \mathcal{R}^M$ *to* N is denoted by R_N.

One-sided population-monotonicity.[7] For all $N, \bar{N}$, all $e = (R, \omega, T) \in \mathcal{OE}^N$, and all $\bar{e} = (\bar{R}, \bar{\omega}, T) \in \mathcal{OE}^{\bar{N}}$, if $N \subseteq \bar{N}$, $R = \bar{R}_N$, $w = \bar{w}_N$, and $z(e) \cdot z(\bar{e}) \geq 0$, then either [for all $i \in N$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

For $N \subseteq M$ let $M \backslash N := \{i \in M \mid i \notin N\}$. Let $R, \bar{R} \in \mathcal{R}^N$, and $j \in N$. If $R_{N\backslash\{j\}} = \bar{R}_{N\backslash\{j\}}$ and $R_j \neq \bar{R}_j$, then we call $\bar{R}$ a *j-deviation from R*.

One-sided replacement-domination.[8] For all N, all $j \in N$, all $e = (R, \omega, T) \in \mathcal{OE}^N$, and all $\bar{e} = (\bar{R}, \omega, T) \in \mathcal{OE}^N$, if $\bar{R}$ is a j-deviation from R and $z(e) \cdot z(\bar{e}) \geq 0$, then either [for all $i \in N\backslash\{j\}$, $\varphi_i(e)\, R_i\, \varphi_i(\bar{e})$] or [for all $i \in N\backslash\{j\}$, $\varphi_i(\bar{e})\, R_i\, \varphi_i(e)$].

As a last property we introduce *replication-invariance*: if an open economy is replicated, *i.e.*, the individual endowments, the outside obligation, and the preference profile are replicated, then the replica of the allocation assigned by the rule for the initial economy equals the allocation assigned by the rule for the replicated economy.

For $e = (R, \omega, T) \in \mathcal{OE}^N$, $k \in \mathbb{N}$, and N' such that $N \subseteq N'$ and $|N'| = k\,|N|$ we partition N' into $|N|$ subsets indexed by $i \in N$ and we refer to the economy in $\mathcal{OE}^N$ such that for all $i \in N$, all of the members of the ith element of the partition have preferences and individual endowments identical to the preferences and the individual endowment of agent i, and in which the outside obligation equals kT, as a *k-replica of e*. We use the shorthand notation $k * e$. Given $x \in X(e)$, we similarly denote by $k * x$ the allocation obtained by k-times replicating x.

Replication-invariance. For all N, all $e \in \mathcal{OE}^N$, and all $k \in \mathbb{N}$, $\varphi(k * e) = k * \varphi(e)$.

The uniform allocation rule satisfies *Pareto efficiency, no-envy in allotments, one-sided population-monotonicity, one-sided replacement-domination,* and *replication invariance*; see for instance Thomson (1995a,1997a). The uniform reallocation rule satisfies *Pareto efficiency, no-envy in net trades, one-sided population-monotonicity, one-sided replacement-domination*, and *replication invariance*; see for instance Klaus et al. (1997) and Klaus (1998). Note that the uniform reallocation rule does not satisfy *no-envy in allotments.*

3 A domain extension for a well-known characterization of the uniform allocation rule

Thomson (1995a) proves that the uniform allocation rule is the unique allocation rule that satisfies *Pareto efficiency, no-envy in allotments, one-sided population-monotonicity,* and *replication-invariance* on the domain of allocation economies

[7] See Thomson (1995a), Klaus, Peters, and Storcken (1997), and Moreno (1996).
[8] See Thomson (1995b,1997a).

with bounded, single-peaked preferences $\mathscr{AE}_b$. We extend this result to the larger domain of allocation economies with single-peaked preferences $\mathscr{AE}$.

Theorem 1. *On $\mathscr{AE}$ the uniform allocation rule is the only allocation rule that satisfies Pareto efficiency, no-envy in allotments, one-sided population-monotonicity, and replication-invariance.*

The proof of Theorem 1 is in parts similar to Thomson's (1995a) proof of the original result on $\mathscr{AE}_b$. Steps 1 and 2 (a) are new. Step 2 (a) is added to the original proof to prove the result on the larger domain $\mathscr{AE}$ while Step 1 is added to facilitate Steps 2 (a) and (b).

Proof. As mentioned earlier, the uniform allocation rule satisfies the properties named in the theorem. To prove the remaining part of the theorem let φ be an allocation rule that satisfies the properties named in the theorem. Let $\mathscr{C} = \{e \mid$ and $\varphi(e) \neq U^a(e)\}$ and suppose, by contradiction, that $\mathscr{C} \neq \emptyset$. In Step 1, we prove that there exists an $e \in \mathscr{C}$ such that $e = (R, \Omega) \in \mathscr{AE}^N$ and $|N| = 2$. In Step 2, we derive a contradiction.

Step 1. Suppose, by contradiction, that for all $e \in \mathscr{C}$ with $e \in \mathscr{AE}^N$, $|N| > 2$.

Let $\bar{e} \in \mathscr{C}$ be such that $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathscr{AE}^{\bar{N}}$ and for all $e \in \mathscr{C}$, $e = (R, \Omega) \in \mathscr{AE}^N$, $|N| \geq |\bar{N}|$. Assume, without loss of generality, that $\bar{N} = \{1, 2, 3, \ldots\}$. Since φ is *Pareto efficient*, it follows that either $z(\bar{e}) > 0$ or $z(\bar{e}) < 0$. Assume, without loss of generality, that $z(\bar{e}) < 0$. Since $\varphi(\bar{e}) \neq U^a(\bar{e})$, there exist agents $j, l \in \bar{N}$, without loss of generality $j = 1$ and $l = 2$, such that $\varphi_1(\bar{e}) < \varphi_2(\bar{e})$ and $p(\bar{R}_2) < \varphi_2(\bar{e})$. By *no-envy in allotments*, agent 2 does not envy agent 1 at $\varphi(\bar{e})$. Hence, $\varphi_1(\bar{e}) \leq r_2(\varphi_2(\bar{e})) < p(\bar{R}_2) < \varphi_2(\bar{e})$. Let $g := \varphi_2(\bar{e}) - r_2(\varphi_2(\bar{e}))$. Next, let $k^* \in \mathbb{N}$ be such that $\frac{\varphi_3(\bar{e})}{k^*} < g$. Let $e^* \in \mathscr{AE}^{N^*}$ be a k^*-replica of $\bar{e}$. Then, $z(e^*) < 0$ and, by *replication-invariance*, $\varphi(e^*)$ is a k^*-replica of $\varphi(\bar{e})$. For $i \in \bar{N}$, we call the $k^* - 1$ agents in e^* that are obtained by replicating agent i, replica of agent i. Furthermore, we call agent i and his $(k^* - 1)$ replica, agents of type i.[9] Starting from e^*, we successively delete all k^* agents of type 3. By *Pareto efficiency* and *no-envy in allotments*, agents of the same type always receive the same allotment. In order to keep the notation simple, while we delete some of the agents of type 3, we denote an allotment of any remaining agent of type $i \in \bar{N}$ by $\varphi_i(\cdot)$.

First, we delete agent 3 from the economy e^*. Denote the economy that is obtained by $e^1 \in \mathscr{AE}^{N^* \backslash \{3\}}$. By *Pareto efficiency* and *one-sided population-monotonicity*, for all $i \in N^* \backslash \{3\}$, $\varphi_i(e^1) \geq \varphi_i(e^*) = \varphi_i(\bar{e})$. The largest allotment an agent of type 1 can receive at e^1 equals $\varphi_1(\bar{e}) + \frac{\varphi_3(\bar{e})}{k^*}$. Since $\frac{\varphi_3(\bar{e})}{k^*} < g$, by *no-envy in allotments,* $\varphi_1(e^1) \leq r_2(\varphi_2(\bar{e})) < \varphi_2(\bar{e}) \leq \varphi_2(e^1)$. Hence, $e^1 \in \mathscr{C}$. Next, let e^2 denote the economy that is obtained by deleting another agent of type 3. Similarly as before, it follows that $e^2 \in \mathscr{C}$. By successively deleting all k^* agents of type 3, we obtain an economy $e^{k^*} \in \mathscr{C}$ that is a k^*-replica

[9] If for agents $i, j \in \bar{N}$, $\bar{R}_i = \bar{R}_j$, we partition the replicas of i, j such that there are exactly $(k^* - 1)$ replica of each agent and k^* agents of each type.

of $\tilde{e} = (\bar{R}_{\bar{N}\setminus\{3\}}, \bar{\Omega}) \in \mathscr{AE}^{\bar{N}\setminus\{3\}}$. Hence, by *replication-invariance*, $\varphi(e^{k^*})$ is a k^*-replica of $\varphi(\tilde{e})$. Thus, $\tilde{e} \in \mathscr{C}$ and $\left|\bar{N}\setminus\{3\}\right| < \left|\bar{N}\right|$. This is a contradiction.

Step 2. Let $\bar{e} \in \mathscr{C}$ be such that $\bar{e} = (\bar{R}, \bar{\Omega}) \in \mathscr{AE}^{\bar{N}}$ and $\left|\bar{N}\right| = 2$.

(a) Assume, without loss of generality, that $\bar{N} = \{1, 2\}$. Since φ is *Pareto efficient*, it follows that either $z(\bar{e}) > 0$ or $z(\bar{e}) < 0$. Assume, without loss of generality, that $z(\bar{e}) < 0$ and, by *no-envy in allotments*, $\varphi_1(\bar{e}) \leq r_2(\varphi_2(\bar{e})) < p(R_2) < \varphi_2(\bar{e})$. Let $g := \varphi_2(\bar{e}) - r_2(\varphi_2(\bar{e}))$. Let $k^* \in \mathbb{N}$ be such that $\frac{\varphi_2(\bar{e})}{k^*} < g$ and $(k^* - 1)\varphi_2(\bar{e}) > k^*(r_2(\varphi_2(\bar{e})) - \varphi_1(\bar{e})) + \varphi_2(\bar{e})$. Let $e^* \in \mathscr{AE}^{N^*}$ be a k^*-replica of $\bar{e}$. Then, $z(e^*) < 0$ and, by *replication-invariance*, $\varphi(e^*)$ is a k^*-replica of $\varphi(\bar{e})$. Starting from e^*, we successively delete all $k^* - 1$ replicas of agent 2. Denote the resulting economy by $\hat{e} \in \mathscr{AE}^{\hat{N}}$. By the same arguments than in Step 1, it follows that $\varphi_1(\hat{e}) \leq r_2(\varphi_2(\bar{e})) < \varphi_2(\bar{e}) \leq \varphi_2(\hat{e})$. After the $k^* - 1$ replicas of agent 2 left the economy, the extra amount $(k^* - 1)\varphi_2(\bar{e})$ has been distributed among the agents of type 1 and agent 2. Hence,

$$\varphi_2(\hat{e}) = \varphi_2(\bar{e}) + [(k^* - 1)\varphi_2(\bar{e}) - k^*(\varphi_1(\hat{e}) - \varphi_1(\bar{e}))].$$

Note that $\varphi_1(\hat{e}) - \varphi_1(\bar{e}) \leq r_2(\varphi_2(\bar{e})) - \varphi_1(\bar{e})$. By the construction of k^* it follows that $k^*(r_2(\varphi_2(\bar{e})) - \varphi_1(\bar{e})) < (k^* - 1)\varphi_2(\bar{e}) - \varphi_2(\bar{e})$. Hence, $k^*(\varphi_1(\hat{e}) - \varphi_1(\bar{e})) \leq k^*(r_2(\varphi_2(\bar{e})) - \varphi_1(\bar{e})) < (k^* - 1)\varphi_2(\bar{e}) - \varphi_2(\bar{e})$ and

$$\begin{aligned} \varphi_2(\hat{e}) &> \varphi_2(\bar{e}) + [(k^* - 1)\varphi_2(\bar{e}) - ((k^* - 1)\varphi_2(\bar{e}) - \varphi_2(\bar{e}))] \\ &> 2\varphi_2(\bar{e}). \end{aligned}$$

Next, we add agent $j \notin \hat{N}$ such that $p(\bar{R}_j) = p(\bar{R}_2)$ and $\bar{R}_j \in \mathscr{R}_b$. Denote the resulting economy by $\tilde{e} \in \mathscr{AE}^{\tilde{N}}$. Note that $z(\tilde{e}) < 0$. By *Pareto efficiency* and *one–sided population-monotonicity*, for all $i \in \hat{N}$, $p(\bar{R}_i) \leq \varphi_i(\tilde{e}) \leq \varphi_i(\hat{e})$. By *Pareto efficiency* and *no-envy in allotments*, agents 2 and j receive the same allotment at $\hat{e}$. So, because $\varphi_2(\hat{e}) > 2\varphi_2(\bar{e})$, $\varphi_2(\tilde{e}) > \varphi_2(\bar{e})$. Thus, $\varphi_2(\tilde{e}) > p(\bar{R}_2)$ and $\varphi_1(\tilde{e}) < \varphi_2(\tilde{e})$. Hence, $\tilde{e} \in \mathscr{C}$.

(b) Note that $r_j(p(\bar{R}_1)) < \infty$ and $p(\bar{R}_1) \leq \varphi_1(\tilde{e})$. Let $g^* := \varphi_2(\tilde{e}) - r_2(\varphi_2(\tilde{e}))$. Let $k^{**} \in \mathbb{N}$ be such that $\frac{\varphi_j(\tilde{e})}{k^{**}k^*} < g^*$ and $(2k^* - 1)\varphi_j(\tilde{e}) > k^{**}k^*(r_j(\varphi_j(\tilde{e})) - \varphi_1(\tilde{e})) + (r_j(p(\bar{R}_1)) - \varphi_j(\tilde{e}))$. Let $e^{**} \in \mathscr{AE}^{N^{**}}$ be a k^{**}-replica of $\tilde{e}$. Then by *replication-invariance*, $\varphi(e^{**})$ is a k^{**}-replica of $\varphi(\tilde{e})$. Starting from e^{**}, we successively delete all k^{**} agents of type 2 and all $k^{**} - 1$ replicas of agent j. Denote the resulting economy by $\check{e} \in \mathscr{AE}^{\check{N}}$. Note that there are $k^{**}k^*$ agents that are either direct replica of agent 1 or replica of replica of agent 1. With a slight abuse of terminology, consider all these agents as agents of type 1. By the same arguments than in Step 1, it follows that $p(\bar{R}_1) \leq \varphi_1(\check{e}) \leq r_j(\varphi_j(\tilde{e})) < \varphi_j(\tilde{e}) \leq \varphi_j(\check{e}) \leq r_j(p(\bar{R}_1))$. After the k^{**} agents of type 2 and all $k^{**} - 1$ replica of agent j left the economy, the extra amount $(2k^* - 1)\varphi_j(\tilde{e})$ has been distributed among the agents of type 1 and agent j. Hence,

$$\varphi_j(\check{e}) = \varphi_j(\tilde{e}) + [(2k^* - 1)\varphi_j(\tilde{e}) - k^{**}k^*(\varphi_1(\check{e}) - \varphi_1(\tilde{e}))].$$

Note that $\varphi_1(\breve{e}) - \varphi_1(\tilde{e}) \leq r_j(p(\overline{R}_1)) - \varphi_1(\tilde{e})$. By the construction of k^{**} it follows that $k^{**}k^*(r_j(p(\overline{R}_1)) - \varphi_1(\tilde{e})) < (2k^* - 1)\varphi_j(\tilde{e}) - (r_j(p(\overline{R}_1)) - \varphi_j(\tilde{e}))$. Hence, $k^{**}k^*(\varphi_1(\breve{e}) - \varphi_1(\tilde{e})) \leq k^{**}k^*(r_j(p(\overline{R}_1)) - \varphi_1(\tilde{e})) < (2k^* - 1)\varphi_j(\tilde{e}) - r_j(p(\overline{R}_1)) + \varphi_j(\tilde{e})$ and

$$\begin{aligned} \varphi_j(\breve{e}) &> \varphi_j(\tilde{e}) + [(2k^* - 1)\varphi_j(\tilde{e}) - (2k^* - 1)\varphi_j(\tilde{e}) + r_j(p(\overline{R}_1)) - \varphi_j(\tilde{e})] \\ &> r_j(p(\overline{R}_1)). \end{aligned}$$

Hence, in contradiction to *no-envy in allotments*, $p(\overline{R}_1) \leq \varphi_1(\breve{e})$ and $\varphi_j(\breve{e}) > r_j(p(\overline{R}_1)) \geq r_j(\varphi_1(\breve{e})) \geq p(\overline{R}_j)$. □

Remark 1. It has not been known whether *replication-invariance* is independent from the other properties in Thomson's (1995a) characterization (Theorem 1 respectively). Similarly, the independence of *replication-invariance* from the other properties in the following theorem (Theorem 2) by Thomson (1997a) has been an open problem. The answer to both questions is given in Klaus (1997) by means of a single allocation rule, unequal to the uniform allocation rule, that satisfies *Pareto efficiency, no-envy in allotments, one-sided population-monotonicity, one-sided replacement-domination,* but not *replication-invariance.*

Theorem 2 (Thomson, 1997a). *On $\mathscr{AE}$ the uniform allocation rule is the only allocation rule that satisfies Pareto efficiency, no-envy in allotments, one-sided replacement-domination, and replication-invariance.*

4 A new characterization of the uniform reallocation rule

As stated in Theorem 1, the uniform allocation rule is the only allocation rule that satisfies *Pareto efficiency*, *no-envy in allotments*, *one-sided population-monotonicity*, and *replication-invariance*. A similar result, proven by Klaus et al. (1997) and Moreno (1996), is true for reallocation rules: the uniform reallocation rule is the only reallocation rule that satisfies *Pareto efficiency*, *weak no-envy in net trades*, and *one-sided population-monotonicity*.[10] The reason why *replication-invariance* is not needed when extending Theorem 1 to the reallocation setting is that *replication-invariance* for reallocation rules is implied by *Pareto efficiency*, *weak no-envy in net trades*, and *one-sided population-monotonicity*: using *one-sided population-monotonicity*, we can "replicate" any economy by adding "clones" of the original agents that have the same preferences and the same endowments. By *weak no-envy in net trades* and *Pareto efficiency*, each of the cloned agents and the original agent receive the same allotments, which, by *one-sided population-monotonicity* and *Pareto efficiency* must be the same as in the original economy. Note that we cannot replicate an allocation economy (and most open economies) by "adding agents".

The following theorem can be seen as an extension of Theorem 2 to the reallocation model.

[10] A stronger version of the theorem that is based on a weaker *one-sided population-monotonicity* condition can be found in Klaus et al. (1997) and Klaus (1998).

Theorem 3. *On $\mathscr{RE}$ the uniform reallocation rule is the only reallocation rule that satisfies Pareto efficiency, no-envy in net trades, one-sided replacement-domination, and replication-invariance.*

The proof of Theorem 3 is in parts similar to Thomson's (1997a) proof of the original result on $\mathscr{AE}$ (Theorem 2). However, extra difficulties in the proof for the reallocation model are caused by possibly negative allotment changes (particularly in the context of *no-envy in net trades*). Thomson (1995b, Theorem 4) conjectures that the uniform reallocation rule is the only rule that satisfies *Pareto efficiency, weak no-envy in net trades, one-sided replacement-domination,* and *replication-invariance.* After the proof of Theorem 3 we show that this conjecture is not true and it is not possible to weaken *no-envy in net trades* in Theorem 3 to *weak no-envy in net trades* (Example 1).

Proof. As mentioned earlier, the uniform reallocation rule satisfies the properties named in the theorem. To prove the remaining part of the theorem let φ be a reallocation rule that satisfies the properties named in the theorem.

Let $e = (R, \omega) \in \mathscr{RE}^N$, and suppose, by contradiction, that $\varphi(e) \neq U^r(e)$. Since φ is *Pareto efficient*, it follows that either $z(e) > 0$ or $z(e) < 0$. Assume, without loss of generality, that $N = \{1, 2, \ldots, n\}$ and $z(e) < 0$. Since $\varphi(e) \neq U^r(e)$, there exists an agent $j \in N$ such that $p(R_j) < \varphi_j(e)$ and $\triangle\varphi_j(e) > \triangle U_j^r(e)$.

Let $N' = \{n+1, \ldots, 2n\}$ and define $e' = (R', \omega') \in \mathscr{RE}^{N \cup N'}$ such that for all $i \in N$, $\omega_i = \omega'_i = \omega'_{n+i}$ and $R_i = R'_i = R'_{i+n}$. Clearly, e' is a 2-replica of e. Hence, by *replication-invariance*, for all $i \in N$, $\varphi_i(e) = \varphi_i(e') = \varphi_{n+i}(e')$. Particularly, it follows that $p(R_j) < \varphi_j(e')$ and $\triangle\varphi_j(e') > \triangle U_j^r(e')$.

Let $\bar{R} \in \mathscr{R}^{N \cup N'}$ be a j-deviation from R' such that $p(R'_j) = p(\bar{R}_j)$ and $0 \, \bar{P}_j \, \varphi_j(e')$. Let $\bar{e} = (\bar{R}, \omega')$ denote this one-sided change of e'. Since $p(\bar{R}_j) = p(\bar{R}_{n+j})$ and $\omega'_j = \omega'_{n+j}$, by *same-sidedness* and *no-envy in net trades*, it follows that $\varphi_j(\bar{e}) = \varphi_{n+j}(\bar{e})$. Suppose that $\varphi_j(\bar{e}) < \varphi_j(e')$. Hence, $\varphi_{n+j}(\bar{e}) < \varphi_{n+j}(e')$ and, by *same-sidedness* and *one-sided replacement-domination*, for all $i \in N \backslash \{j\} \cup N'$, $\varphi_i(\bar{e}) \leq \varphi_i(e')$. Then, by feasibility, it follows that $\sum_{N \cup N'} \omega'_i = \sum_{N \cup N'} \varphi_i(\bar{e}) < \sum_{N \cup N'} \varphi_i(e') = \sum_{N \cup N'} \omega'_i$. This is a contradiction. Similarly, the assumption, $\varphi_j(\bar{e}) > \varphi_j(e')$ yields a contradiction. Hence, $\varphi_j(\bar{e}) = \varphi_j(e')$. Thus, by *same-sidedness* and *one-sided replacement-domination,* for all $i \in N \backslash \{j\} \cup N'$, $\varphi_i(\bar{e}) = \varphi_i(e')$. Hence, $\varphi(\bar{e}) = \varphi(e')$. Then, $\triangle\varphi_j(\bar{e}) > \triangle U_j^r(\bar{e})$ and by feasibility, there exist $k \in N \backslash \{j\} \cup N'$ such that $\triangle\varphi_k(\bar{e}) < \triangle U_k^r(\bar{e})$. By *same-sidedness*, this can only be the case if $p(\bar{R}_k) < U_k^r(\bar{e})$ and $\triangle U_k^r(\bar{e}) = \min\{\triangle U_i^r(\bar{e}) \mid i \in N \cup N'\}$. Thus, $\triangle\varphi_k(\bar{e}) < \triangle U_k^r(\bar{e}) \leq \triangle U_j^r(\bar{e}) < \triangle\varphi_j(\bar{e})$. So, in contradiction to *no-envy in net trades*, $(\omega_j + \triangle\varphi_k(\bar{e}))^+ \, \bar{P}_j \, \varphi_j(\bar{e})$. □

Example 1. The following reallocation rule $\tilde{\varphi}$ satisfies *Pareto efficiency, weak no-envy in net trades, one-sided replacement-domination, and replication-invariance*. If $e = (R, \omega) \in \mathscr{RE}^N$ such that $z(e) < 0$, and for all $i \in D(e)$, $\omega_i = 0$, then

$$\tilde{\varphi}(e) = \begin{cases} p(R_i) & \text{if } i \notin D(e), \\ U_i^r(R_{D(e)}, U_i^a(R_{D(e)}, \sum_{S(e)}(\omega_i - p(R_i))) & \text{if } i \in D(e). \end{cases}$$

For all remaining N and $e = (R, \omega) \in \mathscr{RE}^N$, $\tilde{\varphi}(e) = U^r(e)$. $\Diamond$

We conclude this section with the independence of the axioms in Theorem 3.

The reallocation rule $\tilde{\varphi}$ described in Example 1 satisfies *Pareto efficiency, one-sided replacement-domination,* and *replication-invariance*, but not *no-envy in net trades.*

Example 2. The following no-trade rule φ^0 satisfies *no-envy in net trades, one-sided replacement-domination,* and *replication-invariance*, but not *Pareto efficiency*. For all N and $e = (R, \omega) \in \mathscr{RE}^N$, $\varphi^0(e) = \omega$. $\Diamond$

Example 3. The following absorbing agent reallocation rule $\varphi^{\tilde{a}}$ is similar to the absorbing agent allocation rule φ^a introduced in Klaus (1997). It satisfies *Pareto efficiency, no-envy in net trades,* and *one-sided replacement-domination,* but not *replication-invariance*. Let $e = (R, \omega) \in \mathscr{RE}^N$ be such that $z(e) < 0$ and there exists $j \in D(e)$ such that $p(R_j) > \frac{\Omega}{2}$ and $\omega_j = 0$ (such an agent j must be unique). Then, for this economy we obtain $\varphi^{\tilde{a}}(e)$ from the uniform reallocation $U^r(e)$ by letting agent j absorb the amount $\tilde{a}(e) := \min\{\sum_{S(e)}(\omega_i - p(R_i)), r_j(\frac{\Omega}{2}) - p(R_j)\}$ and subtracting this amount as equally as possible, with the agents' peak amounts as lower bounds, from the uniform reallocation shares $U_i^r(e)$ of the agents $i \in N \backslash \{j\}$. In order to formalize $\varphi^{\tilde{a}}$, let $R' \in \mathscr{R}^N$ be a j-deviation from R such that $p(R_j') = p(R_j) + \tilde{a}(e)$. Then, $\varphi^{\tilde{a}}(e) := U^r(R', U^r(e))$.

For all remaining N and $e = (R, \omega) \in \mathscr{RE}^N$, $\varphi^{\tilde{a}}(e) := U^r(e)$. $\Diamond$

Example 4. The following rule $\tilde{\varphi}^{\tilde{a}}$ is a variation of the absorbing agent rule $\varphi^{\tilde{a}}$. It satisfies *Pareto efficiency, no-envy in net trades,* and *replication-invariance*, but not *one-sided replacement-domination.* Let $e = (R, \omega) \in \mathscr{RE}^N$. If $z(e) < 0$ and there exists $j \in D(e)$ such that $p(R_j) > \frac{\Omega}{2}$ and $\omega_j = 0$, then $\tilde{\varphi}^{\tilde{a}}(e) := \varphi^{\tilde{a}}(e)$.

If e is a k-replica of an economy $\bar{e} \in \mathscr{RE}^{\bar{N}}$ such that $z(\bar{e}) < 0$ and there exists $j \in D(e)$ such that $p(R_j) > \frac{\Omega}{2}$ and $\omega_j = 0$, then $\tilde{\varphi}^{\tilde{a}}(e)$ is a k-replica of $\varphi^{\tilde{a}}(\bar{e})$.

For all remaining N and $e = (R, \omega) \in \mathscr{RE}^N$, $\tilde{\varphi}^{\tilde{a}}(e) := U^r(e)$. $\Diamond$

5 Conclusion

In this article we present allocation and reallocation economies in a unified framework of open economies and add two results to the existing literature on fair allocation and reallocation. Apart from adding new results, a fundamental understanding of results for allocation and reallocation economies will be helpful for future research on open economies. The incompatibility of *no-envy in net trades* with feasibility on the domain of open economies shows that the extension of results for allocation and reallocation economies to open economies might not be straightforward. Apart from Thomson's (1995b) paper, some results for open economies with single-peaked preferences can be found in Herrero (1998a,b) and Schummer and Thomson (1997).

References

Barberà, S., Jackson, M.O., Neme, A. (1997) Strategy-proof allotment rules. *Games and Economic Behavior* 18: 1–21

Benassy, J.P. (1982) *The Economics of Market Equilibrium*. Academic Press, San Diego

Foley, D. (1967) Resource allocation and the public sector. *Yale Economic Essays* 7: 45–98

Herrero, C. (1998a) An Alternative Extension of the Uniform Rule in General Allocation Problems with Single-Peaked Preferences. Working Paper, Universidad de Alicante

Herrero, C. (1998b) Path-Independence in General Allocation Problems with Single-Peaked Preferences. Working Paper, Universidad de Alicante

Klaus, B. (1997) The Role of Replication-Invariance: Two Answers Concerning the Problem of Fair Division when Preferences are Single-Peaked. Working Paper, University of Nebraska at Lincoln, revised November 1999

Klaus, B. (1998) *Fair Allocation and Reallocation: An Axiomatic Study*. Ph.D. Thesis, Maastricht University, The Netherlands, Unigraphic, Maastricht

Klaus, B., Peters, H., Storcken, T. (1997) Reallocation of an infinitely divisible good. *Economic Theory* 10: 305–333

Klaus, B., Peters, H., Storcken, T. (1998) Strategy-proof division with single-peaked preferences and individual endowments. *Social Choice and Welfare* 15: 297–311

Mas-Colell, A. (1992) Equilibrium Theory with Possibly Satiated Preferences. In: Majumdar, M. (ed.) *Equilibrium and Dynamics*, Essays in Honour of D. Gale. McMillan, London, pp 201–213

Moreno, B. (1996) The Uniform Rule in Economies with Single-Peaked Preferences, Endowments and Population Monotonicity. Working Paper, University of Alicante

Moulin, H. (1987) The pure compensation problem: Egalitarianism versus Laissez-Fairism. *Quarterly Journal of Economics* 102: 769–783

Schmeidler, D., Vind K. (1972) Fair net trades. *Econometrica* 40: 637–642

Schummer, J., Thomson, W. (1997) Two derivations of the uniform rule and some applications to bankruptcy. *Economics Letters* 55: 333–337

Sprumont, Y. (1991) The division problem with single-peaked preferences: A characterization of the uniform allocation rule. *Econometrica* 59: 509–519

Thomson, W. (1983a) The fair division of a fixed supply among a growing population. *Mathematics of Operations Research* 8: 319–326

Thomson, W. (1983b) Problems of fair division and the egalitarian Solution. *Journal of Economic Theory* 31: 211–226

Thomson, W. (1995a) Population-monotonic solutions to the problem of fair division when preferences are single-peaked. *Economic Theory* 5: 229–246

Thomson, W. (1995b) Endowment-Monotonicity in Economies with Single-Peaked Preferences. Working Paper, University of Rochester, revised May 1996

Thomson, W. (1995c) Population Monotonic Allocation Rules. In: Barnett W.A., Moulin H., Salles M., Schofield N. (eds.) *Social Choice, Welfare and Ethics.* Cambridge University Press, Cambridge, New York, Melbourne

Thomson, W. (1997a) The replacement principle in economies with single-peaked preferences. *Journal of Economic Theory* 76: 145–168

Thomson, W. (1997b) *The Theory of Fair Allocation*. Princeton University Press (forthcoming)

Thomson, W. (1999) Welfare-domination under preference-replacement: A survey and open questions. *Social Choice and Welfare* 16: 373–394