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The positional power of nodes in digraphs

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Abstract. Many economic and social situations can be represented by a digraph. Both local and global methods to determine the strength or power of all the nodes in a digraph have been proposed in the literature. We propose a new method, where the power of a node is determined by both the number of its successors and the powers of its successors. Our method, called the positional power function, determines a full ranking of the nodes for any digraph. The positional power function can either be determined as the unique solution to a nonhomogeneous system of equations, or as the limit point of an iterative process. The solution can easily be obtained explicitly, which enables us to derive a number of interesting properties of the positional power function. We also consider the Copeland variant of the positional power function. Finally, we extend our method to the class of all weighted graphs.

1 Introduction

Many economic and social situations can be modelled by means of a digraph. A digraph is an irreflexive directed graph consisting of a finite set of nodes and a collection of ordered pairs of these nodes, called arcs or arrows, e.g. see Behzad et al. [1]. An arc from one node to another node represents a dominance relation of the former node over the latter node. For instance, in a

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sports competition each node is a player or team and an arc going from node i to node j means that player i has won a play against player j, e.g., see Moon and Pullman [22], Rubinstein [24], and Laffond et al. [20]. Gilles et al. [13] and van den Brink [4] model a *hierarchical structure* in an organization by a digraph. Within this framework the nodes represent economic agents and an arc going from node i to node j means that agent i has economic power over agent j, e.g., to set trading conditions.

In *social choice theory* the set of nodes represents the set of available alternatives. The problem is that several individuals (voters) may have different, often conflicting, preferences over the available alternatives and that only one alternative can be implemented. The problem to choose the most preferred alternative can be modelled as a digraph by assigning an arc from node i to node j when in a pairwise comparison of the alternatives, alternative i is preferred to j by a majority of the voters. In case the majority voting is decisive for any pair of alternatives the resulting digraph is a so-called tournament, otherwise a so-called weak tournament. For selecting a best alternative on tournaments we refer to David [9], Sen [26], Kano and Sakamoto [17], and on weak tournaments to Peris and Subiza [23], Dutta and Laslier [12], and Duggan and Le Breton [10]. For a general survey, see the monograph of Laslier [21]. In this paper our aim is not just to select a best element but to obtain a full ranking of the nodes in an arbitrary digraph. We will cover both tournaments and weak tournaments. We also show that our method can easily be extended to weighted digraphs, and it is therefore applicable to so-called generalized tournaments.

To rank or measure the power of the nodes in a digraph, several methods have been proposed in the literature. One may distinguish between local and global methods. To determine the power of a node, a local method only uses partial information about the stucture of the digraph. A well-known local method is the *score* of a node, which equals the number of its successors. Other local methods are the *Copeland score* and the *dominance* function. In general, a drawback of local methods is that the power of a node does not depend on the power of the other nodes. These local methods have been axiomatized, see Behzad et al. [1], Rubinstein [24], Henriet [15], Bouyssou [3], van den Brink [4], and van den Brink and Gilles [6], [7].

Global methods use information on the entire structure of the digraph and are typically specified by an iterative procedure. This type of methods goes back to Wei [29] and Kendall [19]. For tournaments these methods have been discussed extensively in Laslier [21]. The method proposed by Wei [29], called the long-path method by Laslier [21], iterates the score vector. Other iterative procedures are the so-called Markov solution proposed by e.g., Daniels [8], and a procedure proposed in Borm, van den Brink and Slikker [2], which is based on the axiomatic dominance function of van den Brink [4]. In Moon and Pullman [22], see also Keener [18], it is shown that only under restrictive assumptions on the digraph, the long-path method converges to a non-zero vector. In particular it must hold that each node must be dominated by at least one other node, excluding for instance digraphs like hierarchies, trees or digraphs with isolated nodes. The Markov solution is only well-defined on the class of generalized tournaments. The procedure of Borm, van den Brink and Slikker [2] gives a unique outcome to any digraph, but has the disadvantage that it only ranks the players in the top cycles. Although this property is often required in case of the social choice problem of choosing a best alternative, it implies that the procedure cannot be used to obtain a full ranking of all the nodes. Both local and global methods will be discussed more extensively in Sect. 2.

In this paper we propose the so-called *positional power* function, a new method for measuring the power of the nodes in a digraph. The positional power of a node is determined by both the number of its successors, as in the local score method, and the power of these successors, as in the global methods. Whereas the global methods proposed so far in the literature yield solutions to a homogeneous system of linear equations, our power vector is a solution to a non-homogeneous system. It will be shown that this solution does not suffer from the drawbacks of the iterative procedures mentioned above. In particular, the system has a unique solution for any digraph and therefore the new method is not restricted to a subclass of digraphs. Moreover, the solution vector gives zero power to a dummy node and a positive power to any node that dominates at least one other node, so that it does not only rank the nodes in the top cycles, but also the other nodes. We also introduce a Copeland version of the positional power function.

This paper is organized as follows. In the next section we discuss both the local and global methods considered in the literature and we show that the solutions generated by these latter methods are solutions of homogeneous systems of linear equations. In Sect. 3 we introduce the new power function as the unique solution of a non-homogeneous system of equations. We also introduce the Copeland variant of this new power function. In Sect. 4 we discuss several properties of the power function and show that the solution of the system of equations can be seen as the limit point of an iterative process. In Sect. 5 we extend the new power function to the class of all weighted graphs.

2 Power functions on digraphs

A directed graph consists of a set of nodes and a set of directed edges or arcs. The set of nodes is denoted by N and consists of a finite number of n elements, indexed by $i = 1, \dots, n$. An arc points from some node $i \in N$ to some node $j \in N$ and is denoted by the ordered pair (i, j). A directed graph on the set N of nodes is denoted by its set A of arcs, i.e., $A \subset N \times N$. If $(i, j) \in A$ we say that node i dominates node j. A directed graph is called *irreflexive* if $(i, i) \notin A$ for every $i \in N$. An irreflexive directed graph is shortly called a *digraph*. A digraph A is said to be a *tournament* if for any two different nodes $i, j \in N$ it holds that either $(i, j) \in A$ or $(j, i) \in A$. A digraph is called to be *transitive* when for any three nodes h, i and j it holds that $(h, j) \in A$ when both $(h, i) \in A$

and $(i,j) \in A$. Clearly, when A is a transitive tournament then A reflects a linear order on the set of nodes. Finally, a digraph A is called empty when $A = \emptyset$ and complete when $A = N \times N \setminus \{(i,i)|i \in N\}$. Throughout this paper $\mathscr{A}$ denotes the collection of all digraphs on a given set N of n nodes, i.e., $\mathscr{A}$ is the collection of all irreflexive directed graphs on N.

A ranking or power function f on the collection $\mathscr{A}$ of digraphs on N assigns for any $A \in \mathscr{A}$ a real number to every node i in N, which can be seen as its strength, as in a sports competition, its power, as in a digraph reflecting a hierarchical structure, or determining its rank, as in a digraph reflecting preferences over a finite set of alternatives. To facilitate the introduction of power functions, we first define for every node $i \in N$ its sets of predecessors and successors in A by

$$P_i^A = \{j \in N|(j,i) \in A\} \quad \text{and} \quad S_i^A = \{j \in N|(i,j) \in A\},$$

respectively, i.e. P_i^A is the set of all nodes by which i is dominated in N and S_i^A is the set of all nodes in N dominated by i. We denote the cardinality of these sets by p_i^A and s_i^A, respectively, i.e. $p_i^A = |P_i^A|$ and $s_i^A = |S_i^A|$, $i \in N$. Observe that when A is a tournament we have for all i and all $h \neq i$ that h belongs to either P_i^A or S_i^A, so that $p_i^A + s_i^A = n - 1$.

Node $i \in A$ is a *dummy node* if it does not dominate any other node; node $i \in A$ is a *top node* if it dominates any node that has at least one predecessor. A power function $f: \mathscr{A} \to \mathbb{R}^n$ satisfies the *dummy node property* if for every $A \in \mathscr{A}$ it holds that $f_i(A) = 0$ if $s_i^A = 0$. Also, f satisfies the *top node property* if for every $A \in \mathscr{A}$ and for every top node $i \in N$, it holds that $f_i(A) \geq f_h(A)$ for all $h \in N$. A function $f: \mathscr{A} \to \mathbb{R}^n$ satisfies the *symmetry* property if $f_i(A) = f_h(A)$ when both $P_i^A = P_h^A$ and $S_i^A = S_h^A$.

The power functions described in the literature fall into two classes: the local methods and the global methods.

2.1 Local power functions

Three local power functions are the well-known *score function*, the *Copeland score function*, and the *dominance function* and are defined as follows.

Definition 2.1. *(i) The score function is the function* $f^s: \mathscr{A} \to \mathbb{R}_+^n$ *given by* $f_i^s(A) = s_i^A$, $i \in N, A \in \mathscr{A}$.

(ii) The Copeland score function is the function $f^{Cs}: \mathscr{A} \to \mathbb{R}_+^n$ *given by* $f_i^{Cs}(A) = s_i^A - p_i^A$, $i \in N, A \in \mathscr{A}$.

(iii) The dominance function is the function $f^d: \mathscr{A} \to \mathbb{R}_+^n$ *given by* $f_i^d(A) = \sum_{j \in S_i^A} \frac{1}{p_j^A}$, $i \in N, A \in \mathscr{A}$.

In van den Brink [4], see also van den Brink and Gilles [7], both the score and dominance functions have been axiomatized. In Rubinstein [24] a characterization of the ranking by the Copeland score function has been given on the subclass of tournaments, see also Henriet [15]. In tournaments the ranking by the score function, given by the numbers of successors of the nodes, is the same as the ranking by the Copeland score function, being the differences of the numbers of successors and predecessors of the nodes. For more properties on the score function we refer to Behzad et al. [1], Delver et al. [11] and van den Brink and Gilles [6], and on the Copeland score function to Bouyssou [3]. The score function and the dominance function have the dummy node, top node, and symmetry property.

2.2 Global power functions

First we consider the *long-path method*, originating from Wei [29] and Kendall [19], see also Daniels [8], Moon and Pullman [22], Saaty [25], Keener [18], Laslier [21], and Slutzki and Volij [27]. Let T^A denote the $n \times n$ *adjacency* matrix T^A of a digraph A with elements $t_{ij} = 1$ if $(i,j) \in A$ and $t_{ij} = 0$ otherwise and let e be equal to the n-vector of ones. For a given digraph A the long-path method considers the sequence $x^t = T^A x^{t-1},\ t = 1, 2, \ldots,$ starting with x^0 equal to e. By definition of T^A the vector x^1 is the score vector s^A, x^2 is the vector that assigns to any node i the scores of all its successors, and so on. This procedure is said to converge if $\lim_{t\to\infty} x^t / \sum_i x_i^t$ is well defined. In case of convergence the limit vector is called the *long-path vector*, denoted LP(A). The procedure is not guaranteed to converge to a reasonable solution. For instance, when A is a transitive tournament, then the procedure converges to the zero vector within a finite number of iterations and so does not give a ranking. Only under severe restrictions the procedure converges to a nonzero solution. More precisely, let $\widetilde{\mathscr{A}}$ be the subset of *strongly connected* digraphs, i.e. $A \in \widetilde{\mathscr{A}}$ if for every two nodes i and j there exists a sequence $i_1, i_2, \ldots, i_k$ such that $i_1 = i$, $i_k = j$ and $(i_h, i_{h+1}) \in A$ for all $h = 1, \ldots, k-1$. Moon and Pullman [22] show that if A is a strongly connected digraph, then the long-path method converges to the unique strictly positive eigenvector (up to normalization) of T^A. This eigenvector corresponds to the highest positive eigenvalue λ^A of T^A. LP(A) is therefore a solution of the homogeneous system of linear equations

$$\lambda^A x = T^A x. \tag{1}$$

The fact that LP(A) is often not well defined when $A \notin \widetilde{\mathscr{A}}$ limits the usefulness of this power concept.

The next procedure, called the Markov procedure, has been proposed by Daniels [8] and others, see Laslier [21], and is given by the iterative system $x^t = \frac{1}{n-1}(T^A + S^A)x^{t-1},\ t = 1, 2, \ldots,$ with $x^0 = e$ and S^A being the $n \times n$ diagonal matrix with ith diagonal element equal to s_i^A. When A is a tournament and thus $p_i^A = n - 1 - s_i^A$ for all i, each column of the matrix $M^A = \frac{1}{n-1}(T^A + S^A)$ sums up to one, and so M^A is a Markov transition matrix. From the elementary theory of stochastic processes it follows that in that case the iterative process has a unique limit point, denoted M(A), being an eigenvector with eigenvalue 1 of the matrix M^A. So, when A is a tournament the *Markov vector* $\mathrm{M}(A)$ is a solution of the homogeneous system of linear equations

$$x = M^A x. \tag{2}$$

However, the iterated Markov power vector of a digraph may not be well defined when the digraph is not a tournament, which also limits its usefulness.[1]

The iterated dominance procedure proposed in Borm et al. [2] works for any digraph. The procedure is given by the iterative system $x^t = \widehat{T}^A x^{t-1}$, $t = 1, 2, \ldots$, with $x^0 = e$ and $\widehat{T}^A$ being the modified adjacency matrix with elements $\widehat{t}^A_{ij} = \frac{1}{p^A_j + 1}$ if $(i, j) \in A$ or when $j = i$, and $\widehat{t}_{ij} = 0$ otherwise. Since each column of the matrix $\widehat{T}^A$ sums up to one, $\widehat{T}^A$ is a Markov transition matrix. Hence, the iterative process has a unique limit point, denoted $\mathrm{ID}(A)$, being an eigenvector with eigenvalue 1 of the matrix $\widehat{T}^A$. So, the *iterated dominance vector* $\mathrm{ID}(A)$ is a solution of the homogeneous system of linear equations

$$x = \widehat{T}^A x. \tag{3}$$

Also this iterative procedure has some drawbacks. First, it should be noticed that for both this process and the Markov process it holds that the procedure converges to an eigenvector with eigenvalue 1 of the corresponding matrix, but that the matrix may not have a unique (normalized) eigenvector with eigenvalue 1. A different initial x^0 may therefore lead to a different solution. Second, the iterated dominance vector only discriminates between the nodes in top cycles. A set K of nodes is a top cycle of digraph A if for any two nodes i and j in K, there exists a sequence $i_1, \ldots, i_\ell$ of nodes such that $i_1 = i$, $i_\ell = j$ and $(i_k, i_{k+1}) \in A$, $k = 1, \ldots, \ell - 1$, and when for any $i \in K$ and $h \notin K$ it holds that $(h, i) \notin A$. Note that there may be multiple top cycles. Any node not being in a top cycle gets value equal to zero, also when it dominates other nodes. Further, the solution does not satisfy the dummy node property. In particular, an isolated node (being a top cycle on its own) gets value 1.

3 The positional power of nodes

The solutions to the iterative procedures described before measure the power of the nodes in a global way. The homogeneous systems of Eqs. (1), (2) and (3) show that in all cases, the power of a node is determined in one way or another by the powers of its successors, which in turn are determined by the powers of their successors, and so on. According to these measures, the power of a node is completely determined by the powers of all nodes within the digraph. On the other hand, the local power functions considered before do not take into account the power of other nodes at all, but are completely

[1] Notice that the Markov procedure can also be applied to generalized tournaments, see Laslier [21], p. 218. However, when restricted to digraphs, a generalized tournament is either a tournament or the complete digraph.

determined by local dominance relations. The power function to be introduced in this paper takes into account both the local and the global influence exercized by a node on other nodes.

The main idea of the new power function is that the power of a node is determined by both the number of its successors, as in the score measure, and the powers of its successors, as in an iterative procedure. In general, for given positive numbers a and c, when node i dominates node j, node i gets a fixed amount c plus $1/a$ times the power of node j. So, the power x_i of node i in a digraph $A \in \mathscr{A}$ is defined as the solution of the system

$$x_i = \sum_{j \in S_i^A} \left(c + \frac{1}{a} x_j \right), \quad i \in N. \tag{4}$$

Notice that the system is not homogeneous, contrary to the systems corresponding to the global methods discussed in Sect. 2.
Rewriting the system of Eq. (4) in matrix notation we obtain

$$x = cs^A + \frac{1}{a} T^A x,$$

or

$$\left(I - \frac{1}{a} T^A \right) x = cs^A, \tag{5}$$

with I the identity matrix of appropriate dimension. The next theorem shows that for every digraph the system of linear equations has a unique nonnegative solution if $a > n - 1$.

Theorem 3.1. *For every digraph $A \in \mathscr{A}$ the system of Eqs. (5) has a unique solution if $a > n - 1$, and this solution is nonnegative. Moreover, it then holds that the matrix $(I - \frac{1}{a} T^A)$ has an inverse and all elements of this inverse are nonnegative.*

Proof. Let b_{ij} be the (i, j)th element of the matrix $B^A = I - \frac{1}{a} T^A$. Since $b_{ii} = 1$ for all i and $b_{ij} \leq 0$ for all $i \neq j$, according to Hawkins and Simon [14] the inverse of B^A exists and is nonnegative iff there exists a nonnegative vector $y \in \mathbb{R}^n$ such that each component of $z = B^A y$ is positive. Take $y = e$. Then, $z_i = \sum_{j=1}^n b_{ij} = 1 - \sum_{\{j \neq i | t_{ij} = 1\}} \frac{1}{a} \geq 1 - \frac{1}{a}(n-1)$, where t_{ij} is the (i, j)-th element of the matrix T^A. Hence, if $a > n - 1$, we have that $z_i > 0$ for all i. Hence, z is strictly positive and therefore the inverse of B^A exists and all elements of the inverse are nonnegative. Since also c and the vector s^A is nonnegative, it follows that system (5) has a unique nonnegative solution.

Theorem 3.1 implies that for $a > n - 1$ the system of Eq. (4) has a unique nonnegative solution x^A given by

$$x^A = c(I - \frac{1}{a} T^A)^{-1} s^A. \tag{6}$$

The system of Eq. (6) shows that the number c only determines the absolute value of the solution, but not the relative powers or the ranking of the nodes. Therefore, without loss of generality we take $c = 1$, so that according to (4), a node gets one unit of power for each of its successors. Moreover, a node gets a fraction $1/a$ of the powers of its successors. With $c = 1$, the system of Eq. (6) further shows that the solution x^A converges to the score vector $f^s(A) = s^A$ when a goes to infinity. To maximize the influence of the power of the successors to the power of a node, we take a equal to n, the smallest feasible integer under the constraint that the solution exists for every digraph.

We now define the positional power vector of a digraph A as the solution to the system

$$x_i = \sum_{j \in S_i^A} \left(1 + \frac{1}{n} x_j\right), \ i \in N. \tag{7}$$

From Eq. (6) we obtain the following definition.

Definition 3.2. *The* positional power function *is the function* $f^p : \mathscr{A} \to \mathbb{R}^n$ *which assigns to every* $A \in \mathscr{A}$ *the solution of (7), i.e.* $f^p(A) = (I - \frac{1}{n} T^A)^{-1} s^A$.

According to Theorem 3.1 the positional power function is well-defined and assigns a nonnegative power vector $f^p(A)$ to any digraph $A \in \mathscr{A}$. This overcomes the drawback of the long-path vector, which is restricted to the subclass $\widetilde{\mathscr{A}}$ of strongly connected digraphs, and the drawback of the Markov solution, which is restricted to (generalized) tournaments.

We remark that instead of taking c equal to one, in some applications other normalizations may be useful. For instance, when A is not the empty set, c can be taken such that the sum of the powers is normalized to one. This is achieved by setting c equal to c^A, where

$$c^A = \left(e^\top \left(I - \frac{1}{n} T^A\right)^{-1} s^A\right)^{-1}.$$

We denote the resulting normalized positional power function by f^{np}. This function can be used to determine whether a collection of nonempty digraphs is balanced, see Herings et al. [16]. A collection $\{A^1, \ldots, A^k\}$ of k nonempty digraphs is said to be balanced if the system

$$\sum_{i=1}^{k} \lambda_i f^{np}(A^i) = e$$

has a nonnegative solution. Balancedness of digraphs can be used to give sufficient conditions for the nonemptiness of the core as solution concept for a nontransferable utility game in graph structure.

As for the score function we may define the Copeland variant of the positional power function. While the score function f^s assigns to a node its number of successors, the Copeland score function assigns to each node of a digraph its number of successors minus its number of predecessors. The function which assigns to every node i the number of predecessors, p_i^A, can be seen as a measure of the local weakness of node i within the digraph A, while the score function is a measure for the local strength or power of node i within A. Notice that $p_i^A = s_i^{A^R}$ for all $i \in N$, where

$$A^R = \{(i,j) \in N \times N | \ (j,i) \in A\}$$

is the reverse of the digraph A. So, the Copeland score vector of the digraph A is equal to $f^{Cs}(A) = s^A - s^{A^R}$. In the same way we define the Copeland variant of the positional power function, called the *Copeland positional power function*, as the difference of the positional power and positional weakness. The positional weakness vector in digraph A is defined to be the solution of the system

$$y_i = \sum_{j \in P_i^A} \left(1 + \frac{1}{n} y_j\right), \ i \in N. \tag{8}$$

Similar as for the score function, the positional weakness of a node in digraph A is equal to the positional power of that node in the reverse A^R of A, i.e., $f^p(A^R)$ is the unique solution to Eq. (8).

Definition 3.3. *The Copeland positional function is the function* $f^{Cp} : \mathscr{A} \rightarrow \mathbb{R}^n$ *which assigns to every* $A \in \mathscr{A}$ *the vector* $f^{Cp}(A) = f^p(A) - f^p(A^R)$.

A positive Copeland positional value of a node means that the node exercizes more power in the digraph than it suffers from the powers being exercized on it, while a negative position means the opposite.

4 Properties

The positional power function satisfies several nice properties. First, it satisfies the dummy node property, i.e. $f_i^p(A) = 0$ if $S_i^A = \emptyset$. Moreover, since x^A is a nonnegative vector it also follows immediately from the system of Eq. (7) that $x_i^A > 0$ when $S_i^A \neq \emptyset$, implying that the positional power function assigns zero power to a node if and only if it is a dummy node. So, the power function f^p also overcomes the drawback of the iterative dominance solution ID(A), which assigns zero power to any node not in a top cycle on the one hand and positive power to an isolated node on the other hand. From the system of Eq. (7) it follows that $f_i^p(A) \geq f_j^p(A)$ if $S_j^A \subset S_i^A$ with strict inequality when S_j^A is a proper subset of S_i^A. This monotonicity property implies that f^p satisfies both the top node property and the symmetry property. It even satisfies the stronger property that any two nodes having the same set of successors have the same power. When A is a tournament there is a unique top cycle and we have that $S_j^A \subset S_i^A \setminus \{j\}$ and thus $f_i^p(A) > f_j^p(A)$ for any i in the top cycle and any j not in the top cycle. So, when applying the positional power function to select a best alternative in a social choice problem, the function selects an alternative from the top cycle. Summarizing, we have the following properties.

Corollary 4.1. *The positional power function $f^p : \mathscr{A} \to \mathbb{R}^n_+$ satisfies for any $A \in \mathscr{A}$ the following properties.*

- *For every node $i \in N$ it holds that $f_i^p(A) > 0$ if and only if $S_i^A \neq \emptyset$.*
- *For every pair of nodes $i, j \in N$ it holds that $f_j^p(A) \geq f_i^p(A)$ if $S_i^A \subset S_j^A$ with equality only when $S_i^A = S_j^A$.*
- *When A is a tournament, $f_i^p(A) > f_j^p(A)$ for any i in the top cycle and any j not in the top cycle.*

Before discussing more properties of f^p we derive the following lemma about the inverse matrix $(I - \frac{1}{n}T^A)^{-1}$. In the sequel we denote this matrix by V^A, with (i, j)th element equal to v_{ij}.

Lemma 4.2. *The elements v_{ij}, $i, j \in N$, have the following properties.*

(i) $v_{ii} = 1 + \sum_{h \in P_i^A} v_{ih}/n$ *and* $v_{ij} = \sum_{h \in P_j^A} v_{ih}/n$ for $j \neq i$.
(ii) $\sum_{j=1}^n (n - s_j^A) v_{ij} = n$.
(iii) $1 \leq v_{ii} \leq 2n/(n+1)$ *and* $0 \leq v_{ij} \leq n/(n+1)$ for $j \neq i$.
(iv) $v_{ij} = 0$ *for* $j \neq i$ $(v_{ii} = 1)$ *if and only if there does not exists an ordered path of arcs from node i to node j (node i).*

Proof. Since $V^A(I - \frac{1}{n}T^A) = I$, we find by rearranging terms that $V^A = I + \frac{1}{n}V^A T^A$. Recalling that the (h, j)th element t_{hj} of the matrix T^A is equal to 1 if $h \in P_j^A$ and 0 otherwise, we obtain property (i). Postmultiplying both sides of the equality $V^A(I - \frac{1}{n}T^A) = I$ by the vector e, we obtain for any $i \in N$

$$\sum_{j=1}^n v_{ij} = 1 + \sum_{j=1}^n \sum_{h \in P_j^A} v_{ih}/n = 1 + \sum_{h=1}^n \sum_{j \in S_h^A} v_{ih}/n = 1 + \sum_{h=1}^n s_h^A\, v_{ih}/n,$$

which, by rearranging terms, yields property (ii). From Theorem 3.1 we already know that $v_{ij} \geq 0$ for all $j \neq i$ and thus from property (i) it follows that $v_{ii} \geq 1$ for all $i \in N$. Since $s_h^A \leq n - 1$ for all h, it follows from property (ii) that

$$\sum_{h=1}^n v_{ih} \leq \sum_{h=1}^n (n - s_h^A) v_{ih} = n.$$

Hence, for $j \neq i$, from property (i) we obtain

$$v_{ij} = \sum_{h \in P_j^A} v_{ih}/n \leq \sum_{h=1}^n v_{ih}/n - v_{ij}/n \leq 1 - v_{ij}/n,$$

because $j \notin P_j^A$. This shows that $v_{ij} \leq n/(n+1)$ for $j \neq i$. Similarly, we obtain

$$v_{ii} \leq 1 + \sum_{h=1}^n v_{ih}/n - v_{ii}/n \leq 2 - v_{ii}/n.$$

To prove property (iv), notice that $0 \leq ((T^A)^k)_{ij} \leq (n-1)^k$ for all k, so $\sum_{k=1}^{\infty} \frac{1}{n^k}(T^A)^k$ exists. Since the product of $I + \sum_{k=1}^{\infty} \frac{1}{n^k}(T^A)^k$ and $I - \frac{1}{n}T^A$ equals I, we obtain that

$$V^A = I + \sum_{k=1}^{\infty} \frac{1}{n^k}(T^A)^k. \tag{9}$$

Clearly, $((T^A)^k)_{ij} > 0$ if and only if there exists at least one ordered path of adjacent arcs of length k from node i to node j. This implies that if there is no path at all from node i to node j we must have that $v_{ij} = 1$ when $i = j$ and $v_{ij} = 0$ when $j \neq i$, and conversely.

The next result follows easily from the previous lemma.

Lemma 4.3. *For any $A \in \mathscr{A}$ it holds that $f^p(A) = n(V^A - I)e$.*

Proof. From Eq. (6) it follows that $f^p(A) = V^A s^A$ and thus

$$f_i^p(A) = \sum_{j=1}^{n} s_j^A v_{ij}, \; i = 1, \ldots, n.$$

From property (ii) of Lemma 4.2 we obtain

$$f_i^p(A) = \sum_{j=1}^{n} n(v_{ij} - 1), \; i = 1, \ldots, n,$$

which proves the lemma.

Since Eq. (9) shows that $V^A - I = \sum_{k=1}^{\infty} \frac{1}{n^{k-1}}(T^A)^k$ it follows from Lemma 4.3 that

$$f^p(A) = \left(\sum_{k=1}^{\infty} \frac{1}{n^{k-1}} \left(T^A\right)^k \right) e$$

and thus that for any starting vector x^0, the power vector $f^p(A)$ is the limit point of the iterative process

$$x^t = T^A e + \frac{1}{n} T^A x^{t-1}, \; t = 1, 2, \ldots. \tag{10}$$

For example, when taking as starting vector $x^0 = 0$, we obtain that $x^1 = s^A$, i.e. the first iteration gives the score vector, which corresponds to the first iteration of the long-path method. However, any next iteration differs from the long-path method because of the fixed term $T^A e = s^A$, giving to a node i a fraction $1/n$ of the current power of its successors plus the fixed amount 1 for each of its successors. For any $i, j = 1, \ldots, n$, the nonnegative number $((T^A)^k)_{ij}/n^{k-1}$, $k \in \mathbb{N}$, is precisely the contribution of node j to the power of node i over all ordered paths of length k in A leading from node i to node j. Remark that a path may contain several cycles and contain a cycle more than once. Adding up all these contributions over $k \in \mathbb{N}$ yields the total contribution v_{ij} of node j to the power of node $i \neq j$ ($v_{ii} - 1$ when $i = j$). Adding up all these total contributions over j yields the positional power of node i in the digraph A. Recall from property (iv) of Lemma 4.2 that v_{ij} ($v_{ii} - 1$ when $i = j$) is positive if and only if there exists at least one ordered path from node i to node j.

The next lemma shows that the positional power function is increasing in A and that adding an arc from h to k increases the power of node h more than the power of any other node.

Lemma 4.4. *Let A and A' be two digraphs such that $A' = A \cup \{(h,k)\}$ for some (h,k) not in A. Then the following properties hold for $f^p(A)$ and $f^p(A')$.*

(i) *For any $i = 1, \ldots, n$, $f_i^p(A') \geq f_i^p(A)$.*
(ii) $f_h^p(A') - f_h^p(A) > \max_{i \in N \setminus \{h\}} f_i^p(A') - f_i^p(A)$.
(iii) *For every $i \neq h$, $f_i^p(A') = f_i^p(A)$ if and only if there is no ordered path of any length in A from node i to node h.*

Proof. Let $E^{h,k} = T^{A'} - T^A$, so element (i,j) of the matrix $E^{h,k}$ is equal to 1 for $(i,j) = (h,k)$ and equal to zero otherwise. Let $\Delta x^A = x^{A'} - x^A$. It holds that $(I - \frac{1}{n}T^A)x^A = s^A$, $(I - \frac{1}{n}T^{A'})x^{A'} = s^{A'}$ and $s^{A'} = s^A + e^h$, where e^h is the hth unit vector in $\mathbb{R}^n$. By subtracting the first equality from the second one we obtain that

$$\left(I - \frac{1}{n}T^{A'}\right)\Delta x^A - \frac{1}{n}E^{h,k}x^A = e^h$$

and so

$$\left(I - \frac{1}{n}T^{A'}\right)\Delta x^A = \frac{1}{n}E^{h,k}x^A + e^h = \left(1 + \frac{x_k^A}{n}\right)e^h.$$

Therefore,

$$\Delta x^A = \alpha V^{A'} e^h,$$

where $\alpha = 1 + \frac{1}{n}x_k^A$. From Theorem 3.1 it follows that $V^{A'}$ is a nonnegative matrix. Since α is positive and e^h is a nonnegative vector, it follows that Δx^A is a nonzero nonnegative vector, which proves property (i).
From $\Delta x^A = \alpha V^{A'} e^h$ it follows that for $i = 1, \ldots, n$,

$$\Delta x_i^A = \alpha v'_{ih},$$

where v'_{ih} is the (i,h)th element of the matrix $V^{A'}$. Applying property (iii) of Lemma 4.2 to $V^{A'}$ we obtain for $i \neq h$

$$\Delta x_h^A = \alpha v'_{hh} \geq \alpha > \alpha v'_{ih} = \Delta x_i^A,$$

which proves property (ii). Finally, when A does not contain an ordered path from node $i \neq h$ to node h, then also A' does not contain such a path. So, according to property (iv) of Lemma 4.2 applied to $V^{A'}$ we have that $v'_{ih} = 0$ and thus $\Delta x_i^A = 0$, which proves property (iii).

Finally, we consider the total positional power being assigned to the nodes in a digraph A. The total positional power of a digraph A, $\sum_{i=1}^{n} f_i^p(A)$, can be interpreted as a measure of the total amount of power being exercized by the nodes of the digraph A.

Lemma 4.5. *The following properties hold for the total power assigned by the positional power function.*

(i) $\sum_{i=1}^{n} f_i^p(A) = ne^\top V^A e - n^2$.
(ii) $|A| \leq \sum_{i=1}^{n} f_i^p(A) \leq n|A|$.
(iii) $\sum_{i=1}^{n} f_i^p(A) = 0$ *if A is the empty digraph, while* $\sum_{i=1}^{n} f_i^p(A) = n^2(n-1)$ *in case A is complete.*

Proof. Clearly, from Lemma 4.3 it follows that

$$\sum_{i=1}^{n} f_i^p(A) = ne^\top (V^A - I)e = ne^\top V^A e - n^2,$$

which shows property (i). By the system of Eq. (7),

$$\sum_{i=1}^{n} f_i^p(A) = \sum_{i=1}^{n} \sum_{j \in S_i^A} (1 + \frac{1}{n} f_j^p(A)) = |A| + \sum_{j=1}^{n} \frac{p_j^A}{n} f_j^p(A),$$

which implies that $\sum_{i=1}^{n} f_i^p(A) \geq |A|$. Rearranging terms gives $\sum_{i=1}^{n}(n - p_i^A) f_i^p(A) = n|A|$, from which it follows that $\sum_{i=1}^{n} f_i^p(A) \leq n|A|$, thereby showing property (ii). Since $p_j^A = 0$ and $|A| = 0$ in case of the empty digraph, and $p_j^A = n-1$ and $|A| = n(n-1)$ in case of the complete digraph, $\sum_{i=1}^{n} f_i^p(A) = 0$ if A is the empty digraph, while in case A is complete

$$\sum_{i=1}^{n} f_i^p(A) = n(n-1) + \sum_{i=1}^{n} \frac{n-1}{n} f_i^p(A)$$

and thus $\sum_{i=1}^{n} f_i^p(A) = n^2(n-1)$, which proves property (iii).

Lemmas 4.4 and 4.5 show that the total positional power of the nodes in a graph is strictly increasing in A and lies between $|A|$ and $n|A|$. It should be noticed that, although the total positional power is increasing in A, it is not like the score function (strictly) increasing in the number of arcs.

The next lemma shows that the sum of the components of the Copeland positional vector of any digraph is equal to zero.

Lemma 4.6. *For any digraph $A \in \mathscr{A}$ it holds that* $\sum_{i=1}^{n} f_i^{Cp}(A) = 0$.

Proof. Since $T^{A^R} = (T^A)^\top$, we have that $V^{A^R} = (I - \frac{1}{n} T^{A^R})^{-1} = (V^A)^\top$. From property (i) of Lemma 4.5 it then follows that $\sum_{i=1}^{n} f_i^{Cp}(A) = \sum_{i=1}^{n}(f_i^p(A) - f_i^p(A^R)) = ne^\top (V^A - (V^A)^\top)e = 0$.

The property that the total power in any digraph is equal to the total power in its reverse digraph also holds for the score function, but not for the dominance function.

5 Weighted digraphs

Until now we restricted ourselves to irreflexive unweighted graphs. In this section we extend our power function to weighted graphs. A weighted graph is given by a set of nodes N and a nonnegative $n \times n$ matrix W, where the (i, j)th element w_{ij} of W denotes the weight on the arc from node i to node j. Notice that we allow for an arc from a node to itself with positive weight w_{ii}. The value w_{ij} measures how strongly node i dominates node j. When $w_{ii} = 0$ for all i and $w_{ij} \in \{0, 1\}$ for all $i \neq j$, a weighted graph corresponds to a digraph.

Some special important cases of weighted graphs are generalized tournaments and ratio-scaled comparison matrices. In a generalized tournament there exists $m > 0$ such that for every $i, j \in N$ with $i \neq j$ it holds that $w_{ii} = 0$ and $w_{ij} + w_{ji} = m$. Generalized tournaments are widely considered in the literature. The matrix W is called a ratio-scaled comparison matrix if $w_{ij}w_{ji} = 1$ for all $i, j \in N$. Saaty [25] and many others later on, see e.g., Vargas and Whittaver [28], applied the long-path method on ratio-scaled comparison matrices in order to rank alternatives.

It is easy to adapt the positional power function to weighted graphs. For given weighted graph with matrix W, we define the positional power vector of W to be equal to the solution of the system of equations

$$x_i = \sum_{j=1}^{n} (w_{ij} + \frac{1}{a^W} w_{ij} x_j), \ \ i \in N. \tag{11}$$

Rewriting the system of Eq. (11) in matrix notation we obtain

$$x = We + \frac{1}{a^W} Wx,$$

or

$$(I - \frac{1}{a^W} W)x = We. \tag{12}$$

For any a^W larger than $\max_{i \in N} \sum_{j=1}^{n} w_{ij}$ it follows by the analogue of the proof of Theorem 3.1 that this system of equations has a unique nonnegative solution.

Theorem 5.1. *For every weighted graph with nonnegative matrix W, the system of equations (12) has a unique nonnegative solution if $a^W > \max_{i \in N} \sum_{j=1}^{n} w_{ij}$. Moreover, it then holds that all elements of the inverse matrix of $(I - \frac{1}{a^W} W)$ are nonnegative.*

A further analysis of the properties of the power function for (special classes) of weighted digraphs remains the subject for further research.

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