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A PARAMETRIC TEST OF THE NEGATIVITY OF THE SUBSTITUTION MATRIX

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SUMMARY

The negativity of the substitution matrix implies that its latent roots are non-positive. When inequality restrictions are tested, standard test statistics such as a likelihood ratio or a Wald test are not χ^2-distributed in large samples. We propose a Wald test for testing the negativity of the substitution matrix. The asymptotic distribution of the statistic is a mixture of χ^2-distributions. The Wald test provides an exact critical value for a given significance level. The problems involved in computing the exact critical value can be avoided by using the upper and lower bound critical values derived by Kodde and Palm (1986). Finally the methods are applied to the empirical results obtained by Barten and Geyskens (1975).

1. INTRODUCTION

An important result in economic theory is the negativity of the substitution matrix S. The semi-negative definiteness of the substitution matrix corresponds to the second-order conditions for optimizing behaviour of economic agents. As a consequence, all latent roots of the substitution matrix must be nonpositive. The difficulty in testing the negativity restrictions is due to the presence of multiple inequality constraints on the latent roots. Whereas equality restrictions from microeconomics, such as homogeneity, adding-up and symmetry have been extensively tested in the literature, the negativity restriction has received far less attention in empirical work. Obviously the complexity of the multiple inequality constraints on nonlinear transformations of the parameters is responsible for the modest attention. A likelihood ratio test of the negativity restriction of S requires parameter estimates which satisfy the inequality constraints. Lau (1978) shows how to impose these constraints on the estimates. Barten and Geyskens (1975) apply Lau's technique in consumer analysis. However, since the estimates must satisfy inequality constraints, the familiar likelihood ratio test is not χ^2-distributed in this case, see e.g. Gouriéroux *et al.* (1980 and 1982). Deaton (1974) proceeds in a different manner. He determines the latent roots of S, which is estimated without imposing the inequality constraints on its roots, and employs a Wald-type test to check whether the positive roots differ significantly from zero. The size of this test is not correct, since estimated roots with a negative value are disregarded and the number of positive latent root estimates is random. As an alternative one might propose to use order statistics to check whether or not the largest latent root differs significantly from zero. This procedure is not always correct since all latent roots must be non-positive. Finally, it is possible to test each latent root separately and to control the overall level of significance by using the Bonferroni inequality (see e.g. Savin, 1980, 1983). However, since

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the individual latent roots will be correlated, the size of the test is not correct asymptotically.

In this paper we show that the Wald or distance test of Kodde and Palm (1986), which is capable of testing multiple nonlinear equality and inequality constraints, solves the problems mentioned above. The procedure does not require estimates subject to the inequality constraints, the size of the test is asymptotically correct and no problems arise with a random number of positive latent root estimates.

The plan of the paper is as follows. In section 2 we deal with the properties of the substitution matrix S in a general problem of optimization. Section 3 outlines the elements of the test required for testing the inequality constraints on S. In section 4 we show how to implement the test to check the negativity of S. A joint test of the negativity and of equality restrictions on S is also discussed. Section 5 reconsiders the empirical results of Barten and Geyskens (1975) using results related to the distance test. In section 6 we summarize the most important conclusions.

2. THE NEGATIVITY OF THE SUBSTITUTION MATRIX

Major parts of economics postulate optimizing behaviour of economic agents as a basis for the analysis. For instance, it is assumed that households maximize utility and firms minimize costs subject to appropriate constraints. The assumption of optimizing behaviour implies a number of restrictions on the model used in the analysis. We will be concerned with the restrictions on the substitution matrix. Consider the maximization of a function $f(x)$ with respect to x, subject to a vector of m independent constraints $g(x) = 0$, where x is a vector of dimension $n (m < n)$. For this problem the Lagrangean is given by $L(x, q) = f(x) - q' g(x)$, where q is a vector of m Lagrange multipliers. A necessary and sufficient second-order condition for a maximum subject to constraints is given by

$$y' A y < 0, \tag{2.1}$$

subject to

$$By = 0, y \neq 0, \tag{2.2}$$

where A is the Hessian matrix of $L(x, q)$ with respect to x and $B = \partial g(x)/\partial x'$, see Samuelson (1947). The substitution matrix $S = A^+ - A^+B'(BA^+B')^{-1}BA^+$, where A^+ denotes the Moore–Penrose generalized inverse of A, see e.g. Rao (1973), is the submatrix corresponding to A of the inverse of the bordered Hessian matrix. The semi-negative definiteness of the substitution matrix, i.e.

$$z'Sz \leqq 0, z \neq 0 \tag{2.3}$$

with the inequality being strict if z is not a linear combination of the columns of B', follows from the second-order conditions for utility maximization. Proofs have been given by Lancaster (1968), Pauwels (1979), Phlips (1974) and Samuelson (1947), among others.

On theoretical grounds the substitution matrix has to satisfy two additional restrictions. First, the symmetry, $S = S'$, is a consequence of Young's theorem. The second restrictions are the adding-up property $BS = 0$ and the homogeneity of S, $SB' = 0$. The conditions of symmetry and negativity, and the homogeneity and adding-up properties, are the so-called integrability conditions which are necessary and sufficient for utility maximization.

3. JOINT TESTING OF EQUALITY AND INEQUALITY CONSTRAINTS

In this section we provide a brief description of tests for the type of restrictions discussed in the preceding section. We first present the distance or Wald test for jointly testing composite hypotheses of nonlinear equality and inequality constraints under H_0. Then we consider the relationships between the distance test and the likelihood ratio test. An advantage of the distance test is that it can be applied when consistent, asymptotically normal estimates of the parameters in the model are available. For the details we refer the reader to Kodde and Palm (1986). Suppose that we are willing to test the p restrictions in

$$H_0: h_1(\theta_0) = 0, h_2(\theta_0) \geqq 0 \text{ against } H_1: h_1(\theta_0) \neq 0,\ h_2(\theta_0) \ngeqq 0 \tag{3.1}$$

where $h_1(\theta_0)$ and $h_2(\theta_0)$ are respectively $q \times 1$ and $(p-q) \times 1$ vector valued functions of the vector of parameters θ evaluated at the true parameter value θ_0. Under H_1 the parameters are not restricted. We assume that θ can be consistently estimated by $\bar{\theta}$, such that the asymptotic distribution is given by

$$T^{1/2}(\bar{\theta} - \theta_0) \overset{A}{\sim} N(0, \Omega), \tag{3.2}$$

where T denotes the sample size and the covariance matrix Ω can be consistently estimated by $\bar{\Omega}$. We transform $h(\theta) = (h_1(\theta)', h_2(\theta)')'$ into a new parameter vector

$$\gamma = (\gamma_1', \gamma_2')' \text{ where } \gamma_i = T^{1/2} h_i(\theta), i \in \{1, 2\}. \tag{3.3}$$

The argument θ in γ has been deleted for the ease of the presentation. When γ is evaluated at $\bar{\theta}$ it will be denoted as $\bar{\gamma}$. The large sample covariance matrix of $\bar{\gamma}$ is given by

$$\Sigma = (\partial h / \partial \theta')\Omega(\partial h' / \partial \theta), \tag{3.4}$$

which is consistently estimated by evaluating (3.4) at $\bar{\theta}$ and $\bar{\Omega}$. In order to obtain the distance test we have to determine the minimum distance from the data (i.e. $\bar{\gamma}$) to the closest feasible point under H_0 and H_1 respectively in the metric of the covariance matrix Σ. The distance to the closest point under H_0 equals

$$D_0 = \min_{\substack{\gamma_1 = 0 \\ \gamma_2 \geqq 0}} (\bar{\gamma} - \gamma)' \Sigma^{-1} (\bar{\gamma} - \gamma), \tag{3.5}$$

and the minimum distance estimator is denoted by $\tilde{\gamma}$. The distance to the closest point under the hypothesis H_1 equals 0, $D_1 = 0$, since the minimum distance estimator equals the unrestricted estimator $\bar{\gamma}$. The distance test is given by

$$D = D_0 - D_1 = D_0. \tag{3.6}$$

Under H_0 the large sample distribution of D is a mixture of χ^2-distributions

$$\sup_{\gamma_2 \geqq 0} Pr(D \geqslant c \mid \Sigma) = \sum_{i=0}^{p-q} Pr[\chi^2(p-i) \geqq c]\, w(p-q, i, \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}), \tag{3.7}$$

where the weights w denote the probability that precisely i of the $p-q$ elements of γ_2 are strictly positive. The covariance matrix in the probability weights is set equal to the conditional covariance matrix of $\bar{\gamma}_2$ given $\bar{\gamma}_1$. For the derivation of the test and the weights w, we refer to Kodde and Palm (1986) and the references therein.

When we only test inequality constraints, $q = 0$ the expressions (3.5) and (3.7) specialize accordingly. Then the weights w in (3.7) depend on Σ. When $\bar{\gamma}$ is an efficient estimate of γ (e.g. a maximum-likelihood estimate) the Wald or distance test (3.6) is asymptotically equivalent to the likelihood ratio (LR) test and the Lagrange multiplier test (LM) or Kuhn–Tucker test (KT). Along the lines of the proof in Gouriéroux *et al.* (1980, 1982), a well-known ordering of these tests for other types of models and hypotheses, KT $\leqq$ LR $\leqq D$, can be shown to hold in the more general case considered in (3.1). Compared with the LR and LM test, the advantages of the D test are threefold: first the computation of D does not require an efficient estimate of Σ. Second, the distance test can also be applied when γ is not efficiently estimated, although then some power will be lost and the equivalence mentioned above does not hold any longer. Third, for nonlinear inequality restrictions the test has local properties only. It is therefore important to evaluate the restrictions in the neighbourhood of the true parameter value θ_0. The fact that the distance test is based on unrestricted parameter estimates only, ensures that in large samples under H_0, the test statistic will indeed be evaluated at a value of θ close to θ_0. Kodde and Palm (1986) give upper and lower bounds for the critical value for jointly testing equality and inequality restrictions for various significance levels. These bounds can be used in all three tests.

4. TESTING THE RESTRICTIONS ON THE SUBSTITUTION MATRIX

In practice it is often possible to derive a consistent, asymptotically normally distributed estimate of the substitution matrix along with a consistent estimate of its asymptotic covariance matrix. The estimate of S may or may not be obtained subject to symmetry, homogeneity and adding-up constraints.

4.1. Testing Negativity

As stated in section 2, the quadratic form in (2.3) has to be semi-negative definite. Let $\bar{S}$ be a consistent symmetric estimator of S such that $T^{1/2}$ vec $(\bar{S}-S_0)\tilde{A}N(0,\Omega)$, with S_0 being the true value of S. We assume that S is not estimated subject to negativity and homogeneity restrictions. Notice, there is no lack of generality in assuming that S is symmetric since we can replace $\bar{S}$ by $[\bar{S}+\bar{S}']/2$ in (2.3). As S has been estimated subject to the symmetry condition, the covariance matrix Ω as given above is singular. When deriving the distribution of the latent roots one has to restrict oneself to the freely varying elements of S. Under H_0, S is semi-negative definite. Therefore all latent roots of S, $\lambda_i, i=1,\ldots,n$, must be smaller than or equal to zero under H_0,

$$H_0: \gamma_i = -T^{1/2}\lambda_i \geqq 0, \quad i=1,\ldots,n. \tag{4.1}$$

In order to apply the distance test we first compute the unrestricted latent roots of $\bar{S}$, which we denote by $\bar{\lambda}_i$, $i=1,\ldots,n$, and we define $\bar{\gamma}'=(\bar{\gamma}_1,\ldots,\bar{\gamma}_n)$ where $\bar{\gamma}_i=-T^{1/2}\bar{\lambda}_i$. The computation of the asymptotic covariance matrix of $\bar{\gamma}$ given in (3.4) requires the partial derivatives of the latent roots with respect to the elements of S. In the appendix we show that the first-order partial derivative of λ_i with respect to vec$'(S)$ equals

$$\partial\lambda_i/\partial\text{vec}'(S) = 2(q_i' \otimes q_i') - \text{vec}'(D_i), \tag{4.2}$$

where q_i denotes the latent vector of S corresponding to λ_i and $\otimes$ is the Kronecker-matrix product and D_i is a diagonal matrix with the squared elements of q_i on the diagonal. The partial derivative in (4.2) must be computed for every latent root. Now the distance test is given by (3.5). We reject the semi-negativity of the substitution matrix if the distance between the unrestricted estimated latent roots and the most favourable point under H_0 exceeds the exact critical value or the upper bound critical value given in Kodde and Palm (1986).

4.2. Testing Negativity and Homogeneity

We first consider testing for negativity of S when S is estimated subject to homogeneity constraints. Then we deal with the joint test of negativity and homogeneity.

Suppose S is estimated subject to $SB' = 0$, where we can regard B as a fixed given matrix. As stated above, $z'Sz < 0$ for z not being a linear combination of B'. Therefore, all latent roots of S, for which the latent vector is not a linear combination of B', must be strictly smaller than zero, $\lambda_i < 0$, $i = 1, \ldots, n-m$. The latent roots which belong to the latent vector manifold of B' have value 0. The distance test is found by applying the following procedure. Compute $\bar{\gamma}_i = -T^{1/2}\bar{\lambda}_i$, $i = 1, \ldots, n-m$ where $\bar{\lambda}_i$'s are the latent roots of S not belonging to the B'-space. Use (4.2) to obtain an estimate of the asymptotic covariance matrix of $\bar{\gamma}$. Finally, determine the distance in (3.5). The null hypothesis has to be rejected if D is larger than the critical value in (3.7), where the number of inequality restrictions equals $n-m$.

A remark ought to be made about the above procedure. The distance test was originally designed to test restrictions of the type $\gamma \geqq 0$ and not $\gamma > 0$. However, this causes no additional difficulties since the difference between the spaces $\gamma \geqq 0$ and $\gamma > 0$ has measure zero.

Next we consider the problem of jointly testing the homogeneity and negativity constraints, for which the null hypothesis is

$$\begin{aligned} H_0 : SB' &= 0 \\ z'Sz &< 0, \ z \text{ is not a linear combination of } B'. \end{aligned} \tag{4.3}$$

Under the alternative hypothesis these restrictions need not be satisfied. The constraint $SB' = 0$ implies $n \times m$ equality restrictions whereas $z'Sz < 0$ implies $n-m$ latent roots of S to be negative. The remaining m roots must be zero.

When unrestricted estimators of B and S are used, SB' will not be zero in small samples and there is a problem which latent roots must be smaller than zero and which roots should be equal to zero. Testing $SB' = 0$ and $z'Sz \leqq 0$ without any restrictions on λ instead of (4.3) remedies the latent root selection problem but introduces dependencies on the restrictions, and the distance test does not apply any more. Instead we transform the null hypothesis into

$$\begin{aligned} H_0 : SB' &= 0 \\ z'CSCz &< 0, \text{ with } z \text{ not being a linear combination of } B', \end{aligned} \tag{4.4}$$

where $C = I - B'(BB')^{-1}B$ is a symmetric matrix of dimension n. Under H_0, $SB' = 0$ so that $CS = SC = S = CSC$, therefore (4.3) and (4.4) are equivalent. The rank of CSC is $n-m$ so that CSC has always m zero latent roots and there is no selection problem. The remaining $n-m$ roots should be negative under H_0.

Let $\bar{S}$ be a consistent symmetric estimator of S and $\bar{B}$ a consistent estimator of B, such that $T^{1/2}(\bar{\theta} - \theta_0) \tilde{A} N(0, \Omega)$, where θ_0 and $\bar{\theta}$ are the true value and an estimate of θ respectively, with $\theta = [\text{vec}'(S), \text{vec}'(B)]'$, and where the covariance matrix Ω is consistently estimated by $\bar{\Omega}$. Define the following magnitudes

$$\gamma_1 = T^{1/2} \text{vec}(BS) \tag{4.5}$$

$$\gamma_2' = (\gamma_{21}, \ldots, \gamma_{2n-m}) \text{ with } \gamma_{2i} = -T^{1/2}\lambda_i, \ i = 1, \ldots, n-m \tag{4.6}$$

$$\gamma = (\gamma_1', \gamma_2')', \tag{4.7}$$

where λ_i denotes the latent roots of CSC which do not belong to the null space. When these magnitudes are evaluated at the consistent estimates $\bar{B}$ and $\bar{S}$ they will be denoted as $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}$ and $\bar{\lambda}_i$ respectively. The distance test for jointly testing the set of equality and inequality restrictions is given in (3.5). The distance test can be computed when a consistent estimate of the covariance matrix Σ is available. The covariance matrix is given by (3.4). In the appendix we provide the formulae for the partial derivative matrices in (3.4). Evaluating the right-hand side of (3.4) at consistent estimates $\bar{\Omega}, \bar{B}$ and $\bar{S}$ yields a consistent estimate Σ. The null hypothesis (4.3) is rejected when the value of D exceeds the critical value of (3.7), where the number of inequality restrictions equals $n-m$ and $q=n\times m$. When B is fixed the procedure specializes accordingly.

In a similar way symmetry can also be tested jointly with negativity and/or homogeneity. Symmetry implies $n(n-1)/2$ equality restrictions on the vector of parameters θ.

5. AN APPLICATION OF THE DISTANCE TEST

Barten and Geyskens (1975) have used two sets of data to illustrate the estimation of the matrix S. They also report the logarithmic likelihood ratio values for various sets of equality and inequality restrictions. The data have been taken from the OECD. The first set refers to the Netherlands, years 1950–1969 and consists of expenditures per capita on five commodity groups. The second set consists of data on expenditures per capita on four commodity groups for the Federal Republic of Germany, years 1950–1968. The change from 1959 to 1960 has been deleted from the sample of Germany (in view of a change in definition of the area in 1960 due to the inclusion of the Saar and West Berlin). The model is the Rotterdam system of consumer demand equations.

Barten and Geyskens report the values given in Table I for the log-likelihood ratio statistic. In addition, we give the number of equality restrictions q and the total number of restrictions $p=q+n-m$ (with n being 5 for the Netherlands and 4 for Germany and $m=1$) and the lower and upper bound values for the critical level of the distance test for size $\alpha=0{\cdot}05$. These values have been obtained by Kodde and Palm (1986).

The likelihood ratio test is asymptotically equivalent to the distance test or Wald test based

Table I. Log-likelihood ratio statistics

H_1	No constraints	Homogeneity	Homogeneity Symmetry
H_0 Homogeneity, Symmetry, Negativity			
Netherlands			
log-likelihood ratio statistic	5·98	2·84	0·036
number of restrictions			
p	15	10	0
q	19	14	4
critical value			
lower bound	22·956	19·045	2·706
upper bound	29·545	23·069	8·761
Germany			
log-likelihood ratio statistic	11·86	9·92	4·10
number of restrictions			
p	10	6	0
q	13	9	3
critical value			
lower bound	19·045	13·401	2·706
upper bound	21·742	16·274	7·045

on efficient estimates of the parameters. Therefore the bounds for the critical level also apply to the likelihood ratio test. If maximum likelihood estimation is not possible the elements of the substitution matrix can for instance be derived from consistent single-equation estimation of the Rotterdam demand system. In that case the distance test can still be applied, although a loss of power may occur compared with using a consistent and efficient estimation procedure. With one exception, the log-likelihood ratio statistics in Table I are smaller than the lower bound, indicating that in line with the conclusion reached by Barten and Geyskens (1975) the null hypothesis cannot be rejected. For Germany, the test of the negativity only, conditionally on homogeneity and symmetry, is inconclusive, as the log-likelihood ratio statistic lies between the lower and upper bound critical values. This finding is in line with the conclusion by Barten and Geyskens (1975) that the negativity hypothesis cannot be firmly rejected for the German data. To reach a conclusion about the significance of the negativity hypothesis in this case, the weights w in (3.7) have to be numerically determined. As the lower and upper bounds coincide when only one inequality is tested (the length of the inconclusive interval increases monotonically with the number of inequality restrictions given the number of equalities), an alternative procedure could consist in testing the inequalities one by one or using a separate induced test based on e.g. Bonferroni's inequality (see e.g. Savin, 1980). Separate testing of the inequalities may also give insight into which roots lead to the inconclusive result.

6. SUMMARY AND CONCLUSIONS

We discussed how the distance or Wald test can be applied to test the negativity of the substitution matrix, possibly jointly with the homogeneity restrictions. The method can be easily extended to test for symmetry. The hypotheses discussed in this paper imply that certain latent roots of the substitution matrix must be smaller than zero. The method tests whether the distance between an unrestricted estimate of the latent roots of S and the most favourable point under H_0 is not significantly different from zero. The large sample distribution of the test statistic is a mixture of χ^2-distributions. The exact critical value of the test in large samples can be determined. In the application presented in section 5, the lower and upper bound critical values derived by Kodde and Palm (1986) appeared to be sufficient to reach a conclusion in all cases but one. Finally, we like to note that a similar procedure can be used to test concavity or quasi-concavity constraints.

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APPENDIX

In this appendix we provide formulae for the first-order partial derivative matrices which are necessary to compute the asymptotic covariance matrix.

First, we derive the sensitivity of the latent roots with respect to a matrix in (4.2), see also Neudecker (1967), Phillips (1982) and Magnus (1985). Let λ be the vector of n roots of S, with S being symmetric. For simplicity, we assume that the roots are distinct. Let Q be the matrix of orthonormal latent vectors of S. Then we have

$$SQ = Q\Lambda \text{ and } Q'Q = I, \tag{A.1}$$

where Λ is a diagonal matrix of the latent roots λ. Let P be the $n \times n^2$ selection matrix such that $P\text{vec}(\Lambda) = \lambda$. In matrix differential form, (A.1) becomes

$$dSQ + SdQ - dQ\Lambda = Qd\Lambda$$
$$Q'dSQ + Q'SdQ - Q'dQ\Lambda = d\Lambda.$$

After using $Q'S = \Lambda Q'$ and vectorization we obtain

$$(Q' \otimes Q')\text{vec}(dS) + [(\text{I} \otimes \Lambda) - (\Lambda \otimes \text{I})](\text{I} \otimes Q')\text{vec}(dQ) = \text{vec}(d\Lambda). \tag{A.2}$$

Premultiplying with P yields

$$P(Q' \otimes Q')\text{vec}(dS) = P\text{vec}(d\Lambda) = d\lambda, \tag{A.3}$$

because $P[(\text{I} \otimes \Lambda) - (\Lambda \otimes \text{I})] = 0$, since the term within brackets is a diagonal matrix with zeros on the diagonal for elements selected by P. By applying the implicit function theorem to (A.3) we find the desired result

$$\partial\lambda/\partial\text{vec}'(S) = P(Q' \otimes Q'). \tag{A.4}$$

The perturbations are not assumed to be symmetric. Application of the chain rule to the i-th now of (A.4) immediately yields the derivative for symmetric perturbations which is given in (4.2).

Next we obtain the partial derivatives required in section 4.2. Since $\gamma_1 = T^{1/2}\,\text{vec}(BS)$. We have $\gamma_1 = T^{1/2}(S \otimes \text{I}_m)\text{vec}(B)$ so that

$$\partial\gamma_1/\partial\text{vec}'(B) = T^{1/2}(S \otimes \text{I}_m). \tag{A.5}$$

Similarly we have

$$\partial\gamma_1/\partial\text{vec}'(S) = T^{1/2}(\text{I}_n \otimes B). \tag{A.6}$$

The partial derivatives of γ_2 with respect to S and B can be obtained from

$$q_1'CSCq_1 = \lambda_1, \tag{A.7}$$

where λ_1 and q_1 are the first non-zero latent root and the corresponding latent vector of CSC respectively. The analysis proceeds analogously for the other non-zero roots. Differentiating (A.7) gives

$$2(dq_1)'CSCq_1 + 2q_1'CS(dC)q_1 + q_1'C(dS)Cq_1 = d\lambda_1. \tag{A.8}$$

Since $q_1'q_1 = 1$ we have $2(dq_1)'q_1 = 0$ so that the first term of (A.8) equals zero because $CSCq_1 = \lambda_1 q_1$. Another useful result is $Cq_1 = q_1$ which follows from

$$\lambda_1 Cq_1 = CCSCq_1 = CSCq_1 = \lambda_1 q_1. \tag{A.9}$$

Because C equals $\text{I} - B'(BB')^{-1}B$, we have

$$dC = -(dB)'(BB')^{-1}B + B'(BB')^{-1}[(dB)B' + B(dB)'](BB')^{-1}B - B'(BB')^{-1}dB. \tag{A.10}$$

Substituting (A.10) into (A.8) and using $Bq_1 = BCq_1 = 0$ along with $Cq_1 = q_1$ we have

$$d\lambda_1 = q_1'(dS)q_1 - 2q_1'SB'(BB')^{-1}(dB)q_1. \tag{A.11}$$

On vectorizing (A.11) applying the implicit function theorem and using the chain rule to account for the symmetry of the perturbations the result emerges

$$\partial\lambda_1/\partial\text{vec}'(S) = 2(q_1' \otimes q_1') - \text{vec}'(D_1)$$

and

$$\partial\lambda_1/\partial \text{vec}'(B) = -2(q_1' \otimes q_1' SB'(BB')^{-1}). \quad (A.12)$$

where D_1 is a diagonal matrix with the squared elements of q_1 on the diagonal.

For the other partial derivatives of the latent roots we substitute the corresponding latent vectors instead of q_1 in (A.12) and we get the result. Notice in order to obtain the partial derivatives with respect to γ we must multiply the formulae in (A.12) by $T^{1/2}$.

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