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ON THE PROPERTIES OF GENERAL EQUILIBRIUM WITH DEFAULT IN ECONOMIES WITH INCOMPLETE MARKETS*

Eduardo A. Rodríguez**

In this paper we study the properties of general equilibrium with default in economies with incomplete markets. It is noted that, in equilibrium, an agent makes two types of comparisons when deciding whether to participate in the credit market: as a lender and as a borrower. As a consequence, the equilibrium can be linked to levels of punishment, perception of default and promised returns. An analysis of equilibrium in the case of economies with two homogeneous types of agents is also presented, from which it can be deduced that in equilibrium under partial default the personal valuations of default for the buyer and the seller are equal.

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1. Introduction

In recent years, the increasing role of capital markets has shown that default should not be underrated when evaluating the orderly functioning of markets. In this sense, the mere possibility or belief in its existence on the part of agents generates immediate effects on both asset prices as well as on the availability of credit, and may even have significant consequences on the agents' consumption possibility set.

When asset markets are incomplete, as occurs in the real world, the lack of instruments in relation to possible future events prevents agents from allocating their resources efficiently based on their preferences and the objective probabilities of their occurrence. Given the impossibility of optimal risk diversification, the possibility of default, when predicted rationally, can contribute to reducing this scarcity. In fact, if agents assume a certain level of punishment for default, the quantity of assets would increase, since each agent could use one or more of them to build a "custom" instrument, each with its price and expected return.

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** *Universidad de Buenos Aires*. E-mail address: edalro@hotmail.com.

Dubey *et al.* (1989, 2000, 2005) and Zame (1993) introduced the idea of the possibility of default in a general equilibrium framework. These studies, as well as several subsequent ones (Zame and Geanakoplos, 2000, Orillo, 2001a, Araujo *et al.*, 2002 and Dubey *et al.*, 2005 among others—see references), prove the existence of equilibrium in this context. However, the demonstration of existence is insufficient in itself for a deeper analysis of its properties. In this sense, a study of agents' optimal individual decision in equilibrium would contribute to the search for qualitative conclusions more general than those presented in the numerical examples contained in some of the aforementioned papers.

Keeping in mind this consideration, an analysis of the properties of general equilibrium with default is presented.

1.1. Main results

The proposed description of general equilibrium with default with incomplete markets enables us to analyze qualitatively the properties of agents' optimal individual decision. Without the technical difficulty of having to deal with an optimization problem which involves a non-differentiable objective function, the equilibrium analysis shows that an agent makes two types of comparisons when making a decision regarding an operation that involves a financial instrument: one as a buyer of assets and another as a seller of assets. In the first case, the agent compares the price to the valuation of the total *effective* asset returns, while in the second case, for the valuation of the returns only the *promised* returns are considered, independent of their effective fulfillment. Regardless of the equilibrium prices vector, if asset punishments differ from one another in a given state, a partial default equilibrium for an asset j implies that all assets with punishments greater than j will be fulfilled in that state while those with punishments lower than j will be fully defaulted.

The proposed characterization also allows us to analyze the relationship between buying and selling assets, punishments, promised payoffs and degree of fulfillment. In this sense, for the existence of equilibrium in asset markets it is necessary that the price of the asset be sufficiently low so that the buyer assumes the risk of default implied by its purchase. On the seller side, the price must be high enough to provide an incentive for taking the risk of a high punishment in the bad state.

In equilibrium, if the transaction is carried out, the price will be equal to the buyer's personal valuation of the portion of the returns that the asset effectively pays. It is definitely the buyer of the asset who ultimately sets the equilibrium price, independent of the seller's personal valuations and promises.

An analysis for economies with two homogenous types of agents shows that if an Arrow asset is subject to partial default, the seller's intertemporal marginal rate of substitution must be lower than the buyer's, while if the promise is fulfilled, the seller's rate must be equal to the buyer's. These properties enable us to verify that, in equilibrium, levels of consumption and default must be such that the personal valuations of the default degree of the asset are equal for the buyer and seller. It is also shown that, in this context, situations of total and partial fulfillment of payments are compatible with finite punishment rates.

1.2. General context and related work

The possibility of default, that is, unfulfillment of a contract, used to be associated with the idea of disequilibrium. The fact that one of the sides of the contract does not deliver the good or goods promised in a given state hinders the fulfillment of the plans made by the other side. However, this relationship between default and disequilibrium is based on the assumption that both sides expect the contract to be fulfilled. If suddenly the side affected were to anticipate a certain degree of default, her plans could be based on this possibility. The idea of equilibrium, viewed as the compatibility of plans, thus begins to sound reasonable, despite the existence of default.

At the end of the 1980s, in a pioneering work, Dubey *et al.* (1989) took this idea much further, setting the discussion within a general equilibrium framework. The main conclusion of this work emphasized the fact that in an economy with incomplete financial markets, the intertemporal allocation of resources could be done more efficiently with the possibility of the existence of default than without it.

In an economy with incomplete financial markets, the possibility of increasing the efficiency of resource allocation is hindered by the lack of appropriate assets to hedge risk in a Pareto efficient way. Thus, it is possible that an agent who wants to allocate her resources in the desired way in one state is simultaneously making promises which are beyond their possibilities in another state. In a context in which there is no possibility of default, asset transactions may not be carried out. However, if the agent had the possibility of defaulting in those states in which she does not have sufficient resources to fulfill her obligations and if in addition, these have a low probability of occurrence, the asset transactions would be possible. If this happens, the efficiency can be increased because the impact of the agent's loss of anticipated profit is smaller due to having assumed a low probability of occurrence of an adverse state, (Zame, 1993). Therefore, by simply facing a punishment, if it is not too high in comparison to the profits in terms of welfare, the agent may be better off defaulting on her obligation in the bad state than staying out of the market. Thus, the default becomes voluntary, depending on the punishment imposed on the side in default.

Clearly, in this situation there is an obvious problem: That which represents an improvement for one agent may represent a worsening for another. An agent who is able to carry out short sales without having to honor her debts and who can go as far into debt as she likes at current interest rates is the same as an agent without any budget constraints, so that the existence of individual equilibrium would be practically impossible.[1] Therefore, it is essential for the existence of the economic problem that some limitation be incorporated into the amount of credit an agent can obtain. This has been done explicitly, both exogenously by Dubey *et al.* (2005) by introducing the idea of "extension of credit," and endogenously by Zame and Geanakoplos (2000) through the requirement of a durable good as collateral.[2]

The models of equilibrium with default with incomplete markets do not differ substantially from the one presented originally in Dubey *et al.* (1989). Its central assumptions can be summarized as follows:

- **Two-period economies:** Time is divided into present and future. There is no possibility of restructuring defaulted obligations, because there is no further division of the period known as "future"[3]. Generally, no discount factors are specified.

1. The only constraint an individual would face in this situation is the amount of total resources available in the economy, an event that the literature discards in equilibrium.
2. Maldonado and Orillo (2005) study the link between models with collateral and those with punishment for default.
3. Orillo (2001) and Araujo *et al.* (2002) study the existence of equilibrium for agents with infinite horizon and commitments backed by collateral. Páscoa and Seghir (2003, 2009) incorporate punishments for default similar to those specified in Dubey *et al.* (1989, 2000, 2005). Within this scheme, Maldonado and Orillo (2003) analyze the problem of infinite horizon in a framework of overlapping generations.

- **Extra-economic punishment:** The only function of the punishment is to serve as an alternative to the cost which the agent must incur if a bad state is revealed. Thus, punishment does not have direct effects on the future participation of the defaulting side in the credit market. The magnitude of the punishment is a parameter which is exogenous to the model and only ensures that the agent fulfills her debt when she is able to do so. For simplicity's sake, punishment is usually understood as linear and separable on the default amount.
- **Expectations of default:** The assets in this type of model are not only represented by their returns, but are also related to an expected degree of default. Thus an asset promises a certain payment in the event of a specific combination of states of nature and degrees of default. If some type of limitation is included, the asset will be identified with a 3-tuple. Equilibrium is reached assuming that the agents' expectations are rational.
- **Anonymous default:** Markets are perfectly competitive, such that each agent sees that her actions have null effect on the aggregate default level. In fact, the assets are thought of as pools such that losses due to default are shared among those who possess the defaulted asset, and the default therefore has a limited impact among the agents. At no time does the owner of a defaulted asset know which agent defaulted. The only information available to the owner is the aggregate degree of default. Thus, an asset is defaulted but an agent is not a defaulter; in other words, there are no "defaulting persons," instead there are "defaulted assets."
- **Single consumption good:**[4] This assumption helps to simplify several issues, the most important being that exchange of goods makes no sense in the second period. Thus, demonstrations are simplified in a sensible way, even though it goes against a natural extension to economies of more than two periods, since in the latter case the assumption of non-exchange of goods in intermediate periods is highly unrealistic.[5] The technical

4. This assumption often appears in the more basic models, not in the more sophisticated ones such as Geanakolos and Zame (2000) and those mentioned in the previous footnote.
5. Dubey *et al.* (1989) point out the fact that in a framework with multiple goods, if an agent fails to deliver a certain good, there arises a problem regarding what prices should be considered for the remaining goods. The answer will depend on whether the defaulting agent has market power or not, or if the group of defaulters is so large that it affects market prices.

advantage of this assumption is acknowledged in many studies, such as Zame (1993), since it allows the demonstration of the existence of equilibrium with default without the need to impose credit constraints on agents.

Once the problem of the existence of equilibrium has been delimited, the following step revolves around its efficiency. It is known that the existence of complete markets is a sufficient condition for the existence of Pareto optimality, while in the case of incomplete markets one can only refer to limited Pareto optimality (Mas-Collel *et al.*, 1995, chapters 16 and 19), generally with a lower degree of efficiency. If the number of assets is lower than the possible states of nature in the future, efficiency can always be lower than the corresponding efficiency of complete markets. It is important to note that the incorporation of new markets does not necessarily increase efficiency when they are incomplete, and may even worsen it (Hart, 1975, example 4). In fact, if all the "markets needed for completion minus one" are incorporated, it is possible that in these new markets no transactions will be generated, maintaining the degree of efficiency unchanged. When there is the possibility of default, the resulting number of actively traded assets is lower than the number of tradeable ones because in default the sale of an asset is not necessarily the opposite of its purchase. The buyer only receives what is delivered to her and the seller also suffers a punishment for what is not delivered. Thus, the relationship between the marginal utility of the purchase and the marginal disutility of the sale can be related to a transaction cost. Assets with returns higher than this cost will be traded, while the rest will not (Dubey *et al.*, 2005). Thus, in equilibrium, a given market may open, but agents simply will not participate in it. The main result of Zame (1993) is, precisely, that the combination of incorporation of new markets with the possibility of default is what enables an improvement in the efficiency of the economy.

1.3. Work plan

After this introduction, the following section contains a review of the original DGS model, which assumes the existence of a single consumption good. We will immediately define the GEI_λ equilibrium, whose properties will be examined in Section 3.

In Section 4, we will concentrate on equilibria in which consumption is not null in all states. First, we will study the comparisons made by the agent in her exercise of optimization. Then we will analyze the connection in equilibrium between degrees of punishment, perception of fulfillment and promised returns with the final position of the agents in a specific asset.

In Section 5, we will assume that agents can be classified into two homogenous types, which will enable us to model their actions through representative agents. In that section, we will study some properties which arise when an agent belonging to a certain group sells an Arrow asset to an agent of the other group and the resulting relationship between the degree of punishment perceived by the issuer of an asset and the level of consumption of both types of agents. Section 6 will present the conclusions.

2. The DGS model

Let's assume a two-period economy with uncertainty. In period 0, H agents trade J assets and a single consumption good. Nature chooses one of the S possible states, which occurs in period 1. All the assets pay on the consumption good in period 1. Following are details on the notation used:

- $s \in S^{*}=\{0\}\cup S$, $S=\{1,...,S\}=$ set of states of nature;
- $h \in H=\{1,...,H\}=$ set of traders;
- $e^h \in \mathbb{R}_{+}^{S*}=$ initial endowments of h. The consumption good is in positive aggregate supply, in other words $e_s = \sum_{h\in H} e_s^h > 0$ for all $s \in S^{*}$;
- $u^h : \mathbb{R}_{+}^{S*} \rightarrow \mathbb{R}$, a utility function on consumption which is smooth, concave and strictly increasing in each of the S^{*} states;
- $j \in J=\{1,...,J\}=$ set of assets;
- $\mathbf{A} = [A_{sj}] \in \mathbb{R}_{+}^{S\times J}$, where A_{sj} is the return on asset j in state s;
- $\lambda_{sj}^{h} =$ default punishment by unit of good on the asset j defaulted by agent h in state s.

To define equilibrium, we consider the macro variables $\bar{p}$, $\bar{\pi}$ and $\bar{\mathbf{K}}$ that each agent perceives. Thus $\bar{p} \in \mathbb{R}_{++}^{S*}$ is the vector of the commodity prices, $\bar{\pi} \in \mathbb{R}_{+}^{J}$ is the vector of asset prices and $\bar{\mathbf{K}} = [K_{sj}] \in \mathbb{R}^{S\times J}$ is a matrix with entries $K_{sj} \in [0,1]$ for all $(s,j) \in S \times J$ which represents the fraction delivered of payments promised by asset j in state s.

The budget set $B^h(\overline{p}, \overline{\pi}, \bar{\mathbf{K}})$ of agent h is given by:

$$B^h(\overline{p},\overline{\pi},\bar{\mathbf{K}}) = \{(x,\theta,\varphi,D) \in \mathbb{R}_+^{S*} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{S\times J}:$$

$$\overline{p}_0(x_0 - e_0^h) + \overline{\pi}(\theta - \varphi) = 0,$$

$$\overline{p}_s(x_s - e_s^h) + \sum\nolimits_j D_{sj} \leq \sum\nolimits_j K_{sj} A_{sj} \theta_j,\ \forall s \in S\}$$

where $x \in \mathbb{R}_+^{S*}$ is the bundle of goods, θ, $\varphi \in \mathbb{R}_+^J$ represent the purchases and sales, respectively, of the J assets and D_{sj} is the total amount effectively delivered by j in s.

The payout of (x, θ, φ, D) in $B^h(\overline{p}, \overline{\pi}, \bar{\mathbf{K}})$ for agent h is:

$$w^h\ (x,\ \theta,\ \varphi,\ D)\ =\ u^h(x) - \sum\nolimits_j \sum\nolimits_s \lambda_{sj}^h \max\{0,\ A_{sj}\, \varphi_j - D_{sj}\}.$$

where $\max\{0,\ A_{sj}\,\varphi_j - D_{sj}\}$ is the default of h on her promise to deliver on asset j in state s. Thus, it is assumed that the punishment is linear and separable in the total defaulted amount.

An GEI_λ equilibrium in a DGS economy is a list $(\bar{p}, \bar{\pi}, \bar{\mathbf{K}}, (\bar{x}^h, \bar{\theta}^h, \bar{\varphi}^h, \bar{D}^h)_{h\in H})$ such that:

[Eq. 0] $\overline{p}_s = 1$ for all $s \in S^*$

[Eq. 1] $(\bar{x}^h, \bar{\theta}^h, \bar{\varphi}^h, \bar{D}^h) \in \arg\max\ w^h\ (x,\theta,\varphi,D)$ on $B^h\ (\bar{p}, \bar{\pi}, \bar{\mathbf{K}})$;

[Eq. 2] $\sum\nolimits_h (\overline{x}^h - \overline{e}^h) = 0;$

[Eq. 3] $\sum\nolimits_h (\bar{\theta}^h - \bar{\varphi}^h) = 0;$

[Eq. 4] $$\bar{K}_{sj} = \begin{cases} \dfrac{\sum_h \bar{D}_{sj}^h}{\sum_h A_{sj} \bar{\varphi}_j^h} & \text{if} \quad \sum_h A_{sj} \bar{\varphi}_j^h > 0 \\ \text{arbitrary} & \text{if} \quad A_{sj} = 0 \end{cases};$$

[Eq. 5] If $\sum_h \bar{\varphi}_j^h = 0 \Rightarrow \bar{K}_{sj} \geq \sum_h t_j^h \mu_{sj}^h$,

where $\mu_{sj}^h = \begin{cases} 1 & \text{if} \quad \partial u^h(\bar{x}^h)/\partial x_s < \lambda_{sj}^h \\ 0 & \text{if} \quad \partial u^h(\bar{x}^h)/\partial x_s > \lambda_{sj}^h \end{cases}$,

$\sum_h t_j^h = 1,\ t_j^h \geq 0$ and if $t_j^h > 0 \Rightarrow$

$(\partial u^h(\bar{x}^h)/\partial x_0)\bar{\pi}_j = \sum_s A_{sj} Min\left\{\lambda_{sj}^h, \partial u^h(\bar{x}^h)/\partial x_s\right\}$.

Condition 0 is just a normalization of prices. Condition 1 indicates that agents must optimize in equilibrium, given the levels of the macro variables. Conditions 2 and 3 require that the goods and assets markets clear.

Condition 4 says that in equilibrium the repayment of each asset must be equal to what all its issuers effectively pay. It is worth emphasizing that, in line with the assumption of perfect competition, each potential buyer of an asset j expects that the default against her will be a proportion of the promised values which is determined by the market and not by her choice of θ_j^h.

Condition 5 considers the situation in which there is no exchange of asset j in equilibrium. In this case, potential lenders do not have market signals on which to base their expectations of default, which is why it is necessary to specify the conjectures they will make so that they are not inconsistent with the incentives the agents will have in equilibrium. This condition indicates that the lender's conjecture is such that she rationally anticipates that borrowers will not pay back anything if their default punishments are sufficiently low, and they will pay back in full if the punishments are sufficiently harsh.

3. Characterization of equilibrium

Let's consider the case proposed originally by Dubey *et al.* (1989), in which it is assumed that $w^h(x, \theta, \varphi, D)$ is quasilinear. In this case, the linear term is the amount of the punishment experienced by the agent, which is linear with respect to the total defaulted. Since it is assumed that in this economy there is a single good, the only variable prices are those of assets. For simplicity, we will use a utility function for the consumption separable among states.

Under these assumptions, the problem of agent h is the following:

Problem P:

$$\underset{(x,\theta,\varphi,D)\in\mathbb{R}_+^{S^*}\times\mathbb{R}_+^J\times\mathbb{R}_+^J\times\mathbb{R}_+^{S^*\times J}}{Max} w^h = u^h(x_0^h)+\sum_{s=1}^S \gamma_s u^h(x_s^h) - \sum_{j=1}^J\sum_{s=1}^S \lambda_{sj}\max\left\{0, A_{sj}\varphi_j^h - D_{sj}^h\right\}$$

$$\text{subject to } \begin{cases}(x_0^h - e_0^h)+\sum_{j=1}^J \overline{\pi}_j(\theta_j^h - \varphi_j^h) = 0\\ (x_s^h - e_s^h)+\sum_{j=1}^J D_{sj}^h \leq \sum_{j=1}^J K_{sj}A_{sj}\theta_j^h \ \ \forall s\in S\end{cases}$$

where we have assumed that $\bar{p}_s = 1$ for all $s \in S^*$ and γ_s represents the probability of occurrence of state s in period 1 viewed from 0.

We know that if the columns of the returns matrix $\mathbf{A} = [A_{sj}] \in \mathbb{R}_+^{S\times J}$, $J < S$, are linearly independent, then there is GEI_λ equilibrium for any vector $\boldsymbol{\lambda} >> 0$ (Dubey *et al.*, 1989, pages 14-19) and therefore a solution to the agent h's optimization problem.

Clearly, when agent h's utility function u^h is strictly increasing in future states, all budget constraints are active in the optimum. It also follows that the equilibrium condition in markets for goods and assets ensures that this situation is verified for all agents in the economy.

The non-differentiability of the objective function of h at points which $A_{sj}\varphi_{sj}^h = D_{sj}^h$ generates disadvantages when trying to apply the Kuhn-Tucker conditions to study the properties of optimal individual decision in equilibrium. Indeed, unless the agent defaults (or pays more than what is owed) in each one of the states, the optimum will occur precisely at one of the points of non-differentiability of w^h. In this sense, the following two propositions taken together formally show that the optimization originally posed by Dubey *et al.* (1989) is equivalent to the problem in which it is assumed that the agents are rewarded for making payments higher than they committed to, but they are unable to do so. Thus, this simple observation enables us to redefine the problem of consumer optimization such that the objective function is differentiable in its entire domain. Therefore, the Kuhn-Tucker conditions are applicable for studying its properties.

Proposition 1. Let $(x^*,\theta^*,\varphi^*,D^*)$ be an equilibrium and u^h be a strictly increasing function in x_s^h for all $s \in S$, then $D_{sj}^{h*} \leq A_{sj}\varphi_j^{h*}$ for all $(s, j) \in S \times J$; in other words, in no future state will agent h pay amounts higher than those promised for the sale of an asset.

Proof: Let's assume that this is not true, so that in the optimum there will be some $(\tilde{s},\tilde{j}) \in S \times J$ such that $D_{\tilde{s}\tilde{j}}^{h*} > A_{\tilde{s}\tilde{j}}\varphi_{\tilde{j}}^{h*}$. Being an optimum, a change in $D_{\tilde{s}\tilde{j}}^{h}$ which verifies the budget constraint for state $\tilde{s}$ should not generate an increase in w^h.

We propose the vector $(x^{**},\theta^*,\varphi^*,D^{**})$, which only differs from the original optimum in the $D_{\tilde{s}\tilde{j}}^{h}$ and $x_{\tilde{s}}^{h}$ variables by an infinitesimal magnitude $\varepsilon > 0$; in other words, $D_{\tilde{s}\tilde{j}}^{h**} = D_{\tilde{s}\tilde{j}}^{h*} - \varepsilon$ and $x_{\tilde{s}}^{h**} = x_{\tilde{s}}^{h*} + \varepsilon$. It must be noted that these new values also satisfy the budget constraint in $\tilde{s}$, the only one in which both variables appear, so $(x^{**},\theta^*,\varphi^*,D^{**})$ is also an admissible vector of the problem. The impact of this change on $D_{\tilde{s}\tilde{j}}^{h}$ and $x_{\tilde{s}}^{h}$ on the utility of agent h can be analyzed in three parts:

1. *On the utility of present consumption*: x_0^{h*} can only vary if any other variable of the budget constraint in $s = 0$ also does. Since $x_{\tilde{s}}^{h}$ and $D_{\tilde{s}\tilde{j}}^{h}$ do not appear in it, and as we have maintained θ_j^h and φ_j^h constant, then the utility of present consumption is not changed by the passage of $(x^*,\theta^*,\varphi^*,D^*)$ to $(x^{**},\theta^*,\varphi^*,D^{**})$.

2. *On the utility of future consumption:*

a) If $s \neq \tilde{s}$: x_s^{h*} can only vary if some variable of the corresponding budget constraint also does. As this does not occur, there is no change in the utility of future consumption in those states.

b) If $s = \tilde{s}$: As $(x_{\tilde{s}}^{h**} - x_{\tilde{s}}^{h*}) = -(D_{\tilde{s}\tilde{j}}^{h**} - D_{\tilde{s}\tilde{j}}^{h*})$, the left side of the budget constraint of state $\tilde{s}$ remains unchanged. As a result of the increasing monotonicity of u^h, the passage of $x_{\tilde{s}}^{h*}$ to $x_{\tilde{s}}^{h**}$ increases the utility of future consumption in state $\tilde{s}$.

3. *On the disutility of punishment:* Because $A_{\tilde{s}\tilde{j}}\varphi_{\tilde{j}}^{h} - D_{\tilde{s}\tilde{j}}^{h} < 0$ is verified at both points, the amount of the punishment remains unaltered by the change and, therefore, so does the disutility for punishment of agent h.

Then $(x^{**},\theta^*,\varphi^*,D^{**})$ generates for h a total utility greater than $(x^*,\theta^*,\varphi^*,D^*)$, with which the latter cannot be an optimum of h.

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Let's consider the following problem:

Problem P':

$$\max_{(x,\theta,\varphi,D)\in\mathbb{R}_+^{S^*}\times\mathbb{R}_+^J\times\mathbb{R}_+^J\times\mathbb{R}_+^{S^*\times J}} \tilde{w}^h = u^h(x_0^h)+\sum_{s=1}^{S}\gamma_s u^h(x_s^h) - \sum_{j=1}^{J}\sum_{s=1}^{S}\lambda_{sj}(A_{sj}\varphi_j^h - D_{sj}^h)$$

$$\text{subject to} \begin{cases} (x_0^h - e_0^h)+\sum_{j=1}^{J}\overline{\pi}_j(\theta_j^h-\varphi_j^h)=0 \\ (x_s^h - e_s^h)+\sum_{j=1}^{J}D_{sj}^h = \sum_{j=1}^{J}K_{sj}A_{sj}\theta_j^h \ \ \forall s\in S \\ D_{sj}^h - A_{sj}\varphi_j^h \leq 0 \ \forall (s,j)\in S\times J \end{cases}$$

which differs from problem P in the function of punishment, the equality of the budget constraints in states $s \in S$ and in the incorporation of the non-negativity constraints of $A_{sj}\varphi_j - D_{sj}$. Thus, while in the original problem the agent does not receive any reward for paying more than her obligations, in this new problem she would receive a reward but is unable to pay more.

At the same time, consider Problem P whith equality in all budget constraints. Let's call this modified problem *Problem* P^*. For application of Proposition 1, if $(x^*,\theta^*,\varphi^*,D^*)$ is optimal in P^*, then it is also optimal in *P'*. The following proposition shows that the reciprocal is also true.

Proposition 2. If $(x^*,\theta^*,\varphi^*,D^*)$ is optimal in *P'*, then it also is in P^*.

Proof: Let there be sets:

$$C = \{(x,\theta,\varphi,D)\in B^h(\overline{p},\overline{\pi},\bar{\mathbf{K}}): \\ \overline{p}_s(x_s - e_s^h)+\sum_j D_{sj} = \sum_j K_{sj}A_{sj}\theta_j \ \ \forall s\in S\}$$

$$M = \{(x,\theta,\varphi,D)\in\mathbb{R}_+^{S^*}\times\mathbb{R}_+^J\times\mathbb{R}_+^J\times\mathbb{R}_+^{S\times J}: \\ D_{sj}^h - A_{sj}\varphi_j^h \leq 0 \ \forall (s,j)\in S\times J\}.$$

By construction we know that $w^h = \tilde{w}^h$ for all $(x,\theta,\varphi,D) \in (C \cap M)$. As $(x^*,\theta^*,\varphi^*,D^*) \in (C \cap M)$ is optimal in P', for all $(x,\theta,\varphi,D) \in (C \cap M)$ it is true that:

$$\max_{C\cap M} \tilde{w}^h = \tilde{w}^h(x^*,\theta^*,\varphi^*,D^*) = w^h(x^*,\theta^*,\varphi^*,D^*) = \max_{C\cap M} w^h.$$

We will demonstrate that $\max_{C} w^h = \max_{C\cap M} w^h$. Let's assume that it is not true, then the optimum in P^* should belong to $(C \sim M)$. However, this contradicts the result of Proposition 1 which requires that the optimum in P^* belong to $(C \cap M)$. Thus $(x^*,\theta^*,\varphi^*,D^*)$ is optimal in P^*.

■

Thus, unlike the original problem, Problem P' has the advantage that it can be addressed through the (necessary) Kuhn-Tucker conditions. This is true given that the new objective function is differentiable in its entire domain and the constraints are all linear, with which the constraint qualification is always verified in the optimum.

Let η_i^h, $i \in S^*$, and δ_{sj}^h, $(s, j) \in S \times J$, be the Lagrange multipliers of the budget constraints and constraints $D_{sj}^h - A_{sj} \leq 0$, respectively. The following is satisfied in the optimum:

$$u^{h\prime}(x_0^{h*}) \leq \eta_0^h, \text{ with equality if } x_0^{h*} > 0; \tag{1}$$

$$\gamma_s u^{h\prime}(x_s^{h*}) \leq \eta_s^h \text{ for all } s \in S, \text{ with equality if } x_s^{h*} > 0; \tag{2}$$

$$\sum\nolimits_{s=1}^{S} \eta_s^h K_{sj} A_{sj} \leq \eta_0^h \overline{\pi}_j \text{ for all } j \in J, \text{ with equality if } \theta_j^* > 0; \tag{3}$$

$$\sum\nolimits_{s=1}^{S} (\lambda_{sj} - \delta_{sj}^h) A_{sj} \geq \eta_0^h \overline{\pi}_j \text{ for all } j \in J, \text{ with equality if } \varphi_j^* > 0; \tag{4}$$

$$\eta_s^h \geq \lambda_{sj} - \delta_{sj}^h \text{ for all } (s, j) \in S \times J, \text{ with equality if } D_{sj}^* > 0; \tag{5}$$

$$(x_0^{h*} - e_0^h) + \sum\nolimits_{j=1}^{J} \overline{\pi}_j (\theta_j^{h*} - \varphi_j^{h*}) = 0,\ \eta_0^h \geq 0; \tag{6}$$

$$(x_s^{h*} - e_s^h) + \sum\nolimits_{j=1}^{J} D_{sj}^{h*} = \sum\nolimits_{j=1}^{J} K_{sj} A_{sj} \theta_j^{h*},\ \eta_s^h \geq 0 \text{ for all } s \in S; \tag{7}$$

$$D_{sj}^{h*} \leq A_{sj}\varphi_j^{h*},\ \delta_{sj}^h \geq 0,\ \delta_{sj}^h (A_{sj}\varphi_j^{h*} - D_{sj}^{h*}) = 0 \text{ for all } (s,j) \in S \times J. \tag{8}$$

Thus, through application of Proposition 2, all optimum points of problem P which satisfy with equality all of the budget constraints verify conditions (1) to (8).[6]

It is evident that the formulation of problem P through conditions (1) to (8) involves the incorporation of the variables δ^h_{sj}. To interpret these new variables it is convenient to write their related constraints as $D^h_{sj} - A_{sj}\varphi^h_j \leq \zeta^h_{sj}$, where ζ^h_{sj} (equal to zero in P') represents the maximum amount that h is permitted to pay in state s above the obligations assumed with the sale of asset j. Thus, δ^h_{sj} indicates the increase in h's maximum utility as a consequence of an increase in the maximum amount allowed for paying j in s. In terms of the original problem P, these new constraints would imply a utility function equal to:

$$\begin{aligned} w^h = \; & u^h(x^h_0) + \sum\nolimits_{s=1}^{S} \gamma_s u^h(x^h_s) \\ & - \sum\nolimits_{j=1}^{J} \sum\nolimits_{s=1}^{S} \lambda_{sj} \max\left\{\zeta^h_{sj}, A_{sj}\varphi^h_j - D^h_{sj}\right\}, \end{aligned}$$

with which δ^h_{sj} indicates the amount by which h's maximum utility would increase if the punishment for default of j in s began to operate at levels greater than ζ^h_{sj} (equal to zero in P^*).

It is evident that for two equally priced assets, agent h will prefer to default the one with a lower punishment. The next two propositions show that this situation is also verified *for all equilibrium prices vectors.*

Proposition 3. If in a state $s' \in S$ the punishments $\lambda_{s'j}$ differ among the different assets, in other words $\lambda_{s'j'} \neq \lambda_{s'j''}$ when $j' \neq j''$, in equilibrium the agent h will partially default at most one asset in s'.

> *Proof*: Suppose that in equilibrium agent h partially defaults on assets j' and j'' in state $s' \in S$. From Condition (5) we know that $\eta^h_{s'} = \lambda_{s'j'} - \delta^h_{s'j'} = \lambda_{s'j''} - \delta^h_{s'j''}$, which through application of Condition (8) implies $\eta^h_{s'} = \lambda_{s'j'} = \lambda_{s'j''}$. Therefore, the punishments are different.
>
> ■

It is also evident in equilibrium that if agent h, seller of assets j' and j'' with $\lambda_{s'j''} < \lambda_{s'j'}$, defaults j' partially in state s', then j'' should

6. Note that with default of j at s, δ_{sj} must be equal to 0, while with fulfillment δ_{sj} can be 0 or have positive values.

not be totally fulfilled in the same state; otherwise h could increase her utility by increasing the degree of fulfillment of j' at the cost of defaulting on j'' by an equal amount. Therefore, from Proposition 3, the default of j'' in s' should be total.

Proposition 4. Let j' and j'' be two assets that agent h defaults in equilibrium in state s'. If the default of j' is total and $\lambda_{s'j''} < \lambda_{s'j'}$, the default of j'' is also total.

> *Proof:* We will demonstrate that there cannot be total default of j' and partial default of j'' with punishments $\lambda_{s'j''} < \lambda_{s'j'}$. Since j' is totally defaulted, from (5) and (8) we have $\eta^h_{s'} \geq \lambda_{s'j'}$, while from the partial default of j'' it follows that $\eta^h_{s'} = \lambda_{s'j''} < \lambda_{s'j'}$, where the inequality is by hypothesis. Both expressions cannot be true simultaneously.
>
> ■

Thus, if in s' agent h totally defaults j^{Δ}, from Proposition 4 it follows that the same will occur with all assets with default punishments lower than j^{Δ}. In addition, we know that if h partially defaults $\overset{\circ}{j}$, all assets j where $\lambda_{s'j^{\circ}} < \lambda_{s'j}$ cannot be defaulted; otherwise, for these assets j either Proposition 3 (if the default of j is partial) or Proposition 4 (if the default is total) would be violated. Thus, if in equilibrium h partially defaults $\overset{\circ}{j}$ it is also proven that for $\lambda_{s'1}<\ldots<\lambda_{s'(j^{\circ}-1)}<\lambda_{s'j^{\circ}}<\lambda_{s'(j^{\circ}+1)}<\ldots<\lambda_{s'J}$, the assets $j \in \{1,\ldots,\overset{\circ}{j}-1\}$ will be totally defaulted, while in the remaining $j \in \{\overset{\circ}{j}+1,\ldots,J\}$ there will be no default whatsoever in s'.

4. Interpretation of the results

We will now focus on the case in which the solution is positive both in the consumption variables x_s, $s \in S$, as well as the multipliers η_i, $i \in S^*$. In this situation, we rewrite inequalities (1) to (5) in the following way:

$$u^{h\prime}(x^h_0) = \eta^h_0 > 0; \tag{9}$$

$$\gamma_s u^{h\prime}(x^h_s) = \eta^h_s > 0 \ \forall s \in S; \tag{10}$$

$$\sum\nolimits_{s=1}^{S} \eta^h_s K_{sj} A_{sj} \leq \eta^h_0 \overline{\pi}_j \ \ \forall j \in J; \tag{11}$$

$$\sum\nolimits_{s=1}^{S} (\lambda_{sj} - \delta^h_{sj}) A_{sj} \geq \eta^h_0 \overline{\pi}_j \ \ \forall j \in J; \tag{12}$$

$$\eta_s^h \geq \lambda_{sj} - \delta_{sj}^h \ \ \forall (s,j) \in S \times J. \tag{13}$$

From inequalities (9), (10) and (11) we obtain:

$$\overline{\pi}_j \geq \sum_{s=1}^{S} \frac{\gamma_s u^{h'}(x_s^h)}{u^{h'}(x_0^h)} K_{sj} A_{sj} \ \ \forall j \in J. \tag{14}$$

On the other hand, combining (12) and (13) and using (9) and (10) we obtain:

$$\overline{\pi}_j \leq \sum_{s=1}^{S} \frac{\gamma_s u^{h'}(x_s^h)}{u^{h'}(x_0^h)} A_{sj} \ \ \forall j \in J. \tag{15}$$

From (14) and (15) it is inferred that:

$$\sum_{s=1}^{S} \frac{\gamma_s u^{h'}(x_s^h)}{u^{h'}(x_0^h)} A_{sj}(1 - K_{sj}) \geq 0 \ \forall j \in J. \tag{16}$$

Note that in this last formula $(1{-}K_{sj})$ is the non-delivered fraction of the payments promised for asset j in state s, such that $A_{sj}\,(1{-}K_{sj})$ is the total default amount on the payment of these returns. Thus, (16) indicates that, in equilibrium, the personal valuation expected of the default amount on the payment of returns of asset j must be non-negative.

With respect to how the agent uses asset price information, it is evident that she makes two types of comparisons when deciding: one as a buyer and the other as a seller. If the agent is considering buying a given asset, inequality (14) will become an equality so the asset price must be equal to her expected valuation[7] of the portion of returns that the asset will *effectively* pay out; in other words, considering a possible default situation.[8] On the other hand, as a seller, the price must not be higher than her expected valuation of the total returns *promised* by the asset,[9] regardless of its effective fulfillment.

7. The term "expected" in this context refers only to the different states of nature, not to the expected default.
8. Note that in the mentioned equation A_{sj} is always multiplied by its respective K_{sj}. If $K_{sj} = 0$ for all $s \in S$, that is total default in all states, the asset price for the buyer is zero because that is her personal valuation of the returns on that asset.
9. Unlike what occurs in (14), equality in Equation (15) is only ensured if the agent defaults on the asset in *all* future states.

In summary, there is a difference in the valuation of the asset by the buyer and the seller when it is partially defaulted even though the expected delivery amount is known in equilibrium by both parties. However, in equilibrium, the price will ultimately be determined by the buyer regardless of the promises and personal valuations of the seller.

In terms of purchase and sale of assets, given a vector of punishments $\tilde{\boldsymbol{\lambda}}$ and matrices $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{A}}$, conditions (11) and (12) say that, in the optimum, agent *i:*

- If $\bar{\pi}_j \begin{Bmatrix} > \\ = \end{Bmatrix} \sum_{s=1}^{S} \gamma_s (u^{i\prime}(x_s^i *) / u^{i\prime}(x_0^i *)) K_{sj} \tilde{A}_{sj} \Rightarrow \theta_j^i * \begin{Bmatrix} = \\ \geq \end{Bmatrix} 0;$
- If $\bar{\pi}_j \begin{Bmatrix} < \\ = \end{Bmatrix} \sum_{s=1}^{S} (\tilde{\lambda}_{sj} - \delta_{sj}^i * / u^{i\prime}(x_0^i *)) \tilde{A}_{sj} \Rightarrow \varphi_j^i * \begin{Bmatrix} = \\ \geq \end{Bmatrix} 0;$

We know that those prices which are above:

$$\sum_{s=1}^{S} u^{i\prime}(x_0^i *)^{-1} (\tilde{\lambda}_{sj} - \delta_{sj}^i *) \tilde{A}_{sj}$$

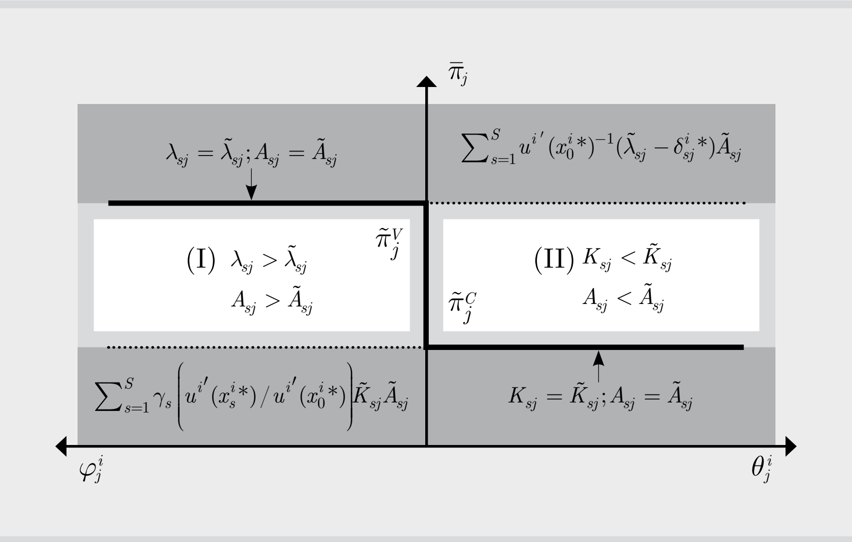
and below:

$$\sum_{s=1}^{S} \gamma_s (u^{i\prime}(x_s^i *) / u^{i\prime}(x_0^i *)) \tilde{K}_{sj} \tilde{A}_{sj}$$

cannot be equilibrium prices since conditions (11) and (12) do not allow that.

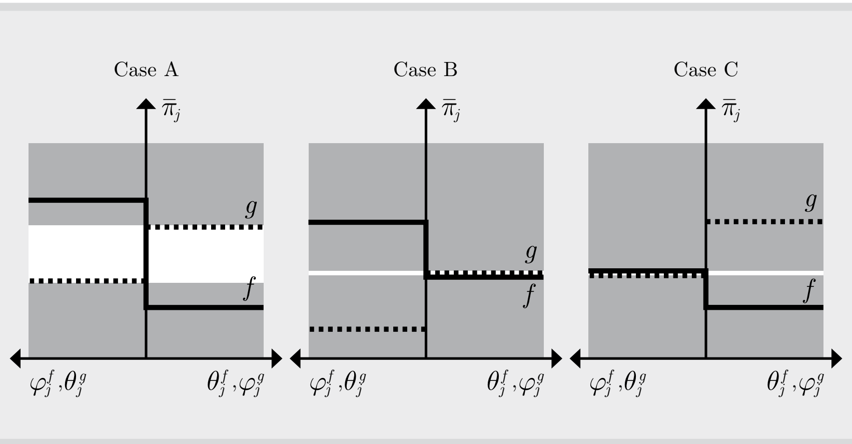
Figure 1 shows the relationship between the quantities traded of an asset j and its price for a given vector (x^*, D^*, δ^*). Because of the aforementioned, prices above $\tilde{\pi}_j^V$ and below $\tilde{\pi}_j^C$ cannot be equilibrium prices because they violate conditions (11) and (12) of agent i (shaded area). Only those points contained in the thick dotted line correspond to optimum traded levels at the price $\tilde{\pi}_j^V \in [\tilde{\pi}_j^C, \tilde{\pi}_j^V]$. Thus, the graph shows that if i is the seller $(\varphi_j > 0)$ she will not deliver asset j for a price lower than $\tilde{\pi}_j^V$ (region I), since the punishment and/or promised return are not sufficiently low. On the other hand, if i is the buyer $(\theta_j > 0)$, she will not acquire the asset for a price higher than $\tilde{\pi}_j^C$ (region II) since the degree of fulfillment and/or the promised return are not high enough.

Figure 1. Prices and trading



From the interaction of two agents f and g we can have, among others, the following cases:

Figure 2. Interaction between agents



In case A the equilibrium cannot be with asset transaction, since the price that *f* demands for the sale is higher than what *g* is willing to pay, while the amount *f* wants to pay for the purchase is much lower than what *g* is willing to give her. In this case, the punishment and return are not sufficiently low for one of the agents and the expected payment is not high enough for the other agent, so the transaction is not carried out.

Case B shows that while the price asked by *f* for the sale is higher than what *g* would be willing to pay for the asset, the same would not occur if *f* is the one who wants to buy and *g* is the one who wants to sell. Thus, since for *f* the price is in accordance with the degree of fulfillment and the payout promised by the asset, while for *g* the same is true both in terms of the punishment and the commitment taken on, the transaction occurs. Case C is similar, but *f* and *g* exchange roles.

5. GEI_λ ECONOMIES WITH TWO HOMOGENEOUS TYPES

In this section we will assume that agents can be classified as one of two homogenous types, represented by agent I or agent II, depending on the group they belong to. Thus, we will analyze what occurs in equilibrium when one of the agents sells an asset to another agent.

We will concentrate on the relationship between an asset's degree of fulfillment, punishment and (positive) consumption of equilibrium of the agents. In this sense, let's assume that agent k sells to h an Arrow asset j^* with positive payout promised in state $\bar{s}$. On the side of agent h (buyer) from conditions (11) and (13) it follows that:

- $$\theta^h_{j^*} > 0 \Rightarrow \sum_{s=1}^{S} \eta^h_s K_{sj^*} A_{sj^*} = \eta^h_0 \overline{\pi}_{j^*} \tag{17}$$

- $$D^h_{\bar{s}j^*} = 0 \text{ (because } \varphi^h_{j^*} = 0) \Rightarrow \eta^h_{\bar{s}} \geq \lambda_{\bar{s}j^*} - \delta^h_{\bar{s}j^*} \tag{18}$$

In regard to agent k (seller), from Condition (4) we have:

- $$\varphi^k_{j^*} > 0 \Rightarrow \sum_{s=1}^{S} (\lambda_{sj^*} - \delta^k_{sj^*}) A_{sj^*} = \eta^k_0 \overline{\pi}_{j^*} \tag{19}$$

and depending on her position with respect to the payout in $\bar{s}$, recalling conditions (5) and (8), it is inferred that in the optimum agent k will satisfy:

a) If she partially defaults:

$$\left.\begin{cases} D^k_{\bar{s}j*} > 0 \Rightarrow \eta^k_{\bar{s}} = \lambda_{\bar{s}j*} - \delta^k_{\bar{s}j*} \\ D^k_{\bar{s}j*} < A_{\bar{s}j*}\varphi^k_{j*} \Rightarrow \delta^k_{\bar{s}j*} = 0 \end{cases}\right\} \Rightarrow \eta^k_{\bar{s}} = \lambda_{\bar{s}j*} \tag{20}$$

b) If she fulfills what was promised:

$$\left.\begin{cases} D^k_{\bar{s}j*} > 0 \Rightarrow \eta^k_{\bar{s}} = \lambda_{\bar{s}j*} - \delta^k_{\bar{s}j*} \\ D^k_{\bar{s}j*} = A_{\bar{s}j*}\varphi^k_{j*} \Rightarrow \delta^k_{\bar{s}j*} \geq 0 \end{cases}\right\} \Rightarrow \eta^k_{\bar{s}} \geq \lambda_{\bar{s}j*} \tag{21}$$

c) If she totally defaults:

$$\left.\begin{cases} D^k_{\bar{s}j*} = 0 \Rightarrow \eta^k_{\bar{s}} \geq \lambda_{\bar{s}j*} - \delta^k_{\bar{s}j*} \\ D^k_{\bar{s}j*} < A_{\bar{s}j*}\varphi^k_{j*} \Rightarrow \delta^k_{\bar{s}j*} = 0 \end{cases}\right\} \Rightarrow \eta^k_{\bar{s}} \geq \lambda_{\bar{s}j*} \tag{22}$$

Regardless of the behavior of k, from equations (17) and (19) the following is always true:

$$\bar{\pi}_{j^*} = \sum\nolimits_{s=1}^{S} \frac{\eta^h_s}{\eta^h_0} K_{sj^*} A_{sj*} = \sum\nolimits_{s=1}^{S} \frac{\lambda_{sj*} - \delta^k_{sj*}}{\eta^k_0} A_{sj*} \tag{23}$$

with which as $A_{\bar{s}j^*} > 0$ and $A_{sj^*} = 0$ for all $s \neq \bar{s}$ the following must also be true:

$$\frac{\eta^h_s}{\eta^h_0} K_{sj*} = \frac{\lambda_{sj*} - \delta^k_{sj*}}{\eta^k_0}. \tag{24}$$

Depending on the degree of asset fulfillment, we will have the following:

a) If asset j^* is partially, but not totally, defaulted in state $\bar{s}$, we know that $0 < K_{\bar{s}j^*} < 1$, and from (24) and recalling that $\eta^k_{\bar{s}} = \lambda_{\bar{s}j^*}$ and $\delta^k_{\bar{s}j^*} = 0$ it is deduced that in equilibrium:

$$K_{\bar{s}j^*} = \frac{\lambda_{\bar{s}j^*} - \delta^k_{\bar{s}j^*}}{\eta^k_0} \frac{\eta^h_0}{\eta^h_{\bar{s}}} = \frac{\eta^k_{\bar{s}}}{\eta^k_0} \frac{\eta^h_0}{\eta^h_{\bar{s}}} < 1 \quad \Rightarrow \quad \frac{\eta^k_{\bar{s}}}{\eta^k_0} < \frac{\eta^h_{\bar{s}}}{\eta^h_0} \tag{25}$$

so that in equilibrium what must occur is:

$$\frac{\lambda^{DP}_{\bar{s}j^*}}{\eta^k_0} = \frac{\eta^h_{\bar{s}}}{\eta^h_0} K_{\bar{s}j^*} \tag{26}$$

Note that, given the concavity of the utility functions of k and h, Equation (26) indicates the existence of a decreasing relationship between punishment and consumption for a given level of default $K_{\bar{s}j^*}$.

Equation (26) also says that, in equilibrium, the personal valuations of greater default on asset j^* in state $\bar{s}$ must be equal for the buyer and the seller. For this purpose, keeping in mind that $\lambda_{\bar{s}j^*}$ represents seller k's marginal disutility for defaulting on j^* in state $\bar{s}$, the left side of (26) measures the value k assigns to the punishment for a marginal default on j^* in $\bar{s}$ in terms of marginal consumption utility in period 0. Recalling that the price of consumer good x is equal to 1, note that if this ratio were less than one, agent k would value greater consumption at 0 more than the disutility she would experience for an increased default on j^* in $\bar{s}$. If this were to occur, k would have incentives to increase the degree of default on this asset in order to use those resources to finance greater present consumption. On the other hand, if this ratio were greater than 1, k would tend to reduce her present consumption and use the resources to fulfill a greater portion of her payments on j^*. According to (26) this punishment-utility ratio should be equal to buyer h's personal valuation of the degree of fulfillment of this asset (right side of the equation).

Note that as in equilibrium agent k verifies $\eta^k_{\bar{s}} = \lambda_{\bar{s}j^*}$ through application of conditions (5) and (8), Equation (26) can be written as $(\eta^k_{\bar{s}}/\eta^k_0) = (\eta^h_{\bar{s}}/\eta^h_0)K_{\bar{s}j^*}$, indicating that the ratio between the intertemporal marginal rates of substitution in consumption between seller k and buyer h must be equal to the degree of fulfillment of asset j^*; in other words, the seller's personal expected valuation of what she promises to pay for the sale of j^* must be equal to the buyer's personal valuation of what she expects to receive from k as payment from this asset.

b) If there is no default in state $\bar{s}$, because k is the only issuer of the asset it is verified that $K_{\bar{s}j^*} = 1$, with which, recalling that $\eta^k_{\bar{s}} \leq \lambda_{\bar{s}j^*}$ we see that:

$$\frac{\eta^h_{\bar{s}}}{\eta^h_0} = \frac{\lambda_{\bar{s}j^*} - \delta^k_{\bar{s}j^*}}{\eta^k_0} = \frac{\eta^k_{\bar{s}}}{\eta^k_0} \tag{27}$$

with which in equilibrium the following has to be true:

$$\lambda^{ND}_{\bar{s}j^*} = \frac{\eta^h_{\bar{s}}}{\eta^h_0}\eta^k_0 + \delta^k_{\bar{s}j} > \frac{\eta^h_{\bar{s}}}{\eta^h_0}\eta^k_0 \tilde{K}_{\bar{s}j^*} = \lambda^{DP}_{\bar{s}j^*} \tag{28}$$

for all $0 < \tilde{K}_{\bar{s}j^*} < 1$. Thus, expression (28) shows that for consumption levels similar to those in case a), the punishment $\lambda^{DP}_{\bar{s}j^*}$ is not sufficient for the inexistence of default on the asset *j'* in state $\bar{s}$. In fact, at punishment levels $\lambda^{DP}_{\bar{s}j^*}$, seller k will always have incentives to increase her consumption, financed by greater default on *j'*.

c) If k, the only issuer of j^*, totally defaults in $\bar{s}$, then $K_{\bar{s}j^*} = 0$[10], with which it follows that:

$$\frac{\eta^h_{\bar{s}}}{\eta^h_0} K_{\bar{s}j^*} = 0 = \frac{\lambda_{\bar{s}j^*} - \delta^k_{\bar{s}j^*}}{\eta^k_0} \leq \frac{\eta^k_{\bar{s}}}{\eta^k_0} \Rightarrow \frac{\eta^k_{\bar{s}}}{\eta^k_0} \geq 0 \tag{29}$$

and remembering that $\delta^k_{\bar{s}j^*} = 0$, $\lambda^{DT}_{\bar{s}j^*}$ must be equal to 0.

From the previous analysis, it can be seen that both situations of total fulfillment and default are compatible in equilibrium with finite rates of punishment. Clearly, equilibrium with partial default is compatible with a lower punishment than that which would correspond to total fulfillment. Equilibrium with total default is a different case, only compatible with null punishment, since a relationship cannot be established a priori between the intertemporal marginal rates of substitution of the agents, as can be done under total fulfillment or partial default. It is worth noting that for economies with three homogeneous types of agents, if two of them are issuers and only one of them totally defaults in $\bar{s}$, then we will have $K_{\bar{s}j^*} < 1$, which brings us to a situation similar to a), since in the latter case the asset j^*, in the aggregate, is partially defaulted.

10. This is not necessarily the case when there are more than two agents, as will be clarified later.

6. Conclusions and possible extensions

In this paper we have studied some properties of GEI_λ equilibrium in a two-period economy with incomplete markets in which default is allowed.

The proposed characterization of equilibrium showed how an agent determines her participation in the credit market. In this sense, we observed that as a buyer, the agent will compare the market price with her personal expected valuation of the total effective returns on the asset. On the other hand, as a seller, the agent will only consider what the asset promises, regardless of whether the payments are made.

Second, we concentrated on the trade of assets, stressing the relationship in equilibrium between degrees of punishment, perception of fulfillment, promised returns and final position in a certain asset. In the first place, if the economy's assets have different punishments in a given state, the maximum number of partially defaulted assets in equilibrium in that state is equal to one. In this sense, it was also shown that all assets with punishment lower than that are totally defaulted; the rest, those with a greater punishment, are fulfilled. With respect to asset prices in equilibrium, if there is trade the prices will be equal to the buyer's personal valuation of the returns she expects to receive from each one of them, regardless of their promised payments. For the buyer, this price is low enough to assume the risk of default which the purchase of the asset implies, while for the seller it is high enough to take the risk of facing a harsh punishment in the bad state.

Finally, we discussed the particular case of economies with two homogeneous types of agents, a buyer and a seller of a given asset. In this sense, we explored the behavior of consumption in a state in which an Arrow asset is defaulted. It is evident that, at least in this case, finite rates of punishment are compatible with degrees of partial and total fulfillment. We have also observed that, in equilibrium, when the default is partial the personal valuations of the degree of default of the asset must be equal for both the buyer and the seller. Thus, the buyer's personal valuation of what the asset is expected to pay must be equal to the ratio between the marginal disutility the seller would experience if an asset is marginally defaulted and its improvement in terms of utility generated by greater present consumption.

The analysis of equilibrium in this paper has focused on the search for qualitative conclusions. However, the proposed characterization of the optimal decision of agents in equilibrium would enable an extension of the analysis to the quantitative field, which implies simulation exercises. In this sense, for the computational effort required to be worthwhile in terms of explaining what is observed in the real world, an interesting contribution would be to extend the analysis of this paper to economies with more than two periods.

However, such an extension is not immediate, as complications begin to arise which cannot be ignored when analyzing the evolution of this type of economy over time.

In the first place, the assumption of the existence of a single consumption good must be eliminated, which implies a considerable increase in the dimensionality of the system of equations to be solved. In the framework of a two-period economy, such an assumption did not imply too many complications because the analysis did not take into account the possibility of goods exchange in the future period. However, when more than two periods are considered, the assumption of non-exchange of goods in intermediate periods is economically unsustainable, so it is essential to incorporate more goods into the analysis. Likewise, the introduction of more consumption goods necessarily requires the inclusion of explicit credit constraint mechanisms to ensure the existence of equilibrium (Dubey *et al.*, 2005).

In this sense, a way to introduce these kinds of constraints is the requirement that a borrower provide collateral. This mechanism, analyzed in Zame and Geanakoplos (2000) for the two-period case as well as in Orillo (2001a, 2001b), Araujo *et al.* (2002) and Páscoa and Seguir (2009) for infinite periods, constitutes an alternative to the introduction of exogenous transversality conditions which tend to prevent the possibility of Ponzi schemes. It is important to point out that a good which acts as collateral should not be only considered as a requirement for obtaining a loan, but also as a durable good, which likewise implies including in the analysis the utility that this good generates for the agent who has it in custody (Zame and Geanakoplos, 2000).

On the other hand, the issue of reputation is another factor which must be considered in this kind of extension. It is to be expected that defaulting agents, in addition to being subject to confiscations or other types of punishments, are restricted from participating in the credit market for a certain period of time. Also, in the event of default, the possibility of renegotiation of the defaulted debt should also be taken into account.

As can be seen, the extension of the DGS model to more than two periods invites treatment of a series of very interesting topics regarding the functioning of financial markets today, which go beyond the objectives of this paper.

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