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Kurtosis transformation and kurtosis ordering

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Abstract: Leptokurtic distributions can be generated by applying certain non-linear transformations to a standard normal random variable. Within this work we derive general conditions for these transformations which guarantee that the generated distributions are ordered with respect to the partial ordering of van Zwet for symmetric distributions and the partial ordering of MacGillivray for arbitrary distributions. In addition, we propose a general power transformation which nests the H-, J- and K-transformations which have already been proposed in the literature. Within this class of power transformations the above mentioned condition can be easily verified and the power can be interpreted as parameter of leptokurtosis.

Keywords: Power kurtosis transformation; leptokurtosis; kurtosis ordering.

1 Introduction

Flexible distribution families which accommodate both skewness and kurtosis can be generated if we transform a standard Gaussian distribution with certain non-linear transformations. One or both tails of the distribution can be shortened or increased by means of separate skewness and kurtosis transformations. If these transformations are parameterized the corresponding parameter can be interpreted as skewness and/or kurtosis parameter. However, this term only makes sense if these parameters actually measure skewness and/or kurtosis of distributions. To verify this we have to prove that general accepted conditions for a skewness and/or kurtosis measures are satisfied. One of these requirements is that a partial skewness and kurtosis ordering is preserved. Whereas van Zwet (1964) proposed kurtosis orderings for symmetrical distributions, Balanda and MacGillivray (1990) introduced kurtosis orderings for arbitrary distributions. For certain transformations it was already shown that these orderings are preserved. Take, for example, the exponential transformation – the so-called H-transformation – which was proposed by Tukey (1977) and extensively discussed by MacGillivray (1992). For other transformations, however, like the K-transformation (Haynes et al.,1997) or the J-transformation (Fischer and Klein, 2003) it has not yet been verified.
Within this work these transformations are embedded in a general class of power transformation. Moreover it is shown that the exponent of this class of power transformation can be understood as kurtosis parameter in the sense of preserving the kurtosis ordering of van Zwet for symmetrical distributions. More general, it is also demonstrated that the kurtosis ordering of van Zwet is preserved for arbitrary (not necessary parameterized) twice differentiable transforms whose ratio of derivatives satisfies certain conditions concerning the monotony.

2 Kurtosis orderings

Van Zwet (1964) introduced a kurtosis ordering (more precise, a partial ordering) on the set of all symmetric, continuous and strictly monotone increasing distributions. In this concept, a symmetric distribution F has less kurtosis than a symmetric distribution G ($F \preceq_S G$), if $G^{-1}(F(x))$ is convex for $x > F^{-1}(0.5)$, where F^{-1} and G^{-1} denote the inverse distribution function (or quantile function) of F and G, respectively. Obviously, $F^{-1}(0.5)$ is the median of the distribution of F.
Balanda and MacGillivray (1990) generalized this partial ordering of van Zwet by using so-called spread functions defined as symmetric differences of quantiles:

$$S_F(u) = F^{-1}(u) - F^{-1}(1-u), \;\; u \geq 0.5.$$

S_F is monotone increasing on $[1/2, 1)$. If F is symmetric, $F^{-1}(u) = -F^{-1}(1-u)$ for $u > 0.5$, so that

$$S_F(u) = 2F^{-1}(u) \;\; u \geq 0.5$$

and

$$S_F^{-1}(x) = F(x/2) \;\text{ for } x > F^{-1}(0.5).$$

This means that (for symmetric distributions) the spread function essentially coincide with the quantile function. In the sense of Balanda and MacGillivray (1990), an arbitrary continuous, monotone increasing distribution function F has less kurtosis than an equally distribution function G ($F \preceq_S G$) if $S_G(S_F^{-1}(x))$ is convex for $x > F^{-1}(0.5)$. If F and G are symmetric distributions,

$$S_G(S_F^{-1}(x)) = 2G^{-1}(2F(x)) = G^{-1}(F(x)) \;\; x \geq F^{-1}(0.5),$$

implying that in this special case the orderings of van Zwet and MacGillivray coincide. This justifies the identical notation $\preceq_S$.

3 Verifying the property of ordering by means of the second derivation

It is well-known that a twice differentiable function is convex if its second derivative is positive. Consequently, we have to investigate the second derivative of

$$S_G(S_F^{-1}(x)) \;\text{ for } x \geq F^{-1}(0.5),$$

provided that it exists. Setting $u = F(x)$, the first derivative is given by

$$a(u) \equiv \frac{\partial S_G(u)}{\partial u} \cdot \frac{1}{S_F'(u)} = \frac{S_G'(u)}{S_F'(u)} \quad \text{for } u \geq 0.5,$$

implying that the second derivative is given by

$$a'(u) = \frac{S_G''(u)S_F'(u) - S_G'(u)S_F''(u)}{S_F'(u)^2} \quad \text{for } u \geq 0.5.$$

$a'(u)$ is positive, if

$$\textbf{(A1)} \quad S''_G(u)S'_F(u) \geq S'_G(u)S''_F(u) \quad \text{for } u \geq 0.5.$$

In terms of density function f and distribution function F,

$$S'_F(u) = \frac{1}{f(F^{-1}(u))} + \frac{1}{f(F^{-1}(1-u))} \quad \text{for } u \geq 0.5$$

and

$$S''_F(u) = \frac{f'(F^{-1}(1-u))}{f(F^{-1}(1-u))^3} - \frac{f'(F^{-1}(u))}{f(F^{-1}(u))^3} \quad \text{for } u \geq 0.5.$$

Note that a necessary condition for the existence of $S''_F(u)$ for $u \geq 0.5$ is that the density f has to be differentiable and strictly positive on $\mathbb{R}$.

Condition (A1) can be simplified for symmetric distributions F and G. Using the so-called score function

$$\psi_F(x) \equiv -\frac{f'(x)}{f(x)}, \quad \text{for } x \in \mathbb{R}$$

we have

$$\textbf{(A2)} \quad \psi_G(G^{-1}(u))\frac{f(F^{-1}(u))}{g(G^{-1}(u))} - \psi_F(F^{-1}(u)) \geq 0 \;\; \text{for } u \geq 0.5.$$

If $\psi_G(x) > 0$ for $x > 0$, equation (A2) is equivalent to

$$\textbf{(A3)} \quad \frac{\psi_F(F^{-1}(u))}{\psi_G(G^{-1}(u))} - \frac{f(F^{-1}(u))}{g(G^{-1}(u))} \leq 0 \;\; \text{for } u > 0.5.$$

This means that the ratio of the score functions has to dominate the ratio of the density functions uniformly for $u > 0.5$.

Example 3.1 (GSH distribution) *The generalized secant hyperbolic (GSH) distribution – which is able to model both thin and fat tails – was introduced by Vaughan (2002) and has density*

$$f_{GSH}(x;t) = c_1(t) \cdot \frac{\exp(c_2(t)x)}{\exp(2c_2(t)x) + 2a(t)\exp(c_2(t)x) + 1}, \qquad x \in \mathbb{R}, t > -\pi \quad (1)$$

with normalizing constants

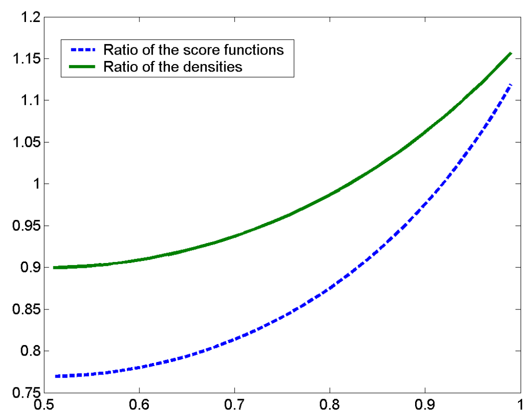
$$\begin{array}{l} a(t) = \cos(t), \quad c_2(t) = \sqrt{\frac{\pi^2 - t^2}{3}}, \; c_1(t) = \frac{\sin(t)}{t} \cdot c_2(t), \;\; \textit{for } -\pi < t \leq 0, \\ a(t) = \cosh(t), \; c_2(t) = \sqrt{\frac{\pi^2 + t^2}{3}}, \; c_1(t) = \frac{\sinh(t)}{t} \cdot c_2(t), \; \textit{for } t > 0 \end{array}.$$

The inverse distribution function is given in closed form by

$$F^{-1}_{GSH}(u;t) = \begin{cases} \frac{1}{c_2(t)} \ln\left(\frac{\sin(tu)}{\sin(t(1-u))}\right) & \textit{für } t \in (-\pi, 0), \\ \frac{\sqrt{3}}{\pi} \ln\left(\frac{u}{1-u}\right) & \textit{für } t = 0, \\ \frac{1}{c_2(t)} \ln\left(\frac{\sinh(tu)}{\sinh(t(1-u))}\right) & \textit{für } t > 0. \end{cases}$$

Klein and Fischer (2003) showed that t is indeed a kurtosis parameter in the sense of van Zwet (1964). For $t_1 = -\pi/2$ and $t_2 = 0.5$ the curve of the ratio of the score functions and of the densities from condition (A3) is exemplarily shown in figure 1, below.

Figure 1: Ratio of score- and density functions



4 A general symmetric kurtosis transformation

Let Z be a random variable which is symmetric around the median 0 and which has continuous distribution function. Define

$$Y \equiv Z \cdot W(Z)$$

where W is a suitable kurtosis transformation. Hoaglin (1983) postulated some plausible requirements to a suitable transformation T of kurtosis. Firstly, W should preserving symmetry, i.e. $\mathrm{W}(z) = \mathrm{W}(-z)$ for $z \in \mathbb{R}$ and we therefore have to discuss W only on the positive axis. Secondly, the initial distribution Z should hardly be transformed in the centre, i.e. $\mathrm{W}(z) \approx z$ for $z \approx 0$. Finally, in order to increase the tails of the distribution, we have to assure that W is accelerated strictly monotone increasing for positive $z > 0$, i.e. $W'(z) > 0$ and $W''(z) > 0$ for $z > 0$. Consequently, W is strictly monotone increasing and convex for $z > 0$. Conversely, a shortening of the tails takes place, either if W is strictly monotone increasing with negative second derivation or if W is not monotone but concave for $z > 0$. Differentiability and monotony imply that $W'(0) = 0$.
Examples which satisfy the aforementioned conditions are:

1. $H(z) = \exp(1/2hz^2) = \exp(1/2z^2)^h$ for $h \in \mathbb{R}$ (Tukey, 1977),
2. $J(z) = \cosh(z)^j$ for $j \in \mathbb{R}$ (Fischer and Klein, 2003),
3. $K(z) = (1 + z^2)^k$ for $k \in \mathbb{R}$ (Tukey, 1964).

The following example shows that the transformations H, J and K are special cases of the general power transformation

$$A(z) \equiv \left(\sum_{i=0}^{\infty} a_i z^{2i} \right)^r \quad \text{for } r \in \mathbb{R}, \tag{2}$$

where the weights a_i, $i = 0, 1, 2, \ldots$ have to be chosen that the limes of the power series $\sum_{i=0}^{\infty} a_i z^{2i}$ exists for all $z \in \mathbb{R}$.

Example:

1. *Tukey's H-transformation can be written as*

$$H(z) = \left(e^{1/2z^2}\right)^h = \left(\sum_{i=0}^{\infty} \frac{1}{2^i i!} z^{2i}\right)^h .$$

It is obvious that $a_i = 1/(2^i i!)$ *in equation (2).*

2. *From*

$$\cosh(z) = 1/2e^z + 1/2e^{-z} = 1/2 \sum_{i=0}^{\infty} \left(\frac{z^i}{i!} + \frac{(-z)^i}{i!}\right) = \sum_{i=0}^{\infty} \frac{1}{2i!} z^{2i}.$$

we see that Fischer and Klein's J-transformation has the power series representation

$$J(z) = \left(\sum_{i=0}^{\infty} \frac{1}{2i!} z^{2i}\right)^j ,$$

that means $a_i = 0$ *for odd* i *and* $a_i = 1/(2i!)$ *for even* i *in equation (2).*

3. *Setting* $a_0 = 1$, $a_1 = 1$ *and* $a_i = 0$, $i > 1$ *in equation (2) leads to Tukey's K-transformation.*

The first and second derivatives of A are

$$A'(z) = r\left(\sum_{i=0}^{\infty} a_i z^{2i}\right)^{r-1} \sum_{i=1}^{\infty} 2ia_i z^{2i-1} = rA(z)C_1(z) \quad \text{with } C_1(z) = \frac{\sum_{i=1}^{\infty} 2ia_i z^{2i-1}}{\sum_{i=0}^{\infty} a_i z^{2i}}$$

and

$$A''(z) = rA(z)((r-1)C_2(z) + C_3(z))$$

with

$$C_2(z) = \left(\frac{\sum_{i=1}^{\infty} 2ia_i z^{2i-1}}{\sum_{i=0}^{\infty} a_i z^{2i}}\right)^2 = C_1(z)^2 \geq 0 \text{ for all } z \in \mathbb{R} \tag{3}$$

and

$$C_3(z) = \frac{\sum_{i=1}^{\infty} 2i(2i-1)a_i z^{2i-2}}{\sum_{i=0}^{\infty} a_i z^{2i}} \tag{4}$$

for $z \geq 0$. Symmetry of A is given by construction. $A(0) = 1$ is satisfied if $a_0 = 1$. $A'(0) = 0$ is satisfied because of $C_1(0) = 0$, if $A(z) \geq 0$ für $z \geq 0$. The monotony condition $A'(z) > 0$ holds, if $r > 0$ and

$$\sum_{i=1}^{\infty} 2ia_i z^{2i-1} \geq 0$$

for $z > 0$. A is tail increasing, if $A''(z) > 0$ which in turn is satisfied, if $r > 0$ and

$$r > 1 - C_3(z)/C_2(z)$$

for all $z > 0$. A transformation A with all this properties will be called a *general power kurtosis transformation* in the sequel. The untransformed distribution (here: standard Gaussian) is obtained for $r = 0$. What remains is to proof that the exponent r can be interpreted as kurtosis parameter in the sense of van Zwet (1964).

5 Kurtosis ordering of general kurtosis transformations

Let Z be a standard Gaussian random variable. Define two random variables

$$Y_i = Z \cdot W_i(Z), \quad i = 1, 2,$$

where W_i, $i = 1, 2$ are kurtosis transformations which are twice differentiable for $z \in \mathbb{R}$ and symmetric around 0. It will be demonstrated in the next theorem that Y_1 has less kurtosis than Y_2 if the condition

$$\textbf{(B)} \qquad \frac{W_2^{(p)}(z)}{W_1^{(p)}(z)} \geq \frac{W_2^{(p-1)}(z)}{W_1^{(p-1)}(z)} \text{ for } z > 0 \text{ and } p = 1, 2$$

holds, provided that $W_i(z) > 0$, $W_i'(z) > 0$ and $W_i''(z) > 0$ for $z > 0$ and $i = 1, 2$. This means that the ratios of the p-derivatives of W_i are monoton increasing in p.

Theorem: *Suppose that Z is a standard normal random variable. Define $Y_i = Z \cdot W_i(Z)$ with distribution function F_i and assume that W_i is twice differentiable on $\mathbb{R}$ with $W_i(z) > 0$, $W_i'(z) > 0$, $W_i''(z) > 0$ for $z > 0$ and symmetric around 0 for $i = 1, 2$. If condition (B) is satisfied for $p = 1, 2$, then $F_1 \preceq_S F_2$.*

Proof: Because of the symmetry of F_i,

$$S_{F_i}(u) = 2F_i^{-1}(u) = 2zW_i(z) \text{ für } z = \Phi^{-1}(u), u \geq 0.5.$$

for $i = 1, 2$. Consequently,

$$S'_{F_i}(\Phi(z)) = 2(W_i(z) + zW_i'(z)) \quad \text{ for } z \geq 0$$

and

$$S''_{F_i}(\Phi(z)) = 2(2W_i'(z) + zW_i''(z)) \quad \text{ for } z \geq 0,$$

$i = 1, 2$. Hence,

$$\begin{aligned}
&1/4\Big[S''_{F_2}(\Phi(z))S'_{F_1}(\Phi(z)) - S''_{F_1}(\Phi(z))S'_{F_2}(\Phi(z))\Big] \\
&= (2W_2'(z) + zW_2''(z))(W_1(z) + zW_1'(z)) - (2W_1'(z) + zW_1''(z))(W_2(z) + zW_2'(z)) \\
&= 2W_2'(z)W_1(z) + 2W_2'(z)zW_1'(z) + zW_2''(z)W_1(z) + z^2W_1'(z)W_2''(z) \\
&\quad -2W_1'(z)W_2(z) - 2W_1'(z)zW_2'(z) - zW_1''(z)W_2(z) - z^2W_2'(z)W_1''(z) \\
&= 2(W_2'(z)W_1(z) - W_1'(z)W_2(z)) + z(W_2''(z)W_1(z) - W_1''(z)W_2(z)) \\
&\quad +z^2(W_2''(z)W_1'(z) - W_1''(z)W_2'(z)),
\end{aligned}$$

so that condition (B) is satisfied, if this term is positive. This is true for $W_1(z) > 0$, $W_1'(z) > 0$, $W_1''(z) > 0$,

$$\frac{W_2'(z)}{W_1'(z)} \geq \frac{W_2(z)}{W_1(z)} \text{ und } \frac{W_2''(z)}{W_1''(z)} \geq \frac{W_2'(z)}{W_1'(z)}$$

for $z > 0$. □

In the special case of a general power kurtosis transformation different exponents define different transformations

$$Y_i = Z \cdot A_i(Z) = \left(\sum_{l=1}^{\infty} a_l z^{2l}\right)^{r_i}, \quad r_i \in \mathbb{R}, \ i = 1, 2.$$

If r is actually a kurtosis parameter, we should be able to show that Y_1 with corresponding parameter r_1 has less kurtosis than Y_2 with corresponding parameter $r_2 > r_1$. For this purpose, condition (B) has to be verified for the ratios of the derivatives of $A_2(z)$ and $A_1(z)$.

Lemma: *Define*

$$A_i(z) \equiv \left(\sum_{l=1}^{\infty} a_l z^{2l}\right)^{r_i}, \quad i = 1, 2, \quad r \in \mathbb{R}$$

and suppose that the power series in brackets converges. Furthermore, assume $A_i'(z) \geq 0$ *and* $A_i''(z) \geq 0$ *for* $z > 0$ *and* $i = 1, 2$. *If* $r_2 > r_1 > 0$,

$$\frac{A_2^{(p)}(z)}{A_1^{(p)}(z)} \geq \frac{A_2^{(p-1)}(z)}{A_1^{(p-1)}(z)} \text{ for } z > 0 \text{ and } p = 1, 2.$$

Proof: It has already been shown that

$$A_i'(z) = r_i A_i(z) C_1(z) \text{ with } C_1(z) = \frac{\sum_{l=1}^{\infty} 2l a_l z^{2l-1}}{\sum_{l=0}^{\infty} a_l z^{2l}} \quad \text{for } i = 1, 2$$

and

$$A_i''(z) = r_i A_i(z)((r_i - 1)C_2(z) + C_3(z)) \quad \text{for } i = 1, 2$$

with $C_2(z) \geq 0$ and $C_3(z)$ as defined in equations (3) and (4). Consequently, for $r_1 > 0$,

$$\frac{A_2'(z)}{A_1'(z)} - \frac{A_2(z)}{A_1(z)} = \frac{A_2(z)}{A_1(z)}\left(\frac{r_2}{r_1} - 1\right) \geq 0,$$

if, and only if $r_2 \geq r_1$. Furthermore, for $r_1 > 0$ we have

$$\frac{A_2''(z)}{A_1''(z)} - \frac{A_2'(z)}{A_1'(z)} = \frac{r_2 A_2(z)}{r_1 A_1(z)}\left(\frac{(r_2 - 1)C_2(z) + C_3(z)}{(r_1 - 1)C_2(z) + C_3(z)} - 1\right) \geq 0$$

if, and only if $r_2 \geq r_1$. □

In that sense the parameter h, j und k of the kurtosis transformations H, J und K can be seen as kurtosis parameters.

6 Summary

Within this work we derived conditions – based on the derivatives of density functions, distribution functions, spread functions and general kurtosis and power kurtosis transformation – under which the kurtosis orderings of van Zwet and MacGillivray, respectively, are preserved. In particular it was shown that the class of general power kurtosis transformation is well suited to generate leptokurtic distributions which are characterized by a well-defined kurtosis parameter.

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