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CONSTRUCTING AND GENERALIZING GIVEN MULTIVARIATE COPULAS: A UNIFYING APPROACH

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SUMMARY

Recently, Liebscher (2006) introduced a general construction scheme of d-variate copulas which generalizes the Archimedean family. Similarly, Morillas (2005) proposed a method to obtain a variety of new copulas from a given d-copula. Both approaches coincide only for the particular subclass of Archimedean copulas. Within this work we present a unifying framework which includes both Liebscher and Morillas copulas as special cases. Above that, more general copulas may be constructed. First examples are given.

Keywords and phrases: construction of d-variate copulas; Archimedean copulas

1 Introduction to d-copulas

Representing the dependence structure of two or more random variables, the popularity of copulas is steadily increasing. Let $[a,b]^d \subseteq \mathbb{R}^d$. A function $K : [a,b]^d \to \mathbb{R}$ is said to be *d-increasing* if its K-volume

$$V_K \equiv \sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\ldots+i_d} K(u_{1i_1}, \ldots, u_{di_d}) \geq 0 \tag{1.1}$$

for all $a \leq u_{i1} \leq u_{i2} \leq b$ and $i = 1, \ldots, d$. If, additionally, $[a,b] = [0,1]$ and K satisfies the boundary conditions

$$K(u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_d) = 0 \quad \text{and} \quad K(1, \ldots, 1, u, 1, \ldots, 1) = u \tag{1.2}$$

for arbitrary $u \in [0,1]$, K is termed as copula and we write C, instead. Putting a different way, let $X_1, \ldots, X_d$ denote d random variables with joint distribution $F(\mathbf{x}) = F(x_1, \ldots, x_d)$ and continuous marginal distribution functions $F_1(x), \ldots, F_d(x)$. According to Sklar's (1959) fundamental theorem, there exists a unique decomposition

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$

of the joint distribution into its marginal distribution functions and the so-called copula

$$C(u_1, \ldots, u_d) = P(U_1 \leq u_1, \ldots, U_d \leq u_d), \quad U_i \equiv F_i(X_i)$$

on $[0,1]^d$ which comprises the information on the underlying dependence structure (For details on copulas we refer to Nelsen, 2006 and Joe, 1997). Finally, if C has dth order derivatives, the d-increasing condition is equivalent to

$$\frac{\partial^d C}{\partial u_1 \dots \partial u_d} \geq 0. \tag{1.3}$$

2 Construction schemes for copulas

There are several construction methods for d-copulas. Among them, the family of Archimedean copulas which enjoys great popularity due to its simple construction. Whereas several generalized Archimedean families emerged in the recent literature (e.g. **XXXXX**), special emphasis will be put on the contributions of Morillas (2005) and Liebscher (2006), henceforth.

2.1 Archimedean copulas

Let $\varphi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing and convex function with $\varphi(1) = 0$ and $\varphi^{[-1]}$ denote the so called *pseudo-inverse* of φ defined by $\varphi^{-1}(t)$ for $0 \leq t \leq \varphi(0)$ and 0 for $\varphi(0) \leq t \leq \infty$. It can be shown (see, e.g. Nelsen, 2006) that $C(u_1, u_2) = \varphi^{[-1]}(\varphi(u_1) + \varphi(u_2))$ defines a class of bivariate copulas, the so-called Archimedean copulas with *additive* generator function φ. Furthermore, if $\varphi(0) = \infty$ the pseudo-inverse describes an ordinary inverse function, and φ is termed as a strict generator. Given a strict generator φ, bivariate Archimedean copulas can be extended to the d-dimensional case ($d \geq 2$): Every function $C : [0,1]^d \to [0,1]$ defined by

$$C(u_1, \dots, u_d) = \varphi^{-1}\Big(\varphi(u_1) + \varphi(u_2) + \dots + \varphi(u_d)\Big) \tag{2.1}$$

is a d-dimensional Archimedean copula if and only if φ^{-1} is completely monotonic on $\mathbb{R}_+$, i.e. if $\varphi^{-1} \in \mathcal{L}_\infty$ with

$$\mathcal{L}_m \equiv \Big\{ \phi : \mathbb{R}_+ \to [0,1] \Big| \, \phi(0) = 1,\, \phi(\infty) = 0,\, (-1)^k \phi^{(k)}(t) \geq 0\,,\, k = 1, \dots, m, \Big\}.$$

Example 2.1. The d-variate Clayton copula arises from $\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1)$ and is given by

$$C^{Cl}(u_1, \dots, u_d) = \Big(u_1^{-\theta} + \dots + u_d^{-\theta} - d + 1\Big)^{-1/\theta},\ \theta > 0. \tag{2.2}$$

Alternatively, Archimedean copulas can be characterized by *multiplicative* generators (see Nelsen, 2006). Setting $\vartheta(t) \equiv \exp(-\varphi(t))$ and $\vartheta^{[-1]}(t) \equiv \varphi^{[-1]}(-\ln t)$, equation (2.1) can be rewritten as

$$C(u_1, \dots, u_d) = \vartheta^{[-1]}\Big(\vartheta(u_1) \cdot \vartheta(u_2) \cdot \dots \cdot \vartheta(u_d)\Big). \tag{2.3}$$

The function ϑ is called *multiplicative* generator of C. Due to the relationship between φ and ϑ, the function $\vartheta : [0,1] \to [0,1]$ is continuous, strictly increasing and concave with $\vartheta(1) = 1$.

2.2 Morillas copulas

Obviously, (2.3) can also be expressed using the independence copula $C^{\perp}(\mathbf{u}) = \prod_{i=1}^{d} u_i$:

$$C(u_1, \dots, u_d) = \vartheta^{[-1]}\Big(C^{\perp}(\vartheta(u_1), \dots, \vartheta(u_d))\Big).$$

Morillas (2005) substitutes $C^{\perp}$ by an arbitrary d-copula $\widetilde{C}$ in order to obtain

$$C_{\vartheta}(u_1, \dots, u_d) = \vartheta^{[-1]}\Big(\widetilde{C}(\vartheta(u_1), \vartheta(u_2), \dots, \vartheta(u_d))\Big) \tag{2.4}$$

and proves that C_{ϑ} is a d-copula if $\vartheta^{[-1]}$ is *absolutely monotonic of order* d on $[0,1]$, i.e. if $\vartheta^{[-1]}(t)$ satisfies

$$(\vartheta^{[-1]})^{(k)}(t) = \frac{d^k \vartheta^{[-1]}(t)}{dt^k} \geq 0$$

for $k = 1, 2, \dots, d$ and $t \in (0,1)$. For detailed properties of C_{ϑ} we refer to Morrilas (2005).

Example 2.2. [Generalized FGM copulas] Possible generator functions ϑ are stated in Morillas (2005, table 1). Notice that not every generator in table 1 is absolutely monotonic for arbitrary $d > 1$. Consider $\vartheta(t) = t^{1/r}$ for $r \geq 1$ (number 2 in table 1), i.e. $\vartheta^{[-1]}(t) = t^r$ which is only absolute monotonic of order $d = 2$ for $r \geq 1$ and assume that $\widetilde{C}$ is a FGM copula, i.e. $\widetilde{C}(u,v) = uv(1 + \theta(1-u)(1-v))$ for $\theta \in [-1,1]$. Hence, with (2.4), the generalized FGM copula

$$C(u, v; \theta, r) = uv\left(1 + \theta(1 - u^{1/r})(1 - v^{1/r})\right)^r$$

results. Extensions to higher dimensions follow immediately.

2.3 Liebscher copulas

Another way of generalizing Archimedean copulas goes back to Liebscher (2006) who introduces d-copulas of the form

$$C(u_1, \dots, u_d) = \Psi\left(\frac{1}{m}\sum_{j=1}^{m} \psi_{j1}(u_1) \cdot \psi_{j2}(u_2) \cdot \ldots \cdot \psi_{jd}(u_d)\right), \tag{2.5}$$

where Ψ and $\psi_{jk} : [0,1] \to [0,1]$ are functions satisfying the following conditions: Firstly, it is assumed that $\Psi^{(d)}$ exist with $\Psi^{(k)}(u) \geq 0$ for $k = 1, 2, \dots, d$ and $u \in [0,1]$, and that $\Psi(0) = 0$. Secondly, ψ_{jk} is assumed to be differentiable and monotonely increasing with $\psi_{jk}(0) = 0$ and $\psi_{jk}(1) = 1$ for all k, j. Thirdly, Liebscher's construction requires that

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m} \psi_{jk}(v)\right) = v \quad \text{for } k = 1, 2, \dots, d \text{ and } v \in [0,1].$$

These conditions guarantee that C defined in (2.5) is actual a copula (see Theorem 4.1 in Liebscher, 2006). It is easily verified that the approaches of Morillas (2005) and Liebscher (2006) coincide only if $m = 1$, $\vartheta_{11}^{[-1]} = \vartheta_{12}^{[-1]} = \ldots \vartheta_{1d}^{[-1]} = \Psi$ in (2.5) and $C_\vartheta = C^\perp$ in (2.4) which corresponds to the generalized (multiplicative) Archimedean case.

In addition, Liebscher (2006) provided a general method who to obtain appropriate functions ψ_{jk}. Assume that $h_{jk} : [0, 1] \to [0, 1]$ for $j = 1, \ldots, m$ and $k = 1, \ldots, d$ are differentiable and bijective functions such that $h'_{jk}(u) > 0$ for $u \in (0, 1)$, $h_{jk}(0) = 0$ and $h_{jk}(1) = 1$. Further assume that $mu = h_{1k}(u) + \ldots + h_{mk}(u)$ holds for each $k = 1, \ldots, d$. Let $\psi = \Psi^{-1}$ be the inverse function of Ψ which is assumed to be differentiable. An appropriate choice is then given by $\psi_{jk}(u) = h_{jk}(\psi(u))$, since $\psi'_{jk}(u) = h'_{jk}(\psi(u)) \cdot \psi'(u) > 0$.

Example 2.3. Consider $d = m = 2$ and define for $\alpha, \beta \in [1, 2]$

$$h_{11}(u) \equiv u^\alpha, \quad h_{21}(u) \equiv 2u - u^\alpha, \quad h_{12}(u) \equiv u^\beta, \quad h_{22}(u) \equiv 2u - u^\beta.$$

Together with $\Psi(t) \equiv t^r$, e.g. $\psi(u) = t^{1/r}$ for $r \geq 1$, the corresponding Liebscher copula is

$$C(u, v) = \left(0.5 \left[u^{\alpha/r}(2v^{1/r} - v^{\alpha/r}) + u^{\beta/r}(2v^{1/r} - v^{\beta/r})\right]\right)^r. \tag{2.6}$$

3 A unifying approach

The key idea of Morillas (2005) was to replace the independence copula (which is implicitly assumed within the multiplicative generalized Archimedean framework) by an arbitrary copula C and to proof that the new function is a copula, too. Having a closer look at (2.5), one might be tempted to replace the product by an arbitrary d-copula in order to extend Liebscher's proposal, at the one hand and to nest Morillas' proposal as second special case, at the other hand. Indeed, in the next section it will be shown that the new function is again a d-copula.

3.1 The main result

For reasons of clarity, consider first $d = 2$ but arbitrary m. In accordance to Liebscher we assume that $\Psi^{(k)}(u) \geq 0$ for $k = 0, \ldots, d$ with $\Psi(0) = 0$. Moreover, ψ_{ij} is presumed to be differentiable and monotone increasing with $\psi_{ij}(0) = 0$, $\psi_{ij}(1) = 1$ and, in order to guarantee the boundary conditions, that

$$\Psi\left(\frac{1}{m}\sum_{j=1}^{m}\psi_{jk}(v)\right) = v.$$

For an arbitrary copula C with existing copula density we define

$$K(u, v) \equiv \Psi\left(\frac{1}{m}\sum_{j=1}^{m} C\left(\psi_{j1}(u), \psi_{j2}(v)\right)\right). \tag{3.1}$$

Obviously,

$$K(u,0) = \Psi\left(\frac{1}{m}\sum_{j=1}^{m}\Big[C(\psi_{j1}(u),\psi_{j2}(0))\Big]\right) = \Psi\left(\frac{1}{m}\sum_{j=1}^{m}\Big[C(\psi_{j1}(u),0)\Big]\right) = \Psi(0) = 0.$$

Similar, $K(0,v) = 0$ and

$$\begin{aligned} K(u,1) &= \Psi\left(\frac{1}{m}\sum_{j=1}^{m}\Big[C(\psi_{j1}(u),\psi_{j2}(1))\Big]\right) = \Psi\left(\frac{1}{m}\sum_{j=1}^{m}\Big[C(\psi_{j1}(u),1)\Big]\right) \\ &= \Psi\left(\frac{1}{m}\sum_{j=1}^{m}\psi_{j1}(u)\right) = u. \end{aligned}$$

Assuming differentiablity of C and neglecting the arguments of the following functions,

$$\frac{\partial K}{\partial u} = \Psi' \cdot \frac{1}{m} \cdot \left[\sum_{j=1}^{m}\frac{\partial C}{\partial u}\cdot \psi'_{j1}\right]$$

and

$$\begin{aligned} k \equiv \frac{\partial^2 K}{\partial u \partial v} &= \Psi'' \cdot \frac{1}{m} \cdot \left[\sum_{j=1}^{m}\frac{\partial C}{\partial u}\cdot \psi'_{j1}\right] + \Psi' \cdot \frac{1}{m}\cdot\frac{\partial}{\partial v}\left[\sum_{j=1}^{m}\frac{\partial C}{\partial u}\cdot \psi'_{j1}\right] \\ &= \Psi'' \cdot \frac{1}{m} \cdot \left[\sum_{j=1}^{m}\frac{\partial C}{\partial u}\cdot \psi'_{j1}\right] + \Psi' \cdot \frac{1}{m}\cdot\left[\sum_{j=1}^{m}\frac{\partial^2 C}{\partial u \partial v}\cdot \psi'_{j1}\cdot\psi'_{j2}\right]. \end{aligned}$$

Positivity follows from the assumptions above and because

$$0 \le \frac{\partial C}{\partial u}, \frac{\partial C}{\partial v} \le 1,$$

which is stated, e.g. by Drouet-Mari & Kotz (2001, p. 67).

Example 3.1 (Continuation of example 2.3)**.** Again, consider $d = m = 2$ and $h_{11}(u) \equiv u^{\alpha}$, $h_{21}(u) \equiv 2u - u^{\alpha}$, $h_{12}(u) \equiv u^{\beta}$, $h_{22}(u) \equiv 2u - u^{\beta}$ for $\alpha, \beta \in [1,2]$ together with $\Psi(t) \equiv t^r$ e.g. $\psi(u) = t^{1/r}$ for $r \ge 1$. Plugging the FGM copula into (3.1),

$$\begin{aligned} C(u,v;\alpha,\beta,\theta) &= 2^{-r}\left(u^{\frac{\alpha}{r}}\left(2\,v^{\frac{1}{r}} - v^{\frac{\alpha}{r}}\right)\left[1+\theta\left(1-u^{\frac{\alpha}{r}}\right)\left(1-2\,v^{\frac{1}{r}}+v^{\frac{\alpha}{r}}\right)\right]\right. \\ &\quad \left. +u^{\frac{\beta}{r}}\left(2\,v^{\frac{1}{r}} - v^{\frac{\beta}{r}}\right)\left[1+\theta\left(1-u^{\frac{\beta}{r}}\right)\left(1-2\,v^{\frac{1}{r}}+v^{\frac{\beta}{r}}\right)\right]\right)^{r}. \end{aligned}$$

Setting $\theta = 0$, the copula from example 2.3 is obtained.

Example 3.2 (Continuation of example 2.2)**.** Consider $m = 2$, $\Psi(t) = t^r$ for $r \geq 1$ and $\psi_{ij}(t) = t^{1/r}$. It follows from the last proof that the result also holds if two different copulas C_1 and C_2 are considered instead of C. Consequently, assuming C_i to be a FGM copula with parameter θ_i for $i = 1, 2$, the following generalized FGM copula results:

$$C(u, v; \theta, r) = uv \left[\left(1 + \theta_1(1 - u^{1/r})(1 - v^{1/r})\right)^r + \left(1 + \theta_2(1 - u^{1/r})(1 - v^{1/r})\right)^r \right].$$

Setting $\theta_2 = 0$, the copula from example 2.2 is obtained.

3.2 The multivariate case

The copula from (3.1) can be extended to higher dimensions. Under the above assumptions on Ψ and ψ_{ij} define

$$K(u_1, \ldots, u_d) = \Psi \left(\frac{1}{m} \sum_{j=1}^{m} C\left(\psi_{j1}(u_1), \ldots, \psi_{j2}(u_d)\right) \right), \tag{3.2}$$

where $m \geq 1$ and $d \geq 2$. Now,

$$\begin{aligned} & K(u_1, \ldots, u_{k-1}, 0, u_{k+1}, \ldots, u_d) = \\ = \; & \Psi \left(\frac{1}{m} \sum_{j=1}^{m} \Big[C(\psi_{j1}(u_1), \ldots, \psi_{jk-1}(u_{k-1}), \psi_{jk}(0), \psi_{jk+1}(u_{k+1}), \ldots, \psi_{jd}(u_d)) \Big] \right) \\ = \; & \Psi \left(\frac{1}{m} \sum_{j=1}^{m} \Big[C(\psi_{j1}(u_1), \ldots, \psi_{jk-1}(u_{k-1}), 0, \psi_{jk+1}(u_{k+1}), \ldots, \psi_{jd}(u_d)) \Big] \right) = \Psi(0) = 0 \end{aligned}$$

and

$$\begin{aligned} & K(1, \ldots, 1, u_k, 1, \ldots, 1) = \\ = \; & \Psi \left(\frac{1}{m} \sum_{j=1}^{m} \Big[C(\psi_{j1}(1), \ldots, \psi_{jk-1}(1), \psi_{jk}(u_k), \psi_{jk+1}(1), \ldots, \psi_{jd}(1)) \Big] \right) \\ = \; & \Psi \left(\frac{1}{m} \sum_{j=1}^{m} \Big[C(1, \ldots, 1, \psi_{jk}(u_k), 1, \ldots, 1)) \Big] \right) = \Psi \left(\frac{1}{m} \sum_{j=1}^{m} \psi_{jk}(u_k) \right) = u_k \end{aligned}$$

such that the boundary conditions are valid.

Neglecting the arguments of the functions and presuming differentiability,

$$k(\mathbf{u}) \equiv k(u_1, \ldots, u_d) = \frac{1}{m} \sum_{i=1}^{d} \left(\sum_{l=1}^{\binom{d-1}{i-1}} \Psi^{(i)} \sum_{j=1}^{m} \frac{\partial^{d+1-i} C}{\partial u_1 \prod_{v \in M_i(l)} \partial u_v} \psi'_{j1} \prod_{v \in M_i(l)} \psi'_{jv} \right), \quad (3.3)$$

where $M_i(l) \subseteq \{2, \ldots, d\}$ with $|M_i(l)| = d - i$. This is a density because $\psi'_{ij}(u) \geq 0$, $\Psi^{(i)} \geq 0$ (by assumption) and $\frac{\partial^i C}{\prod_i \partial u_i}$ exists under certain assumptions (see Joe, 1997, p. 15). Note that the copulas used in the function Ψ can be different.

Example 3.3. Assume $d = 3$ and arbitrary m. Further, using $M_1(1) = \{2, 3\}$, $M_2(1) = \{2\}$, $M_2(2) = \{3\}$ and $M_3 = \emptyset$ in (3.3) the copula density is

$$\begin{aligned} k(\mathbf{u}) &= \frac{\partial}{\partial u_1 \partial u_2 \partial u_3} K(\mathbf{u}) \\ &= \Psi' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^3 C}{\partial u_1 \partial u_2 \partial u_3} \psi'_{j1} \psi'_{j2} \psi_{j3} \right) + \Psi'' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^2 C}{\partial u_1 \partial u_2} \psi'_{j1} \psi'_{j2} \right) \\ &+ \Psi'' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^2 C}{\partial u_1 \partial u_3} \psi'_{j1} \psi'_{j3} \right) + \Psi''' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial C}{\partial u_1} \psi'_{j1} \right). \end{aligned}$$

Example 3.4. For $d = 4$, m arbitrary and with $M_1(1) = \{2, 3, 4\}$, $M_2(1) = \{3, 4\}$, $M_2(2) = \{2, 4\}$, $M_2(3) = \{2, 3\}$, $M_3(1) = \{2\}$, $M_3(2) = \{3\}$, $M_3(3) = \{4\}$, $M_4 = \emptyset$ in (3.3), the copula density $k(\mathbf{u})$ is given by

$$\begin{aligned} k(\mathbf{u}) &= \frac{\partial}{\partial u_1 \partial u_2 \partial u_3 \partial u_4} K(\mathbf{u}) \\ &= \Psi' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^4 C}{\partial u_1 \partial u_2 \partial u_3 \partial u_4} \psi'_{j1} \psi'_{j2} \psi'_{j3} \psi'_{j4} \right) + \Psi'' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^3 C}{\partial u_1 \partial u_3 \partial u_4} \psi'_{j1} \psi'_{j3} \psi'_{j4} \right) \\ &+ \Psi'' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^3 C}{\partial u_1 \partial u_2 \partial u_4} \psi'_{j1} \psi'_{j2} \psi'_{j4} \right) + \Psi'' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^3 C}{\partial u_1 \partial u_2 \partial u_3} \psi'_{j1} \psi'_{j2} \psi'_{j3} \right) \\ &+ \Psi''' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^2 C}{\partial u_1 \partial u_2} \psi'_{j1} \psi'_{j2} \right) + \Psi''' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^2 C}{\partial u_1 \partial u_3} \psi'_{j1} \psi'_{j3} \right) \\ &+ \Psi''' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial^2 C}{\partial u_1 \partial u_4} \psi'_{j1} \psi'_{j4} \right) + \Psi'''' \cdot \frac{1}{m} \left(\sum_{j=1}^{m} \frac{\partial C}{\partial u_1} \psi'_{j1} \right). \end{aligned}$$

4 Parametric candidates for Ψ and h

So far, we introduced a very general copula representation which might be used, on the one hand, to construct general d-copulas themselves but, on the other hand, to generalize given d-copulas. Beside of the copula C itself, two functions Ψ and h have to be specified. We conclude this work with some remarks on how to construct these functions.

4.1 Construction of parametric Ψ-functions

Suitable function Ψ on $[0,1]$ with $\Psi(0) = 0$ and $\Psi(1) = 1$ are required to be absolute monotonic of order k. Trivially, this holds if Ψ is absolute monotonic for any order. According to Feller (1950, p. 249), absolute monotonic and continuous functions admit the representation

$$u(x) = p_0 + p_1 x + p_2 x^2 + \ldots, \quad 0 \leq x < 1$$

for non-negative coefficients p_j. In order to ensure that $u(0) = 0$ and $u(1) = 1$, it follows that $p_0 = 0$ und $p_0 + p_1 + p_2 + \ldots = 1$. Hence, $\Psi(x)$ can be derived from a probability generation function of a discrete distribution (up to an additive and a scaling constant). For instance, for a given $n \in \mathbb{N}$ and $p \in (0,1]$, the probability generation function of a binomial distribution is $p(t) = ((1-p) + pt)^n$ and one obtains

$$\Psi^B(x; p, n) = \frac{((1-p) + px)^n - (1-p)}{p}.$$

The probability generation function of the geometric distribution is given by $p(t) = \frac{p}{1-qt}$, the corresponding Ψ-function is given by

$$\Psi^G(x; p) = \frac{\frac{p}{1-qx} - p}{\frac{p}{1-q} - p} = \frac{(1-q)x}{1-qx}.$$

Similarly, from the Poisson distribution, $p(t) = \exp(-\lambda + \lambda t)$ for $\lambda > 0$ and

$$\Psi^P(x; \lambda) = \frac{e^{-\lambda + \lambda x} - e^{-\lambda}}{1 - e^{-\lambda}} = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1}.$$

Finally, if X has probability mass function

$$P(X = x) = \frac{1}{\cosh(1) x!} \quad \text{for} \quad x = 0, 2, 4, 6, \ldots, \infty$$

we arrive at

$$p(t) = \frac{\cosh(t)}{\cosh(1)}, \quad \text{i.e.} \quad \Psi(x) = \frac{\cosh(x) - \cosh(0)}{\cosh(1) - \cosh(0)}$$

which can be easily generalized to

$$\Psi(x; \alpha) = \frac{\cosh(x^\alpha) - \cosh(0)}{\cosh(1) - \cosh(0)}, \quad \alpha > 0.$$

Of course, other choices for h can be obtained from alternative probability distributions. Moreover, every inverse of Ψ itself is again an admissible function.

4.2 Construction of parametric h-functions

Following Liebscher's construction sketched in subsection 2.3, any suitable differentiable and monotone increasing function h with $h(0) = 0$ and $h(1) = 1$ with $h'(x) \leq 2$ for $x \in [0, 1]$ may be serve as appropriate candidate. Here we present a very simple way to obtain such functions h: Starting from a random variable X on $(a, b) \supseteq [0, 1]$ with distribution function F and (existing) density function f we define

$$h(t) = \frac{F(t) - F(0)}{F(1) - F(0)}$$

which is easily seen to satisfy the above-mentioned requirements if $f(t) \leq 2F(1) - 2F(0)$ is guaranteed. Moreover, $h^{-1}(t)$ is well-defined and also a possible candidate. Possible choices together with the corresponding parameter restrictions are subject to table 1, below.

Parameter	$h(x)$	Distribution	$F(x)$
$\alpha \in [1, 2]$	x^α	Power	x^α
$\delta > 0.3183$	$\frac{x+\sin(x/c)c}{1+\sin(1/c)c}$	Cosine	$\frac{\pi+x/c+\sin(x/c)}{2\pi}$
$\delta \geq 1$	$\frac{x\sqrt{\delta^2-x^2}+\arcsin(x/\delta)\delta^2}{\sqrt{\delta^2-1}+\arcsin(1/\delta)\delta^2}$	Semicircular	$\frac{1}{2} + \frac{x\sqrt{\delta^2-x^2}+\delta^2 \arcsin(x/\delta)}{\pi}$
$\alpha \in [-0.5, 1]$	$\frac{(a+1)x}{1+ax}$	Nameless	$\frac{(a+1)x}{1+ax}$
$c < 2.5138$	$\frac{\ln(1+cx)}{\ln(1+c)}$	Bradford	$\frac{\ln(1+cx)}{\ln(1+c)}$
$0 < \lambda < 1.5936$	$\frac{1-\exp(-\lambda x)}{1-\exp(-\lambda)}$	Exponential	$1 - \exp(-\lambda x)$
$\sigma > 0.4044$	$\frac{\Phi(x,0,\sigma)-\Phi(0,0,\sigma)}{\Phi(1,0,\sigma)-\Phi(0,0,\sigma)}$	Normal	$\Phi(x, 0, \sigma)$
$0.9728 < c < 3.2599$	$\frac{1-\exp(-x^c)}{1-\exp(-1)}$	Weibull	$1 - \exp(-x^c)$
$0 < c < 6.8954$	$\frac{1-(0.5(1+\exp(-x)))^{-c}}{1-2^c(1+e^{-1})^{-c}}$	Gen. Logistic	$(1 + \exp(-x))^{-c}$
$\delta > 0.3102$	$\frac{\exp(-\exp(-x/\delta))-1/e}{\exp(-\exp(-1/\delta))-1/e}$	Gumbel	$\exp(-\exp(-x/\delta))$
$\delta > 0.5366$	$\frac{4\arctan(\exp(\pi x/(2\delta)))-\pi}{4\arctan(\exp(\pi/(2\delta)))-\pi}$	Hyp.Secant	$\frac{2}{\pi} \arctan(\exp(\pi x/(2\delta)))$
$(a, b) \in \mathcal{A}^*$	$abx^{a-1}(1 - x^a)^{b-1}$	Kumaraswany	$abx^{a-1}(1 - x^a)^{b-1}$

Table 1: Different parametric generator functions with

$$\mathcal{A}^* \equiv \left\{ (a, b) | a \geq 1,\ b \geq 1, \quad ab \left(\frac{a-1}{ab-1}\right)^{1-1/a} \left(1 - \frac{a-1}{ab-1}\right)^b \leq 2 \right\}$$

In order to roughly compare the flexibility of these h-functions, we conclude with figure 1, where the possible "range" of all transformations (for different parameter constellations) is numerically displayed.

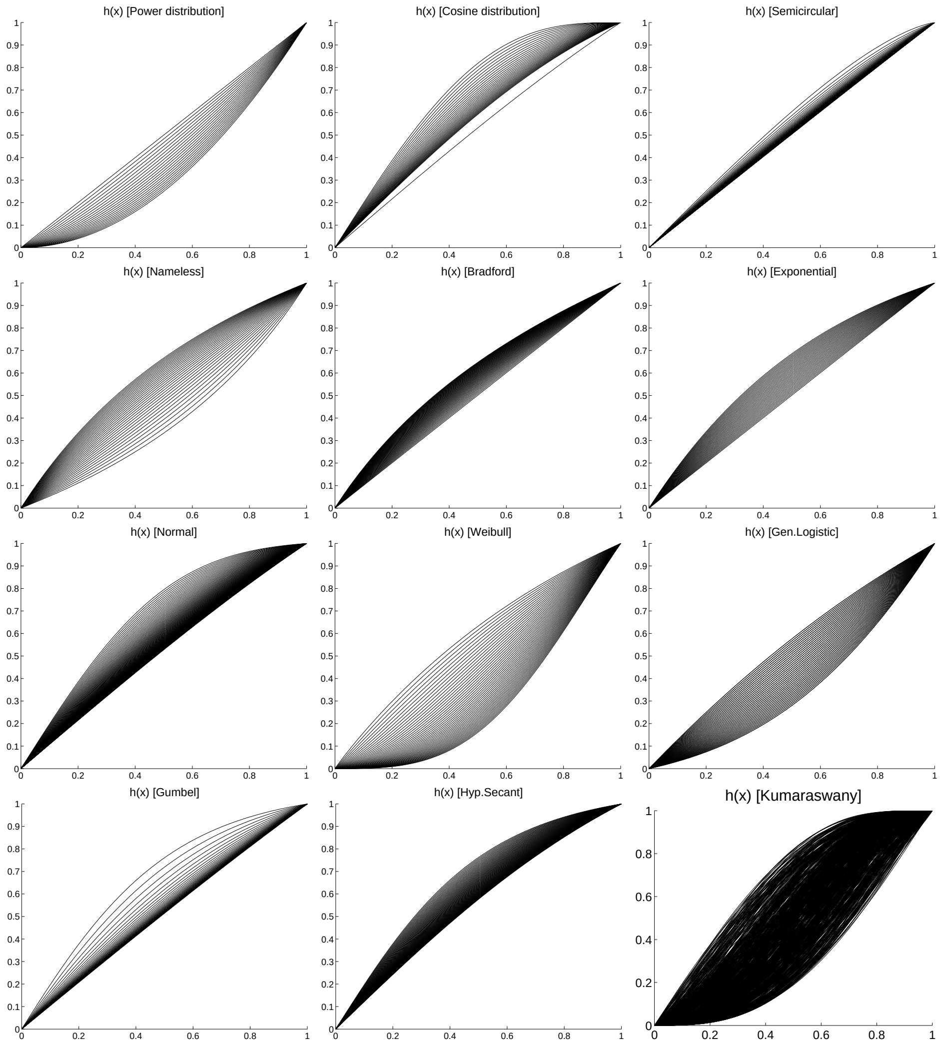

Figure 1: Different h-functions.

5 Summary

The contribution of this paper is a very general construction scheme of d-copulas which generalizes the recent proposals of Morillas (2005) and Liebscher (2006). Given m d-copulas of the same type (i.e from the same copula family) or, possibly, from different families, we show how to combine these copulas to a new copulas by means of certain generator functions which have been adopted from Liebscher (2006). Liebscher's framework is recovered if these "parent copulas" correspond to independence copulas, Morrillas's framework if $m = 1$. We also show how to construct such generator functions. Finally, some examples are given.

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