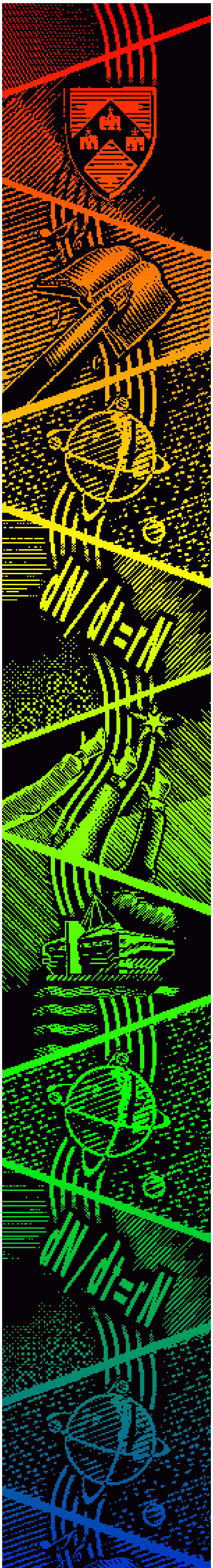

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A Theory of Non-Exclusive Real Options with First-Mover Advantages

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A Theory of Non-Exclusive Real Options with First-Mover Advantages*

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Abstract

This paper analyses the exercise decision of non-exclusive real options in a two-player setting. A general model of non-exclusive real options, allowing the underlying asset to follow any strong Markov process is developed, thus extending the existing literature, which is mainly based on one-dimensional geometric Brownian motion. For games with a first-mover advantage it is proved that an equilibrium with the rent-equalisation property exists. As an example, a duopoly where two firms can adopt a new technology, whose profitability follows a two-dimensional, correlated geometric Brownian motion is studied.
Keywords: Timing games, Real options, Rent equalisation, Technology adoption
JEL classification: C73, D81, O32

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1 Introduction

Ever since the seminal contributions of Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994), the standard approach in investment appraisal has been to view projects as (real) options, the valuation and optimal exercise decision of which have, consequently, to be determined by applying the principles of (financial) option pricing established by Black/Scholes/Merton.Contrary to their financial counterparts, however, real options are typically non-exclusive. This introduces a game theoretic dimension that is absent in standard option pricing models.

The literature on non-exclusive options has, so far, been relatively sparse. This is mainly due to the mathematical intricacies of timing games in continuous time. The main problem is that, in continuous time, the "time instant immediately after time t" is not well-defined (cf. Simon (1987a), Simon (1987b), and Simon and Stinchcombe (1989)).

Many situations in which non-exclusive real options arise have a so-called "first mover advantage". Consider, for example, two firms that have the option to adopt a new technology. The firm that adopts first can have an advantage over its rival due to additional profits that may accrue from the technological innovation. Such a model has been analysed in a deterministic, continuous time setting by Fudenberg and Tirole (1985). They show that, in equilibrium, the two firms will try to preempt each other. In fact, equilibrium strategies are such that both firms' discounted profit streams equal that of the case in which they are not the first firm to invest. In other words, in equilibrium there is *rent equalisation*.

In the game analysed in Fudenberg and Tirole (1985) a particular coordination problem arises, as there are situations where the first mover advantage leads to an environment where it is profitable for one – and only one – firm to invest. In continuous time there is no easy way to deal with this problem. Essentially there are only two possibilities: either firm always invests, whereas the other firm never invests. Both constitute a (pure strategy) equilibrium. These ar, however, not intuitively appealing. How and why would firms coordinate on either equilibrium? By using a technique from optimal control theory Fudenberg and Tirole (1985) introduce the possibility of a mixed strategy, which leads to the result that each firm invests with probability 1/2, which is an intuitively appealing result. Furthermore, joint investment occurs with zero probability.

These ideas have also been applied – explicitly or implicitly – to game-theoretic extensions of real option models.[1] The technique used in most models is a direct,

[1] See, for example, Smets (1991), Grenadier (2000), Huisman (2001), Huisman and Kort (1999),

or simplified, application of the concepts of Fudenberg and Tirole (1985). In fact, in many cases the coordination problem mentioned earlier is dealt with by simply assuming that firms invest with equal probability and that joint investment is impossible, i.e. firms toss a fair coin to solve the coordination problem.

By default, however, non-exclusive real option models deal with uncertainty and, hence, with stopping times instead of deterministic time. Therefore, much of the analysis in Fudenberg and Tirole (1985) is not directly applicable to models with uncertainty. Furthermore, the actual technique used by Fudenberg and Tirole (1985) does not have an intuitive interpretation and is merely a tool to obtain results in continuous time. A first step towards a formal analysis of non-exclusive real options is provided by Murto (2004), who considers exit in a duopoly with declining profitability. In that paper, however, the coordination problem does not arise and an equilibrium in pure strategies can be found. Murto (2004) uses ideas introduced by Dutta and Rustichini (1995). In this framework, the profitability of each firm depends (deterministically) on how many firms are present in the industry and a random part, which follows a geometric Brownian motion (GBM). The firms then each choose a stopping set and exit as soon as the GBM hits their stopping set.

In this paper, the Dutta and Rustichini (1995) and Murto (2004) framework is extended in several ways. Firstly, I adapt and embed the method of Fudenberg and Tirole (1985) to solve the coordination problem in the basic set-up of Murto (2004). Secondly, I prove existence of equilibrium for non-exclusive real options where the underlying asset follows a general, possibly higher dimensional, Markov process. It is shown that there exists an equilibrium in which the rent-equalisation principle holds for this general class of games. Finally, the equilibrium results are applied to a situation where two firms can invest in a project. The set-up is similar to Huisman (2001, Chapter 7), i.e. each firm's profits consists of a deterministic part, which depends on the number of firms having invested, and a random part. The novelty here is that each firm's profits is subject to different, but possibly correlated, GBMs. This introduces an asymmetry in an otherwise symmetric model. It is shown that there are three possible investment scenarios. Two in which one firm acts as if it were an exogenously determined Stackelberg leader whereas the other firm acts as a Stackelberg follower, and one in which both firms try to preempt each other. In the latter case it holds that joint investment occurs with probability zero. A similar result is well-known for models based on a one-dimensional GBM and is basically due to the continuous sample paths of GBM. However, the probability with which each firm invests is not equal to 0.5 (a.s.). This result indicates that one has to be

Weeds (2002), and Thijssen et al. (2006).

careful with imposing exogenous assumptions on the solution to the coordination problem (like the "coin toss" mentioned earlier).

1.1 An illustrative example

To obtain some insight in the problem at hand, consider the following example. It is a basic version of models analysed in Smets (1991), Dixit and Pindyck (1994, Section 9.3), and Huisman (2001, Chapter 7). There are two symmetric firms, both of whom can invest in a new technology by investing a fixed cost $I > 0$. The profits accruing from this project are driven by two main factors: an underlying geometric Brownian motion, $(Y_t)_{t \geq 0}$, and the "investment status" of each firm. Let $\pi_{kl}(Y)$, $k, l = 0, 1$, denote the profits of a firm where k indicates its investment status ($k = 1$ if invested, $k = 0$ otherwise) and l denotes the competitor's investment status. For all Y it is assumed that $\pi_{10}(Y) > \pi_{11}(Y) > \pi_{00}(Y) \geq \pi_{01}(Y)$. Furthermore, it is assumed that there is a *first mover advantage*: $\pi_{10}(Y) - \pi_{00}(Y) > \pi_{11}(Y) - \pi_{00}(Y)$. What are equilibrium investment strategies?

The way this question is answered is by drawing an analogy with a Stackelberg model. There are, basically, three possibilities: a firm invest first (becomes *leader*), does not invest first (becomes *follower*), or both firms invest simultaneously. The expected discounted profits for these scenarios, if first investment takes place at time $t \geq 0$ equal $L(Y_t)$, $F(Y_t)$, and $M(Y_t)$, respectively. A typical plot is given in Figure 1.

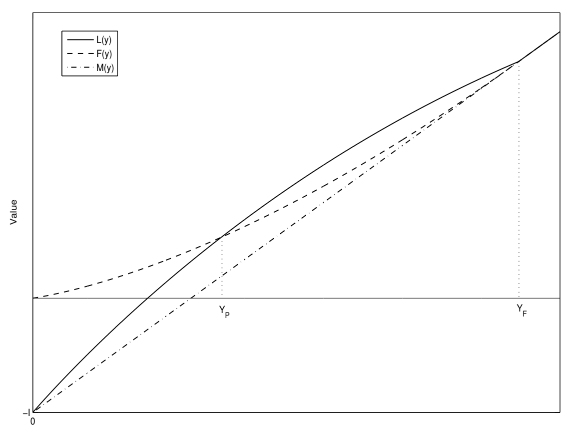

Figure 1: Payoff functions.

In Figure 1, the value Y_F denotes the optimal investment trigger for the follower and Y_P denotes the "preemption point", i.e. the point where each firm prefers to be the leader, rather than the follower. In a subgame perfect equilibrium, both firms will immediately invest as soon as $Y_t \geq Y_F$. Conversely, for $Y_t < Y_P$, neither firm will invest. The interesting region is $Y_t \in [Y_P, Y_F]$.

There are two asymmetric equilibria, namely where either firm always invests as soon as $[Y_P, Y_F]$ is reached and the other never invests. There is, however, no *a priori* reason why firms would coordinate on these equilibria. In order to construct a symmetric equilibrium, one needs what Fudenberg and Tirole (1985) call a "sequence of atoms". This can be thought of as an infinitesimal version of the "grab-the-dollar" game. In each round, firm i invests with a probability α_i, until at least one firm invests. In this way, the possibility of a "coordination mistake" – both firms investing simultaneously – is not ruled out *ex-ante*.[2] It turns out that in equilibrium

$$\alpha^i(Y) = \frac{L(Y) - F(Y)}{L(Y) - M(Y)}.$$

In particular, this implies that at Y_P either firm invests with probability 1/2, and simultaneous investment does not occur (a.s.). Furthermore, the expected discounted profit of both firms equals $F(Y)$, regardless of whether they invest first or not. In other words, *rent-equalisation* takes place in equilibrium.

1.2 Contribution and overview of the paper

The contribution of the paper is two-fold. On the methodological front it presents an intuitively appealing way to analyse non-exclusive (real) options. The driving idea behind the development is to separate – as much as possible – the optimal stopping problems involved in standard (real) options analysis and the game theoretic analysis. Essentially I propose a setting where players use pure strategies to determine *when* they act, but use mixed strategies to determine *what* happens at the time they decide to act. A second contribution is an application of the methodology to investment under uncertainty. In particular, the tools developed in the paper allow for the incorporation of higher dimensional stochastic processes, which leads to several new insights in the investment problem under uncertainty.

Early results in the literature on non-exclusive real options are often obtained by applying the method developed in Fudenberg and Tirole (1985) *as if* it were a deterministic problem.[3] However, due to the stochastic nature of Y, one should be dealing with stopping times. In this paper, an attempt is made to use the ideas from

[2] Joint investment is referred to as a mistake, because neither firm wants it to happen, since the leader and follower values are both larger than the value of simultaneous investment.

[3] A notable exception is Lambrecht and Perraudin (2003), who analyse an incomplete information game and, therefore, use Bayesian Nash equilibrium.

Fudenberg and Tirole (1985) to develop a notion of "subgame perfect equilibrium", where time is essentially stochastic. In this way one can replicate the results from the literature in an appropriate framework. In addition, the coordination device from Fudenberg and Tirole (1985) is given a new interpretation, which makes it appealing for use in the analysis of non-exclusive real options. A further difference with Fudenberg and Tirole (1985) is to replace the use of distribution functions as part of the players' strategies by stopping sets, as suggested by Dutta and Rustichini (1995). The strategy and equilibrium concepts are developed in Section 2.

In Section 3 the existence of a symmetric subgame perfect equilibrium is proved for non-exclusive real options with a first-mover advantage, where the underlying uncertainty follows a d-dimensional strong Markov process. For such non-exclusive real options there can exist a subset of the state space where each player wants to preempt the other. This is called the *preemption region*. It is in this region that the coordination device actually comes into play. It is shown that the principle of rent-equalisation applies in equilibrium in the preemption region. That is, the strategies are chosen such that their expected payoff equals the expected payoff they would get if they were not the first player to exercise for sure. More in particular, it is easily shown that the situation where each player exercises with probability 1/2 in the preemption region is a pathological case if the stochastic process has continuous sample paths. This indicates that caution is needed when making (exogenous) assumptions on coordination as frequently happens in the literature (see, for example, Grenadier (1996), Grenadier (2000), and Weeds (2002)).

Finally, a numerical example is presented in Section 1.1. This example is a straightforward extension of Smets (1991), Dixit and Pindyck (1994, Chapter 9) and Huisman (2001, Chapter 7). I consider a model of technology adoption by firms, where players are completely symmetric up to the uncertainty that they face. Both firms' profits are affected by a geometric Brownian motion with equal trend and volatility. The novelty, however, is that the two processes are not perfectly correlated. The most important consequence of this asymmetry is that it is not *a priori* clear that preemption indeed takes place. In fact, a simulation study shows that the expected time to first investment increases (roughly) linearly in the instantaneous correlation. This implies that the more asymmetric the firms are, the sooner investment takes place (in expectation). The probability of preemption occurring in equilibrium, on the other hand, is (roughly) parabolic in the correlation. This implies, in essence, that the competitive pressure in the market is higher when firms are either more or less correlated.

2 Strategies, Payoffs, and Equilibrium

Throughout this section it is assumed that there are two players, indexed by $i \in \{1, 2\}$. The two players each hold an option of either the call or the put type. The aim of this section is to define an equilibrium that is the stochastic continuous time analogue of a subgame perfect Nash equilibrium. From a game theoretic point of view, the main problem in continuous time modelling is the absence of a well-defined notion of "immediately after time t" (cf. Simon and Stinchcombe (1989)). Dutta and Rustichini (1995) solve this problem by viewing time as being parameterised by two variables.

Definition 1. *Time* is the two-dimensional set $T = \mathbb{R}_+ \times \mathbb{Z}_+$, endowed with the lexicographic ordering, denoted by $\geq_L$, and the standard topology induced by $\geq_L$.

That is, a typical time element is a duplet $t = (s, z) \in T$, which consists of a continuous and a discrete part. In the remainder, s refers to the continuous and z to the discrete component. The continuous component s can be thought of as "real time", on which the underlying uncertainty works, whereas z represents "coordination time", which is used by players to coordinate their actions.

2.1 The underlying asset

There are two stochastic processes that influence the payoffs of players. The first is an exogenously given stochastic process – denoted by Y – which represent the evolution of the "underlying asset" of the non-exclusive option. The second process – denoted by X – is endogenously determined by players' strategies and describes the evolution of the *sate of play*. Its state space is the set $\Xi = \{(0,0), (1,0), (0,1), (1,1)\}$. For $x \in \Xi$, it holds that

$$x_i = \begin{cases} 1 & \text{if Player } i \text{ has exercised the option} \\ 0 & \text{otherwise} \end{cases}, \quad i = 1, 2.$$

More formally, let $(\Omega, \mathcal{F})$ be a measurable space, with a given filtration $(\mathcal{F}_t)_{t \geq_L (0,0)}$. Note that, here, a filtration is a sequence of σ-fields, such that

$$\mathcal{F}_{(s,z)} \subseteq \mathcal{F}_{(s',z')} \subseteq \mathcal{F},$$

if $(s, z) \leq_L (s', z')$.

For all $y \in \mathbb{R}^d$, let P_y be a probability measure on $(\Omega, \mathcal{F})$ and let $(Y_t)_{t \geq_L (0,0)}$ be a strong Markov process defined on $(\Omega, \mathcal{F}, P_y)$, such that

1. $(Y_t)_{t \geq_L (0,0)}$ is adapted to $(\mathcal{F}_t)_{t \geq_L (0,0)}$,

2. $(Y_t)_{t \geq_L (0,0)}$ takes values in $(\mathbb{R}^d, \mathcal{B}^d)$, where $\mathcal{B}^d$ is the d-dimensional Borel σ-field,

3. for all $s \in \mathbb{R}_+$ it holds that $Y_{(s,0)} = Y_{(s,1)} = \cdots \equiv Y_s$, and

4. $Y_{(0,0)} = y$, P_y-a.s.

So, in essence, $(Y_t)_{t \geq_L (0,0)}$ can be created from a stochastic process in continuous time, $(Y_s)_{s \in \mathbb{R}_+}$ – whose sample paths are right-continuous and left-continuous over stopping times – extended to T, such that condition 3 holds. The process $(Y_t)_{t \geq_L (0,0)}$ represents the evolution of the value of the asset underlying the real option. For further reference, define the stopping time

$$\tau_y(A) := \inf\{s \in \mathbb{R}_+ | Y_{(s,z)} \in A, Y_{(0,0)} = y, P_y\text{-a.s.}\},$$

for all $A \in \mathcal{B}^d$. If there is no confusion possible as to the value of y the subscript is dropped.

2.2 Strategies and the coordination mechanism

For every $y \in \mathbb{R}^d$ and $x \in \Xi$, a strategy for Player i will consist of two ingredients, namely a set $S^i_{y,x} \subseteq \mathbb{R}^d_+$, and a probability $\alpha^i_{y,x} \in (0,1]$. The set $S^i_{y,x}$ is referred to as Player i's *stopping set*, given the initial values (y,x).[4] The probability $\alpha^i_{y,x}$ describes Player i's *exercise intensity* at time $\tau(S^i_{y,x})$. The probabilities $\alpha^1_{y,x}$ and $\alpha^2_{y,x}$ together determine the evolution of a process $(X_t)_{t \geq_L (0,0)}$, which tracks the exercise status of both players.

Definition 2. A *Markov strategy* $\sigma^i = (\sigma^i_{y,x})_{(y,x) \in \mathbb{R}^d \times \Xi}$ for Player i, $i = 1,2$, specifies for all $y \in \mathbb{R}^d_+$ and all $x \in \Xi$ a pair $\sigma^i_{y,x} = (S^i_{y,x}, \alpha^i_{y,x})$, where $S^i_{y,x} \subseteq \mathbb{R}^d_+$ and $\alpha^i_{y,x} \in (0,1]$, such that

1. $S^i_{y,x} = \emptyset$, if $x_i = 1$;

2. $\alpha^i_{y,x} = 1$, if $x_j = 1$.

These two conditions are merely regularity conditions. The former ensures that a player can exercise the option only once. The latter condition is imposed in recognition of the fact that, if Player j has already exercised, Player i faces a decision-theoretic problem and there is no need for coordination. As a further convention, it

[4] Note that it is not required, *a priori*, that the stopping set is connected. So, the model is rich enough to include situations where the optimal exercise rule is not of the standard "optimal exercise trigger" form.

is assumed that $\tau_y(\emptyset) = \infty$. The set of Markov strategies for player i is denoted by $\mathcal{S}^i$. It is important to note that, typically, $\alpha^i_{y,x}$ can – and will – depend on $Y_{\tau(S^i_{y,x})}$.

Let $\sigma = (\sigma^1, \sigma^2) \in \mathcal{S}^1 \times \mathcal{S}^2$. For all $y \in \mathbb{R}^d$ and $x \in \Xi$, the exercise intensities $\alpha^1_{y,x}$ and $\alpha^2_{y,x}$ will determine the evolution of a process $\left(X^{\sigma_{y,x}}_t\right)_{t \geq_L (0,0)}$ describing the evolution of the exercise status of both players in the following way. First of all, $\left(X^{\sigma_{y,x}}_t\right)_{t \geq_L (0,0)}$ is operating in discrete time:

$$X^{\sigma_{y,x}}_{(s,0)} = \lim_{z \to \infty} X^{\sigma_{y,x}}_{(s-,z)}, \tag{1}$$

for all $s \in \mathbb{R}^d$.

Let $\tau_y := \tau(S^1_{y,x}) \wedge \tau(S^2_{y,x})$. For all $s < \tau_y$ no player exercises the option and, therefore,

$$X^{\sigma_{y,x}}_{(s,z)} = x, \quad \text{for all } z \in \mathbb{Z}_+ \text{ and } s < \tau_{y,x}. \tag{2}$$

At τ_y, the players start playing a game in coordination time to determine who exercises the option at time τ_y. In each round of this coordination game, Player i, $i = \{1, 2\}$, exercises the option with probability $\alpha^i_{y,x}$. Play continues until *at least one* player exercises the option, which happens at time

$$\zeta_{y,x} := \inf\{z \in \mathbb{Z}_+ | X^{\sigma_{y,x}}_{(\tau_y,z)} \in \mathcal{A}\},$$

where $\mathcal{A} = \{(1, 0), (0, 1), (1, 1)\}$ is the set of absorbing states. After that, the state of play changes and, hence, a new subgame starts. Therefore,

$$X^{\sigma_{y,x}}_{(\tau_y,z)} = X^{\sigma_{y,x}}_{(\tau_y,\zeta_{y,x})}, \quad \text{for all } z > \zeta_{y,x}, \text{ and} \tag{3}$$

$$X^{\sigma_{y,x}}_{(s,z)} = X^{\sigma_{y,x}}_{(\tau_y,\zeta_{y,x})}, \quad \text{for all } s > \tau_y \text{ and } z \in \mathbb{Z}_+. \tag{4}$$

At time τ_y, the state of play changes with constant transition probabilities. In particular, if $x = (0, 0)$, the transition probabilities for $(X^{\sigma_{y,x}}_{(\tau_y,z)})_{z \in \mathbb{N}}$ are denoted by $Q^{\sigma_{y,x}}(\cdot)$ and defined as follows:

$$\begin{aligned}
Q^{\sigma_{y,x}}\left(X^{\sigma_{y,x}}_{(\tau_{y,x},z)} = (0,0) | X^{\sigma_{y,x}}_{(\tau_{y,x},z-1)} \notin \mathcal{A}\right) &= (1 - \alpha^1_{y,x})(1 - \alpha^2_{y,x}) \\
Q^{\sigma_{y,x}}\left(X^{\sigma_{y,x}}_{(\tau_{y,x},z)} = (1,0) | X^{\sigma_{y,x}}_{(\tau_{y,x},z-1)} \notin \mathcal{A}\right) &= \alpha^1_{y,x}(1 - \alpha^2_{y,x}) \\
Q^{\sigma_{y,x}}\left(X^{\sigma_{y,x}}_{(\tau_{y,x},z)} = (0,1) | X^{\sigma_{y,x}}_{(\tau_{y,x},z-1)} \notin \mathcal{A}\right) &= (1 - \alpha^1_{y,x})\alpha^2_{y,x} \\
Q^{\sigma_{y,x}}\left(X^{\sigma_{y,x}}_{(\tau_{y,x},z)} = (1,1) | X^{\sigma_{y,x}}_{(\tau_{y,x},z-1)} \notin \mathcal{A}\right) &= \alpha^1_{y,x}\alpha^2_{y,x} \\
Q^{\sigma_{y,x}}\left(X^{\sigma_{y,x}}_{(\tau_{y,x},z)} = X^{\sigma_{y,x}}_{(\tau_{y,x},z-1)} | X^{\sigma_{y,x}}_{(\tau_{y,x},z-1)} \in \mathcal{A}\right) &= 1.
\end{aligned} \tag{5}$$

So, the process $\left(X^{\sigma_{y,x}}_t\right)_{t \geq_L (0,0)}$ is essentially a Markov chain, which is supported by some probability measure $P^{\sigma_{y,x}}$ on $(\Omega, \mathcal{F})$, such that $X^{\sigma_{y,x}}_{(0,0)} = x$, $P^{\sigma_{y,x}}$-a.s. From the transition probabilities in (5) it then follows that

$$\begin{aligned} p_{10}^{\sigma_{y,x}} &\equiv P^{\sigma_{y,x}}\big(X_{(\tau_y+,0)}^{\sigma_{y,x}}=(1,0)\big)=\frac{\alpha_{y,x}^1(1-\alpha_{y,x}^2)}{\alpha_{y,x}^1+\alpha_{y,x}^2-\alpha_{y,x}^1\alpha_{y,x}^2} \\ p_{01}^{\sigma_{y,x}} &\equiv P^{\sigma_{y,x}}\big(X_{(\tau_y+,0)}^{\sigma_{y,x}}=(0,1)\big)=\frac{(1-\alpha_{y,x}^1)\alpha_{y,x}^2}{\alpha_{y,x}^1+\alpha_{y,x}^2-\alpha_{y,x}^1\alpha_{y,x}^2} \\ p_{11}^{\sigma_{y,x}} &\equiv P^{\sigma_{y,x}}\big(X_{(\tau_y+,0)}^{\sigma_{y,x}}=(1,1)\big)=\frac{\alpha_{y,x}^1\alpha_{y,x}^2}{\alpha_{y,x}^1+\alpha_{y,x}^2-\alpha_{y,x}^1\alpha_{y,x}^2}. \end{aligned} \tag{6}$$

It is easy to see that $P^{\sigma_{y,x}}(\zeta_{y,x}<\infty)=1$, so that play in coordination time is finite, $P^{\sigma_{y,x}}$-a.s. In fact, this construction allows for the exercise intensities to be interpreted as mixed strategies in a particular normal form game, where each player has two pure strategies, namely *exercise* and *don't exercise*. The probabilities needed to compute the expected payoffs in this normal form game follow from the probability measure in (6). This re-interpretation will be crucial in the equilibrium existence proof in Section 3.

For $x=(1,0)$ and $x=(0,1)$, one player, say Player j, has already exercised the option. This implies that $\tau_y=\tau(S_{y,x}^i)$, and, consequently, that $\alpha_{y,x}^j=0$. At time $\tau(S_{y,x}^i)$, the resulting Markov chain has only one absorbing state, namely $(1,1)$, which is reached in finite time as well, because $\alpha_{y,x}^i=1$. A final remark on the construction of Markov strategies and the resulting evolution of $\big(X_t^{\sigma_{y,x}}\big)_{t\geq_L(0,0)}$ is that players essentially use pure strategies to determine *when* they act (they choose a stopping set), but use mixed strategies to determine *what* happens via the exercise intensity.

To summarise we get the following definition.

Definition 3. Let $\sigma=(\sigma_1,\sigma_2)\in\mathcal{S}^1\times\mathcal{S}^2$ and $(y,x)\in\mathbb{R}^d\times\Xi$. The *exercise process induced by* σ is a process $\big(X_t^{\sigma_{y,x}}\big)_{t\geq_L(0,0)}$, which satisfies (1)–(5), with induced probability measure $P^{\sigma_{y,x}}$. Furthermore, $X_{(0,0)}^{\sigma}=x$, $P_{y,x}^{\sigma}$-a.s.

The class of all induced exercise processes and the family of induced probability measures are denoted by

$$\mathcal{X}=\Big(\big(X_t^{\sigma_{y,x}}\big)_{t\geq_L(0,0)}\Big)_{(y,x)\in\mathbb{R}^d\times\Xi}^{\sigma\in\mathcal{S}^1\times\mathcal{S}^2}\quad\text{and}\quad\mathcal{P}=\big(P^{\sigma_{y,x}}\big)_{(y,x)\in\mathbb{R}^d\times\Xi}^{\sigma\in\mathcal{S}^1\times\mathcal{S}^2},$$

respectively.

2.2.1 Payoffs and Equilibrium

Given the strategies $\sigma=(\sigma_1,\sigma_2)$, the starting point $(y,x)\in\mathbb{R}^d\times\Xi$, and the state of the stochastic processes $(Y_t,X_t^{\sigma_{y,x}})_{t\in T}$ the *instantaneous payoff* to Player i at time t is given by $V^i_{X_t^{\sigma_{y,x}}}(Y_t) \in \mathbb{R}$. For all $x \in \Xi$ it is assumed that $V^i_x(\cdot)$ is strictly increasing. Since Y only changes in the continuous "real time" dimension of time, this formulation implies that, for given x, payoffs are realised in real time only. Note that the instantaneous payoffs are assumed to be Markovian in the sense that they only depend on the current state of the processes Y and $X^{\sigma_{y,x}}$.

In addition to the instantaneous payoffs, it is assumes that Player i incurs a sunk cost $I^i > 0$, if she exercises the option. Finally, players discount payoffs according to a discount factor $\left(\Lambda^i_t\right)_{t \geq_L (0,0)}$, which is adapted to $(\mathcal{F}_t)_{t \geq_L (0,0)}$, and constant over (discrete) "coordination time" (just like the process $(Y_t)_{t \geq_L (0,0)}$). A non-exclusive real option game can now be defined as follows.

Definition 4. A two-player *non-exclusive real option game* (NERO) is a collection $\Gamma = \Big((Y_t)_{t \geq_L (0,0)}, \big(\mathcal{S}^i, V^i, I^i, \left(\Lambda^i_t\right)_{t \geq_L (0,0)}\big)_{i=1,2}, (\mathcal{X}, \mathcal{P})\Big)$.

For $x = (0,0)$, the *expected discounted payoff* of the strategies $(\sigma^1, \sigma^2) \in \mathcal{S}^1 \times \mathcal{S}^2$ to Player 1, under P_y and $P^{\sigma_{y,x}}$ is then equal to

$$
\begin{aligned}
V^1_{y,x}(\sigma^1_{y,x}, \sigma^2_{y,x}) = \mathbb{E}^{P_y}\Big[& \int_0^{\tau_y} \Lambda^1_t V^1_{00}(Y_t)dt \\
& + \mathbb{1}_{\tau_y \equiv \tau(S^1_{y,x}) < \tau(S^2_{y,x})}\Big(\int_{\tau_y}^{\tau(S^2_{Y_{\tau_y},(1,0)})} \Lambda^1_t V^1_{10}(Y_t)dt \\
& + \int_{\tau(S^2_{Y_{\tau_y},(1,0)})}^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt - \Lambda^1_{\tau_y} I^1\Big) \\
& + \mathbb{1}_{\tau_y \equiv \tau(S^2_{y,x}) < \tau(S^1_{y,x})}\Big(\int_{\tau_y}^{\tau(S^1_{Y_{\tau_y},(0,1)})} \Lambda^1_t V^1_{01}(Y_t)dt \\
& + \int_{\tau(S^1_{Y_{\tau_y},(0,1)})}^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt - \Lambda^i_{\tau(S^1_{Y_{\tau_y},(0,1)})} I^1\Big) \\
& + \mathbb{1}_{\tau_y \equiv \tau(S^1_{y,x}) = \tau(S^2_{y,x})} W^1_{Y_{\tau_y}}(\sigma^1_{y,x}, \sigma^2_{y,x})\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
W^1_{y,x}(\sigma^1_{y,x}, \sigma^2_{y,x}) = p^{\sigma_{y,x}}_{11} \mathbb{E}^{P_y}\Big[& \int_0^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt - I^1\Big] \\
& + p^{\sigma_{y,x}}_{10} \mathbb{E}^{P_y}\Big[\int_0^{\tau(S^2_{y,(1,0)})} \Lambda^1_t V^1_{10}(Y_t)dt \\
& + \int_{\tau(S^2_{y,(1,0)})}^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt - I^1\Big] \\
& + p^{\sigma_{y,x}}_{01} \mathbb{E}^{P_y}\Big[\int_0^{\tau(S^1_{y,(0,1)})} \Lambda^1_t V^1_{01}(Y_t)dt \\
& + \int_{\tau(S^1_{y,(0,1)})}^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt - \Lambda^1_{\tau(S^1_{y,(0,1)})} I^1\Big].
\end{aligned}
$$

For $x \in \{(1,0),(0,1),(1,1)\}$, the expected discounted payoffs are

$$\begin{aligned} V^1_{y,(1,0)}(\sigma^1,\sigma^2) =& \mathbb{E}^{P_y}\Big[\int_0^{\tau(S^2_{y,(1,0)})} \Lambda^1_t V^1_{10}(Y_t)dt \\ &+ \int_{\tau(S^2_{y,(1,0)})}^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt\Big], \\ V^1_{y,(0,1)}(\sigma^1,\sigma^2) =& \mathbb{E}^{P_y}\Big[\int_0^{\tau(S^1_{y,(0,1)})} \Lambda^1_t V^1_{01}(Y_t)dt \\ &+ \int_{\tau(S^1_{y,(0,1)})}^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt - \Lambda^1_{\tau(S^1_{y,(0,1)})} I^1\Big], \end{aligned}$$

and

$$V^1_{y,(1,1)}(\sigma^1,\sigma^2) = \mathbb{E}^{P_y}\Big[\int_0^{\infty} \Lambda^1_t V^1_{11}(Y_t)dt\Big],$$

respectively. The payoffs for Player 2 are defined equivalently.

A subgame perfect equilibrium is now readily defined as follows.

Definition 5. Let Γ be a two-player NERO. A collection of strategies $(\bar{\sigma}^1,\bar{\sigma}^2) \in \mathcal{S}^1 \times \mathcal{S}^2$ constitutes a *subgame perfect equilibrium* (SPE) if it prescribes a Nash equilibrium for all $(y,x) \in \mathbb{R}^d_+ \times \Xi$, i.e.

$$\forall_{i\in\{1,2\}}\forall_{\sigma^i\in\mathcal{S}^i}\forall_{(y,x)\in\mathbb{R}^d_+\times\Xi} : V^i_{y,x}(\bar{\sigma}^i,\bar{\sigma}^j) \geq V^i_{y,x}(\sigma^i,\bar{\sigma}^j).$$

△

Note that in standard extensive form games, the notion of subgame perfectness is defined over time. Due to the strong Markov property of $(Y_t)_{t\geq_L(0,0)}$, the definition of SPE above could equivalently be defined over stopping times. The definition over states, however, is more convenient due to the Markovian nature of the problem.

3 NEROs with a First Mover Advantage

In this section American-type perpetual non-exclusive call options are studied. In the analysis that follows, several particular discounted payoff functions play an important role. For $x = (0,1)$, the *follower value*, $F^i(y)$, for Player i is defined to be the value of her optimal stopping problem, given $Y_{(0,0)} = y$ equals y. That is,

$$\begin{aligned} F^i(y) &= \sup_{\tau\in\mathcal{M}} \mathbb{E}^{P_y}\Big[\int_0^{\tau} \Lambda^i_t V^i_{01}(Y_t)dt + \int_{\tau}^{\infty} \Lambda^i_t V^i_{11}(Y_t)dt - \Lambda^i_\tau I^i\Big] \\ &= \sup_{\tau\in\mathcal{M}} \mathbb{E}^{P_y}\Big[\int_0^{\tau} \Lambda^i_t V^i_{01}(Y_t)dt + \Lambda^i_\tau \mathbb{E}^{P_{Y_\tau}}\Big(\int_0^{\infty} \Lambda^i_t V^i_{11}(Y_t)dt - I^i\Big)\Big], \end{aligned} \tag{7}$$

where $\mathcal{M}$ is the set of Markov times adapted to $(\mathcal{F}_t)_{t\geq L(0,0)}$. Let $S_F^i(y)$ denote the optimal stopping set resulting from (7).[5]

Let $x = (0,0)$. The *leader value*, $L^i(y)$, for Player i is the expected discounted payoff stream if Player i exercises the option at time $t = (0,0)$, with $Y_{(0,0)} = y$, given that Player j exercises the option at time $\tau(S_F^j(y))$. That is,

$$L^i(y) = \mathbb{E}^{P_y}\Big[\int_0^{\tau(S_F^j(y))} \Lambda_t^i V_{10}^i(Y_t)dt + \int_{\tau(S_F^j(y))}^{\infty} \Lambda_t^i V_{11}^i(Y_t)dt - I^i\Big]. \tag{8}$$

Furthermore, let $S_L^i(y)$ be the optimal stopping set of the problem

$$\begin{aligned}\bar{L}^i(y) = \sup_{\tau\in\mathcal{M}} \mathbb{E}^{P_y}\Big[&\int_0^{\tau} \Lambda_t^i V_{00}^i(Y_t)dt + \int_{\tau}^{\tau(S_F^j(Y_\tau))} \Lambda_t^i V_{10}^i(Y_t)dt \\ &+ \int_{\tau(S_F^j(Y_\tau))}^{\infty} \Lambda_t^i V_{11}(Y_t)dt - \Lambda_\tau^i I^i\Big] \\ = \sup_{\tau\in\mathcal{M}} \mathbb{E}^{P_y}\Big[&\int_0^{\tau} \Lambda_t^i V_{00}^i(Y_t)dt + \Lambda_\tau^i L^i(Y_\tau)\Big].\end{aligned} \tag{9}$$

The optimal stopping time τ in (9) is the time at which Player i would exercise the option, if she knew that Player j could not preempt her. Preemption may take place, however in the region $S_P^i(y) := \{y \in \mathbb{R}^d | L^i(y) \geq F^i(y)\}$.

The Markovian nature of $(Y_t)_{t\geq L(0,0)}$, together with the infinite horizon have an important implication for the optimal stopping sets $S_F^i(y)$ and $S_L^i(y)$. Intuitively speaking, the process $(Y_t)_{t\geq L(0,0)}$ always "starts afresh". Consider the optimal stopping problem for the leader. Fixing $\omega \in \Omega$ and following the sample path $t \mapsto Y_t(\omega)$ one can observe the following. Evaluating $L^i(Y_t(\omega))$ gives Player i enough information to optimally decide whether to exercise the option or to wait. In other words, the state space $\mathbb{R}^d$ can be split in a set C, the *continuation set*, and a stopping set $D = \mathbb{R}^d \setminus C$, where the option is exercised. The sets C and D are independent of the starting point y of the process $(Y_t)_{t\geq L(0,0)}$. So, $S_F^i(y) = S_F^i$ and $S_L^i(y) = S_L^i$, for all $y \in \mathbb{R}^d$. So, in order to find a subgame perfect equilibrium, we only have to find a pair (S^i, α^i) which induces a Nash equilibrium for all $x \in \Xi$, which greatly simplifies the analysis that follows. Note that, of course, the actual optimal stopping time does depend on the starting point, so the optimal leader value does as well. The following assumption is made on the optimal stopping sets.

Assumption 1. *For all players i, the optimal stopping sets S_F^i and S_L^i are non-empty.*

[5]Note that it is possible that $S_F^i(y) = \emptyset$.

Thirdly, let $M^i(y)$ denote the expected discounted value to Player i if $x = (1,1)$ and $Y_{(0,0)} = y$, i.e.

$$M^i(y) = \mathbb{E}^{P_y}\Big[\int_0^\infty \Lambda_t^i V_{11}^i(Y_t)dt - I^i\Big]. \tag{10}$$

Let $S_M^i = \{y \in \mathbb{R}^d | M^i(y) \geq F^i(y)\}$ be the set of payoffs where simultaneous investment has a higher expected discounted payoff than the follower value. Finally, let $S_N^i = (S_P^i \cup S_L^i \cup S_F^i)^c$.

The following assumptions are made with respect to the instantaneous payoff functions and the optimal stopping sets.

Assumption 2. *For every Player i, $i \in \{1,2\}$, it holds that*

1. $V_{kl}^i : \mathbb{R}^d \to \mathbb{R}$ *is continuous for all* $k, l = 0, 1$,
2. $S_F^i \subseteq S_L^i \subseteq S_P^i$.

The second condition ensures that, for each player, there are values for y, where she wants to be leader rather than follower. In other words, there is a first mover advantage.

For further reference, let $\bar{S}_P^i = S_P^i \backslash S_L^i$ (which could be an empty set) and

$$\varphi^i(y) = \frac{L^i(y) - F^i(y)}{L^i(y) - M^i(y)},$$

for all $y \in \mathbb{R}^d$, such that $L^i(y) \neq M^i(y)$.

Theorem 1. *Let G be a two-player NERO satisfying Assumptions 1 and 2. Let $y \in \mathbb{R}_+^d$ and $x \in \Xi$. Then $\bar{\sigma} = (\bar{\sigma}_{y,x}^1, \bar{\sigma}_{y,x}^2)_{(y,x) \in \mathbb{R}^d \times \Xi} \in \mathcal{S}^1 \times \mathcal{S}^2$, with $\bar{\sigma}_{y,x}^i = (\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i)$ constitutes a SPE, where*

$$(\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i) = \begin{cases} \begin{cases} \big(S_P^i, \varphi^j(y)\big) & \text{if } y \in S_P^1 \cap S_P^2 \\ (S_L^i, 1) & \text{otherwise} \end{cases}, & \text{if } x = (0,0), \\ (\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i) = (S_F^i, 1), & \text{if } x = (0,1), \text{ and} \\ (\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i) = (\emptyset, 0), & \text{otherwise.} \end{cases}$$

Proof. Let $y \in \mathbb{R}^d$. First, note that, for all $y \in \mathbb{R}^d$,

$$\begin{aligned} M^i(y) &= \mathbb{E}^{P_y}\Big[\int_0^\infty \Lambda_t^i V_{11}^i(Y_t)dt - I^i\Big] \\ &\leq \sup_{\tau \in \mathcal{M}} \mathbb{E}^{P_y}\Big[\int_0^\tau \Lambda_t^i V_{01}^i(Y_t)dt + \int_\tau^\infty \Lambda_t^i V_{11}^i(Y_t)dt - \Lambda_\tau^i I^i\Big] \\ &= F^i(y), \end{aligned}$$

since Player i can always choose $\tau = 0$. Hence, $S^i_M \subseteq S^i_F$ and $S^i_M = \{y \in \mathbb{R}^d | M^i(y) = F^i(y)\}$. Also, the case where $x_i = 1$ is trivial. Consider the following cases.

1. $x_i = 0$, $x_j = 1$.
Since Player j has already exercised the option, Player i faces the decision theoretic problem (7). The optimal stopping set for this problem is S^i_F. The definition of Markov strategies prescribes that $\alpha^i_{y,x} = 1$. So, $(\bar{S}^i_{y,x}, \bar{\alpha}^i_{y,x}) = (S^i_F, 1)$ is a weakly dominant strategy.

2. $x = (0,0)$, $y \in S^i_L$.
The optimal stopping problem (9) takes into account that Player j exercises as soon as S^j_F, which is a weakly dominant strategy for Player j. Therefore, if $y \in S^i_L$, exercising immediately is a weakly dominant strategy, which is implemented by $(\bar{S}^i_{y,x}, \bar{\alpha}^i_{y,x}) = (S^i_L, 1)$.

3. $x = (0,0)$, $y \in \bar{S}^i_P \cap \bar{S}^j_P$.
In this case it holds that $\tau_y = 0$. In order to determine whether it pays for Player i to deviate from $\bar{\sigma}^i_{y,x}$ to $\tilde{\sigma}^i_{y,x} = (\tilde{S}^i_{y,x}, \tilde{\alpha}^i_{y,x})$, two possible deviations have to be considered:

   1. Player i wants to exercise immediately, by choosing a stopping set $\tilde{S}^i_{y,x}$ such that $\tau(\tilde{S}^i_{y,x}) = 0$, but chooses $\tilde{\alpha}^i_{y,x} \neq \bar{\alpha}^i_{y,x}$, and
   2. Player j chooses a stopping set $\tilde{S}^i_{y,x}$, such that $\tau(\tilde{S}^i_{y,x}) \neq 0$.

Consider the former deviation. In this case, the players are effectively playing the game depicted in Figure 2. The cell (*continue*, *continue*) has no payoff, since this

	Exercise	Continue
Exercise	$M^1(y), M^2(y)$	$L^1(y), F^2(y)$
Continue	$F^1(y), L^2(y)$	

Figure 2: The coordination game.

outcome is impossible under the Markov chain (5). When $\sigma^i_{y,x} = (S^i_{y,x}, \alpha^i_{y,x})$ and $\sigma^j_{y,x} = (S^j_{y,x}, \alpha^j_{y,x})$ are such that $\tau(S^i_{y,x}) = \tau(S^j_{y,x})$, P_y-a.s., then $V^i_{y,x}(\sigma^i_{y,x}, \sigma^j_{y,x}) = W^i_{y,x}(\sigma^i, \sigma^j)$. Note that

$$W^i_{y,x}\big((S^i_{y,x}, 1), (S^j_{y,x}, \alpha^j_{y,x})\big) = (1 - \alpha^j_{y,x})L^i(y) + \alpha^j_{y,x} M^i(y),$$

and

$$W^i_{y,x}\big((S^i_{y,x},0),(S^j_{y,x},\alpha^j_{y,x})\big) = F^i(y),$$

for all $\alpha^j_{y,x} \in [0,1]$. We have that

$$W^i_{y,x}\big((S^i_{y,x},1),(S^j_{y,x},\alpha^j_{y,x})\big) > W^i_{y,x}\big((S^i_{y,x},0),(S^j_{y,x},\alpha^j_{y,x})\big) \iff \alpha^j_{y,x} < \varphi^i(y).$$

In other words, the best-response correspondences, $\big(B^i(\alpha^j_{y,x}), B^j(\alpha^i_{y,x})\big)$, for both players can be depicted as in Figure 3. The point $\big(\varphi^j(y),\varphi^i(y)\big)$ is the only mixed

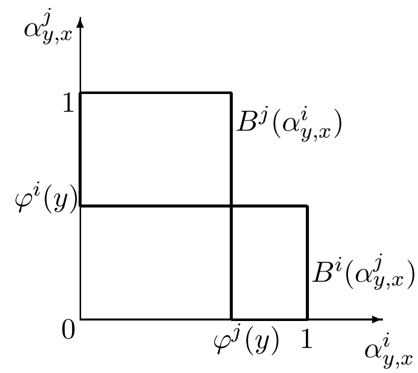

Figure 3: Best response correspondences.

strategy Nash equilibrium[6] and, hence, unilateral deviations do not lead to higher expected utility.

Furthermore, note that Player j's strategy is such that Player i is indifferent between exercising and not exercising. This immediately implies that the strategies $\bar{\sigma}$ lead to an expected payoff

$$V^i_{y,x}(\bar{\sigma}^i_{y,x},\bar{\sigma}^j_{y,x}) = W^i_{y,x}((\bar{S}^i_{y,x},0),\bar{\sigma}^j_{y,x}) = F^i(y). \tag{11}$$

On the other hand, let $\tilde{\sigma}^i \neq \bar{\sigma}^i$ be such that $\tau(\tilde{S}^i_{y,x}) \neq 0$. Then Player j exercises immediately, and Player i faces the optimal stopping problem (7). Therefore,

$$V^i_{y,x}(\tilde{\sigma}^i_{y,x},\bar{\sigma}^j_{y,x}) = V^i_{y,(0,1)}\big((S^i_F,1),\bar{\sigma}^j_{y,(0,1)}\big) = F^i(y).$$

So, Player i has no incentive to deviate.

4. $x=(0,0)$, $y \in (S^i_N \cup \bar{S}^i_P) \cap S^j_L$.
In this case $\tau_y = \tau(\bar{S}^j_{y,x},\bar{\alpha}^j_{y,x}) = 0$. Moreover, Player j invests with probability $\bar{\alpha}^j_{y,x} = 1$. Note that

$$V^i_{y,x}(\bar{\sigma}^1_{y,x},\bar{\sigma}^2_{y,x}) = V^i_{y,(0,1)}(\bar{\sigma}^1_{y,(0,1)},\bar{\sigma}^2_{y,(0,1)}) = F^i(y).$$

[6]Note that there are two pure-strategy equilibria as well; one where Player i becomes leader and Player j follower with probability one, and the symmetric counter-part.

Suppose that Player i deviates to $\tilde{\sigma}^i_{y,x} = (\tilde{S}^i_{y,x}, \tilde{\alpha}^i_{y,x})$, with $\tau(\tilde{S}^i_{y,x}) = 0$. For all $\tilde{\alpha}^i_{y,x} \in (0,1]$, it then follows from (6) that

$$\begin{aligned} V^i_{y,x}(\tilde{\sigma}^1_{y,x}, \bar{\sigma}^2_{y,x}) &= W^i_{y,x}(\tilde{\sigma}^1_{y,x}, \bar{\sigma}^2_{y,x}) \\ &= 0 \cdot L^i(y) + (1 - \tilde{a}^1_{y,x})F^i(y) + \tilde{\alpha}^i_{y,x} M^i(y) \\ &\leq M^i(y). \end{aligned}$$

For any deviation such that $\tau(\tilde{S}^i_{y,x}) \neq 0$, it follows that

$$V^i_{y,x}(\bar{\sigma}^1_{y,x}, \bar{\sigma}^2_{y,x}) = V^i_{y,(0,1)}(\bar{\sigma}^1_{y,(0,1)}, \bar{\sigma}^2_{y,(0,1)}) = F^i(y).$$

So, Player i has no incentive to deviate.

5. $x = (0,0)$, $y \in \bar{S}^i_P \cap (S^j_N \cup \bar{S}^j_P)$.
Note that, in this case τ_y is the first hitting time of $\Theta = S^i_L \cup S^j_L \cup (\bar{S}^1_P \cap \bar{S}^2_P)$. So, what remains to be shown is that, for all elements of Θ, waiting until Θ is hit is an equilibrium. From the above analysis it follows that, under $\bar{\sigma}$, the expected payoff to Player i equals

$$\begin{aligned} V^i_{y,x}(\bar{\sigma}^i_{y,x}, \bar{\sigma}^j_{y,x}) &= \mathbb{E}^{P_y}\Big[\int_0^{\tau_y} \Lambda^i_t V_{00} dt + \Lambda^i_{\tau_{y,x}} V^i_{Y_{\tau_y},x}(\bar{\sigma}^i_{Y_{\tau_y},x}, \bar{\sigma}^j_{Y_{\tau_y},x})\Big] \\ &= \begin{cases} \mathbb{E}^{P_y}\Big[\int_0^{\tau_y} \Lambda^i_t V_{00} dt + \Lambda^i_{\tau_y} L^i(Y_{\tau_y})\Big] & \text{if } \tau_y = \tau_y(S^i_L), \\ \mathbb{E}^{P_y}\Big[\int_0^{\tau_y} \Lambda^i_t V_{00} dt + \Lambda^i_{\tau_y} F^i(Y_{\tau_y})\Big] & \text{otherwise.} \end{cases} \end{aligned}$$

The only deviation $\tilde{\sigma}^i$ of $\bar{\sigma}^i$ that could possibly lead to a higher payoff has $\tilde{S}^i_{y,x}$ such that $\tau(\tilde{S}^i_{y,x}) < \tau_y(\Theta)$. But, by definition, $\tau(\tilde{S}^i_{y,x})$ does not solve (9). Hence, waiting to exercise the option is weakly dominant. ∎

As a corollary, suppose that firms are symmetric, i.e. $V^1_{kl} = V^2_{kl} \equiv V_{kl}$, all $k, l \in \{0,1\}$, and $I^1 = I^2 \equiv I$. In that case all stopping regions are the same and the following result follows immediately from Theorem 1.

Corollary 1. *Let G be a symmetric two-player NERO satisfying Assumptions 1 and 2. Let $y \in \mathbb{R}^d_+$ and $x \in \Xi$. Then $\bar{\sigma} = (\bar{\sigma}^1_{y,x}, \bar{\sigma}^2_{y,x})_{(y,x) \in \mathbb{R}^d \times \Xi} \in \mathcal{S}^1 \times \mathcal{S}^2$, with $\bar{\sigma}^i_{y,x} = (\bar{S}^i_{y,x}, \bar{\alpha}^i_{y,x})$ constitutes a SPE, where*

$$(\bar{S}^i{}_{y,x}, \bar{\alpha}^i_{y,x}) = \begin{cases} \begin{cases} (S_P, \varphi(y)) & \text{if } y \in S_P \\ (S_F, 1) & \text{otherwise} \end{cases}, & \text{if } x = (0,0), \\ (\bar{S}^i_{y,x}, \bar{\alpha}^i_{y,x}) = (S_F, 1), & \text{if } x_i = 0,\ x_j = 1, \text{ and} \\ (\bar{S}^i_{y,x}, \bar{\alpha}^i_{y,x}) = (\emptyset, 0), & \text{otherwise.} \end{cases}$$

In the case that $(Y_t)_{t \geq L(0,0)}$ is one-dimensional, a typical plot of the payoff functions would look like Figure 1, where $y < Y_F$, $S^F = [Y_F, \infty)$, $S_N = [0, Y_P)$, and $\bar{S}_P = [Y_P, Y_F)$. If, in addition, $x = (0,0)$, $y \in S_N$, and $(Y_t)_{t \geq L(0,0)}$ has continuous sample paths, then $\tau_y = \tau_y(\bar{S}_P)$ and each player exercises the option at time τ_y with probability $p_{10}^{\sigma_{y,x}} = p_{01}^{\sigma_{y,x}} = \frac{1}{2}$, since

$$\alpha_{\tau_{y,x}} = \frac{L(Y_P) - F(Y_P)}{L(Y_P) - M(Y_P)} = 0.$$

Note that if $(Y_t)_{t \geq L(0,0)}$ exhibits jumps, there can be a positive probability that both players exercise simultaneously in the preemption region.

From Theorem 1 it follows that there are four possible exercise scenarios, depending on which subset of the state-space $\mathbb{R}^d$ is reached first. For a two-dimensional underlying asset, the regions could look like in Figure 4. Firstly, it is possible that

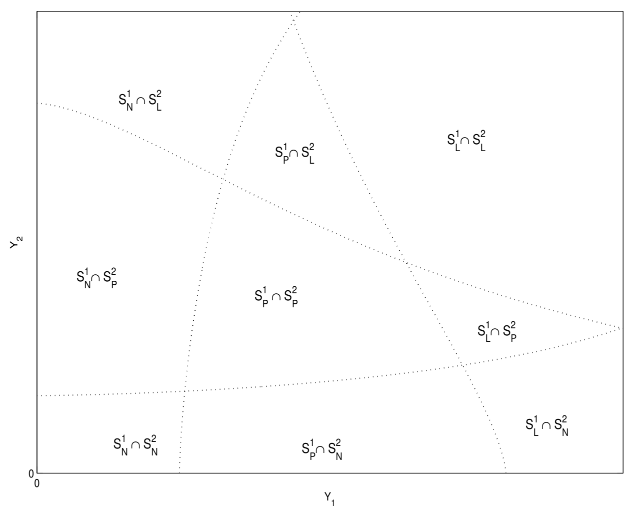

Figure 4: Exercise regions with a two-dimensional underlying asset.

both players exercise simultaneously if $Y_\tau \in S^1_L \cap S^2_L$. In such cases the option is so deep in the money that it becomes optimal for each player to exercise regardless of the actions of the other player. Since strategic considerations do not play a role in this scenario I refer to it as *monopolistic simultaneous exercise*. Note that, if the sample paths of Y are continuous and $Y_0 \notin S^1_L \cap S^2_L$, this scenario will not occur (a.s.).

Secondly, either player may exercise at her optimal time. This happens if $Y_\tau \in S^i_L \cap (S^j_L)^c$. In this case, Player i exercises the option as if she was an exogenously determined leader. It is referred to as *monopolistic sequential exercise*.

If $Y_\tau \in \bar{S}_P^1 \cap \bar{S}_P^2$,[7] neither player finds the option deep enough in the money to optimally exercise. Due to the threat of preemption, however, at least one player exercises along every equilibrium path. Therefore, this set is called the *preemption region.* Note that from (11) it follows that, in equilibrium, expected payoffs in the preemption region are equal to the follower payoffs for both players. This phenomenon is called *rent-equalisation.* So, *ex ante*, both players expect their follower payoff. In every realisation, however, three possible outcomes occur. In two outcomes one player exercises first, while the other players waits until the option is deep enough in the money to be the follower. These scenarios are called *preemptive sequential exercise.* Finally, it is possible that both players exercise simultaneously (if $p_{11} > 0$). In that case there is *preemptive simultaneous exercise.*

4 Technology Adoption in an Industry with Asymmetric Uncertainty

Consider a market with two firms, both of whom have an option to invest in a new technology. It is assumed that the discounted profit stream of Firm i equals

$$\pi^i(y, x) = \Lambda^i V_x^i(y),$$

and that uncertainty is driven by a two-dimensional geometric Brownian motion (GBM),

$$\begin{bmatrix} dY_1/Y_1 \\ dY_2/Y_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix},$$

where z_1 and z_2 are independent Wiener processes. Let $\sigma_k^2 = \sigma_{k1}^2 + \sigma_{k2}^2$, $k = 1, 2$, be the total instantaneous variance of Y_k. The instantaneous correlation between Y_1 and Y_2 equals

$$\rho = \frac{1}{dt}\mathbb{E}^{P_y}\Big(\frac{dY_1}{Y_1}\frac{dY_2}{Y_2}\Big) = \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2}.$$

The discount factor for Firm i, $i = 1, 2$, is assumed to follow

$$\frac{d\Lambda^i}{\Lambda^i} = -\mu_{\Lambda^i} dt - \sigma_{\Lambda_1^i} dz_1 - \sigma_{\Lambda_2^i} dz_2.$$

The instantaneous profits are taken to be linear in y_i, so $V_{kl}^i(Y) = D_{kl}Y_i$, for all $Y \in \mathbb{R}_+^2$, and $k, l \in \{0, 1\}$. Note that the deterministic parts are assumed to be equal for both firms and are taken such that

1. $D_{10} > D_{11} > D_{00} \geq D_{01}$,

[7] Recall from the proof of Theorem 1 that $\bar{S}_P^i = S_P^i \backslash S_L^i$.

2. $D_{10} - D_{00} > D_{11} - D_{01}$.

These are standard assumptions to model a game with a first-mover advantage (cf. Huisman (2001)). Finally, sunk costs are the same for both firms and equal to $I > 0$. That is, firms are symmetric up to the uncertainty that underlies their profits – which, in turn, are correlated – and their discount factors.

The uncertainty driving the discounted profit stream to Firm i follows the SDE

$$\begin{aligned}\frac{d\Lambda^i Y_i}{\Lambda^i Y_i} =& \frac{dY_i}{Y_i} + \frac{d\Lambda^i}{\Lambda^i} + \frac{dY_i}{Y_i}\frac{d\Lambda^i}{\Lambda^i} \\ =& -(\mu_{\Lambda^i} + \sigma_{i1}\sigma_{\Lambda_1^i} + \sigma_{i2}\sigma_{\Lambda_2^i} - \mu_i)dt \\ & + (\sigma_{i1} - \sigma_{\Lambda_1^i})dz_1 + (\sigma_{i2} - \sigma_{\Lambda_2^i})dz_2 \\ \equiv& -\delta_{ii}dt + (\sigma_{i1} - \sigma_{\Lambda_1^i})dz_1 + (\sigma_{i2} - \sigma_{\Lambda_2^i})dz_2,\end{aligned} \tag{12}$$

where δ_{ii} is the convenience yield of Firm i with respect to Y_i. Also define the convenience yield of Firm i with respect to Y_j,

$$\delta_{ij} = \mu_{\Lambda^i} + \sigma_{\Lambda_1^i}\sigma_{j1} + \sigma_{\Lambda_2^i}\sigma_{j2} - \mu_j.$$

It is assumed throughout that $\delta_{ii} > 0$ and $\delta_{ij} > 0$. From (12) it follows that the value of simultaneous exercising of the option to Firm i equals

$$M^i(y) = \mathbb{E}^{P_y}\Big[\int_0^\infty \Lambda_t^i D_{11} Y_{it} dt\Big] = \frac{D_{11}}{\delta_{ii}} y_i - I.$$

4.1 Follower Value

First, I derive the follower value, $F^i(Y_i)$, for Firm i. Note that this value does not depend on Y_j, since Firm j has already exercised her option. The value of exercising the option – the option's "strike price" – at $y \in \mathbb{R}_+^2$ is $M^i(y)$. Denote the value the option to exercise at y by $C_F^i(y)$. Then, the no-arbitrage value (relative to Λ^i) of $C_F^i(\cdot)$ should satisfy (cf. Cochrane (2005)),

$$\begin{aligned}& \Lambda^i D_{01} Y_i dt + \mathbb{E}^{P_y}[d\Lambda^i C_F^i] = 0 \\ \Longleftrightarrow\ & D_{01} Y_i dt + \mathbb{E}^{P_y}[dC_F^i] + \mathbb{E}^{P_y}\Big[C_F^i \frac{d\Lambda^i}{\Lambda^i}\Big] = -\mathbb{E}^{P_y}\Big[\frac{d\Lambda^i}{\Lambda^i} dC_F^i\Big].\end{aligned} \tag{13}$$

From Ito's lemma it follows that

$$\begin{aligned}dC_F^i =& \frac{\partial C_F^i}{\partial Y_i} dY_i + \frac{1}{2}\frac{\partial^2 C_F^i}{\partial Y_i^2}(dY_i)^2 \\ =& \Big(\frac{1}{2}\frac{\partial^2 C_F^i}{\partial Y_i^2}\sigma_i^2 Y_i^2 + \mu_i \frac{\partial C_F^i}{\partial Y_i} Y_i + D_{01} Y_i\Big) dt \\ & + \frac{\partial C_F^i}{\partial Y_i} Y_i(\sigma_{i1} dz_1 + \sigma_{i2} dz_2).\end{aligned}$$

Substitution in (13) gives a second order PDE with general solution

$$C_F^i(y) = A_F^i y_i^{\beta_{ii}} + B_F^i y_i^{\gamma_{ii}} + \frac{D_{01}}{\delta_{ii}} y_i,$$

where A_F^i and B_F^i are constants and $\beta_{ii} > 1$ and $\gamma_{ii} < 0$ are the roots of the equation

$$\mathcal{Q}_{ii}(\xi) \equiv \frac{1}{2}\sigma_i^2 \xi(\xi - 1) + (\mu_{\Lambda^i} - \delta_{ii})\xi - \mu_{\Lambda^i} = 0.$$

Under the standard boundary condition, $\lim_{y_i \downarrow 0} C_F^i(y) = 0$, and the value-matching and smooth-pasting conditions (cf. Peskir and Shiryaev (2006)) it is obtained that $S_F^i = \{Y \in \mathbb{R}_+^2 | Y_i \in [Y_F^i, \infty)\}$, where

$$Y_F^i = \frac{\beta_{ii}}{\beta_{ii} - 1} \frac{\delta_{ii}}{D_{11} - D_{01}} I.$$

The follower value is then equal to

$$F^i(y) = \begin{cases} \frac{1}{\beta_{ii}\delta_{ii}} (Y_F^i)^{1-\beta_{ii}} y_i^{\beta_{ii}} + \frac{D_{01}}{\delta_{ii}} y_i & \text{if } y_i < Y_F^i \\ \frac{D_{11}}{\delta_{ii}} y_i - I & \text{if } y_i \geq Y_F^i. \end{cases}$$

4.2 Leader Value

Having established the value for the follower I now turn to the leader value. In deriving the leader value I assume that Firm j cannot invest before Firm i. Therefore, the leader value for Firm i can only be computed when $y_j < Y_F^j$. If Firm i becomes the leader, then, by definition, Firm j becomes the follower. The exercise decision of Firm j, which depends on y_j, as we saw, influences the profit of Firm i. Hence, her leader value depends on y_i and y_j.

The no-arbitrage value of $L^i(y)$ follows

$$D_{10} Y_i dt + \mathbb{E}^{P_y}[dL^i] + \mathbb{E}^{P_y}\Big[L^i \frac{d\Lambda^i}{\Lambda^i}\Big] = -\mathbb{E}^{P_y}\Big[\frac{d\Lambda^i}{\Lambda^i} dL^i\Big]. \tag{14}$$

From Ito's lemma it follows that[8]

$$\begin{aligned} dL^i =& L_i^i dY_i + L_j^i dY_j + \frac{1}{2} L_{ii}^i dY_i^2 + \frac{1}{2} L_{jj}^i dY_j^2 + L_{ij}^i dY_i dY_j \\ =& \Big(\frac{1}{2}\sigma_i^2 Y_i^2 L_{ii}^i + \frac{1}{2}\sigma_j^2 Y_j^2 L_{jj}^i + \rho Y_i Y_j L_{ij}^i + \mu_i Y_i L_i^i + \mu_j Y_j L_j^i\Big) dt \\ &+ (\sigma_{i1} Y_i L_i^i + \sigma_{j1} Y_j L_j^i) dz_1 + (\sigma_{i2} Y_i L_i^i + \sigma_{j2} Y_j L_j^i) dz_2 \end{aligned}$$

Substitution in (14) leads to a second order PDE with general solution

$$L^i(y) = A_{ii}^L y_i^{\beta_{ii}} + B_{ii}^L y_i^{\gamma_{ii}} + A_{ij}^L y_j^{\beta_{ij}} + B_{ij}^L y_j^{\gamma_{ij}} + \frac{D_{10}}{\delta_{ii}} y_i - I,$$

[8] For $f(x_1, x_2)$, let $f_i(\cdot) = \frac{\partial f(\cdot)}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j}$.

where A^L_{ii}, $A^L_{i,j}$, B^L_{ii}, and B^L_{ij} are constants and $\beta_{ij} > 1$ and $\gamma_{ij} < 0$ are the roots of the quadratic equation

$$\mathcal{Q}_{ij}(\xi) \equiv \frac{1}{2}\sigma_j^2\xi(\xi-1) + (\mu_{\Lambda^i} - \delta_{ij})\xi - \mu_{\Lambda^i} = 0.$$

If $y_j = 0$, then the threshold Y^j_F will never be reached and, hence, Firm i will receive D_{10} over the time interval $[0,\infty)$. This leads to the boundary condition $\lim_{y_j \downarrow 0} L^i(y) = \frac{D_{10}}{\delta_{ii}} y_i - I$. This implies that $A^L_{ii} = B^L_{ij} = 0$. Also, if $y_i = 0$, then Firm i only incurs the sunk costs. This leads to the boundary condition $\lim_{y_i \downarrow 0} L^i(y) = -I$, which implies $B^L_{ii} = 0$. Finally, if $y_j = Y^j_F$, then both firms exercise simultaneously. Therefore, another boundary condition is given by $L^i(y_i, y^j_F) = M^i(y) = \frac{D_{11}}{\delta_{ii}} y_i - I$. Solving for A_{ij} then gives

$$L^i(y) = \frac{D_{10}}{\delta_{ii}} y_i - \frac{D_{10} - D_{11}}{\delta_{ii}} \Big(\frac{y_j}{Y^j_F}\Big)^{\beta_{ij}} y_i - I. \tag{15}$$

Note that the second term in (15) is a correction for the possibility that Firm j may exercise its option as well at some time in the future.

The value function in (15) is the strike price of the option to Firm i of becoming the leader and can, therefore, be used to derive the optimal stopping set S^i_L in the following way. For $y \in S^i_N$, let $C^i(y)$ denote the option value of Firm i of exercising the option with strike price governed by (15). The no-arbitrage value of $C^i(y)$ follows

$$D_{00} Y_i dt + \mathbb{E}^{P_y}[dC^i] + \mathbb{E}^{P_y}\Big[C^i \frac{d\Lambda^i}{\Lambda^i}\Big] = -\mathbb{E}^{P_y}\Big[\frac{d\Lambda^i}{\Lambda^i} dC^i\Big]. \tag{16}$$

From Ito's lemma it follows that

$$\begin{aligned} dC^i =& C^i_i dY_i + C^i_j dY_j + \frac{1}{2} C^i_{ii} dY_i^2 + \frac{1}{2} C^i_{jj} dY_j^2 + C^i_{ij} dY_i dY_j \\ =& \big(\frac{1}{2}\sigma_i^2 Y_i^2 C^i_{ii} + \frac{1}{2}\sigma_j^2 Y_j^2 C^i_{jj} + \rho Y_i Y_j C^i_{ij} + \mu_i Y_i C^i_i + \mu_j Y_j C^i_j\big) dt \\ &+ (\sigma_{i1} Y_i C^i_i + \sigma_{j1} Y_j C^i_j) dz_1 + (\sigma_{i2} Y_i C^i_i + \sigma_{j2} Y_j C^i_j) dz_2 \end{aligned}$$

Substitution in (16) leads to a second order PDE with general solution

$$C^i(y) = A_{ii} y_i^{\beta_{ii}} + B_{ii} y_i^{\gamma_{ii}} + A_{ij} y_j^{\beta_{ij}} + B_{ij} y_j^{\gamma_{ij}} + \frac{D_{00}}{\delta_{ii}} y_i,$$

where A_{ii}, $A_{i,j}$, B_{ii}, and B_{ij} are constants. If $y = (0,0)$, then the sets S^i_L, S^j_L, S^i_F, and S^j_F will never be reached and, hence, Firm i will receive D_{00} over the time interval $[0,\infty)$. This leads to the boundary condition $\lim_{y \downarrow (0,0)} C^i(y) = \frac{D_{00}}{\delta_{ii}} y_i$. This implies that $B_{ii} = B_{ij} = 0$. The value-matching and smooth-pasting conditions, in this case, are

$$\begin{cases} C^i(y) = L^i(y) \\ \frac{\partial C^i(y)}{\partial y_i} = \frac{\partial L^i(y)}{\partial y_i} \\ \frac{\partial C^i(y)}{\partial y_j} = \frac{\partial L^i(y)}{\partial y_j} \end{cases}$$

Solving this system of equations leads to the optimal stopping set

$$S_L^i = \{Y \in \mathbb{R}_+^2 | Y_j \leq Y_F^j, Y_i \in [Y_L^i(Y_j), \infty)\}$$

where

$$Y_L^i(Y_j) = \frac{\beta_{ii}\delta_{ii}}{(\beta_{ii}-1)(D_{10}-D_{00}) + (D_{10}-D_{11})(Y_j/Y_F^j)^{\beta_{ij}}} I.$$

The following lemma establishes the existence of a first-mover advantage.

Lemma 1. *For $i \in \{1,2\}$ and $Y_j \leq Y_F^j$, it holds that $S_F^i \subset S_L^i$.*

Proof. Let $Y_j = Y_F^j$. Since $D_{10} \geq D_{11}$, and $D_{10} - D_{00} > D_{11} - D_{01}$ it immediately follows that

$$\begin{aligned}
Y_L^i(Y_F^j) &= \frac{\beta_{ii}\delta_{ii}}{(\beta_{ii}-1)(D_{10}-D_{00}) + (D_{10}-D_{00})} I \\
&\leq \frac{\beta_{ii}}{\beta_{ii}-1} \frac{\delta_{ii}}{D_{10}-D_{00}} I \\
&< \frac{\beta_{ii}}{\beta_{ii}-1} \frac{\delta_{ii}}{D_{11}-D_{01}} I = Y_F^i.
\end{aligned}$$

Furthermore, it is easy to see that $\frac{\partial Y_L^i(Y_j)}{\partial Y_j} \leq 0$, for $Y_j < Y_F^j$. Hence, for all $Y_j \leq Y_F^j$, it holds that $Y_L^i(Y_j) < Y_F^i$. ■

For every $Y_j \leq Y_F^j$, let $Y_P^i(Y_j)$ be the solution of the equation $L^i(Y_P^i(Y_j), Y_j) = F^i(Y_P^i(Y_j))$. It is then easy to see that

$$S_P^i = \{Y \in \mathbb{R}_+^2 | Y_j \leq Y_F^j, Y_i \in [Y_P^i(Y_j), \infty)\}.$$

By construction of (15) it holds that $Y_P^i(Y_j) \leq Y_L^i(Y_j)$, for all $Y_j \leq Y_F^j$. Therefore, it holds that $S_P^i \subseteq S_L^i$. In other words, the conditions in Assumptions 1–2 are satisfied and Theorem 1 provides an equilibrium for this NERO.

A further result can be obtained. Recall that $\bar{S}_P^i = S_P^i \backslash S_L^i$ is the preemption region. The following lemma establishes that this region is non-empty.

Lemma 2. *It holds that $\bar{S}_P^1 \cap \bar{S}_P^2 \neq \emptyset$.*

Proof. Let $\mathcal{A} = [0, Y_L^1(Y_F^2)] \times [0, Y_L^2(Y_F^1)]$ and define the function $f : \mathcal{A} \to \mathbb{R}^2$, where, for $i = 1, 2$, $f_i(y) = F^i(y_i) - L^i(y)$. Note that for $i = 1, 2$ and $Y_j \in [0, Y_L^2(Y_F^1)]$, by Lemma 1, it holds that

$$f_i(Y_L^i(Y_F^j), y_j) < F^i(Y_F^i) - L^i(Y_F^i, y^j) = 0 \tag{17}$$

$$f_i(0, y_j) = I > 0 \tag{18}$$

Since $\mathcal{A}$ is a convex and compact set, and f is a continuous function, there exists a stationary point on $\mathcal{A}$ (cf. Eaves (1971)), i.e.

$$\exists_{y^* \in \mathcal{A}} \forall_{y \in \mathcal{A}} : y f(y^*) \leq y^* f(y^*). \tag{19}$$

Let $i \in \{1, 2\}$. Suppose that $y_i^* > 0$. Then there exists $\varepsilon > 0$, such that $y = y^* - \varepsilon e_i \in \mathcal{A}$, where e_i is the i-th unit vector. From (19) it then follows that

$$y f(y^*) - y^* f(y^*) = -\varepsilon f_i(y^*) \leq 0 \iff f_i(y^*) \geq 0. \tag{20}$$

Similarly, if $y_i^* < Y_L^i(Y_F^j)$, there exists $\varepsilon > 0$, such that $y = y^* + \varepsilon e_i \in \mathcal{A}$. Therefore,

$$y f(y^*) - y^* f(y^*) = \varepsilon f_i(y^*) \leq 0 \iff f_i(y^*) \leq 0. \tag{21}$$

Hence, from (20) and (21) it follows that $f(y^*) = 0$, if $y^* \in \mathcal{A} \backslash \partial \mathcal{A}$.

Suppose that $y_i^* = 0$, and let $y \in \mathcal{A}$ be such that $y_j = y_j^*$. Then (19) implies that $(y - y^*) f(y^*) = y_i f_i(y^*) \leq 0 \iff f_i(y^*) \leq 0$, which contradicts (18). Similarly, supposing that $y_i^* = Y_L^i(Y_F^j)$, and taking $y \in \mathcal{A}$ such that $y_i = y_i^*$, it is obtained that $(y - y^*) f(y^*) = (y_i - y_i^*) f_i(y^*) \leq 0 \iff f_i(y^*) \geq 0$, which contradicts (17). Hence, there exists $y^* \in \mathcal{A} \backslash \partial \mathcal{A}$, such that $L^i(y^*) = F^i(y_i^*)$, $i = 1, 2$. ■

4.3 A Numerical Illustration

Consider the case with payoffs, sunk costs, and parameters as given in Table 1. It is assumed that both firms have the same discount factor, $\Lambda^1 = \Lambda^2 \equiv \Lambda$. It

$(D_{10}, D_{11}, D_{00}, D_{01}) = (8, 5, 3, 1)$	$I = 100$
$\mu_\Lambda = 0.04$	$\sigma_\Lambda = (0.05, 0.05)$
$\mu_Y = (0.03, 0.03)$	$\Sigma_Y = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}$

Table 1: Model characteristics.

is, furthermore, assumed that Firm 1 is only influenced by z_1, whereas Firm 2's payoffs are influenced by both shocks. This could correspond to a situation where Firm 1 is a domestic firm, with an option to invest in a new product, and Firm 2 is a foreign firm with a similar option. The Wiener process z_2 can represent, for example, exchange rate risk.

The resulting optimal stopping regions and the preemption region are depicted in Figure 4. Starting at $Y_0 = y \in S_N^1 \cap S_N^2$, note that, since Y has continuous sample paths, in equilibrium there is always one firm which does not exercise the option at time $\tau_y(\Theta)$, a.s.[9] It is, however, not the case that in the preemption region both

[9] Recall that $\Theta = S_L^i \cup S_L^j \cup (\bar{S}_P^1 \cap \bar{S}_P^2)$.

firms exercise with probability 0.5, as is the case in papers where it is assumed that firms toss a fair coin to determine who exercises first in the preemption region.[10] In fact, conditional on $\tau_y(\Theta) = \tau_y(\bar{S}_P^1 \cap \bar{S}_P^2)$, the probability that both firms exercise with probability 0.5 is equal to 0. This is the case since $P(X^{\sigma_{y,x}}_{(\tau_y,\zeta_{y,x})} = (1,0)) = P(X^{\sigma_{y,x}}_{(\tau_y,\zeta_{y,x})} = (0,1))$ only if $\varphi^1(Y_{\tau_y(\Theta)}) = \varphi^2(Y_{\tau_y(\Theta)}) = 0$, because there is always one firm for whom $\varphi^i(Y_{\tau_y(\Theta)}) = 0$. There is only one point where this happens, namely at the intersection of $Y_P^1(Y_2)$ and $Y_P^2(Y_1)$. Due to absolute continuity, this point is reached with zero probability.

So, three investment scenarios can occur with positive probability. These are (i) monopolistic sequential investment (by either firm), (ii) preemptive sequential investment (by either firm), and (iii) preemptive simultaneous investment. The precise probabilities of each scenario occurring depend on the underlying fundamentals. This analysis shows that is difficult to judge the competitiveness of a market based on *ex post* observed investment behaviour. For example, simultaneous investment is not necessarily a sign of tacit collusion. It could very well happen in a preemptive environment.

To examine the influence of the instantaneous correlation between z_1 and z_2 on investment timing and competition, the following simulation experiment is conducted. The same parameter values as before are chosen, with $\sigma_{\Lambda^i,j} = 0.05$, all $i, j = 1, 2$, $\sigma_{11} = 0.2$, and $\sigma_{12} = 0$. This time, however, it is assumed that $\sigma_2 = 0.2$, but that the loadings on z_1 and z_2 vary with ρ. So, firms are symmetric, but for the loading of the respective risk they face on the factors z_1 and z_2.

For every value $\rho \in (-1, 1)$, 1,000 sample paths of Y are generated. The starting point of the process Y is always taken to be $(0.15, 0.15)$. In Figure 5, the expected time to (first) investment and the probability of preemption (as opposed to a scenario where either firm acts as if it were a monopolist) are plotted as functions of ρ.

The correlation between shocks can indicate what kind of industry is being investigated. If shocks are perfectly negatively correlated, for example, one might be dealing with a situation where both firms offer goods that are perfect substitutes. The profit of one firm goes up when the profit of the other goes down. Conversely, if shocks are perfectly positively correlated, one might be looking at an industry with homogenous goods. If the shocks are uncorrelated, the two firms seem to operate in unconnected markets.

Firstly, the left-panel of Figure 5 indicates that the expected time to first investment increases with ρ. So, the more integrated the industry, the longer it takes – in expectation – until (first) investment takes place. The right-panel of Figure 5

[10] See, for example, Grenadier (1996) or Weeds (2002)

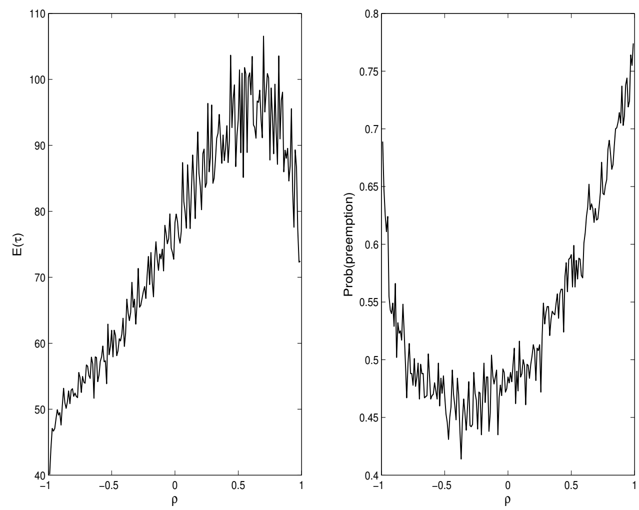

Figure 5: Expected investment time (left-panel) and probability of preemption (right-panel) as a function of the instantaneous correlation ρ.

shows the probability of investment scenarios (ii) or (iii) occurring. That is, the probability that $Y_{\tau_y} \in \bar{S}_P^1 \cap \bar{S}_P^2$. Intuitively, it measures the preemptive pressure in the industry. It appears that preemptive pressure is lowest if the two markets are uncorrelated. This is of course what one would expect. It is, however, striking that there is no one-to-one relation between the expected time of first investment and the preemptive pressure in the industry. If, namely, goods become more substitutable ($-1 < \rho < 0$ and decreasing) expected time to investment goes down, while preemptive pressure goes up. This is what one would intuitively expect. If, however, goods become more homogenous ($0 < \rho < 1$ and increasing) expected time goes up even though preemptive pressure increases.

5 Conclusion

In this paper a general model for two-player non-exclusive real option games with first mover advantages has been introduced. The strategy and equilibrium concepts are based on ideas from Dutta and Rustichini (1995) and Fudenberg and Tirole (1985). The advantage of using a coordination device in the spirit of Fudenberg and Tirole (1985) is that it allows one to endogenously solve for a particular coordination problem that often arises in preemption games. The basic idea is that if a coordination problem arises, the two players engage in a game in "coordination time", which leads to an absorbing Markov chain. The probabilities with which each player exercises the option are then simply given by the limit distribution of this chain. The main result of the paper, Theorem 1, proves the validity of the rent-equalisation principle in NEROs with first-mover advantages, where uncertainty is governed by a strong Markovian stochastic process.

Most of the present literature on game-theoretic real option models assumes that uncertainty is represented by a one-dimensional geometric Brownian motion. This paper shows that the qualitative results change significantly if a two-dimensional GBM is used. In fact, in much of the literature on non-exclusive real options the coordination device is not used, but exogenous assumptions are made on the resolution of the coordination problem instead. Usually it is argued that a fair coin is tossed and each player exercises with probability 1/2. It seems that such assumptions are based on Fudenberg and Tirole (1985) who show that this is the case in the particular (deterministic, symmetric players) model they study. In a purely symmetric model with non-exclusive real options and one-dimensional GBM this is indeed true. With a two-dimensional GBM, however, the coordination problem arises as well and, in equilibrium, neither player exercises with probability 1/2 (a.s.). Furthermore, both players exercise with unequal probability (a.s.). It is still the case, however, that both players do not exercise simultaneously (a.s.) as is a standard result (or indeed assumption) in the current literature. This is due to the continuous sample paths of GBM.

The analysis in this paper opens up several avenues for future research. Firstly, the actual behaviour of the model for particular stochastic processes can be examined. Of particular interest would be the situation where Y follows a jump-diffusion process. In the models currently studied in the literature the probability of both players jointly exercising is zero, due to continuity of sample paths. This property would be lost in jump-diffusion model. This might consequently lead to an additional value of waiting.

Secondly, the model in Section 4 could be used to analyse specific economic problems. A straightforward one is the question whether currency unions, or currency pegging, accelerates investment. In the setting of Section 4 one can think of two firms, a domestic one and a foreign one. The domestic firm is exposed to one source of risk, say product-market risk due to demand fluctuations, whereas the foreign firm is also exposed to exchange rate risk. A monetary union would take away the latter source of risk and lead to a duopoly as analysed in Huisman (2001, Chapter 7). The expected first and second exercise times could be simulated and a welfare analysis could be made to compare both situations.

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