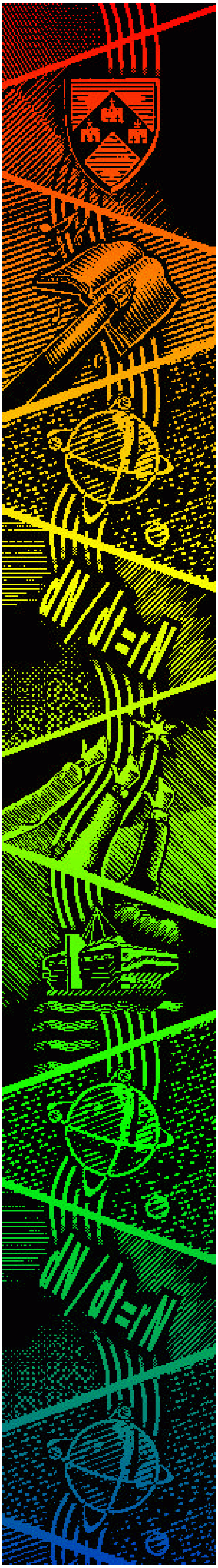

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Stability and Equilibrium in Decision Rules: An Application to Duopoly

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Abstract

This paper analyses an indefinitely-repeated Cournot duopoly. Firms select simple dynamic decision rules which, taken together, comprise a first-order linear difference equation system. A boundedly-rational objective function is assumed, by which the firm's payoff is its profit at the point of convergence, if any. Stable Nash equilibria are characterised and located in output space, stability in this context being equivalent to subgame-perfection. Comparable results are derived for a conventional discounted-profit objective function, where this equivalence does not hold, but where stability may nevertheless be of intrinsic interest. In either context, stability is incompatible with joint profit maximisation.

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1 Introduction and overview

In this paper we analyse an indefinitely-repeated Cournot duopoly, within a framework motivated by the idea of bounded rationality. Characteristic of this is the proposition that computational simplicity is valuable to agents. In our model, simplicity is an issue at two levels. The first, and more familiar, is that of the firm's strategy. Among the simplest types of strategy, in this dynamic context, is the Reaction Function, which gives the firm's current output as a time-independent function of its rival's immediately previous output. Similarly, a Supply Function has as its only argument the immediately previous market price. Strategies of this kind have received much attention in the literature[1], perhaps reflecting a widespread, if tacit, assumption that agents value simplicity.

But the second level is that of the objective function, in terms of which firms identify their (mutually) optimal strategies. Even if both firms' strategies are very simple, the implied sequence of outputs can be highly complicated and, therefore, difficult to evaluate. So firms might be forced, or choose, to simplify this task. One possibility, which we explore in this paper, is that a boundedly-rational firm evaluates an output sequence only at its point of convergence, if any. In the case of a convergent sequence, this is effectively equivalent to a fully-rational firm with a zero discount rate, using a limit-of-mean-profit criterion. But it is stronger than this criterion, in that it places a zero value on non-convergent sequences.

Our analysis focuses on the asymptotic stability of outputs, in a Nash equilibrium. Given the assumed objective function, stability here is equivalent to subgame-perfection. For a fairly simple class of strategy, to be defined shortly, we show that almost any profitable output point can be supported as a stable, boundedly-rational equilibrium. Notably excluded, however, is the duopoly contract curve, comprising points of mutual profit maximisation.

We also analyse stability in the fully-rational, discounted profit case. As already noted, our boundedly-rational equilibrium is largely equivalent to the limiting (zero discounting) case. But in the fully-rational context stability does not imply subgame-perfection. Indeed, we find here that

[1] See, for example, Stanford (1986a), Klemperer and Meyer (1989), or Grossman (1981).

linear Reaction Functions, for example, can support stable equilibria where subgame-perfection is known to require rather more complex strategies. The lower the discount rate, the larger is the set of output points thus supportable. This raises the possibility of stability constituting a weak form of equilibrium refinement for fully-rational firms constrained to the simplest of strategies.

For the purposes of exposition, the paper deals first with the fully-rational case. Here, we are interested in strategy pairs which satisfy three criteria:

- *Equilibrium*, i.e., the mutual optimality of each firm's strategy;

- *Stationarity* of the generated output sequence, which means simply that outputs (and thus the price) are constant from one period to the next;

- *Stability*, i.e., re-convergence to the stationary point following any deviation from it.

Taking only the first two criteria, we might ask whether a given output vector can be supported as a stationary equilibrium, i.e., as the stationary output sequence of a Nash equilibrium strategy pair. It may easily be shown that, given concave profit functions, any profitable output point can be thus supported (Proposition 1), and that this requires only (linear) Reaction Functions.

Subgame-perfection is, of course, more demanding. Thus, Stanford (1986a) demonstrates that Reaction Functions (linear or otherwise) can support subgame-perfect equilibria only at the standard Cournot equilibrium point. Friedman and Samuelson (1994) use the term *single-period recall function* (SPRF) to describe a strategy which gives a firm's output as a function of the immediately previous outputs of both firms, and of which the Reaction Function (as defined here) is a special case. They identify a class of continuous, non-linear, SPRFs capable of supporting a subgame-perfect equilibrium at any profitable output point.[2] This complements Stanford's result, and also that of Robson (1986), who shows that linear SPRFs can support such equilibria only

[2] It is well known that subgame-perfect equilibria can be sustained by discontinuous trigger strategies, as in Friedman (1971) and Abreu (1986). However, Friedman and Samuelson (1994) showed that continuous versions of such strategies could be found.

at the Cournot point. Such results are important because they tell us something about the degree of strategic complexity required to support (subgame-perfect) equilibria. And complexity is of interest, perhaps, because we can more plausibly imagine real firms using simple rules than complex ones. As already suggested, this idea could be articulated in terms of bounded rationality, by supposing that the use of a more complex rule is more demanding on a firm's limited or costly computational resources.

The assumption in this paper is that both firms adopt linear SPRFs which, as noted, are insufficient for a subgame-perfect equilibrium other than at the Cournot point. However, our main interest is not subgame-perfection as such, but rather the third criterion listed above, i.e., that of stability. We find (Proposition 2) that linear SPRFs can support stable equilibria at a wide range of output points. Notably excluded, however, are points of mutual profit-maximisation, i.e., the contract curve. At best, in the limiting case of a zero discount rate, equilibria here may be "semi-stable" in that there is re-convergence, following any deviation, but generally to some other point, off the contract curve. We also consider two special cases of linear SPRF. The first is a Reaction Function; here the stable equilibria comprise a cross-shaped set containing the two firms' Cournot (contemporaneous) best-response curves. The second is a Supply Function; here the stable equilibria form a curved band running close to, but strictly above, the contract curve. In each case, the size of the corresponding set is positively related to the per-period discount factor, i.e., negatively related to the discount or interest rate.

We then similarly analyse a boundedly-rational equilibrium. Just as in the fully-rational case, stationary equilibria may be found at any profitable output point (Proposition 3). The significance of stability, in this context, is that it is here equivalent to subgame-perfection (Propositions 4a and 4b). Stable equilibria, while more widespread than in the fully-rational case, are not quite ubiquitous (Proposition 5). Again notably excluded are points on the contract curve. Equilibria here are, at best, semi-stable. But they cannot be subgame-perfect.

The paper is structured as follows. In section 2 we define and elaborate our assumed class of strategies. Section 3 outlines our assumptions concerning the firm's profit function. Conventional, fully-rational, stationary equilibria are characterised in section 4. The stability condition is introduced in section 5, and applied to the fully-rational equilibrium (Proposition 2). Finally, section 6 contains our analysis (Propositions 3-5) of boundedly-rational equilibria.

2 The Decision Rule

The model comprises two firms producing an homogeneous good. In each period $(t = 0,1,2,..)$, each firm $(i = 1,2)$ produces an output level $x_{i,t}$, giving an output vector $x_t = (x_{1,t}, x_{2,t})$. We assume no restriction on the output space other than that it is real, so that the set of feasible output vectors is $\mathrm{X} \equiv \{ x_t \mid x_t \in \mathbb{R}^2 \}$. Let $\chi = \langle x_0, x_1, x_2, .. \rangle$ denote an entire sequence of such output vectors. We describe as *stationary* (at $x \in \mathrm{X}$) any output sequence χ such that $x_t = x$ for all *t*.

Governing each firm's behaviour is a time-independent decision rule which makes $x_{i,t}$ a linear function of the outputs of both firms in the immediately preceding period. This *first-order linear decision rule* (FOLDR) is defined by a real triple $\rho_i = (a_i, b_i, c_i)$ such that:

$$x_{i,t+1} = a_i x_{i,t} + b_i x_{j,t} + c_i \qquad (i \neq j)$$

This is a special (i.e., linear) case of what Friedman and Samuelson (1994) call a *single-period recall function*. Two subclasses of FOLDR which will be of interest are:

(i) A *Reaction Function* (RF), defined by $a_i = 0$, in which firm i's output is a function only of firm j's (immediate past) output.

(ii) A *Supply Function* (SF), defined by $a_i = b_i$, in which firm i's output is, in effect, a function only of the (immediate past) market-clearing price.

At the outset $(t=0)$ each firm adopts a *strategy* $\sigma_i = \langle x_{i,0}, \rho_i \rangle$ comprising an initial output and a

FOLDR. The strategy pair $\sigma=(\sigma_1,\sigma_2)$ generates a unique output sequence χ_σ which we call its *trajectory*. The trajectory of σ is stationary at $x=(x_1,x_2)$ only if, for each i :[3]

$$x_i \;=\; a_i x_i \,+\, b_i x_j \,+\, c_i \tag{1}$$

Given ρ_i, let $S_i(\rho_i)$ denote the set of all $x \in X$ satisfying (1). We call this the *stationary set* for ρ_i. A *stationary solution* for $\rho=(\rho_1,\rho_2)$ is any:

$$x \in S(\rho) \;\equiv\; S_1(\rho_1) \cap S_2(\rho_2)$$

A strategy pair $\sigma=(\sigma_1,\sigma_2)$ can be equivalently represented as $\sigma=\langle x_0,\rho\rangle$. From the definition of $S(\rho)$, it follows that the trajectory of $\sigma=\langle x_0,\rho\rangle$ is stationary if and only if $x_0 \in S(\rho)$.

We describe as *regular* any ρ_i for which $a_i \neq 1$, in which case (1) may be rewritten as:

$$x_i \;=\; \beta x_j \;+\; \gamma_i \tag{2}$$

where:

$$\beta_i \;\equiv\; \frac{b_i}{1-a_i} \qquad \text{and} \qquad \gamma_i \;\equiv\; \frac{c_i}{1-a_i}$$

We similarly describe as regular any $\rho=(\rho_1,\rho_2)$ of which each ρ_i is regular. For a regular ρ, therefore, $S(\rho)$ comprises solutions to (2) for each $i=1,2$ simultaneously. There will be exactly one such solution, unless $\beta_1\beta_2 = 1$. In that case the two stationary sets will either coincide (if $\gamma_i = \beta_i\gamma_j$), giving infinitely many stationary solutions, or fail to intersect, giving none.

[3] Notice that we use x with a single subscript to denote either the (timeless) output of a given firm (i or j), or the output vector in a given time period (t). While convenient, this is potentially ambiguous if the subscript is a numeral. Hereafter, single subscripts 1 or 2 refer only to the firms, while 0 refers to the initial time period.

3 Profit in a single period

We consider equilibria within $\Omega \subset X^+ \subset X$, where:

$$X^+ \equiv \{x \in X \mid x_1, x_2 \geq 0\} \qquad \text{and} \qquad \Omega \equiv \{x \in X^+ \mid \pi_1(x), \pi_2(x) > 0\}$$

For any $x_t \in X^+$, the profit for firm i in period t is given by:

$$\pi_i(x_t) = x_{i,t} f(x_{1,t} + x_{2,t}) - g_i(x_{i,t})$$

where $f(x_{1,t} + x_{2,t})$ is the market-clearing price and $g_i(x_{i,t})$ is firm i's total cost. We assume that each of these functions is differentiable, and that $f < 0$ for all $x \in \Omega$. We also assume that, within X^+, $\pi_i(.)$ is quasiconcave .

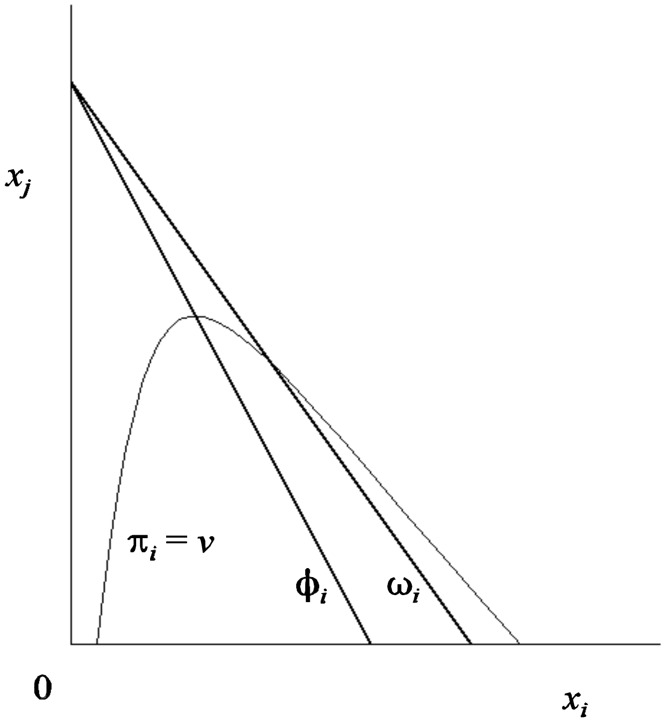

Figure 1: A typical iso-profit contour

Figure 1 shows a typical profit contour for firm *i*. Its gradient is:

$$s_i \equiv \frac{dx_j}{dx_i}\Big|_{\pi_i = v} = \frac{g_i' - f}{x_i f'} - 1$$

Two particular points on the contour may be noted. One is where $s_i = 0$ and thus:

$$g_i' = f + x_i f'$$

so that firm *i*'s marginal cost is equal to its marginal revenue, given x_j. The other is where $s_i = -1$ and thus marginal cost is equal to price. Figure 1 shows each respective locus, ϕ_i and ω_i. The former corresponds to the standard Cournot (contemporaneous) best-response function. The latter, which we term the *Walrasian locus*, corresponds to price-taking profit maximisation.

These assumptions, and the representation in Figure 1, are quite conventional. However, since our decision rules do not constrain output trajectories to X^+, and since we need firms to be able to evaluate any (non-equilibrium) trajectory, then we have to make some further, unconventional assumptions. We shall therefore assume that $f(.)$ and $g_i(.)$ are defined for all $x \in X$, in such a way as to maintain quasiconcavity of each $\pi_i(x)$ everywhere. Consistent with this are, for example, the extensions defined by:

$$f(x) = f(\max\{0, x_1\} + \max\{0, x_2\}) \qquad g_i(x_i) = g_i(\max\{0, x_i\})$$

An alternative approach would be to restrict the output trajectories by truncating the firms' decision rules as, for example:

$$x_{i,t+1} = \max.\{\, 0 \,,\, a_i x_{i,t} + b_i x_{j,t} + c_i \,\}$$

Such an approach is adopted by Stanford (1986b). For our purposes, however, the former approach is simpler and more tractable.

4 Stationary, fully-rational equilibria

In a fully-rational equilibrium, each firm's strategy maximises the discounted sequence of that firm's profits, given the strategy of its rival. For firm j this is:

$$\Pi_j(\chi) \equiv \sum_{t=0}^{\infty} \delta^t \pi_j(x_t)$$

The discount factor $\delta \in (0,1)$ we assume to be common to the two firms; this simplifies the analysis, but is not crucial to it. Consider some strategy pair $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2)$ such that $\chi_{\hat{\sigma}}$ is stationary at $\hat{x}$. Thus for each firm i:

$$\hat{x}_{i,0} = \hat{x}_i \qquad \text{and} \qquad \hat{x}_i = \hat{a}_i \hat{x}_i + \hat{b}_i \hat{x}_j + \hat{c}_i \tag{3}$$

Given $\hat{\sigma}_i$, under what conditions is $\hat{\sigma}_j$ optimal for firm j?

Suppose firstly that $\hat{b}_i = 0$. Since $\chi_{\hat{\sigma}}$ is stationary, it follows that $x_{i,0} = \hat{x}_i$ and $\hat{\rho}_i = (\hat{a}_i, 0, (1-\hat{a}_i)\hat{x}_i)$, i.e., that firm i's output is independently stationary at $\hat{x}_i$. But then $\hat{\sigma}_j$ is optimal if and only if $\hat{x}$ maximises each $\pi_j(x_t)$ subject to $x_{i,t} = \hat{x}_i$. A necessary condition for this is $s_j(\hat{x}) = 0$, i.e., that the slope of firm j's iso-profit contour be zero. This is also sufficient, given quasiconcavity of $\pi_j(.)$. We can thus verify the existence of a repeated Cournot equilibrium, where each firm's output is independently stationary at $x \in \Omega$ defined by $s_1(x) = s_2(x) = 0$.

Now suppose instead that $\hat{b}_i \neq 0$. Given that $\hat{\sigma}_i$ satisfies (3), $\hat{\sigma}_j$ is optimal if and only if $\chi_{\hat{\sigma}}$ maximises $\Pi_j(\chi)$ subject to, for all t:

$$x_{i,0} = \hat{x}_{i,0} \qquad \text{and} \qquad x_{i,t+1} = \hat{a}_i x_{i,t} + \hat{b}_i x_{j,t} + \hat{c}_i$$

Given these constraints, and given that $\chi_{\hat{\sigma}}$ is stationary at $\hat{x}$, then for any T:

$$\frac{\partial \Pi_j}{\partial x_{j,T}} = -\delta^T \frac{\partial \pi_j(\hat{x})}{\partial x_i} \left[s_j(\hat{x}) - \delta \hat{b}_i \sum_{t=T}^{\infty} (\delta \hat{a}_i)^{t-T} \right] \tag{4}$$

For $\hat{\sigma}_j$ to be optimal it is necessary that (4) is zero for every *T*. That is:

$$-1 < \delta\hat{a}_i < 1 \quad \text{and} \quad s_j(\hat{x}) = \frac{\delta\hat{b}_i}{1-\delta\hat{a}_i} \tag{5}$$

Note that (5) also characterises the optimal $\hat{\sigma}_j$ given $-1 < \delta\hat{a}_i < 1$ and $\hat{b}_i = 0$, and therefore some of the Cournot equilibrium strategies.

For an illustrative interpretation of (5), suppose that $\hat{\rho}_i$ is an RF, so that $\hat{a}_i = 0$ and the first part of the condition is satisfied. Then any single-period output perturbation by firm *j* induces an output response by firm *i* in, and only in, the following period. The ratio of the latter to the former is $\hat{b}_i$, here the gradient of $S_i(\hat{\rho}_i)$. The second part of (5) then requires that the net (discounted) effect on firm *j*'s profit is zero. In the limiting case with zero discounting $(\delta = 1)$, this implies that firm *j*'s iso-profit contour is tangent to $S_i(\hat{\rho}_i)$.

For FOLDRs in general, (5) is most easily interpreted in this limiting case, where the first part of the condition is that $\hat{a}_i \in (-1,1)$. This implies that following any single-period perturbation in firm *j*'s output, firm *i*'s output will re-converge to $\hat{x}_i$. If this were not the case, then firm *j* could thereby induce, for example, an indefinite (unlimited) reduction in firm *i*'s output. Given that $\hat{a}_i \in (-1,1)$, the cumulative total of all subsequent output responses by firm *i* converges to $\hat{\beta}_i$, as a proportion of firm *j*'s initial perturbation. For marginal perturbations, the second part of (5) requires that the net (in this case, undiscounted) effect on firm *j*'s profit is zero. Again, in this limiting case this implies that firm *j*'s iso-profit contour is tangent to $S_i(\hat{\rho}_i)$.

In addition to being necessary, (5) is also sufficient for optimality if $\Pi_j(\chi)$ is concave, which would follow from concavity of $\pi_j(.)$. It would also be sufficient were $\Pi_j(\chi)$ only quasiconcave, but this does not follow from quasiconcavity of $\pi_j(.)$. Given a sufficiently strong assumption of this kind, stationary equilibria are ubiquitous in Ω, as stated in the following Proposition.[4]

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[4] Note the formal similarity between this proposition and the ubiquity of consistent-conjectures equilibria, as demonstrated by Klemperer and Meyer (1988), and similarly central to which is the tangency of the (conjectured) reaction function of one firm to an iso-profit contour of the other.

Proposition 1 Given concavity of $\pi_i(.)$, then for any $x \in \Omega$ and any $\delta \in (0,1)$ there exists an equilibrium $\sigma = \langle x_0, \rho \rangle$ such that χ_σ is stationary at *x*.

Proof: Given *x* and δ, put $x_0 = x$, and construct ρ to satisfy (1) and (5), i.e,:

$$\delta a_i \in (-1,1) \qquad b_i = (1-\delta a_i)\frac{s_j(x)}{\delta} \qquad c_i = (1-a_i)x_i - b_i x_j$$

QED

Note that among the ρ satisfying (1) and (5) there is always an RF-pair, for which:

$$b_i = \frac{s_j(x)}{\delta} \tag{6}$$

Thus, stationary RF equilibria are ubiquitous. This is not true, however, of SF equilibria. From the second condition in (5), an equilibrium in SFs requires that, for each firm *i*:

$$a_i = b_i = \frac{s_j(x)}{\delta(1+s_j(x))} \tag{7}$$

But then the first condition in (5) can then be satisfied only if $s_i(x) > -½$ for each firm. This restriction will be illustrated in the next section, where we derive the conditions under which such equilibria are also asymptotically stable.

To conclude this section, however, we should comment on one aspect of our equilibrium analysis. In deriving (5), we considered the effect on firm *j*'s (discounted) profits of a perturbation in $x_{j,T}$ for any single *T*. Thus, (5) is necessary for a *broad* equilibrium in which $\hat{\sigma}_j$ is optimal, for firm *j*, among all possible trajectories consistent with $\hat{\sigma}_i$. However, if firm *j* is restricted to strategies of the form $\sigma_j = \langle x_{j,0}, \rho_j \rangle$ then it cannot independently vary any single $x_{j,T}$. We might therefore wish to consider a correspondingly *narrow* equilibrium, in which $\hat{\sigma}_j$ is optimal only among all σ_j. But it is easily seen that (5) is necessary also for $(\hat{\sigma}_1, \hat{\sigma}_2)$ to be a narrow equilibrium. Consider alternative strategies σ_j comprising $x_{j,0} = x_j$ and $\rho_j = (a_j, 0, (1-a_j)x_j)$ each of which, in effect, fixes firm *j*'s output at $x_{j,t} = x_j$. Among these strategies:

$$\frac{d\Pi_j}{dx_j} = \sum_{t=T}^{\infty} \frac{\partial \Pi_j}{\partial x_{j,T}} \tag{8}$$

Given $\hat{\sigma}_i$, the selection of such a strategy with $x_j = \hat{x}_j$ gives the same stationary trajectory, and the same payoff to firm *j*, as does the selection of $\hat{\sigma}_j$. So for $\hat{\sigma}_j$ to be optimal among σ_j it is necessary for (8) to be zero, when evaluated at $\hat{x}$. Given (4), this requires (5) as before.

5 Stability

As shown by Robson (1986), and as noted section 1, FOLDRs cannot support subgame-perfect, fully-rational equilibria other than at the Cournot point. But in this section we show that FOLDRs can support stable equilibria over a wide range of output points, this size of this set being negatively related to the discount rate. We also characterise and locate the sets of such equilibria supported, respectively, by RFs and SFs. Stability may have some intrinsic interest in the fully-rational context, for example as a weak form of equilibrium refinement. But, in any case, our results here provide a useful background to our discussion of boundedly-rational equilibria, in section 6, where stability and subgame-perfection are equivalent.

The FOLDR pair ρ is (asymptotically) stable if and only if the trajectory of every $\sigma = \langle x_0, \rho \rangle$ converges to a common $x \in \mathrm{X}$, which must therefore be its unique stationary solution $\mathrm{S}(\rho)$. Necessary and sufficient for this is that each of its two eigenvalues is within the unit circle. These are given by:

$$\lambda(\rho) = a \pm \left[a^2 - a_1 a_2 + b_1 b_2\right]^{½} \tag{9}$$

where: $$a \equiv \frac{a_1 + a_2}{2}$$

Thus for ρ to be stable requires at least that $a \in (-1,1)$. Note that, since $a^2 \geq a_1 a_2$, the eigenvalues will be real if $b_1 b_2$ is non-negative. One instance of this is where $b_i = 0$ for either firm, in which case the eigenvalues are just $\{a_1, a_2\}$.

In section 2 we noted the possibility of a regular ρ with infinitely many stationary solutions, where the two firms' (linear) stationary sets coincide, so that $\beta_1\beta_2 = 1$. Here, from (9):

$$\lambda(\rho) \quad = \quad 1 \text{ and } 2a-1$$

The first eigenvalue $\lambda(\rho) = 1$ applies to each of the stationary solutions. The other eigenvalue $\lambda(\rho) = 2a-1$ applies to any other trajectory, all of which are linear and mutually parallel. If $a \in (0,1)$ then they all converge, but not to the same stationary solution. We will describe such a ρ as *semi-stable*. It will be of significance in what follows.

We now characterise and locate sets of stable, stationary equilibria. Consider firstly a pair of RFs which, from (9), is stable if and only if:

$$-1 \; < \; b_1 b_2 \; < \; 1$$

From (6), a stationary equilibrium in RFs is therefore stable if and only if:

$$-\delta^2 \; < \; s(x) \; < \; \delta^2 \qquad \text{where} \quad s(x) \equiv s_1(x) s_2(x) . \tag{10}$$

Figure 2 sketches this in the special case of a linear demand function $p = 1 - x_1 - x_2$ and identical quadratic costs $g(x_i) = x_i^2$. It shows the set Ω, bounded above by the (linear) zero-profit contours which intersect at the output vector $(1/3, 1/3)$, here denoted *z*. The output vector corresponding to the Walrasian (price-taking) equilibrium is $(1/4, 1/4)$, denoted *w*. The Cournot equilibrium is at $(1/5, 1/5)$, denoted *c*, and joint profit maximisation is achieved at $(1/6, 1/6)$, denoted *m*. The curve passing through *m* is the contract curve, comprising points of mutual profit maximisation. It is characterised by the mutual tangency of the two firms' iso-profit contours, and thus by $s(x) = 1$. This is true also of the curve passing through *w*, but which comprises points of mutual (local) profit minimisation. We call this the *anti-contract curve*. The figure also shows, for each firm, the loci identified in section 3, which in this special case are linear. The Walrasian loci, ω_i, are the lines intersecting at *w*. The Cournot loci, ϕ_i, are the lines intersecting at *c*. These subdivide Ω into four main areas: two off-diagonal areas in which $s(x)<0$, and two on-diagonal areas in which $s(x)>0$. (This refers to the positive diagonal, connecting *z* to the origin.)

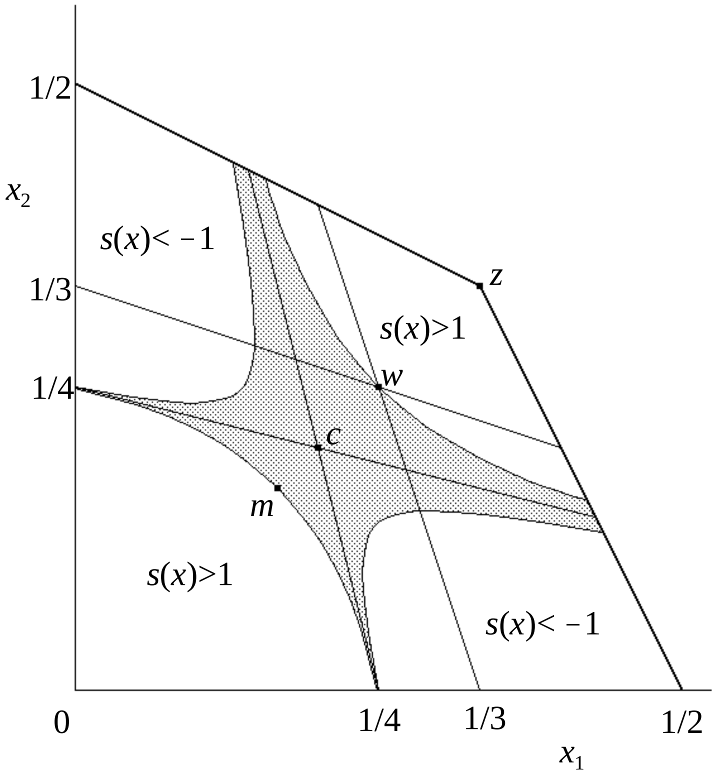

Figure 2: Stationary equilibria with stable RFs

The set of stationary equilibria in stable RFs, i.e., output points satisfying (10), is a strict subset of Ω. At its largest, as $\delta \rightarrow 1$, it approaches the shaded region in Figure 2. It always lies strictly above the contract curve and below the anti-contract curve, along each of which $s(x)=1$. Off the diagonal, it is bounded by curves along which $s(x)=-1$. At its smallest, as $\delta \rightarrow 0$, it reduces to the cross formed by $\phi_1 \cup \phi_2$, which includes the Cournot point *c*. Each ϕ_i is characterised by $s_i(x)=0$, and so satisfies (10) for any positive δ.

Figure 3 sketches the corresponding region for Supply Functions. Recall from the previous section that SFs, unlike RFs, can support stationary equilibria only where $s_i(x) > -½$ for each firm. The relevant (linear) boundaries, below which this is satisfied, are shown intersecting above *c*.

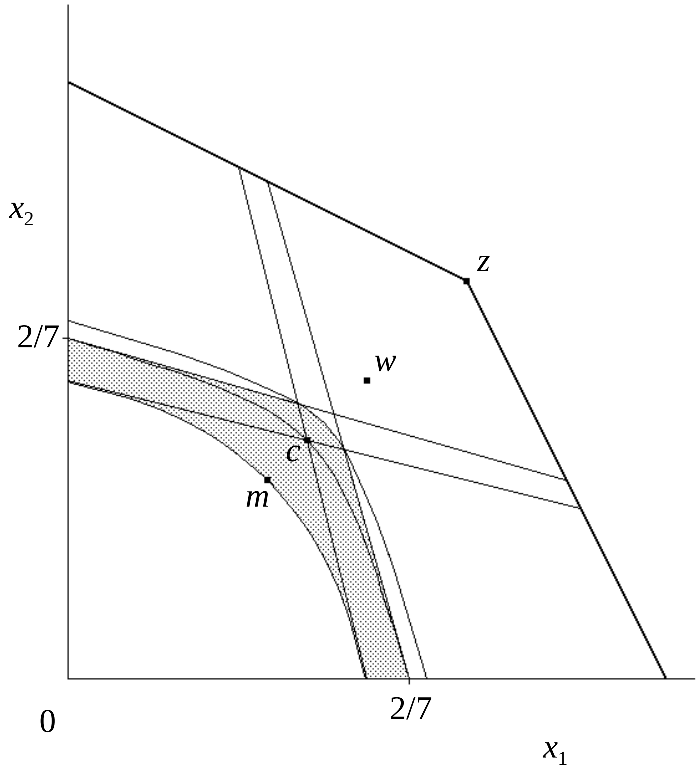

Figure 3: Stationary equilibria with stable SFs

From (9), a pair of SFs is stable if and only if:

$$-1 < b_1 + b_2 < 1$$

From (7), a stationary equilibrium in SFs is therefore stable if and only if:

$$-\delta < r(x) < \delta \qquad \text{where} \qquad r(x) \equiv \frac{s_1(x)}{1+s_1(x)} + \frac{s_2(x)}{1+s_2(x)}$$

At its largest, as $\delta \to 1$, the set of stable, stationary SF equilibria approaches the shaded region in Figure 3. It always lies strictly above the contract curve, along which $r(x) = 1$, and below curve shown passing above *c*, along which $r(x) = -1$. At its smallest, as $\delta \to 0$, it reduces to the curve shown passing through *c*, along which $r(x) = 0$.

Having looked in particular at RFs and SFs, we now consider FOLDRs in general. For any $\rho = (\rho_1, \rho_2)$ satisfying the equilibrium condition (5), the eigenvalues may be written as:

$$\lambda(\rho) \;=\; a \pm \left[a^2 - a_1 a_2 + \frac{s(x)}{\delta^2}(1-\delta a_1)(1-\delta a_2) \right]^{½} \tag{11}$$

where $\delta a_i \in (-1,1)$. We have found that output points on the contract curve cannot be supported as stationary equilibria either in stable RF pairs or in stable SF pairs. It is readily established that such points cannot be supported by stable FOLDRs at all. Suppose that $s(x) = 1$, as on the contract curve. From (11) it follows that:

$$\lambda(\rho) \;=\; \frac{1}{\delta} \quad \text{and} \quad 2a - \frac{1}{\delta}$$

whereby at least one eigenvalue $(1/\delta)$ is not within the unit circle. Although at any point on the contract curve there exist stationary equilibria in FOLDRs, none of these is stable. In the limiting case $(\delta = 1)$ there exist stationary equilibria, with $a \in (0,1)$, which are semi-stable. Here, any deviation from the stationary output trajectory will be followed by re-convergence, but in general to some other stationary solution, off the contract curve.

So $s(x) \neq 1$ is a requirement for stability of stationary equilibria, notably excluding the contract curve. The following is more specific.

Proposition 2 For there to exist an equilibrium $\sigma = \langle x_0, \rho \rangle$ such that ρ is stable and χ_σ is stationary at $x \in \Omega$, it is necessary and (if each $\pi_j(.)$ is concave) sufficient that $s(x) < \delta^2$.

Proof: We first demonstrate necessity. Assume an equilibrium $\sigma = \langle x_0, \rho \rangle$ such that χ_σ is stationary at *x*. Suppose that $s(x) > 0$. Then from (11) the eigenvalues are real, in which case ρ is stable only if:

$$|a| + \left[a^2 - a_1 a_2 + \frac{s(x)}{\delta^2}(1-\delta a_1)(1-\delta a_2) \right]^{½} < 1$$

which can be re-arranged to:

$$\frac{s(x)}{\delta^2} < \frac{1 + a_1 a_2 - 2|a|}{(1-\delta a_1)(1-\delta a_2)}$$

Stability implies that $a \in (-1,1)$, and therefore that the right-hand side cannot exceed unity.[5] So stability here requires that $s(x) < \delta^2$.

We now demonstrate sufficiency. Given $\delta \in (0,1)$ and $x \in \Omega$ such that $s(x) < \delta^2$, put $x_0 = x$ and construct a stable ρ satisfying (1) and (5). As in the proof of Proposition 1, put:

$$b_i = (1-\delta a_i)\frac{s_j(x)}{\delta} \qquad\qquad c_i = (1-a_i)x_i - b_i x_j$$

Put also $a = 0$ so that, from (11), ρ is stable if:

$$-\delta^2 < \delta^2 a_i^2 + (1-\delta^2 a_i^2)\, s(x) < \delta^2$$

This may be viewed as is a generalisation of (10), and interpreted as requiring the $(\delta^2 a_i^2)$–weighted average of 1 and $s(x)$ to lie within the open interval $(-\delta^2, \delta^2)$. It is straightforward to verify (e.g., graphically) that, given any $\delta \in (0,1)$ and $s(x) < \delta^2$, there always exist $\delta a_i \in (-1,1)$ to satisfy this.

QED

So $s(x) < \delta^2$ is a necessary and (given concavity) sufficient condition for the existence of a stable,

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[5] This can be confirmed by considering the arithmetic difference between the (positive) denominator and the numerator, and verifying that this must be non-negative. For any given *a*, this difference is minimised by setting $a_1 = a_2 = a$, given which it is:

$$2(|a| - \delta a) - (1-\delta^2)a^2$$

This expression is a quadratic in each of its restrictions to, respectively, non-negative and non-positive values of *a*. As such, it is straightforward to verify that it is non-negative for any $a \in (-1,1)$.

stationary equilibrium at *x*. The set of output points satisfying this can be visualised in Figure 2. It comprises the set of stable RF equilibria, plus the two off-diagonal areas in which $s(x) \le 0$. At its smallest, as $\delta \to 0$, it reduces to just these two areas (which together include the Cournot loci). Elsewhere in Ω, i.e., where $s(x) > 0$, it has the same boundaries as the set of stable equilibria in RFs. At its largest, as $\delta \to 1$, these are the contract curve and the anti-contract curve.

6 Subgame-perfection and a boundedly-rational equilibrium

In this section we analyse a boundedly-rational equilibrium, where stability and subgame-perfection are equivalent. An equilibrium is subgame-perfect if the mutually–optimal strategies would remain so, following any deviation in either firm's output in any period. Stability also concerns the consequences of output deviation, but not in any directly normative sense and, in the fully-rational context, we have seen that the two criteria are not generally co-extensive. However, they may be more closely related in the limiting case of zero discounting $(\delta = 1)$. Consider an equilibrium trajectory where the criterion of optimality, for each firm, is its 'long-run' profit. If ρ is stable, then following any deviation output re-converges to the same long-run, mutually-optimal point. A formal proposition of this kind is demonstrated by Stanford (1986b): with zero-discounting, a stable equilibrium in RFs must be subgame perfect. We now define a FOLDR equilibrium in which stability is not only sufficient, but also necessary, for subgame-perfection.

Assume that each firm *j* seeks to maximise:

$$U_j(\chi) \equiv \begin{cases} \pi_j(x) & \text{if } \chi \text{ converges to some } x \in X \\ 0 & \text{otherwise} \end{cases}$$

The first part of this definition is equivalent to a limit-of-mean-profit objective function. This criterion is commonly used, for example by Stanford, to represent zero discounting. Any output sequence with bounded profits is evaluated as the limit of its mean per-period profit. In the case of a convergent sequence, this is just the value to which each period's profit converges.

Although our approach has been to allow unbounded output sequences, it is consistent with this for profit to be bounded, which is all that the limit-of-mean-profit criterion requires. So we could follow Stanford in applying this criterion. However, even when suitably bounded, the computation and evaluation of a non-convergent trajectory can be very difficult. A firm might therefore economise on computational time and expense by simply attaching a zero value to non-convergent output trajectories, as in the second part of the above definition.[6] The first part of the definition could similarly reflect computational constraints, rather than zero-discounting as such, for convergent, but non-stationary, trajectories. So our objective function as a whole could be taken to represent a form of bounded-rationality on the part of the firm.

Given this objective function, the characterisation of a stationary equilibrium at x is identical to that of a convergent equilibrium, i.e., an equilibrium with a trajectory converging to x. In this context, the former is just a special case of the latter. So we can more generally locate output points at which there are convergent equilibria, among which there will always be a stationary equilibrium. Furthermore, for an equilibrium with a stable ρ the initial output vector x_0 is irrelevant. A stable equilibrium can be unambiguously located in output space by the unique stationary solution $\mathrm{S}(\rho)$, in the knowledge that this locates a convergent equilibrium for any x_0. Indeed, this is why such an equilibrium is subgame-perfect. These observations will be elaborated more formally below. Firstly, though, we characterise a boundedly-rational equilibrium, by considering some $\hat{\sigma}=(\hat{\sigma}_1,\hat{\sigma}_2)$ such that $\chi_{\hat{\sigma}}$ converges to $\tilde{x}$. Given $\hat{\sigma}_i$, under what conditions is $\hat{\sigma}_j$ optimal for firm j?

Suppose firstly that $\hat{b}_i=0$, so that firm i's output converges to $\tilde{x}_i$, irrespective of firm j's strategy. Thus any output trajectory which firm j can induce through its own strategy choice will converge, if at all, only to some $x\in \mathrm{X}$ such that $x_i=\tilde{x}_i$. But firm j can induce a trajectory converging to any such x, simply by choosing an appropriate convergent trajectory for its own output, i.e.:

$$a_j\in(-1,1) \qquad b_j=0 \qquad c_j = (1-a_j)x_j$$

[6] This is similar in spirit to an assumption made by Klemperer and Meyer (1989, p.1247), i.e., that firms' payoffs are zero when no unique equilibrium exists for given supply functions.

It follows that $\hat{\sigma}_j$ is optimal, in terms of the boundedly-rational objective function, if and only if $\tilde{x}$ maximises $\pi_j(x)$ subject to $x_i = \tilde{x}_i$. Necessary and (given quasiconcavity) sufficient for this is that $s_j(\tilde{x}) = 0$. The reasoning here closely parallels that in section 4, and it is straightforward to verify the existence of a stationary Cournot equilibrium, strategically identical to that in the fully-rational case. In this boundedly-rational case we may also verify the existence of equilibrium trajectories converging to, but not necessarily stationary at, the Cournot point.

Now suppose instead that $\hat{b}_i \neq 0$. Any trajectory which firm *j* can induce through its own strategy choice will converge, if at all, only to some $x \in S_i(\hat{\rho}_i)$. But firm *j* can induce a trajectory converging to any such *x*, as confirmed in the following Lemma:

Lemma 1 For any ρ_i such that $b_i \neq 0$, and for any $x \in S_i(\rho_i)$, there exists some ρ_j such that $\rho = (\rho_1, \rho_2)$ is stable and $S(\rho) = \{x\}$.

Proof: Given ρ_i and *x*, construct a stable ρ satisfying (1), e.g.:

$$a_j = -a_i \qquad b_j = -a_j^2 / b_i \qquad c_j = (1 - a_j)x_j - b_j x_i$$

QED

Note that $b_i \neq 0$ is crucial to this proposition. If instead $b_i = 0$, as in a Cournot equilibrium, then the eigenvalues are $\{a_1, a_2\}$, so that firm *j* cannot independently ensure stability.

So, given that $b_i \neq 0$, firm *j* can induce a trajectory converging to, and only to, any $x \in S_i(\hat{\rho}_i)$. It follows that $\hat{\sigma}_j$ is optimal, in terms of the boundedly-rational objective function, if and only if $\tilde{x}$ solves:

$$\text{max. } \pi_j(x) \qquad \text{subject to:} \quad x_i = \hat{a}_i x_i + \hat{b}_i x_j + \hat{c}_i$$

Necessary and (given our assumption of quasiconcavity) sufficient for this is that:

$$a_i \neq 1 \quad \text{and} \quad s_j(\tilde{x}) = \frac{\hat{b}_i}{1 - \hat{a}_i} \equiv \hat{\beta}_i \tag{12}$$

which may be compared with (5), the corresponding condition for fully-rational equilibrium. The second part of (12) confirms the equivalence to the limiting case of zero discounting $(\delta = 1)$. However the first part, that $\hat{\rho}_i$ be regular, is considerably weaker than the limiting case of the corresponding part of (5), which is that $\hat{a}_i \in (-1,1)$. This reflects our strong assumption that the boundedly-rational firm places a zero value on any non-convergent trajectory.

Note that (12) also characterises any (convergent) Cournot equilibrium in which $\hat{\rho}_i$ is regular, and therefore where $\hat{\sigma}_j$ is optimal if and only if $\tilde{x}$ maximises $\pi_j(x)$ subject to $x \in S_i(\hat{\rho}_i)$. A Cournot equilibrium in which $\hat{\rho}_i = (1,0,0)$ cannot be thus characterised, since here $S_i(\hat{\rho}_i) = X$.

From (12) we can confirm that stationary, boundedly-rational equilibria are ubiquitous.

Proposition 3 For any $x \in \Omega$ there exists a boundedly-rational equilibrium $\sigma = \langle x_0, \rho \rangle$ such that χ_σ is stationary at x.

Proof: Given x, put $x_0 = x$, and construct ρ to satisfy (1) and (12), i.e.:

$$a_i \neq 1 \qquad b_i = (1-a_i)s_j(x) \qquad c_i = (1-a_i)x_i - b_i x_j$$

QED

Among the equilibria characterised by (12) are those which are (perhaps) non-stationary, but in which $\hat{\rho}$ is stable, with a unique stationary solution at $\tilde{x}$. Such cases are really equilibria in decision rules $(\hat{\rho}_1, \hat{\rho}_2)$ rather than in strategies $(\hat{\sigma}_1, \hat{\sigma}_2)$ as such. Furthermore, they are subgame-perfect. An equilibrium $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2)$ is subgame-perfect if and only if at any future time t, the adopted strategies remain mutually optimal from any feasible position, whether or not on the equilibrium trajectory. In the present context, this requires that at any time t, $\hat{\rho}_j$ is optimal given $\hat{\rho}_i$, and given any feasible x_t. But all features of the market are time-independent, including the objective functions. So this is equivalent simply to requiring that $(\hat{\rho}_1, \hat{\rho}_2)$ are mutually optimal, regardless of x_0.

Proposition 4a If $\hat{\sigma} = \langle \hat{x}_0, \hat{\rho} \rangle$ is a boundedly-rational equilibrium, and if $\hat{\rho}$ is stable, then any $\sigma = \langle x_0, \hat{\rho} \rangle$ is a subgame-perfect boundedly-rational equilibrium.

Proof: If $\hat{\rho}$ is stable then the trajectory of any $\sigma = \langle x_0, \hat{\rho} \rangle$ converges to $\tilde{x}$ which by assumption, for each firm *j*, maximises $\pi_j(x)$ subject to $x \in S_i(\hat{\rho}_i)$. (Note that this applies also to the Cournot equilibrium, where stability entails that $\hat{\rho}$ is regular.) Given $\hat{\rho}_i$, therefore, firm *j* can induce no better trajectory than this. So $\sigma = \langle x_0, \hat{\rho} \rangle$ is a boundedly-rational equilibrium. Since the same argument applies at any subsequent time *t*, for any given x_t, the equilibrium is also subgame-perfect.

QED

This is similar to the proposition demonstrated for RFs under zero-discounting by Stanford (1986b). Given our boundedly-rational objective function, however, the following converse proposition is also true.

Proposition 4b A boundedly-rational equilibrium $\hat{\sigma} = \langle \hat{x}_0, \hat{\rho} \rangle$, with a trajectory converging to $\tilde{x}$, is subgame-perfect only if $\hat{\rho}$ is stable.

Proof: Assume that $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)$ is not stable. We require to show that the equilibrium is therefore not subgame-perfect.

Suppose firstly that $\hat{b}_i \neq 0$ for some firm *i*. If $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)$ is not stable then there is some $x_0 \neq \hat{x}_0$ such that the trajectory of $\sigma = \langle x_0, \hat{\rho} \rangle$ does not converge to $\tilde{x}$. But from Lemma 1 there exists some ρ, with $\rho_i = \hat{\rho}_i$, such that the trajectory of $\sigma = \langle x_0, \rho \rangle$ does converge to $\tilde{x}$. So, given x_0, $\hat{\rho}_j$ is not the best response to $\hat{\rho}_i$. Thus, the equilibrium is not subgame-perfect.

Suppose instead that $\hat{b}_i = 0$ for each firm *i*, and therefore that the eigenvalues are $\{\hat{a}_1, \hat{a}_2\}$. If $\hat{\rho}$ is not stable, then $\hat{a}_j \notin (-1,1)$ for at least one firm *j*. Lemma 1 does not apply here, since the other eigenvalue is $\hat{a}_i$. However, irrespective of the value of $\hat{a}_i$, there is some x_0 such that the trajectory of $\sigma = \langle x_0, \hat{\rho} \rangle$ does not converge to $\tilde{x}$, but would do so were

$a_j \in (-1,1)$. Any $x_0 \neq \tilde{x}$ such that $x_{i,0} = \tilde{x}_i$ is of this type, these being the eigenvectors corresponding to the eigenvalue a_j. So for at least one firm *j* there exists some x_0, given which $\hat{\rho}_j$ is not the best response to $\hat{\rho}_i$. Thus, the equilibrium is not subgame-perfect.

QED

So a boundedly-rational equilibrium $\hat{\sigma} = \langle \hat{x}_0, \hat{\rho} \rangle$ is subgame-perfect, and thus essentially an equilibrium in decision rules, if and only if $\hat{\rho}$ is stable. It now remains to locate the set of such equilibria. We know, from Proposition 3, that a stationary equilibrium can be found at any profitable output point. But this is not so for stable equilibria. There are boundedly-rational equilibria with stationary (and otherwise convergent) trajectories for which $\hat{\rho}$ cannot be stable. Significantly, these occur on the contract curve. Consider an equilibrium $\hat{\sigma} = \langle \hat{x}_0, \hat{\rho} \rangle$, with a trajectory converging to some $\tilde{x}$ on the contract curve, i.e., such that $s(\tilde{x}) = 1$. From (12) it follows that $\hat{\beta}_1 \hat{\beta}_2 = 1$, i.e., that the two firms' stationary sets coincide. So there are infinitely many stationary solutions other than $\tilde{x}$, and $\hat{\rho}$ is not stable. At best, i.e., if $\hat{a} \in (0,1)$, $\hat{\rho}$ is semi-stable. But this does not suffice for subgame-perfection. There exists some x_0 (arbitrarily close to $\hat{x}_0$) such that the trajectory of $\langle x_0, \hat{\rho} \rangle$ converges to some stationary solution other than $\tilde{x}$, and less profitable for at least one firm *j*. But from Lemma 1 we know that, given $\hat{\rho}_i$, there is some $\rho_j \neq \hat{\rho}_j$ which does ensure convergence to $\tilde{x}$. So, given such an x_0, $(\hat{\rho}_1, \hat{\rho}_2)$ are not mutually optimal.

Just as in the fully-rational case, therefore, $s(x) \neq 1$ is necessary for a stable, boundedly-rational equilibrium to be located at *x*. In this case, we may show that it is also sufficient.

Proposition 5 For there to exist a boundedly-rational equilibrium $\sigma = \langle x_0, \rho \rangle$ such that ρ is stable and χ_σ converges to $x \in \Omega$, it is necessary and sufficient that $s(x) \neq 1$.

Proof: Necessity has already been demonstrated. To demonstrate sufficiency, construct a stable ρ which, given x, satisfies (1) and (12). As for Proposition 3, put:

$$b_i = (1-a_i) s_j(x) \qquad c_i = (1-a_i) x_i - b_i x_j$$

Put also $a=0$, so that from (11) ρ is stable if:

$$-1 \;<\; a_i^2 + (1-a_i^2)\,s(x) \;<\; 1$$

Given that $s(x)\neq 1$, it is always possible to find some $a_i \neq 1$ which satisfies this.

QED

So there is a stable boundedly-rational equilibrium located (i.e., with its stationary solution) at any profitable output point except on the contract curve and the anti-contact curve. This near-ubiquity reflects our strong assumption that non-convergent trajectories have zero value, and therefore that each firm's FOLDR need only be regular. By contrast, in the limiting case $(\delta=1)$ of the stationary fully-rational equilibrium the corresponding existence condition is $s(x)<1$, which additionally excludes output points below the contract curve and above the anti-contract curve.

The sets of stable RF and SF boundedly-rational equilibria can similarly be deduced from the limiting cases of their respective stationary, fully-rational counterparts. For RFs this is straightforward; in Figure 2 the relevant set is represented by the shaded region. For SFs note that the second condition in (12) requires that, for each firm *i*:

$$a_i \;=\; b_i = \frac{s_j(x)}{1+s_j(x)}$$

Given which the first condition in (12) is automatically satisfied. So, in the boundedly-rational case, the (linear) boundaries shown in Figure 3 do not apply. The set of stable SF equilibria is just the curved region bounded below by the contract curve.

To summarise: we have established that stable, boundedly-rational equilibria in FOLDRs can be found almost anywhere in (positive-profit) output space. Excluded, however, are points of mutual profit-maximisation on the contract curve. There are boundedly-rational equilibria with outputs stationary at such points, but these are not stable and, therefore, not subgame-perfect.

References

Abreu D, (1986); “Extremal equilibria of oligopolistic supergames”, *Journal of Economic Theory* 39:191-225

Friedman J, (1971); “A noncooperative equilibrium for supergames”, *Review of Economic Studies* 38:1-12

Friedman J, and L Samuelson, (1994); “Continuous reaction functions in duopolies”, *Games and Economic Behavior* 6:55-82

Grossman S, (1981); “Nash equilibrium in the industrial organization of markets with large fixed costs”, *Econometrica* 49:1149-77

Kelmperer P, and M Meyer, (1988); “Consistent conjectures equilibria”, *Economics Letters* 27:111-15

Kelmperer P, and M Meyer, (1989); “Supply function equilibria in oligopoly under uncertainty”, *Econometrica* 57:1243-77

Robson A, (1986); “The existence of Nash equilibria in reaction functions for dynamic models of oligopoly”, *International Economic Review* 27:539-44

Stanford W, (1986a); “Subgame perfect reaction function equilibria in discounted duopoly supergames are trivial”, *Journal of Economic Theory* 39:226-232

Stanford W, (1986b); “On continuous reaction function equilibria in duopoly supergames with mean payoffs”, *Journal of Economic Theory* 39:233-250