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Correcting Market Failure Due to Interdependent Preferences: When Is Piecemeal Policy Possible?

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Abstract

Generally, implementation of Pigovian taxes to correct for market failure requires an enormous set of information. For each commodity-person combination a different tax is required to correct the resulting market inefficiency. In this paper, we analyse interdependent preferences and inefficiency of the market solution with the aim of finding conditions justifying simple rules for such taxes. We examine the utility possibility curve and Scitovsky community indifference curve, allowing for general utility interdependence and agent heterogeneity. In particular we show the equivalence of taxes derived from the Marshallian and compensated demand approaches. We move on to analyse the welfare cost of consumption externalities and show that it decomposes into part due to individuals choosing suboptimal quantities and part due to individuals using valuations that are not socially optimal. We show what forms of externality can justify simple policy corrections. In particular, we analyse the conditions which are required for the market failure to be corrected by: 1) specific indirect ad valorem taxes on commodities, 2) the same proportional tax rate on every commodity, 3) a proportional income tax rate on each individual. The conditions are related to the restrictions necessary to have H synthetic consumers without externalities who replicate behaviour of individuals with externalities. An example with two individuals and three goods concludes the paper.

Keywords: Consumption externalities, Piecemeal policy.

JEL classification: D62, D11.

1 Introduction

Usually some taxation method is suggested for the correction of externalities. These ideas are based on Pigovian transfers and Lindahl pricing in which the taxes serve to replace private marginal rates of substitution by social marginal rates of substitution (Lindahl (1919), Bergstrom (1970), Milleron (1972),). However in general different commodity and person specific taxes are necessary in each market and the appropriate tax rate depends on the particular Pareto optimal allocation of commodities which is under consideration. The tax authority needs full information on preferences and technology to implement Pigovian taxes, Lindahl distributive mechanisms do not require this but require restricted forms of externality to be able to achieve efficiency (Bergstrom (1977), Tian (2004)).

Hence there is interest in finding conditions under which simple taxes can be used. In reality we have a mixture of personal income taxation, general sales taxes at a more or less uniform rate (VAT) and specific commodity indirect taxes e.g. on alcohol and tobacco. Because these tax rates are not differentiated by

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personal spending patterns, they are feasible to administer. Some research tries to define conditions under which piecemeal policy is possible, in which the appropriate correction in one market is independent of activity in other markets. There is some looseness in this idea-for example if only one market is subject to externalities, it means that taxation is directed only to transactions in that market. If several markets are subject to consumption externalities it might mean that the taxes necessary to restore Pareto optimality in one market are unaffected by exogenous shocks in other markets e.g. supply shocks. In the context of a distorted sector in which consumer and producer prices differ, Jewitt (1981) and Blackorby, Donaldson & Schworm (1991) find that piecemeal policy is only justified if the set of efficient points can be described by a relation between two aggregates of commodities corresponding to the distorted and non-distorted sectors. However the reasons for the distortion are not modelled. In particular, the distortion is not an endogenous function of quantities as it is with consumer externalities.

Characterising corrective tax rules with general interdependent preferences and finding what sort of preference restrictions must be imposed if the necessary taxes are to take simple forms are still open research issues. Kooreman & Schoonbeek (2004) analyse consumption externalities in a setting with a fixed income distribution. After showing that Pareto improvements are generally possible starting from a market solution with consumption externalities[1], they consider an example in which individual preferences have the linear expenditure form[2] and the consumption externalities enter through the subsistence terms. Imposing particular assumptions (identical preferences, equal incomes, the subsistence term for commodity i depends only on the total consumption of that commodity i by all other consumers) they compute the welfare losses due to externalities as the deviation from an equal utility distribution Pareto optimum and then compute the commodity taxes that will make the market solution and Pareto optimum coincide.

In this paper we explore the implications of general interdependent preferences and the theoretical properties which are required if piecemeal policy is to be possible. With general preferences, we start by showing that a Pareto optimum corresponding to a given utility distribution can be reached through a decentralised market system using Pigovian taxes on compensated demands with consumption externalities. We characterise the system of Pigovian taxes required for this task. Then we show that a Pareto optimum corresponding to a given income distribution can also be achieved as a market solution from Marshallian demands with Pigovian taxes. We then show that the compensated and Marshallian approaches are equivalent which gives us a measure of the welfare cost of the externalities defined in terms of the expenditure functions of the

[1] So long as there is at least one good such that all individuals value the consumption of that good by each other individual at less than its market value.

[2] Following up a suggestion of Pollak(1970) and Kapteyn et al (1997).

individuals. This cost decomposes into part due to the wrong pricing of goods in the market solution, and part due to individuals consuming the wrong quantities to implement the Pareto optimum. The equivalence of the compensated and Marshallian approaches is also of interest because the market solution is a Nash equilibrium in mutual best responses but the strategic interpretation of the two is different since in one case the individuals environment has prices and income, and in the other prices and a given utility level.

Next we try to analyse when piecemeal policy is justified and proceed by defining necessary conditions on preferences under which the individual compensated demand for any good under externalities depends only on prices, individual utility and the consumption by other individuals of that particular good. This is the scenario in which it most likely that simple taxes will succeed. It turns out that the externalities **must** enter individual preferences and expenditure functions as a form of subsistence level/cost. In a sense the linear expenditure type system with the form of externalities Kooreman & Schoonbeek use is one of the forms that must prevail if piecemeal policy is to be justified.

Given this form of preferences we find further restrictions under which simple taxes will work. In particular if externalities take the form of varying in a linear way with the total consumption of all other individuals of each good (we call this case linear popular no spillover externalities) then when:

- Only one good has an external effect, the Pigovian taxes on any individual are identical for other goods. However for the externality inducing good the tax on any individual depends on the relative strength of externalities between other individuals.

- For every individual the strength of externality is equal for each good then for any pair of goods the ratio of the tax rates of the two goods is equal for all individuals (and it is equal to the strength of the externality on the two goods). If the social and private marginal valuations of individuals coincides then for each good every individual faces the same Pigovian tax rate, it is as if there are specific indirect ad valorem taxes on commodities

- For each individual there is a common strength of externality for every good which differs by individual: each individual pays the same proportional tax rate on every commodity so this is equivalent to a proportional income tax rate.

- For every individual and every commodity there is a common strength of externality. Here the proportional income tax rate of each individual is actually at the same rate for all individuals.

Interestingly in this case the compensated Nash equilibrium aggregate demands will also tend to satisfy the usual Slutsky symmetry and negative semidefiniteness restrictions so that in the aggregate it may be as if there are synthetic consumers whose behaviour without externalities replicates aggregate demand. Finally we illustrate the results by computing the compensated and Marshallian Pareto optima, Nash equilibria, Pigovian taxes and the social cost of the externalities through an example with two individuals and three commodities which is designed to show the relations between the general concepts explicitly.

The plan of the paper is to review the dual of the Pareto optimality problem without externalities (Section 3). We then use this dual formulation in the presence of consumption externalities to define the appropriate taxes in general and then show how individuals facing these taxes interact to produce a Nash equilibrium which is Pareto optimal. First we compute the Pigovian taxes required to ensure that the market solution (a compensated Nash equilibrium) yields a Pareto optimum with a given utility distribution (Section 4.2). Then we consider Pigovian taxation with interdependent preferences and fixed welfare weights (Section 4.3). We establish links between the compensated and the Marshallian approach (Section 4.4). We compute the welfare cost of the externalities (Section 4.5). Next we discuss the piecemeal policy issue (Section 5). An example with 3 goods and 2 individuals concludes the paper.

2 The Setting

We work with H individuals indexed h; h has preferences given by $u_h(x_h)$ in the absence of externalities where x_h is a consumption allocation of n commodities. With externalities we write $u_h(x_h, x_{\sim h})$ where $x_{\sim h}$ is an ordered list of the consumption allocation of all individuals other than h. We represent the resource constraint of the system by a linear function

$$\sum_i p_i \sum_h x_{ih} = y$$

We can interpret this as a linear transformation locus for the economy, in this case p_i represents both the market price and average cost of the ith commodity which is in perfectly elastic supply. Alternatively it may represent a market budget constraint for a group of H individuals, so p_i is the market price for good i and y represents the disposable resources of the group. For example the group may be a family or a team within an organisation. In the family case we think of a family model in which the H individuals are each family members and the family has exogenous resources y which can be allocated to purchase consumption goods for different family members.

3 Pareto Optimality Without Externalities

When there are no externalities, the problem of finding a Pareto optimal allocation is related to the ideas of utility possibility curves and Scitovsky community preference fields Gorman (1953). Given a fixed utility distribution $(\overline{u}_1, ..\overline{u}_H)$, the Scitovsky community indifference curve is:

$$X(\overline{u}_1, ..\overline{u}_H) = \min[X_1, ..X_n | u_h(x_h) \geq \overline{u}_h, \sum x_h \leq X]. \tag{1}$$

giving the minimum amounts of the aggregate quantities of goods that place each individual on his prescribed indifference curve. (Here min and max operators are understood in a vector sense). For each set of aggregate quantities a distribution of commodities between consumers is implicitly defined which just allows attainment of the utility distribution.

Define the utility possibility curve for fixed aggregate quantities of goods $(X_1, ..X_m)$ by:

$$U(X_1, ..X_m) = \max[u_1, ..u_n | U_h(x_h) \geq u_h, \sum x_h \leq X] \tag{2}$$

The utility possibility curve indicates the maximum level of utility that an individual can achieve given the utility level of the others.

Gorman(1953) shows that $X = (X_1, ..X_m) \in X(\overline{u}_1, ..\overline{u}_H)$ iff $(\overline{u}_1, ..\overline{u}_H) \in U(X_1, ..X_m)$. With the linear resource constraint (2) becomes

$$U(p, y) = \max[u_1, ..u_n | U_h(x_h) \geq \overline{u}_h, \sum x_h \leq X, \sum p_i X_i \leq y] \tag{3}$$

Under regularity conditions[3] a point $\overline{u}_1, ..\overline{u}_H$ is in $U(p, y)$ iff the group resources are just sufficient to reach this utility distribution. Analogously to (1) we have

$$\begin{aligned} y &= G(p, \overline{u}_1, ..\overline{u}_H) \\ &= \min[\sum p_i X_i | U_h(x_h) \geq \overline{u}_h, \sum_h x_{ih} \leq X_i] \\ &= \min[\sum p_i \sum x_{i_h} | U_h(x_h) \geq \overline{u}_h, h = 1..H] \\ &= \sum g_h(p, \overline{u}_h) \qquad (4) \end{aligned}$$

where $g_h(p, \overline{u}_h)$ is the expenditure function of individual h. This implies that the group can decentralise the task of attaining a particular Pareto optimal utility distribution by allocating y_h of the group resources to h and leaving h to make their own choices. From a version of Hotelling's lemma we can define the aggregate compensated demand

$$F_i(p, \overline{u}_1, ..\overline{u}_H) = \partial G(p, \overline{u}_1, ..\overline{u}_H)/\partial p_i = \partial \sum g_h(p, \overline{u}_h)/\partial p_i$$

and so the aggregate compensated demands inherit the properties of the individual compensated demands; in particular they have a symmetric and negative semidefinite Jacobian with respect to p and are homogeneous of degree zero in p.

[3] $u(.)$ is continuous, strictly quasiconcave and locally nonsatiated.

4 Pareto Optimality with Externalities

4.1 With a fixed utility distribution

With general externalities[4] $u_h(x_h, x_{\sim h})$ the problem of attaining a Pareto optimum has the form

$$\min[\sum p_i \sum x_{i_h} | u_h(x_h, x_{\sim h}) \geq \overline{u}_h, h = 1..H]$$

The first order conditions[5] are

$$\begin{aligned} p_i &= \sum_k \lambda_k^c \frac{\partial u_k}{\partial x_{ih}} && (5) \\ u_h(x_h, x_{\sim h}) &= \overline{u}_h, h - 1..H && (6) \end{aligned}$$

where λ_k^c is the marginal social cost of individual k. That is all marginal effects of an individuals consumption must be taken into account so that in particular (4) is lost.

4.1.1 Implementation of a compensated Pareto optimum

If each individual were set the task of reaching a fixed utility level at minimum cost

$$\min[\sum p_i x_{i_h} | u_h(x_h, x_{\sim h}) \geq \overline{u}_h] \quad (7)$$

they would set

$$\begin{aligned} p_i &= \mu_h^c \frac{\partial U_h(x_h, x_{\sim h})}{\partial x_{ih}} && (8) \\ U_h(x_h, x_{\sim h}) &= \overline{u}_h \end{aligned}$$

and would ignore the effect of their consumption on others. Solving (8) for one individual gives the compensated reaction curves

$$x_{ih}^c = X_{ih}^c(p, \overline{u}_h, x_{\sim h})$$

and solving these equations in turn yields the compensated Nash equilibrium (CNE) demands :

$$x_{ih}^{NEc} = X_{ih}^{NEc}(p, \overline{u}_1, ..\overline{u}_H) \quad (9)$$

However we could introduce individual and commodity specific Pigovian taxes π_{ih} so that the cost to h of a unit of good i becomes $p_i \pi_{ih}$ and (8) becomes

$$\begin{aligned} p_i \pi_{ih}^c &= \mu_h^c \frac{\partial u_h}{\partial x_{ih}} && (10) \\ U_h(x_h, x_{\sim h}) &= \overline{u}_h. \end{aligned}$$

[4] We continue to assume $u_h()$ is strictly quasi concave and nonsatiated in x_h, and continuous in all variables.
[5] Given our assumptions these are also sufficient.

or

$$\begin{aligned} \frac{p_i \pi^c_{ih}}{p_n \pi^c_{nh}} &= \frac{\partial u_h / \partial x_{ih}}{\partial u_h / \partial x_{nh}} \qquad i = 1..n-1 \\ U_h(x_h, x_{\text{-}h}) &= \overline{u}_h. \end{aligned} \tag{11}$$

The compensated Pigovian taxes must be selected to yield (5) which leads to (see Appendix A.1):

$$\frac{\pi^c_{ih}}{\pi^c_{nh}} = \sum_k \left(\lambda^c_k \frac{\partial u_k}{\partial x_{nk}} / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}} \right) \left(\frac{\partial u_k}{\partial x_{nh}} / \frac{\partial u_k}{\partial x_{nk}} \right) - \sum_{k \neq h} \left(\lambda^c_k \frac{\partial u_k}{\partial x_{nk}} / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}} \right) \frac{\pi^c_{ik}}{\pi^c_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}} \right) \tag{12}$$

which in turn can be written for good i as

$$\begin{bmatrix} \pi^c_{i1}/\pi^c_{n1} \\ \vdots \\ \pi^c_{iH}/\pi^c_{nH} \end{bmatrix} = \begin{bmatrix} p_n \left[\sum_k \eta^i_{1k}\right] / \left[\lambda^c_1 \partial u_1/\partial x_{n1}\right] \\ \vdots \\ p_n \left[\sum_k \eta^i_{Hk}\right] / \left[\lambda^c_H \partial u_H/\partial x_{nH}\right] \end{bmatrix} \tag{13}$$

where η^i_{hk} is the hkth element of the inverse of the HxH matrix

$$\begin{bmatrix} 1 & \partial u_2/\partial x_{i1}/\partial u_2/\partial x_{i2} & ... & \partial u_H/\partial x_{i1}/\partial u_H/\partial x_{iH} \\ \partial u_1/\partial x_{i2}/\partial u_1/\partial x_{i1} & 1 & ... & \vdots \\ \vdots & ... & ... & \partial u_H/\partial x_{iH-1}/\partial u_H/\partial x_{iH} \\ \partial u_1/\partial x_{iH}/\partial u_1/\partial x_{i1} & \partial u_2/\partial x_{iH}/\partial u_2/\partial x_{i2} & ... & 1 \end{bmatrix}$$

This matrix will be nonsingular if at the Pareto optimal point in question the marginal externalities for good i are independent-there is sufficient diversity between individuals for no one individual to be affected by externalities in a way which is a linear combination of the external effects for other individuals.

It is evident that for each individual h, the Pigovian taxes π_{ih} are only determined up to a factor of proportionality, they serve to equate the social and private marginal rates of substitution[6]. Substituting (13) into (11) gives (5). This means that if we solve all the $(n+1)H$ equations in (11) for the unknowns x_{ih} and λ^c_h with π^c_{ih} defined by (13) then it is equivalent to solving the system of equations (5). In other words a Nash equilibrium in which each individual faces the corrected prices, given consumption of the other individuals and a given utility level replicates the socially optimal way of achieving the same utility distribution.

If these taxes are used as fixed numbers then decentralised choice will lead to individuals choosing the Pareto optimal consumption bundle so long as the decentralised choice problem remains well defined. One difficulty is that the tax could be non-positive which would make the effective market price of the good in question negative. This could occur if x_{ih} has such a strong positive effect on the utility of other individuals that it is efficient to pay h per unit of consumption of i. But then since h is nonsatiated, there will be no decentralised solution-h will choose an infinite consumption of good i. We rule this out. Apart from

[6]So they can be scaled to yield zero tax revenue in aggregate.

this case, the decentralised choice problem is well behaved under Pigovian taxation given the regularity assumptions-the first order conditions will characterise the individuals best choice.

4.2 With fixed welfare weights

Instead of obtaining a Pareto optimum through minimising the aggregate cost of financing the utility distribution, the Pareto optimum could be defined by maximising a linear combination of individual utilities subject to the aggregate budget constraint.

$$\begin{aligned} & \max \sum \beta_h U_h(x_h, x_{\sim h}) && (14) \\ s.t. \quad : \quad & \sum \sum p_i x_{ih} = y \end{aligned}$$

leading to

$$\begin{aligned} \lambda^m p_i &= \sum_k^m \beta_h \frac{\partial U_k}{\partial x_{ih}} && (15) \\ \sum_h \sum_i p_i x_{ih} &= y \end{aligned}$$

The corresponding individual problem for some income distribution $(y_1, ...y_H)$ is

$$\begin{aligned} & \max U_h(x_h, x_{\sim h}) && (16) \\ s.t. \quad : \quad & \sum p_i x_{ih} = y_h \end{aligned}$$

Solving (16) for one individual gives the Marshallian reaction curves (Kooreman & Schoonbeek)

$$x_{ih}^m = X_{ih}^m(p, y_h, x_{\sim h})$$

and solving these equations in turn yields the Marshallian Nash equilibrium (MNE) demands:

$$x_{ih}^{NEm} = X_{ih}^{NEm}(p, y_1...y_H) \tag{17}$$

There are then two reasons for lack of equivalence between the Pareto optimum and the market solution. Firstly private and social marginal rates of substitution (mrs) differ so in the market solution individuals ignore the effects of their consumption on others; secondly the income distribution may not align with the particular welfare weights that are being used.

4.2.1 Implementation of a Marshallian Pareto optimum

If we introduced Pigovian taxes on each person and good π_{ih}^m, the first order conditions in the market solution would become

$$\begin{aligned} \frac{p_i \pi_{ih}^m}{p_n \pi_{nh}^m} &= \frac{\partial u_h / \partial x_{ih}}{\partial u_h / \partial x_{nh}} \qquad i = 1..n-1 \\ \sum_i p_i x_{ih} &= y_h \end{aligned} \tag{18}$$

and if we want these to replicate (15) the taxes must satisfy (see Appendix A.2)

$$\frac{\pi_{ih}^m}{\pi_{nh}^m} = \sum_k \beta_k \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{nh}} / \frac{\partial u_k}{\partial x_{nk}} \right) / \beta_h \frac{\partial u_h}{\partial x_{nh}} - \sum_{k \neq h} \beta_k \frac{\pi_{ik}^m}{\pi_{nk}^m} \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}} \right) / \beta_h \frac{\partial u_h}{\partial x_{nh}} \tag{19}$$

which for good i can be written

$$\begin{bmatrix} \pi_{i1}^m / \pi_{n1}^m \\ \vdots \\ \pi_{iH}^m / \pi_{nH}^m \end{bmatrix} = \begin{bmatrix} \lambda^m p_n \left[\sum_k \eta_{1k}^i \right] / \left[\beta_1 \partial u_1 / \partial x_{n1} \right] \\ \vdots \\ \lambda^m p_n \left[\sum_k \eta_{Hk}^i \right] / \left[\beta_H \partial u_H / \partial x_{nH} \right] \end{bmatrix} \tag{20}$$

where the η's have the same definition as previously. Thus the Marshallian Pigovian taxes coincide with the compensated Pigovian prices if

$$\beta_h / \lambda^m = \lambda_h^c$$

In addition the income distribution must be chosen to match the welfare weights i.e. if we define the individual utility levels achieved in the Marshallian Pareto optimum and in the market solution with Marshallian Pigovian taxes by $v_h^{PO}(p, y, \beta_1, ...\beta_H)$ and $v_h^m(\pi^m \cdot p, y_1, ..y_H)$ respectively then $y_1..y_H$ must be chosen so that

$$v_h^{PO}(p, y, \beta_1, ...\beta_H) = v_h^m(\pi^m \cdot p, y_1, ..y_H) \qquad h = 1..H \tag{21}$$

giving $y_h(\pi^m, p, y, \beta_1, ...\beta_H)$. To ensure that all resources are exactly consumed, the taxes must be scaled to give zero tax revenue.

4.3 Equivalence between Marshallian and Compensated Nash Equilibria

There are links between the compensated and Marshallian Nash equilibria. From standard theory we know that if

$$\begin{aligned} g_h(p, \overline{u}_h, x_{\tilde{}h}) &= \min_{x_h} \{ p \cdot x_h u_h(x_h, x_{\tilde{}h}) = \overline{u}_h \} \\ v_h(p, y_h, x_{\tilde{}h}) &= \max_{x_h} \{ u(x_h, x_{\tilde{}h}) | p x_h = y_h \} \end{aligned}$$

then as identities in $p, \overline{u}_h, y_h, x_{\sim h}$

$$
\begin{aligned}
g_h(p, v_h(p, y_h, x_{\sim h}), x_{\sim h}) &= y_h \\
v_h(p, g_h(p, \overline{u}_h, x_{\sim h}), x_{\sim h}) &= \overline{u}_h
\end{aligned}
$$

Thus if y_h gives a maximal utility level of $\overline{u}_h$ then y_h is the minimal cost incurred to reach the utility level $\overline{u}_h$. Setting $x_{\sim h}$ to be its Nash equilibrium demand function

$$
\begin{aligned}
g_h(p, v_h(p, y_h, X^{NEm}_{\sim h}(p, y_1..y_H)), X^{NEm}_{\sim h}(p, y_1..y_H)) &= y_h = p \cdot F_h(p, y_1..y_H) \\
v_h(p, g_h(p, \overline{u}_h, X^{NEc}_{\sim h}(p, \overline{u}_1...\overline{u}_H)), X^{NEc}_{\sim h}(p, \overline{u}_1...\overline{u}_H)) &= \overline{u}_h
\end{aligned}
$$

We also know from standard theory that the Marshallian and compensated demands satisfy associated identities:

$$
\begin{aligned}
X^c_{ih}(p, \overline{u}_h, x_{\sim h}) &= X^m_{ih}(p, g_h(p, \overline{u}_h, x_{\sim h}), x_{\sim h}) \\
X^m_{ih}(p, y_h, x_{\sim h}) &= X^c_{ih}(p, v_h(p, y_h, x_{\sim h}), x_{\sim h})
\end{aligned}
$$

so evaluated at the relevant Nash equilibrium demands

$$
\begin{aligned}
X^c_{ih}(p, \overline{u}_h, x_{\sim h}) &= X^m_{ih}(p, g_h(p, \overline{u}_h, X^{NEm}_{\sim h}(p, g_1..g_H)), X^{NEm}_{\sim h}(p, g_1..g_H)) \\
X^m_{ih}(p, y_h, x_{\sim h}) &= X^c_{ih}(p, v_h(p, y_h, X^{NEc}_{\sim h}(p, v_1...v_H)), X^{NEc}_{\sim h}(p, v_1...v_H))
\end{aligned}
$$

From this we deduce that

(i) the Nash equilibrium demands for each good and individual in the CNE with $p, k_1...k_H$ have identical values to the MNE demands with p and individual incomes y_h set to the costs of each individual h of purchasing the goods in the CNE (i.e. g_h)

(ii) the Nash equilibrium demands for each good and individual in the MNE with $p, y_1...y_H$ have identical values to the CNE demands with p and individual utilities $\overline{u}_h$ set to the utility levels of each individual h in the MNE

Thus for any CNE there is an associated income distribution generating a MNE which yields the utility distribution of the CNE with identical demands for each individual and good.

From this we can deduce equivalence of the Marshallian and compensated Pareto optima since these can be represented as Nash equilibria with Pigovian taxes. Comparing(11) and (18) the solutions will coincide if there are suitable links between the utility distribution, the aggregate resources, the income distribution and the welfare weights. There is a consumption allocation x_{ih} which solves (11) and attains the utility distribution $\overline{u}_1 ... \overline{u}_H$ at minimum aggregate cost. Setting $y_h = \Sigma_i p_i x_{ih}$ the allocation x_{ih} also solves (18). Then with $y = \Sigma y_h$ there are values of β_h for which the allocation x_{ih} also solves.(15). This reflects Gorman's result that the consumption allocation is in the Scitovsky community indifference curve if and only if the utility distribution defining the Scitovsky community indifference curve is attainable from that consumption allocation. This result extends to the case of externalities. There is also a form of (4)

$$\begin{aligned} G(p, \overline{u}_1, ..\overline{u}_H) &= \sum_{ih} \pi^c_{ih} p_i X^{NEc}_{ih}(\pi^c \cdot p, \overline{u}_1, ..\overline{u}_H) \\ &= \sum_h g_h(\pi^c \cdot p, \overline{u}_h, X^{NEc}_{\text{-}h}(\pi^c \cdot p, \overline{u}_1, ..\overline{u}_H)) \end{aligned}$$

In the sequel we concentrate on the compensated demand scenario because of the equivalence between this and the Marshallian scenario outlined above.

The Pigovian taxes are endogenous in that they depend on all quantities consumed by each individual in the particular Pareto optimum. They vary by both commodity and individual.

There are some general properties of the Pigovian taxes: using

$$\frac{\pi^c_{ih}}{\pi^c_{nh}} = \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{nh}} / \frac{\partial u_k}{\partial x_{nk}} \right) / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}} - \sum_{k \neq h} \lambda^c_k \frac{\pi^c_{ik}}{\pi^c_{nk}} \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}} \right) / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}}$$

suppose that there are no external effects in good n

$$\frac{\partial u_k}{\partial x_{nh}} = 0 \qquad k \neq h$$

Then

$$\sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{nh}} / \frac{\partial u_k}{\partial x_{nk}} \right) / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}} = 1$$

and

$$\frac{\pi^c_{ih}}{\pi^c_{nh}} = 1 - \sum_{k \neq h} \lambda^c_k \frac{\pi_{ik}}{\pi_{nk}} \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}} \right) / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}} \tag{22}$$

Second suppose that in addition good i has no external effects so that

$$\frac{\partial u_k}{\partial x_{ih}} = 0 \qquad k \neq h$$

The Pigovian taxes become

$$\frac{\pi^c_{ih}}{\pi^c_{nh}} = 1$$

so no correction is required in the ith market.

4.4 The Welfare Cost Of the Externality

We can use the framework of consumer surplus to compute the welfare cost of consumption externalities. Private choices have a cost to h of

$$g_h(p, \overline{u}_h, x_{\tilde{}h}) = \min[\sum p_i x_{i_h} | U_h(x_h, x_{\tilde{}h}) \geq u_h]$$

which in the Nash equilibrium is $g_h(p, \overline{u}_h, x^{NE}_{\tilde{}h}) = \widetilde{g}_h(p, \overline{u}_1, ...\overline{u}_H)$. The aggregate cost of attaining $\overline{u}_1, ...\overline{u}_H$ with decentralised decisions is then

$$\sum \widetilde{g}_h(p, \overline{u}_1, ...\overline{u}_H) = \sum_{ih} p_i x^{NE}_{ih}$$

With Pigovian taxes, the efficient way of attaining $\overline{u}_1, ...\overline{u}_H$ has a cost of

$$\sum g_h(\pi p, \overline{u}_h, x^{PO}_{\tilde{}h}) = \sum_{ih} \pi_{ih} p_i x^{NE}_{ih}(\pi_{ih} p_i)$$

Hence the welfare cost can be measured by

$$\begin{aligned} C &= \sum g_h(\pi p, \overline{u}_h, x^{PO}_{\tilde{}h}) - \sum \widetilde{g}_h(p, \overline{u}_1, ...\overline{u}_H) \qquad (23) \\ &= \sum_{ih} p_i x^{PO}_{ih} - \sum_{ih} p_i x^{NE}_{ih}(p_i) \\ &= \sum_{ih} p_i (x^{PO}_{ih} - x^{NE}_{ih}(\pi_{ih} p_i)) + \sum_{ih} p_i (x^{NE}_{ih}(\pi_{ih} p_i) - x^{NE}_{ih}(p_i)) \end{aligned}$$

The welfare cost can be decomposed into part corresponding to the resource misallocation arising from the incorrect choice of quantities and part arising from the misvaluation of commodities. In Figure 1 the two parts of the welfare cost are outlined in bold.

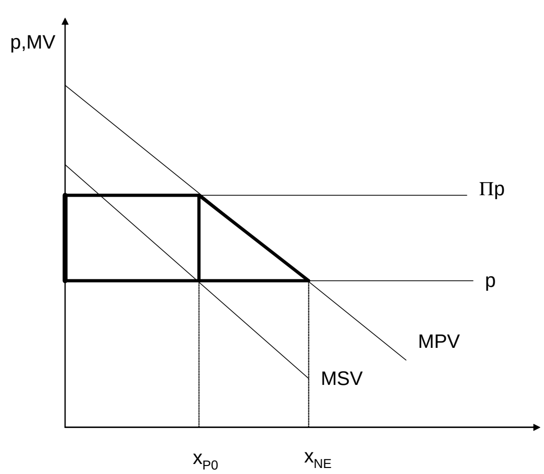

Figure 1: Welfare cost decomposition

5 Piecemeal Policy

5.1 Compensated Nash Equilibrium with No Spillovers

Piecemeal policy involves the idea of being able to correct for market failure in one market in a way which is independent of conditions in other markets. For example, if there is only one commodity which exhibits externalities then piecemeal policy is possible if the Pigovian taxes in all the other markets are equal to zero. More generally, piecemeal policy is valid if the Pigovian taxes in one market are invariant to changes in conditions (in either preferences or prices) in other markets, or in wealth. Both these ideas involve some notion of independence of the external effect in different markets.

A natural place to start is to consider the case in which any individual demand for any good only has external effects corresponding to consumption of the same good by other individuals. For example one consumers spending on say mobile phones is only influenced by the behaviour of others through their spending on mobile phones[7]. This is close to a necessary condition for piecemeal policy to be possible. From (13) for π_{ih}/π_{nh} to be independent of quantities consumed of goods other than i, n requires that $(\partial u_k/\partial x_{ih})/(\partial u_k/\partial x_{ik})$ be independent of $x_{jk}, j \neq i, n$ meaning that $u_k()$ in Nash equilibrium is separable in commodities.

Define *externalities with no commodity spillovers* to exist if each individuals compensated reaction curve for any good i depends only on prices, utility and the consumption of other individuals of that good

$$x_{ih}^c = X_{ih}^c(p, \overline{u}_h, x_{i1}, ..x_{ih-1}, x_{ih+1}, ..x_{iH}) \tag{24}$$

The first question is then what are the individual preferences that generate this form of compensated demand? Since $X_{ih}^c(\cdot)$ must be homogeneous of degree zero in p, it follows that we must be able to write the expenditure function as

$$g_h(p, \overline{u}_h, x_{\text{-}h}) = \sum p_i X_{ih}^c(\cdot)$$

and moreover by differentiating (24) and appealing to Hotelling's rule, that

$$\frac{\partial X_{ih}^c}{\partial p_j} = \frac{\partial X_{jh}^c}{\partial p_i}$$

[7] This is impossible in the case of Marshallian demands: it would make each individuals demand for any good depend on prices, individual income and the demand for that good by each other individual.

$$\widehat{x}_{ih} = f_{ih}(p, y_h, x_{i1}, ..x_{ih-1}, x_{ih+1}, ..x_{iH})$$

However this is inconsistent with the budget constraint; to hold for all p, M_h needs the identity

$$\sum p_i f_{ih}(p, y_h, x_{i1}, ..x_{ih-1}, x_{ih+1}, ..x_{iH}) = y_h$$

and differentiating through wrt x_{ik} implies that $\partial f_{ih}/\partial x_{ik} = 0$ identically or that there can actually be no externality. Essentially all goods compete for consumer income.

so that for $i \neq j$ $\partial X^c_{ih}/\partial p_j$ must be independent of the externality effects $x_{i1}, ..x_{ih-1}, x_{ih+1}, ..x_{iH}$ because the LHS must be independent of $x_{i1}..x_{iH}$ and the RHS independent of $x_{j1}..x_{jH}$. That is $\partial^2 X^c_{ih}/\partial p_j \partial x_{ik} = 0$ so that

$$X^c_{ih}(\cdot) = \phi_{ih}(p_i, \overline{u}_h, x_{i1}, ..x_{ih-1}, x_{ih+1}, ..x_{iH}) + \psi_{ih}(p, \overline{u}_h)$$

Integrating this over p_i leads to an individual expenditure function of the form

$$g_h(p, \overline{u}_h, x_{\char126 h}) = \sum_i A_{ih}(p_i, \overline{u}_h, x_{i1}, ..x_{ih-1}, x_{ih+1}, ..x_{iH}) + B_h(p, \overline{u}_h) \tag{25}$$

so that $h's$ compensated demands have the form

$$x_{ih} = \frac{\partial A_{ih}(p_i, \overline{u}_h, x_{i1}..x_{iH})}{\partial p_i} + \frac{\partial B_h(p, \overline{u}_h)}{\partial p_i} \tag{26}$$

This has the implication that in the compensated demands there are no spillovers of externalities between commodities. We need $A_i(\cdot)$ to be homogeneous of degree one in p_i and so effectively it is linear in p_i;and $B(\cdot)$ also to be homogeneous of degree one in p. Imposing this leads to

$$g_h(p, \overline{u}_h, x) = \sum_i p_i \widetilde{A}_{ih}(\overline{u}_h, x_{i1}, ..x_{ih-1}, x_{ih+1}, ..x_{iH}) + B_h(p, \overline{u}_h) \tag{27}$$

This has a relation to a Klein-Rubin linear expenditure system of preferences; in fact it extends this by allowing the subsistence level to vary with the standard of living of the consumer $\overline{u}_h$. Interestingly Pollak (1976), Kapteyn et al (1997) and Kooreman & Schoonbeek (2004) focus on interdependence in the Klein-Rubin utility function, interpreting the subsistence parameter as a linear function of the quantities consumed by other individuals.

5.1.1 The Linear No Spillover Case

A special form of (27) makes the externalities work only through a linear combination of the consumption of others (Kapteyn et al (1997)).

$$u_h(x_h, x_{\char126 h}) = u_h(x_{1h} + \sum_{k \neq h} w_{1kh} x_{1k}, .., x_{nh} + \sum_{k \neq h} w_{nkh} x_{nk})$$

Then

$$g_h(p, \overline{u}_h, x) = -\sum_i p_i \sum_{k \neq h} w_{ikh} x_{ik} + B_h(p, \overline{u}_h)$$

Note that $B_h()$ itself can be interpreted as an expenditure function: it must be concave and homogeneous of degree one in p and represents an arbitrary form of base utility. Here the compensated demands are

$$x_{ih} = -\sum_{k \neq h} w_{ikh} x_{ik} + \frac{\partial B_h(p, \overline{u}_h)}{\partial p_i}$$

The strength of the external effect is independent of prices and of the standard of living.

A further specialisation arises if $h's$ preferences react only to the total consumption of others of each good and in a way that is independent of the standard of living. Empirically examples would be congestion goods (public transport) and network goods (mobile telephones). In this case when marginal external effects are constants, δ_{ih}, individual preferences have the form

$$u_h(x_h, x_{\check{}h}) = u_h(x_{1h} + \delta_{1h} \sum_{k \neq h} x_{1k}, .., x_{nh} + \delta_{nh} \sum_{k \neq h} x_{nk})$$

and the compensated demands become

$$x_{ih} = -\delta_{ih} \sum_{k \neq h} x_{ik} + \frac{\partial B_h(p, \overline{u}_h)}{\partial p_i} \tag{28}$$

so it is the total consumption of others which affects an individual's compensated demand. We call this linear popular no spillover externalities. Kooreman & Schoonbeek use a linear expenditure system which has this form.

The Pigovian Taxes For linear no spillover externalities from (13) the taxes are given by

$$\begin{bmatrix} \pi^c_{i1}/\pi^c_{n1} \\ \vdots \\ \pi^c_{iH}/\pi^c_{nH} \end{bmatrix} = \begin{bmatrix} p_n \left[\sum_k \eta^i_{1k}\right] / \left[\lambda^c_1 \partial u_1/\partial x_{n1}\right] \\ \vdots \\ p_n \left[\sum_k \eta^i_{Hk}\right] / \left[\lambda^c_H \partial u_H/\partial x_{nH}\right] \end{bmatrix} \tag{29}$$

where $[\eta^i_{hk}]$ is a matrix of constants being the inverse of

$$\begin{bmatrix} 1 & w_{i21} & ... & w_{iH1} \\ w_{i12} & 1 & & . \\ . & . & . & w_{iHH-1} \\ w_{i1H} & w_{i2H} & & 1 \end{bmatrix}$$

The effect of linearity of externalities is that $[\eta^i_{hk}]$ is independent of prices or the level of individual income or utility. In the popular case the matrix simplifies to

$$A = \begin{bmatrix} 1 & \delta_{i2} & ... & \delta_{iH} \\ \delta_{i1} & 1 & & . \\ . & . & . & \delta_{iH} \\ \delta_{i1} & \delta_{i2} & & 1 \end{bmatrix}$$

and then it can be shown that (see appendix A3)

$$\sum_k \eta^i_{hk} = \prod_{k \neq h} (\delta_{ik} - 1)/\det(A) \tag{30}$$

For each commodity-person, the tax factors into a product of a person specific term, common to all commodities, and a person-commodity specific term. The person specific tax $(p_n/(\lambda^c_h \partial u_h/\partial x_{nh}))$ reflects the difference between the market and social marginal valuations of individual h at the Pareto optimum as measured through the marginal utilities of the last good. The commodity specific part depends only on the strength of the various externalities δ_{ih} and it is independent of prices or the level of utility. This is due to the linearity of the externality effect. Moreover for any commodity the ratio of the Pigovain taxes on any two individuals h, h' is independent of the externalities imposed by any other individual

$$(\pi^c_{ih'}/\pi^c_{nh'})\,/\,(\pi^c_{ih}/\pi^c_{nh}) = [(\delta_{ih'}-1)/(\delta_{ih'}-1)]\,/\,[\lambda^c_{h'}\partial u_{h'}/\partial x_{nh'}/\lambda^c_{h'}\partial u_h/\partial x_{nh}]$$

so there is an independence of irrelevant externalities property[8]. Restricted types of popular externalities will generate commonly observed taxation regimes, for example a personal income tax system or a system of specific indirect commodity taxes. In particular if

- Only good 1 has an external effect ($\delta_{ih} = 0,\ i > 1$)

The Pigovian taxes on any individual are identical for goods $i > 2$, however for good 1 the tax on individual h depends on the relative strength of externalities between other individuals.

- For every individual the strength of externality is equal for each good ($\delta_{ih} = \delta_i$)

For any pair of goods the ratio of the tax rates of the two goods is equal for all individuals (and it is equal to the strength of the externality on the two goods). If the social and private marginal valuations of individuals coincide then for each good every individual faces the same tax rate, it is as if there are specific indirect ad valorem taxes on commodities

- For each individual there is a common strength of externality for every good ($\delta_{ih} = \delta_h$)

Each individual pays the same proportional tax rate on every commodity so this is equivalent to a proportional income tax rate.

- For every individual and every commodity there is a common strength of externality ($\delta_{ih} = \delta$)

The proportional income tax rate of each individual is actually at the same rate for all individuals.

The Compensated Nash Equilibrium The compensated reaction curves (28) can be solved commodity by commodity for the compensated Nash equilibrium quantities.

$$\begin{bmatrix} 1 & \delta_{i1} & \dots & \delta_{i1} \\ \delta_{i2} & 1 & & \delta_{i2} \\ & & \cdot & \\ \delta_{iH} & & \delta_{iH} & 1 \end{bmatrix} \begin{bmatrix} x_{i1} \\ \\ \\ x_{iH} \end{bmatrix} = \begin{bmatrix} \frac{\partial B_1(p,\overline{u}_1)}{\partial p_i} \\ \\ \\ \frac{\partial B_H(p,\overline{u}_H)}{\partial p_i} \end{bmatrix}$$

[8] The structure of the taxes in (30) and this property would also hold if $u_h = U_h(x_{1h}, ...x_{nh}, \Sigma_{k\neq h}x_{1k}, ...\Sigma_{k\neq h}x_{nk})$.

leading to

$$x_{ih}^{NE} = \sum_k \eta_{hk}(\delta_{ih}) \frac{\partial B_k(p, \overline{u}_k)}{\partial p_i}$$

and aggregate Nash equilibrium compensated demands

$$X_i^{NE} = \sum_k (\sum_h \eta_{hk}(\delta_{ih})) \frac{\partial B_k(p, \overline{u}_k)}{\partial p_i} \tag{31}$$

(31) is of special interest since each term $B_h(p, \overline{u}_h)$ has all the properties of an expenditure function. In particular it has a negative semidefinite Jacobian, so if

$$(\sum_h \eta_{hk}(\delta_{ih})) > 0$$

the Jacobian of the aggregate compensated Nash equilibrium demands will also satisfy the sign restrictions of negative semidefiniteness. Note that the aggregate demand may fail to have a negative semidefinite Jacobian if some or all of the coefficients η_{hk} are negative. The aggregate compensated Nash equilibrium demand have a symmetric Jacobian if

$$\sum_k (\sum_h \eta_{hk}(\delta_{ih})) \frac{\partial^2 B_k(p, \overline{u}_k)}{\partial p_i \partial p_j} = \sum_k (\sum_h \eta_{hk}(\delta_{jh})) \frac{\partial^2 B_k(p, \overline{u}_k)}{\partial p_j \partial p_i}$$

A clear case when this holds is if $\delta_{ih} = \delta_h$ for all i: the case of common externality effects across commodities. Then we can think of these aggregate demands as coming from H synthetic consumers where the kth consumer has an expenditure function

$$(\sum_h \eta_{hk}(\delta_h)) B_k(p, \overline{u}_k)$$

exhibiting no externalities (recall each $B_k()$ has the properties of an expenditure function) and whose compensated demand is

$$z_{ik} = (\sum_h \eta_{hk}(\delta_h)) \frac{\partial B_k(p, \overline{u}_k)}{\partial p_i} \tag{32}$$

6 A 3 Commodity, 2 Individual Example

6.1 The utilities

As in Pollak, Kapteyn and Kooreman & Schoonbeek, we take an LES utility function[9] for $h = 1, 2$:

$$u_h(\mathbf{x}_1, \mathbf{x}_2) = \sum_{i=1}^{3} \alpha_{ih} \log(x_{ih} + \delta_{ih} x_{ik}), k \neq h, \sum_{i=1}^{3} \alpha_{ih} = 1 \tag{33}$$

where $\mathbf{x}_h = (x_{1h}, x_{2h}, x_{3h})$. The individual expenditure function is

$$g_h(p, \overline{u}_h, x_{\sim h}) = -\sum_i p_i \delta_{ih} x_{ik} + \exp(\overline{u}_h) \prod \left(\frac{\alpha_{ih}}{p_i} \right)^{-\alpha_{ih}}, k \neq h$$

[9]Detailed calculations are available on request.

6.2 Demands in a Compensated Pareto Optimum

For $i = 1, 2, 3$ and $h, k = 1, 2$ $h \neq k$

$$x_{ih}^{POc} = \frac{\alpha_{ih}\lambda_h^c}{p_i(1-\delta_{ik})} - \frac{\alpha_{ik}\delta_{ih}\lambda_k^c}{p_i(1-\delta_{ih})} \tag{34}$$

where:

$$\lambda_h^c = \exp(\overline{u}_h)\prod_{i=1}^{3}(\frac{1-\delta_{ik}\delta_{ih}}{1-\delta_{ik}})^{-\alpha_{ih}}\prod_{i=1}^{3}\left[\frac{\alpha_{ih}}{p_i}\right]^{-\alpha_{ih}} \tag{35}$$

Thus:

$$\begin{aligned} x_{ih}^{POc} &= -\frac{\alpha_{ik}\delta_{ih}}{p_i(1-\delta_{ih})}(\exp(\overline{u}_k)\prod_{i=1}^{3}\left(\frac{1-\delta_{ik}}{1-\delta_{ik}\delta_{ih}}\right)^{\alpha_{ik}}\prod_{i=1}^{3}\left[\frac{\alpha_{ik}}{p_i}\right]^{-\alpha_{ik}}) \\ &+\frac{\alpha_{ih}}{p_i(1-\delta_{ik})}(\exp(\overline{u}_h)\prod_{i=1}^{3}\left(\frac{1-\delta_{ik}}{1-\delta_{ik}\delta_{ih}}\right)^{\alpha_{ih}}\prod_{i=1}^{3}\left[\frac{\alpha_{ih}}{p_i}\right]^{-\alpha_{ih}}) \end{aligned} \tag{36}$$

The cost to each individual h of attaining the utility level $\overline{u}_h$ is

$$\begin{aligned} y_h^c &= \left(\sum_i \frac{\alpha_{ih}}{(1-\delta_{ik})}\right)(\exp(\overline{u}_h)\prod_{i=1}^{3}\left(\frac{1-\delta_{ik}}{1-\delta_{ik}\delta_{ih}}\right)^{\alpha_{ih}}\prod_{i=1}^{3}\left[\frac{\alpha_{ih}}{p_i}\right]^{-\alpha_{ih}}) \\ &-\left(\sum_i \frac{\alpha_{ik}\delta_{ih}}{(1-\delta_{ih})}\right)(\exp(\overline{u}_k)\prod_{i=1}^{3}\left(\frac{1-\delta_{ik}}{1-\delta_{ik}\delta_{ih}}\right)^{\alpha_{ik}}\prod_{i=1}^{3}\left[\frac{\alpha_{ik}}{p_i}\right]^{-\alpha_{ik}}) \end{aligned}$$

6.3 Compensated Demands in a Nash Equilibrium

On the other hand in the market solution, the compensated Nash equilibrium demands for each individual and good are

$$x_{ih}^{NEc} = \exp(\overline{u}_h)\frac{\alpha_{ih}}{p_i}\prod_{i=1}^{3}\left[\frac{\alpha_{ih}}{p_i}\right]^{-\alpha_{ih}} + \exp(\overline{u}_k)\frac{\alpha_{ik}}{p_i}\frac{\delta_{ih}}{(1-\delta_{ih}\delta_{ik})}\prod_{i=1}^{3}\left[\frac{\alpha_{ik}}{p_i}\right]^{-\alpha_{ik}} \tag{37}$$

6.4 Compensated Pigovian Prices

Applying equation (13) the Pigovian taxes are given by

$$\pi_{ih}^c = \frac{(1-\delta_{ik})(1-\delta_{3h}\delta_{3k})}{(1-\delta_{ih}\delta_{ik})(1-\delta_{3k})}, \pi_{3h}^c = 1 \tag{38}$$

for each $h \neq k$. The terms in good 3 represent the effects of the marginal valuation of the individual for good n. For positive Pigovian prices we require either $0 \leq \delta_{ik} < 1$ for all i, k or $\delta_{ik} > 1$ for all i, k.

6.5 Compensated Demands with Pigovian Pricing

Replacing the prices p_i in (38) with the tax corrected prices $\pi_{ih}^c p_i$ the Nash equilibrium demands become

$$\begin{aligned} x_{ih}^{NEc} &= \exp(\overline{u}_h)\frac{\alpha_{ih}}{p_i}(\pi_{ih}^c)^{-1}\prod_{i=1}^{2}(\pi_{ih}^c)^{\alpha_{ih}}\prod_{i=1}^{3}\left[\frac{\alpha_{ih}}{p_i}\right]^{-\alpha_{ih}} + \\ &\exp(\overline{u}_k)\frac{\alpha_{ik}}{p_i}(\pi_{ik}^c)^{-1}\frac{\delta_{ih}}{(1-\delta_{ih}\delta_{ik})}\prod_{i=1}^{2}(\pi_{ik}^c)^{\alpha_{ik}}\prod_{i=1}^{3}\left[\frac{\alpha_{ik}}{p_i}\right]^{-\alpha_{ik}} \end{aligned} \tag{39}$$

Putting (38) in (39) we get the Pareto optimal demand stated in (36) so these price corrections do eliminate the market failure.

The tax revenue is defined by

$$\sum_{\iota h}(\pi_{ih}-1)p_i x_{ih}^{NEc}$$

6.6 Demands in a Marshallian Pareto Optimum

For given aggregate resources y and with a welfare weights $\beta, 1-\beta$ respectively for individuals 1 and 2 the Pareto optimal demands are

$$x_{ih}^{POm} = \frac{\alpha_{ih} y \beta_h}{p_i(1-\delta_{ik})} - \frac{\alpha_{ik}\delta_{ih}\beta_k y}{p_i(1-\delta_{ih})}$$

where $\beta_1 = \beta, \beta_2 = 1-\beta$. In a particular Pareto optimum the costs to the two individuals are

$$\begin{aligned} y_1 &= y\left[\beta\sum_{\iota}\frac{\alpha_{i1}}{(1-\delta_{i2})} - (1-\beta)\sum_{\iota}\frac{\alpha_{i2}\delta_{i1}}{(1-\delta_{i1})}\right] \\ y_2 &= y\left[(1-\beta)\sum_{\iota}\frac{\alpha_{i2}}{(1-\delta_{i1})} - \beta\sum_{\iota}\frac{\alpha_{i1}\delta_{i2}}{(1-\delta_{i1})}\right] \end{aligned}$$

which defines the income distribution necessary to sustain the Pareto optimum in markets. Note that the sum of these is equal to y.

6.7 Marshallian Demands in a Nash Equilibrium with Fixed Income Distribution

We derive these from the more general equations (41) below where all $\pi_{ih}^m = 1$.

6.8 Marshallian Pigovian Prices

Using (20) these are given by

$$\pi_{ih}^m = \frac{(1-\delta_{ik})(1-\delta_{3h}\delta_{3k})}{(1-\delta_{ih}\delta_{ik})(1-\delta_{3k})} \tag{40}$$

Notice that the Marshallian Pigovian prices are equal to the compensated Pigovian prices.

6.9 Marshallian Demands in a Nash Equilibrium with Fixed Income Distribution and Pigovian Pricing

The Marshallian Nash equilibrium demands with Pigovian pricing have the form

$$x_{1h}^{NEm} = [(y_k+\delta_{2k}y_h)\delta_{1h}A_{1k} - (\delta_{2h}y_k+y_h)A_{1h} - (y_k+\delta_{3k}y_h)\delta_{1h}A_{2k} + (\delta_{3h}y_k+y_h)A_{2h} + A_3 y_h]/(p_1 D) \tag{41}$$

$$x_{2h}^{NEm} = [(\delta_{1k}y_h+y_k)\delta_{2h}A_{1h} - (y_h+\delta_{1h}y_k)A_{1k} - (y_h+\delta_{3h}y_k)A_{4h} - (\delta_{3k}y_h+y_k)\delta_{2h}A_{4k} + A_5 y_h]/(p_2 D)$$

$$x_{3h}^{NEm} = [(\delta_{1h}y_k+y_h)A_{2k} - \delta_{3h}(y_k+\delta_{1k}y_h)A_{2h} - (y_k+\delta_{2k}y_h)\delta_{3h}A_{4h} + (\delta_{2h}y_k+y_h)A_{4k} + A_6 y_h]/(p_3 D)$$

where for $h \neq k, h, k = 1, 2$

$$\begin{aligned}
A_{1h} &= -\alpha_{1h}\alpha_{2k}(1-\delta_{3h}\delta_{3k})\pi_{1k}^{M}\pi_{2h}^{M} \\
A_{2h} &= -\alpha_{1h}\alpha_{3k}(1-\delta_{2h}\delta_{2k})\pi_{1k}^{M}\pi_{2h}^{M}\pi_{2k}^{M} \\
A_{4h} &= -\alpha_{2h}\alpha_{3k}(1-\delta_{1h}\delta_{1k})\pi_{1k}^{M}\pi_{1h}^{M}\pi_{2k}^{M} \\
A_{3} &= \alpha_{1h}\alpha_{1k}(1-\delta_{2h}\delta_{2k})(1-\delta_{3h}\delta_{3k})\pi_{2k}^{M}\pi_{2h}^{M} \\
A_{5} &= \alpha_{2h}\alpha_{2k}(1-\delta_{1h}\delta_{1k})(1-\delta_{3h}\delta_{3k})\pi_{1k}^{M}\pi_{1h}^{M} \\
A_{6} &= \alpha_{3h}\alpha_{3k}(1-\delta_{2h}\delta_{2k})(1-\delta_{1h}\delta_{1k})\pi_{1k}^{M}\pi_{1h}^{M}\pi_{2k}^{M}\pi_{2h}^{M}
\end{aligned}$$

and

$$\begin{aligned}
D = {} & (1-\delta_{22}\delta_{21})(1-\delta_{11}\delta_{12})\alpha_{32}\alpha_{31} \\
& +\alpha_{11}(1-\delta_{22}\delta_{21})(1-\delta_{12}\delta_{31})\alpha_{31}\pi_{12}^{m} + \alpha_{12}(1-\delta_{22}\delta_{21})(1-\delta_{11}\delta_{32})\alpha_{31}\pi_{11}^{m} + \\
& \alpha_{11}\alpha_{12}(1-\delta_{22}\delta_{21})(1-\delta_{31}\delta_{32})\pi_{21}^{m} + \alpha_{21}(1-\delta_{11}\delta_{12})(1-\delta_{31}\delta_{22})\alpha_{32}\pi_{11}^{m}\pi_{12}^{m} + \\
& \alpha_{12}\alpha_{21}(1-\delta_{11}\delta_{12})(1-\delta_{31}\delta_{32})\pi_{11}^{m}\pi_{12}^{m} + \alpha_{22}(1-\delta_{11}\delta_{12})(1-\delta_{32}\delta_{21})\alpha_{31}\pi_{11}^{m} + \\
& \alpha_{22}\alpha_{21}(1-\delta_{21}\delta_{12})(1-\delta_{31}\delta_{32})\pi_{21}^{m}\pi_{12}^{m} + \alpha_{22}\alpha_{21}(1-\delta_{11}\delta_{12})(1-\delta_{32}\delta_{31})\pi_{11}^{m}\pi_{12}^{m})
\end{aligned}$$

To see the equivalence between these demands and those in a Pareto optimum for given aggregate resources y and relative welfare weights β on the two individuals, define individual incomes by the expenditure of each individual in the particular Marshallian Pareto optimum:

$$\begin{aligned}
y_1 &= \sum_{i=1}^{3} p_i x_{i1}^{POm} \\
&= y\left[\beta \sum_{\iota} \frac{\alpha_{i1}}{(1-\delta_{i2})} - (1-\beta)\sum_{\iota} \frac{\alpha_{i2}\delta_{i1}}{(1-\delta_{i1})}\right] \\
y_2 &= \sum_{i=1}^{3} p_i x_{i2}^{POm} \\
&= y\left[(1-\beta) \sum_{\iota} \frac{\alpha_{i2}}{(1-\delta_{i1})} - \beta\sum_{\iota} \frac{\alpha_{i1}\delta_{i2}}{(1-\delta_{i1})}\right]
\end{aligned}$$

Substitute these into the Marshallian Nash equilibrium demands with Pigovian pricing to get

$$x_{ih}^{NEm} = X_{ih}(\pi \cdot p, y, y\beta_h)$$

(with $\beta_1 = \beta, \beta_2 = 1-\beta$) in the form

$$x_{ih}^{NEm} = \frac{N_{ihy}y + N_{ihy\beta}y\beta_h}{p_i D} \tag{42}$$

Detailed calculation shows that

$$\begin{aligned} N_{ihy} &= A_{ih}X + B_{ih} \\ N_{ihy\beta} &= A_{ih}Y \end{aligned}$$

where

$$\begin{aligned}
A_{11} &= \frac{(\delta_{32}\delta_{31}-1)^3(-1+\delta_{22})(\delta_{21}-1)(\alpha_{11}(\delta_{11}-1)+\alpha_{12}\delta_{11}(\delta_{12}-1))}{(\delta_{32}-1)(-1+\delta_{22}\delta_{21})(\delta_{31}-1)} \\
A_{12} &= -\frac{(\delta_{32}\delta_{31}-1)^3(-1+\delta_{22})(\delta_{21}-1)(\alpha_{12}(\delta_{12}-1)+\alpha_{11}\delta_{12}(\delta_{11}-1))}{(\delta_{32}-1)(-1+\delta_{22}\delta_{21})(\delta_{31}-1)} \\
A_{21} &= \frac{(\delta_{32}\delta_{31}-1)^3(1-\delta_{12})(\delta_{11}-1)(\alpha_{21}(\delta_{21}-1)+\alpha_{22}\delta_{21}(\delta_{22}-1))}{(\delta_{32}-1)(1-\delta_{22}\delta_{21})(\delta_{31}-1)} \\
A_{22} &= -\frac{(\delta_{32}\delta_{31}-1)^3(-1+\delta_{12})(\delta_{11}-1)(\alpha_{22}(\delta_{22}-1)+\alpha_{21}\delta_{22}(\delta_{21}-1))}{(\delta_{32}-1)(-1+\delta_{22}\delta_{21})(\delta_{31}-1)} \\
A_{31} &= -\frac{(\delta_{32}\delta_{31}-1)^3((1-\delta_{32}\delta_{31})\alpha_{31}+\delta_{31}(\alpha_{21}+\alpha_{11}-\alpha_{12}-\alpha_{22}))(1-\delta_{22})(\delta_{21}-1)(\delta_{12}-1)(1-\delta_{11})}{(1-\delta_{11}\delta_{12})(\delta_{32}-1)^2(1-\delta_{22}\delta_{21})(\delta_{31}-1)^2} \\
A_{32} &= \frac{(\delta_{32}\delta_{31}-1)^3((1-\delta_{32}\delta_{31})\alpha_{32}+\delta_{32}(\alpha_{12}+\alpha_{22}-\alpha_{21}-\alpha_{11}))(1-\delta_{22})(\delta_{21}-1)(\delta_{12}-1)(1-\delta_{11})}{(1-\delta_{11}\delta_{12})(\delta_{32}-1)^2(1-\delta_{22}\delta_{21})(\delta_{31}-1)^2} \\
X &= -\frac{\alpha_{12}\delta_{11}}{(1-\delta_{11})}-\frac{\alpha_{22}\delta_{21}}{(1-\delta_{21})}-\frac{\alpha_{32}\delta_{31}}{(1-\delta_{31})} \\
Y &= \frac{\alpha_{11}}{1-\delta_{12}}+\frac{\alpha_{12}\delta_{11}}{1-\delta_{11}}+\frac{\alpha_{21}}{1-\delta_{22}}+\frac{\alpha_{22}\delta_{21}}{1-\delta_{21}}+\frac{\alpha_{31}}{1-\delta_{32}}+\frac{\alpha_{32}\delta_{31}}{1-\delta_{31}}
\end{aligned}$$

The functions B_{ih} are dependent on δ and α and do not justify the space to display here[10]. The ratio Y/D is relatively simple:

$$Y/D = -\frac{(\delta_{32}-1)(-1+\delta_{11}\delta_{12})(\delta_{31}-1)(-1+\delta_{22}\delta_{21})}{(\delta_{32}\delta_{31}-1)^3(\delta_{21}-1)(-1+\delta_{22})(\delta_{12}-1)(-1+\delta_{11})}$$

so that $N_{ihy\beta}/D$ reduces to the coefficient of βy in Marshallian Pareto optimal demand. It is more tedious to show that the coefficient N_{ihy}/D also reduces to the coefficient of y in the Marshallian Pareto optimal demand but in fact it does.

6.10 Equivalence of the Compensated and Marshallian Pareto optimum

Here we show that for any CPO with a utility distribution k_1, k_2 there is an aggregate resource level and welfare weights in which the Marshallian Pareto optimal demands coincide with the compensated Pareto optimal demands.

[10] Details are available on request.

Given a compensated Pareto optimum we can compute the aggregate resources that it requires by summing the values of the individuals expenditure functions at the CPO:

$$\begin{aligned} y &= \sum_h g_h(p, \overline{u}_1, \overline{u}_2) = \\ & \sum_h \exp(\overline{u}_h) \prod_{i=1}^{3} \left(\frac{1-\delta_{ik}}{1-\delta_{ik}\delta_{ih}} \right)^{\alpha_{ih}} \prod_{i=1}^{3} \left[\frac{\alpha_{ih}}{p_i} \right]^{-\alpha_{ih}}, k \neq h \end{aligned} \tag{43}$$

Notice that without externalities the feasible utility distribution from given income y that makes say u_1 is maximal gives 1 all the aggregate income y and 2 has nothing. But with externalities this may not be true-1 may be better off from 2 having some of the income if there are positive externalities.In the Marshallian demands at the Pareto optimum corresponding to y, β, replace y by the expression (43) and then take the compensated and Marshallian demands for the first good by the first individual, equate them and solve for β giving

$$\beta = exp(\overline{u}_1) \prod_{i=1}^{3} \left(\frac{1-\delta_{ik}}{1-\delta_{i2}\delta_{i1}} \right)^{\alpha_{i1}} \prod_{i=1}^{3} \left[\frac{\alpha_{i1}}{p_i} \right]^{-\alpha_{i1}} / \sum_h \exp(\overline{u}_h) \prod_{i=1}^{3} \left(\frac{1-\delta_{ik}}{1-\delta_{ik}\delta_{ih}} \right)^{\alpha_{ih}} \prod_{i=1}^{3} \left[\frac{\alpha_{ih}}{p_i} \right]^{-\alpha_{ih}}$$

which has the interpretation of the share of the total cost of attaining the Pareto optimum attributable to individual 1.

With these values of y and β, $x_{ih}^{POc} = x_{ih}^{POm}$ for each pair of values i, h.

6.11 Social Cost

The social cost of the externality SC is the sum of losses incurred by each individual, in turn the individual losses are the sum of the losses on each commodity which depend on the strength of preference for the commodity and on the strength of the externality

$$SC = \sum_{h=1}^{2} \left[\exp(\overline{u}_h) \prod_{i=1}^{3} \left[\frac{\alpha_{ih}}{p_i} \right]^{-\alpha_{ih}} \left[\sum_{i=1}^{2} \alpha_{ih} \frac{(\delta_{ik}-1)}{(\delta_{ih}\delta_{ik}-1)} + \alpha_{3h} \left(\frac{(\delta_{3k}-1)}{(\delta_{3h}\delta_{3k}-1)} - 1 \right) \right] - \prod_{i=1}^{3} \left[\frac{(\delta_{ik}-1)}{(\delta_{ih}\delta_{ik}-1)} \right]^{-\alpha_{ih}} \right]$$

7 Conclusions

In this paper we use the ideas of a utility possibility curve and a Scitovsky community indifference curve to implement a Pareto optimum when there are consumption externalities. We use the long established idea of Pigovian taxation to analyse this, focussing on situations in which piecemeal policy is possible in the sense that corrective tax policy in one market is largely independent of tax policy in other markets. We show that for this to be possible individual preferences must have a form in which the externality enters as an adjustment to a subsistence term in individual utility or cost. This is interesting since for other reasons the literature has suggested modelling interdependent preferences in this way. We show that if the externality has this form and also enters only through the sum of the consumption of other individuals and linearly then various simple tax systems can be used to correct for the externalities. In particular the correct taxes on one good can be computed independently of other goods, and for cases where there is further restriction on the form of the externality (especially across individuals or across goods) the commonly used taxes such as specific excise taxes or a personal income tax can implement a Pareto optimum. This provides a justification for concentrating on these forms of preferences, of course the other justification is empirical-such preferences are likely to arise with network of congestion goods. Our results generalise those in the literature on characterising Pigovian taxes and provide the link to piecemeal policy. We also give a decomposition of the welfare cost of externalities. On the positive economics side we find conditions under which the market solution for compensated demands (which has the form of a Nash equilibrium due to the consumption interdependence) will either be downward sloping or have a symmetric Jacobian with respect to prices. Finally we examine a two individual, three commodity example to see how the taxes look and the features of the welfare cost of the externalities.

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A Appendix

A.1 Deriving Pigovian Taxes with fixed utility distribution

From (5)

$$\lambda_h^c \frac{\partial u_h}{\partial x_{ih}} / \sum_k \lambda_k^c \frac{\partial u_k}{\partial x_{nh}} = \frac{p_i}{p_n} - \sum_{k \neq h} \lambda_k^c \frac{\partial u_k}{\partial x_{ih}} / \sum_k \lambda_k^c \frac{\partial u_k}{\partial x_{nh}}$$

Solve for $\partial u_h / \partial x_{ih}$:

$$\frac{\partial u_h}{\partial x_{ih}} = \frac{1}{\lambda_h} \frac{p_i}{p_n} \sum_k \lambda_k^c \frac{\partial u_k}{\partial x_{nh}} - \frac{1}{\lambda_h} \sum_{k \neq h} \lambda_k^c \frac{\partial u_k}{\partial x_{ih}}$$

and put this in

$$\frac{\partial u_h(x_h, x_{\bar{h}})}{\partial x_{ih}} = \frac{\pi_{ih}^c p_i}{\pi_{nh}^c p_n} \frac{\partial u_h(x_h, x_{\bar{h}})}{\partial x_{nh}}$$

$$
\begin{aligned}
\frac{\pi^c_{ih}}{\pi^c_{nh}} &= \frac{\sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nh}}}{\lambda^c_h \frac{\partial U_h(x_h, x_{-h})}{\partial x_{nh}}} - \frac{p_n}{p_i}\frac{\sum_{k\neq h} \lambda^c_k \frac{\partial u_k}{\partial x_{ih}}}{\lambda^c_h \frac{\partial U_h(x_h, x_{-h})}{\partial x_{nh}}} \\
&= \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nh}} / \left(\lambda_h \frac{\partial u_h}{\partial x_{nh}}\right) - \sum_{k\neq h} \lambda^c_k \frac{\pi_{ik}}{\pi_{nk}} \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}}\right) / \left(\lambda^c_h \frac{\partial u_h}{\partial x_{nh}}\right) \qquad \text{(A1)} \\
&= \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{nh}} / \frac{\partial u_k}{\partial x_{nk}}\right) / \left(\lambda^c_h \frac{\partial u_h}{\partial x_{nh}}\right) - \sum_{k\neq h} \lambda^c_k \frac{\pi_{ik}}{\pi_{nk}} \frac{\partial u_k}{\partial x_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}}\right) / \left(\lambda^c_h \frac{\partial u_h}{\partial x_{nh}}\right) \\
&= \sum_k \left(\lambda^c_k \frac{\partial u_k}{\partial x_{nk}} / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}}\right) \left(\frac{\partial u_k}{\partial x_{nh}} / \frac{\partial u_k}{\partial x_{nk}}\right) - \sum_{k\neq h} \left(\lambda^c_k \frac{\partial u_k}{\partial x_{nk}} / \lambda^c_h \frac{\partial u_h}{\partial x_{nh}}\right) \frac{\pi_{ik}}{\pi_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}}\right)
\end{aligned}
$$

Note that in the linear no spillover case this becomes

$$
\frac{\pi^c_{ih}}{\pi^c_{nh}} = \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nk}} \delta_{nk} / \left(\lambda_h \frac{\partial u_h}{\partial x_{nh}}\right) - \sum_{k\neq h} \lambda^c_k \frac{\partial u_k}{\partial x_{nk}} \frac{\pi_{ik}}{\pi_{nk}} \delta_{ik} / \left(\lambda_h \frac{\partial u_h}{\partial x_{nh}}\right)
$$

Rearranging (44) and writing the system of equations for good i

$$
\lambda^c_h \frac{\partial u_h}{\partial x_{nh}} \frac{\pi^c_{ih}}{\pi^c_{nh}} + \sum_{k\neq h} \left(\lambda^c_k \frac{\partial u_k}{\partial x_{nk}}\right) \frac{\pi_{ik}}{\pi_{nk}} \left(\frac{\partial u_k}{\partial x_{ih}} / \frac{\partial u_k}{\partial x_{ik}}\right) = \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nh}}
$$

$$
\begin{bmatrix}
1 & \frac{\partial u_2}{\partial x_{i1}} / \frac{\partial u_2}{\partial x_{i2}} & \dots & \frac{\partial u_H}{\partial x_{i1}} / \frac{\partial u_H}{\partial x_{iH}} \\
\frac{\partial u_1}{\partial x_{i2}} / \frac{\partial u_1}{\partial x_{i1}} & 1 & \dots & \vdots \\
\dots & \dots & \ddots & \frac{\partial u_H}{\partial x_{iH-1}} / \frac{\partial u_H}{\partial x_{iH}} \\
\frac{\partial u_1}{\partial x_{iH}} / \frac{\partial u_1}{\partial x_{i1}} & \frac{\partial u_2}{\partial x_{iH}} / \frac{\partial u_2}{\partial x_{i2}} & \dots & 1
\end{bmatrix}
\begin{bmatrix}
\lambda^c_1 \partial u_1/\partial x_{n1} & \dots & \dots & 0 \\
\vdots & \lambda^c_2 \partial u_2/\partial x_{n2} & & \vdots \\
\vdots & \dots & \ddots & \vdots \\
0 & \dots & \dots & \lambda^c_H \partial u_H/\partial x_{nH}
\end{bmatrix}
$$

$$
\cdot \begin{bmatrix} \pi_{i1}/\pi_{n1} \\ \vdots \\ \pi_{iH}/\pi_{nH} \end{bmatrix} = \begin{bmatrix} \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{n1}} \\ \vdots \\ \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nH}} \end{bmatrix} = \begin{bmatrix} p_n \\ \vdots \\ p_n \end{bmatrix}
$$

$$
\begin{bmatrix} \pi_{i1}/\pi_{n1} \\ \vdots \\ \pi_{iH}/\pi_{nH} \end{bmatrix} = \begin{bmatrix}
\lambda^c_1 \partial u_1/\partial x_{n1} & \dots & \dots & 0 \\
\vdots & \lambda^c_2 \partial u_2/\partial x_{n2} & & \vdots \\
\vdots & & \ddots & \vdots \\
0 & \dots & \dots & \lambda^c_H \partial u_H/\partial x_{nH}
\end{bmatrix}^{-1}
$$

$$
\begin{bmatrix}
1 & \frac{\partial u_2}{\partial x_{i1}} / \frac{\partial u_2}{\partial x_{i2}} & \dots & \frac{\partial u_H}{\partial x_{i1}} / \frac{\partial u_H}{\partial x_{iH}} \\
\frac{\partial u_1}{\partial x_{i2}} / \frac{\partial u_1}{\partial x_{i1}} & 1 & & \vdots \\
\vdots & \dots & \ddots & \frac{\partial u_H}{\partial x_{iH-1}} / \frac{\partial u_H}{\partial x_{iH}} \\
\frac{\partial u_1}{\partial x_{iH}} / \frac{\partial u_1}{\partial x_{i1}} & \frac{\partial u_2}{\partial x_{iH}} / \frac{\partial u_2}{\partial x_{i2}} & \dots & 1
\end{bmatrix}^{-1}
\begin{bmatrix} \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{n1}} \\ \vdots \\ \sum_k \lambda^c_k \frac{\partial u_k}{\partial x_{nH}} \end{bmatrix} = \begin{bmatrix} \left[p_n \sum_h \eta^i_{1h}\right] / \lambda^c_1 \partial u_1/\partial x_{n1} \\ \vdots \\ \vdots \\ \left[p_n \sum_h \eta^i_{Hh}\right] / \lambda^c_H \partial u_H/\partial x_{nH} \end{bmatrix}
$$

where η^i_{1h} are the elements of the inverse of the matrix of marginal externality effects with $1's$ on the diagonal.

A.2 Deriving Pigovian Taxes with welfare weights and fixed income distribution

ThePareto optimum has

$$
\frac{p_i}{p_n} = \frac{\sum_k \beta_k \partial u_k/\partial x_{ih}}{\sum_k \beta_k \partial u_k/\partial x_{nh}} \qquad \text{(A2)}
$$

$$\frac{\pi_{ih}p_i}{\pi_{nh}p_n} = \frac{\partial u_h/\partial x_{ih}}{\partial u_h/\partial x_{nh}} \Rightarrow \partial u_h/\partial x_{ih} = \frac{\pi_{ih}p_i}{\pi_{nh}p_n}\partial u_h/\partial x_{nh} \tag{A3}$$

$$\frac{p_n}{p_i} = \frac{\pi_{ik}}{\pi_{nk}}\frac{\partial u_k/\partial x_{nk}}{\partial u_k/\partial x_{ik}} \qquad \text{for any } k \tag{A4}$$

Solve (A2) for $\partial u_h/\partial x_{ih}$:

$$\frac{\partial u_h}{\partial x_{ih}} = \frac{1}{\beta_h}\frac{p_i}{p_n}\sum_k \beta_k \frac{\partial u_k}{\partial x_{nh}} - \frac{1}{\beta_h}\sum_{k\neq h}\beta_k\frac{\partial u_k}{\partial x_{ih}}$$

Use (A3)

$$\frac{\pi_{ih}p_i}{\pi_{nh}p_n}\partial u_h/\partial x_{nh} = \frac{1}{\beta_h}\frac{p_i}{p_n}\sum_k \beta_k \frac{\partial u_k}{\partial x_{nh}} - \frac{1}{\beta_h}\sum_{k\neq h}\beta_k\frac{\partial u_k}{\partial x_{ih}}$$

Solve for π_{ih}/π_{nh}

$$\begin{aligned}\frac{\pi_{ih}^m}{\pi_{nh}^m} &= \frac{\sum_k \beta_k \frac{\partial u_k}{\partial x_{nh}}}{\beta_h \partial u_h/\partial x_{nh}} - \frac{p_n}{p_i}\frac{\sum_{k\neq h}\beta_k\frac{\partial u_k}{\partial x_{ih}}}{\beta_h\partial u_h/\partial x_{nh}} \\ &= \sum_k \beta_k\frac{\partial u_k}{\partial x_{nh}}/\beta_h\frac{\partial u_h}{\partial x_{nh}} - \sum_{k\neq h}\beta_k\frac{\partial u_k}{\partial x_{nk}}\frac{\pi_{ik}}{\pi_{nk}}\left(\frac{\partial u_k}{\partial x_{ih}}/\frac{\partial u_k}{\partial x_{ik}}\right)/\beta_h\frac{\partial u_h}{\partial x_{nh}} \end{aligned} \tag{A5}$$

Rearrange (A5)

$$\frac{\pi_{ih}^c}{\pi_{nh}^c}\beta_h\frac{\partial u_h}{\partial x_{nh}} + \sum_{k\neq h}\beta_k\frac{\partial u_k}{\partial x_{nk}}\frac{\pi_{ik}}{\pi_{nk}}\left(\frac{\partial u_k}{\partial x_{ih}}/\frac{\partial u_k}{\partial x_{ik}}\right) = \sum_k\beta_k\frac{\partial u_k}{\partial x_{nh}} \tag{A6}$$

In matrix notation (A6) is

$$\begin{bmatrix} 1 & \partial u_2/\partial x_{i1}/\partial u_2/\partial x_{i2} & ... & \partial u_H/\partial x_{i1}/\partial u_H/\partial x_{iH} \\ \partial u_1/\partial x_{i2}/\partial u_1/\partial x_{i1} & 1 & & \vdots \\ \vdots & & \ddots & \partial u_H/\partial x_{iH-1}/\partial u_H/\partial x_{iH} \\ \partial u_1/\partial x_{iH}/\partial u_1/\partial x_{i1} & \partial u_2/\partial x_{iH}/\partial u_2/\partial x_{i2} & ... & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} \beta_1\partial u_1/\partial x_{n1} & ... & ... & 0 \\ \vdots & \beta_2\partial u_2/\partial x_{n2} & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & ... & ... & \beta_H\partial u_H/\partial x_{nH} \end{bmatrix} \begin{bmatrix} \pi_{i1}/\pi_{n1} \\ \vdots \\ \pi_{iH}/\pi_{nH} \end{bmatrix} = \begin{bmatrix} \sum_k \beta_k \partial u_k/\partial x_{n1} \\ \vdots \\ \sum_k \beta_k\partial u_k/\partial x_{nH} \end{bmatrix} =$$

$$\begin{bmatrix} \lambda p_n \\ \vdots \\ \lambda p_n \end{bmatrix}$$

from (15).

The solution is then

$$\begin{bmatrix} \pi_{i1}/\pi_{n1} \\ \vdots \\ \pi_{iH}/\pi_{nH} \end{bmatrix} = \begin{bmatrix} \lambda p_n \left[\sum_k \eta_{1k}^i\right]/\left[\beta_1\partial u_1/\partial x_{n1}\right] \\ \vdots \\ \lambda p_n\left[\sum_k \eta_{Hk}^i\right]/\left[\beta_H\partial u_H/\partial x_{nH}\right] \end{bmatrix} \tag{A7}$$

A.3 Pigovian Taxes With Popular No Spillover Externalities

Let

$$B = \begin{bmatrix} e_1 & 1 & ... & 1 \\ 1 & e_2 & ... & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & ... & e_H \end{bmatrix}$$

We assert that the inverse of A is C :

$$c_{ij} = -\Pi_{k\neq i,j}(e_k - 1)/\det(B) \qquad i \neq j \tag{44}$$

$$\begin{aligned} c_{jj} &= -e_i c_{ij} - \Sigma_{k\neq i,j} c_{kj} \\ &= \left[e_i \Pi_{k\neq i,j}(e_k - 1) + \Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1)\right]/\det(B) \qquad \text{for any } i \neq j, \text{ each } j \end{aligned} \tag{45}$$

$$\begin{aligned} \det(B) &= e_j c_{jj} + \Sigma_{i\neq j} c_{ij} \qquad \text{for any } j \\ &= (e_j e_i - 1)\Pi_{k\neq i,j}(e_k - 1) + (e_j - 1)\Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1) \end{aligned}$$

To verify this note that

$$BC = \begin{bmatrix} e_1 c_{11} + \Sigma_{k\neq 1} c_{k1} & e_1 c_{12} + \Sigma_{k\neq 1} c_{k2} & ... & e_1 c_{1H} + \Sigma_{k\neq 1} c_{kH} \\ e_2 c_{21} + \Sigma_{k\neq 2} c_{k1} & e_2 c_{22} + \Sigma_{k\neq 2} c_{k2} & ... & e_2 c_{2H} + \Sigma_{k\neq 2} c_{kH} \\ \vdots & & \ddots & \vdots \\ e_H c_{H1} + \Sigma_{k\neq H} c_{k1} & e_H c_{H2} + \Sigma_{k\neq H} c_{k2} & ... & e_H c_{HH} + \Sigma_{k\neq H} c_{kH} \end{bmatrix}$$

A typical off diagonal term of BC has the form

$$\begin{aligned} & e_i c_{ij} + \Sigma_{k\neq i} c_{kj} \\ = \; & e_i c_{ij} + \Sigma_{k\neq i,j} c_{kj} + c_{jj} \\ = \; & -e_i \Pi_{k\neq i,j}(e_k - 1)/\det(B) - \Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1)/\det(B) + \left[e_i \Pi_{k\neq i,j}(e_k - 1) + \Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1)\right]/\det(B) \\ = \; & 0 \end{aligned}$$

whilst a typical diagonal term has the form

$$\begin{aligned} & e_j c_{jj} + \Sigma_{k\neq j} c_{kj} \\ = \; & e_j \left[e_i \Pi_{k\neq i,j}(e_k - 1) + \Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1)\right]/\det(B) - \Sigma_{k\neq j}\Pi_{l\neq k,j}(e_l - 1)/\det(B) \\ = \; & \left[e_j e_i \Pi_{k\neq i,j}(e_k - 1) + e_j \Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1) - \Sigma_{k\neq j}\Pi_{l\neq k,j}(e_l - 1)\right]/\det(B) \\ = \; & \left[e_j e_i \Pi_{k\neq i,j}(e_k - 1) + e_j \Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1) - \Sigma_{k\neq j,i}\Pi_{l\neq k,j}(e_l - 1) - \Pi_{l\neq i,j}(e_l - 1)\right]/\det(B) \\ = \; & \left[e_j e_i \Pi_{k\neq i,j}(e_k - 1) + (e_j - 1)\Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1) - \Pi_{l\neq i,j}(e_l - 1)\right]/\det(B) \\ = \; & \left[(e_j e_i - 1)\Pi_{k\neq i,j}(e_k - 1) + (e_j - 1)\Sigma_{k\neq i,j}\Pi_{l\neq k,j}(e_l - 1)\right]/\det(B) \\ = \; & 1 \end{aligned}$$

Hence the inverse of B is indeed given by C.

The matrix we are actually interested in is

$$A = \begin{bmatrix} 1 & \delta_2 & ... & \delta_H \\ \delta_1 & 1 & & . \\ . & . & . & \delta_H \\ \delta_1 & \delta_2 & & 1 \end{bmatrix} = \begin{bmatrix} 1/\delta_1 & 1 & ... & 1 \\ 1 & 1/\delta_2 & ... & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & ... & 1/\delta_H \end{bmatrix} \begin{bmatrix} \delta_1 & 0 & ... & 0 \\ 0 & \delta_2 & ... & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & ... & \delta_H \end{bmatrix}$$

derived from B by setting $e_i = 1/\delta_i$

$$A = B \begin{bmatrix} \delta_1 & 0 & ... & 0 \\ 0 & \delta_2 & ... & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & ... & \delta_H \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \begin{bmatrix} \delta_1 & 0 & ... & 0 \\ 0 & \delta_2 & ... & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & ... & \delta_H \end{bmatrix}^{-1} C \\ &= \begin{bmatrix} 1/\delta_1 & 1 & ... & 1 \\ 1 & 1/\delta_2 & ... & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & ... & 1/\delta_H \end{bmatrix} C \\ &= \begin{bmatrix} c_{11}/\delta_1 & c_{12}/\delta_1 & ... & c_{1H}/\delta_1 \\ \vdots & & \ddots & \vdots \\ c_{H1}/\delta_H & c_{H2}/\delta_H & ... & c_{HH}/\delta_H \end{bmatrix} \end{aligned}$$

From (44) and (45)

$$\begin{aligned} c_{ij}/\delta_i &= \frac{\Pi_{k\neq i,j}((\delta_k - 1)/\delta_k}{\delta_i \det(B)} \\ c_{jj}/\delta_j &= \frac{-(1/\delta_i)\Pi_{k\neq i,j}((\delta_k - 1)/\delta_k - \Sigma_{k\neq i,j}\Pi_{l\neq k,j}((\delta_l - 1/\delta_l)}{\delta_j \det(B)} \end{aligned}$$

The Pigovian tax term is the sum of the terms in any row of

$$\begin{bmatrix} c_{11}/\delta_1 & c_{12}/\delta_1 & ... & c_{1H}/\delta_1 \\ \vdots & & \ddots & \vdots \\ c_{H1}/\delta_H & c_{H2}/\delta_H & ... & c_{HH}/\delta_H \end{bmatrix}$$

To evaluate $\left[c_{ii} + \Sigma_{k\neq i} c_{ik}\right]/\delta_i$ consider

$$\begin{aligned} c_{ii} + \Sigma_{k\neq i} c_{ik} &= -e_k c_{ki} - \Sigma_{l\neq k,i} c_{li} + \Sigma_{l\neq i} c_{il} \\ &= \left[e_k \Pi_{l\neq k,i}(e_l - 1) + \Sigma_{l\neq k,i}\Pi_{j\neq l,i}(e_j - 1) - \Sigma_{l\neq i}\Pi_{j\neq l,i}(e_j - 1)\right]/\det(B) \\ &= \left[e_k \Pi_{l\neq k,i}(e_l - 1) + \Sigma_{l\neq k,i}\Pi_{j\neq l,i}(e_j - 1) - \Sigma_{l\neq i,k}\Pi_{j\neq l,i}(e_j - 1) - \Pi_{j\neq k,i}(e_j - 1)\right]/\det(B) \\ &= \left[e_k \Pi_{j\neq k,i}(e_j - 1) + \Sigma_{l\neq k,i}\Pi_{j\neq l,i}(e_j - 1) - \Sigma_{l\neq i,k}\Pi_{j\neq l,i}(e_j - 1) - \Pi_{j\neq k,i}(e_j - 1)\right]/\det(B) \\ &= \left[(e_k - 1)\Pi_{j\neq k,i}(e_j - 1)\right]/\det(B) \\ &= \left[\Pi_{j\neq i}(e_j - 1)\right]/\det(B) \end{aligned}$$

Replacing e_j by $1/\delta_j$

$$\begin{aligned} [c_{ii} + \Sigma_{k \neq i} c_{ik}] / \delta_i &= \Pi_{j \neq i} (\frac{\delta_j - 1}{\delta_j}) / (\delta_i \det(B)) \\ &= \frac{\Pi_{j \neq i} (\delta_j - 1)}{\det(A)} \end{aligned}$$