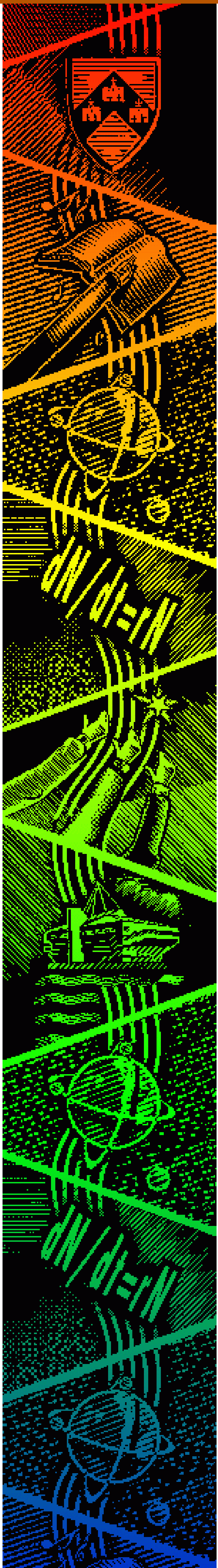

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An Extended Structural Credit Risk Model

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AN EXTENDED STRUCTURAL CREDIT RISK MODEL (forthcoming in the Icfai Journal of Financial Risk Management; all copyrights rest with the Icfai University press)

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Abstract

This paper presents an extended structural credit risk model that provides closed form solutions for fixed and floating coupon bonds and credit default swaps. This structural model is an "extended" one in the following sense. It allows for the default free term structure to be driven by the a multi-factor Gaussian model, rather than by a single factor one. Expected default occurs as a latent diffusion process first hits the default barrier, but the diffusion process is not the value of the firm's assets. Default can be "expected" or "unexpected". Liquidity risk is correlated with credit risk. It is not necessary to disentangle the risk of unexpected default from liquidity risk. A tractable and accurate recovery assumption is proposed.

Key words: structural credit risk model, Vasicek model, Gaussian term structure model, bond pricing, credit default swap pricing, unexpected default, liquidity risk.

JEL classification: G13.

1 Introduction and literature

This paper presents a tractable extended structural credit risk model that provides closed form solutions to price defaultable fixed and floating rate bonds and credit default swaps. This model extends the structural credit risk models in the literature.

Most structural models assume that the default-free yield curve is described by the one factor Vasicek model (1977). This does not seem satisfactory since the literature has clearly documented that multi-factor models are needed to describe the dynamics of the default-free yield curve, see e.g. Dai-Singleton (2000). Bakshi-Madan-Zhang (2006) recently found that two latent factors driving default-free yields also enhance the empirical fit of their defaultable bond pricing model. Similarly Hubner and Pascal (2004) suggest that a two-factor default-free term structure model may be appropriate also to price defaultable bonds. Even the reduced form credit risk models typically assume that default free yields are driven by two stochastic factors, see e.g. Driessen (2005). Thus the structural model in this paper assumes that the default-free yield curve is described by the three-factor Gaussian model of Babbs and Nowman (1999) which seems to fit the US Treasury yield curve quite well. A fully general Gaussian model as in Langetieg (1980) and Dai-Singleton (2002) could equally be assumed without affecting model tractability.

Most structural credit risk models assume that the firm's assets value follows a price process, although this assumption seems difficult to reconcile with the fact that the firm's assets value is not observable. Essentially the whole firm has usually been assumed to be a traded asset or to be perfectly equivalent to a replicating portfolio involving the firm's stock, see e.g. Ericsson (1998). Anyway such assumption has provided a number of interesting corporate finance theoretical insights. When assuming that the firm is a traded asset, prominent models that accommodated stochastic interest rates, such as Bris and de Varenne (1997), Schobel (1999) or Hubner and Francois (2004), have also made tractable assumptions about the firm's payout policy (typically no payout) and about the dynamics of the default barrier. Essentially the expected growth rate of both the default barrier and firm's assets value were driven by stochastic default-free interest rates. In this paper, which is merely concerned about pricing defaultable bonds and credit derivatives, we do not assume that the firm is a traded asset. Default still coincides with the time a diffusion process hits a barrier and the pricing model still provides closed form solutions. The "distance from default" process is latent, but it can at any time be inferred from observed bond market prices and credit derivatives. The model is akin to a reduced form credit risk model in so far as it does not use equity market information.

Most structural credit risk models do not explicitly attempt to price liquidity risk and unexpected default. The concept of unexpected default is familiar from the reduced form credit risk pricing literature and Cathcart and El-Jahel (2003) propose that an issuer's default may be expected, if triggered by the hitting of a default barrier, or "unexpected", if triggered by a Poisson-type event. The model of this paper extends this insight to the case where the stochastic intensities that drive unexpected default and liquidity risk have arbitrary correlation with the factors that drive the default-free yield curve and the barrier hitting diffusion process.

The model makes a recovery assumption that proves both very tractable and quite realistic. A simple approximation in computing the bond recovery value simplifies the bond pricing model by capturing with a single latent factor both the risk of unexpected default and liquidity risk. Thus we need not disentangle the risk of unexpected default from liquidity risk, both of which drive the short term yield spreads of defaultable bonds.

Finally, most structural credit risk models, with the notable exception of Longstaff and Schwartz (1995) do not provide closed form solutions for defaultable floating rate bonds. The model in this paper provides such closed form solutions, which are much simpler than those in Longstaff and Schwartz (1995).

In recent years structural credit risk models have been extended in a number of ways, but the focus of this paper is closer to those models that accommodate a stochastic default-free yield curve, since this seems a key requirement for practical bond pricing purposes. Among such a subset of structural models, those of Longstaff and Schwartz (1995), Bris and de Varenne (1997), Schobel (1999), Cathcart and El-Jahel (1999, 2003), Dufresne and Goldstein (2001), Hubner and Pascal (2004) stand out. Like in Cathcart and El-Jahel (1999, 2003) in this paper the barrier hitting diffusion process is not the value of the firm's assets. Like in Longstaff and Schwartz (1995), Bris and de Varenne (1997), Schobel (1999), Dufresne and Goldstein (2001), Hubner and Pascal (2004), the barrier hitting diffusion process is instantaneously correlated with the instantaneous interest rate. Like in Cathcart and El-Jahel (2003) default can expected or unexpected. Like in Longstaff-Neis-Mittal (2005) both credit and liquidity risk are modelled.

For clarity of exposition the following sections present increasingly general formulations of the bond pricing model. The conclusions follow.

2 The basic model with constant interest rates

This section introduces the model in the most basic setting. For now we assume that the default-free short interest rate r is constant over time, only to relax this assumption later. The assumptions are similar to those in the literature. As in Cathcart and El-Jahel (1998, 2003) default risk is triggered by a latent process S. S is not the value of a traded asset and we leave it un-identified. Although we do not observe S, we can infer S from observed bond prices. In the risk-neutral world S follows the process

$$dS = S \cdot (\mu - \lambda_s \sigma) \cdot dt + S \cdot \sigma \cdot dw_s \quad (1)$$

where μ, λ_s, σ are constant and dw_s is the differential of a Wiener process. μ is the growth rate of S in the real world, λ_s is the market price of S-risk, σ is the volatility parameter. Default occurs the first time S hits the barrier level K from above. The fact that S is latent and is not the value of the firm's assets entails that, unlike most other structural credit risk models, this model does not make use of equity market information for estimation or calibration purposes. Instead, like reduced form credit risk models, this model only makes use of bond and credit derivative market information for parameter calibration or estimation. For pricing purposes all we need to know is the magnitude of $\ln\left(\frac{S}{K}\right)$, not also the identity of S and K. $\ln\left(\frac{S}{K}\right)$ is a measure of "distance from default" that can be inferred from the prices of bonds and credit derivatives. As S is a non-identified latent factor, nothing prevents us from applying the model also to price the credit risk of sovereign or sub-sovereign bonds or government agencies. In other words, by renouncing to assign a theoretical interpretation to S and K, we gain flexibility and still have a viable pricing model.

An issuer, which may be a firm or a government or an agency, has issued a discount bond with face value of 1, with maturity T and with market value $D(t,T)$ at time t. $D(t,T)$ depends on S and time. For now we make the "recovery of Treasury assumption", thus the bond recovery value a fraction π (with $0 \leq \pi \leq 1$) of the bond face value and π is received at time T, which is the maturity date in the bond contract. From past literature we know that under these assumptions

$$D(t,T) = e^{-r(T-t)} \cdot (\pi + (1-\pi) \cdot P(t,T)) \tag{2}$$

where

$$\begin{aligned} P(t,T) &= N\left(\frac{\ln\left(\frac{S}{K}\right) + \left(\mu - \lambda_s \sigma - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - \left(\frac{S}{K}\right)^{\left(1-\frac{2(\mu-\lambda_S\sigma)}{\sigma^2}\right)} N\left(\frac{\ln\left(\frac{K}{S}\right) + \left(\mu - \lambda_s \sigma - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{3}$$

$N(d)$ is the cumulative standard normal distribution function with upper limit of integration equal to d. $P(t,T)$ is the survival probability in the risk-neutral world over the period $]t,T]$ when default can only occur as S hits the barrier K.

3 The model when interest rates are stochastic

This section generalises the previous bond pricing model by introducing stochastic interest rates. Following Babbs and Nowman (1999), we now assume that the default-free short interest rate r is driven by three Gaussian latent factors. Thus

$$r = \boldsymbol{\iota}' \cdot \mathbf{x}. \tag{4}$$

where $\boldsymbol{\iota} = (1, 1, 1)$ and $\mathbf{x} = (x_1, x_2, x_3)'$. x_1, x_2, x_3 are three latent factors whose respective risk-neutral processes are

$$dx_1 = k_1 (\theta_1 - x_1) dt + \sigma_1 dw_1 \tag{5}$$

$$dx_2 = k_1 (\theta_2 - x_2) dt + \sigma_2 dw_2 \tag{6}$$

$$dx_3 = k_3 (\theta_3 - x_3) dt + \sigma_3 dw_3 \tag{7}$$

where k_i, θ_i, σ_i are constant and dw_i are differentials of Wiener processes for $i = 1, 2, 3$. S is instantaneously correlated with x_1, x_2, x_3, i.e.

$$dw_s \cdot dw_1 = \rho_1 dt, dw_s \cdot dw_2 = \rho_2 dt, dw_s \cdot dw_3 = \rho_3 dt.$$

Moreover x_1, x_2, x_3 are also correlated, i.e.

$$dw_1 \cdot dw_2 = \rho_{1,2} dt, dw_1 \cdot dw_3 = \rho_{1,3} dt, dw_2 \cdot dw_3 = \rho_{2,3} dt.$$

Employing a similar notation we can write $\rho_{1,1} = 1$, $\rho_{2,2} = 1$ and $\rho_{3,3} = 1$. As the three factors (x_1, x_2, x_3) are latent, we set $\theta_2 = \theta_3 = 0$, which are conditions equivalent to those in Babbs and Nowman (1999) to guarantee the econometric identification of the model. This term structure model is essentially the one put forward by Babbs and Nowman (1999). This choice is also motivated by the good empirical performance of multi-factor Gaussian models in fitting the US yield curve, as documented in Dai-Singleton (2002). Moreover Gaussian models do not suffer from the admissibility restrictions that affect general affine model specifications as explained in Duffie-Kan (1996). On the other hand even a flat and constant yield curve may be good enough when pricing credit default swaps, so that this extended structural model is of more interest to price defaultable bonds than credit default swaps.

In this setting the value of a zero coupon bond is denoted as $D(S, \mathbf{x}, t)$ or more simply as D. $Z(\mathbf{x}, t)$ denotes the value of a default-free zero coupon bond with the same maturity and face value as $D(S, \mathbf{x}, t)$. For now we retain the "recovery of Treasury" assumption, in keeping with other structural models that assume a stochastic default free term structure, see e.g. Longstaff and Schwartz (1995) or Cathcart and El-Jahel (1998). Then the absence of arbitrage opportunities implies that D now satisfies the following equation

$$\begin{aligned}
&\frac{\partial D}{\partial t} + \frac{\partial^2 D}{\partial S^2}\sigma^2 S^2 + \frac{\partial D}{\partial S}\left(\mu - \lambda_S \sigma\right) S - \left(x_1 + x_2 + x_3\right) D \qquad (8) \\
&+ \frac{\partial^2 D}{\partial x_1 \partial S}\rho_1 \sigma_1 \sigma S + \frac{\partial^2 D}{\partial x_2 \partial S}\rho_2 \sigma_2 \sigma S + \frac{\partial^2 D}{\partial x_3 \partial S}\rho_3 \sigma_3 \sigma S \\
&+ \frac{\partial^2 D}{\partial x_1 \partial x_2}\rho_{1,2}\sigma_2 \sigma_1 + \frac{\partial^2 D}{\partial x_1 \partial x_3}\rho_{3,1}\sigma_3 \sigma_1 + \frac{\partial^2 D}{\partial x_3 \partial x_2}\rho_{3,2}\sigma_3 \sigma_2 \\
&+ \frac{\partial^2 D}{\partial x_3^2}\sigma_3^2 + \frac{\partial D}{\partial x_3}k_3\left(\theta_3 - x_3\right) + \frac{\partial^2 D}{\partial x_2^2}\sigma_2^2 + \frac{\partial D}{\partial x_2}k_2\left(\theta_2 - x_2\right) + \frac{\partial^2 D}{\partial x_1^2}\sigma_1^2 + \frac{\partial D}{\partial x_1}k_1\left(\theta_1 - x_1\right) = 0
\end{aligned}$$

subject to

$$D(S \to \infty, \mathbf{x}, t) \to Z(\mathbf{x}, t) \tag{9}$$

$$D(K, \mathbf{x}, t) = Z(\mathbf{x}, t) \cdot \pi \tag{10}$$

$$D(S, \mathbf{x}, T) = 1. \tag{11}$$

The first condition states the as $S \to \infty$ default becomes impossible and the value $D(S, \mathbf{x}, t)$ of the defaultable bond approaches the default-free value $Z(\mathbf{x}, t)$. The second condition states that when $S = K$ default is triggered and the bond value equals the recovery value $Z(\mathbf{x}, t) \cdot \pi$ according to the "recovery of Treasury" assumption. The last condition is the usual terminal condition for a bond with face value of 1. The solution to equation 8 and to its conditions is

$$D(S, \mathbf{x}, t) = Z(\mathbf{x}, t) \cdot \left(\pi + (1 - \pi) \cdot P^T(t, T)\right) \tag{12}$$

where

$$Z(\mathbf{x}, t) = \exp\left(A(t, T) - \sum_{i=1}^{3} x_i \cdot B_i(t, T)\right)$$
$$B_i(t, T) = \frac{1 - e^{-k_i(T-t)}}{k_i}$$

and $A(t, T)$ solves the ODE

$$\frac{\partial A(t,T)}{\partial t} + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} B_i(t,T)\, B_j(t,T)\, \rho_{i,j}\sigma_i\sigma_j = 0 \tag{13}$$

and

$$P^T(t,T) = N\left(\frac{\ln\left(\frac{S_t}{K}\right) + \sigma\int_t^T \lambda(u)\,du}{\sigma\sqrt{T-t}}\right) - \left(\frac{S_t}{K}\right)^{-\frac{2}{\sigma}\frac{\int_t^T \lambda(u)du}{T-t}} N\left(\frac{\ln\left(\frac{K}{S_t}\right) + \sigma\int_t^T \lambda(u)\,du}{\sigma\sqrt{T-t}}\right) \tag{14}$$

where $N(d)$ is the cumulative standard normal distribution function with upper limit of integration equal to d and with

$$\int_t^T \lambda(u)\,du = \left(\frac{m}{\sigma} - \lambda_s - \frac{1}{2}\sigma\right)(T-t) - \sum_{i=1}^{3}\frac{\rho_i\sigma_i}{k_i}\left(T - t - \frac{1-e^{-k_i(T-t)}}{k_i}\right).$$

The solution to ODE 13 can be quickly computed numerically through the Rounge-Kutta method or in closed form. $P^T(t,T)$ is the survival probability over the period $[t,T]$ in a world that is forward risk neutral with respect to $Z(\mathbf{x},t)$. We notice that when $\rho_1 = \rho_2 = \rho_3 = 0$, $P^T(t,T)$ in equation 14 becomes equal to the more familiar survival probability in the risk-neutral world given in equation 3. In this setting the credit spread implied by the defaultable zero coupon bond value $D(S,\mathbf{x},t)$ is $-\frac{\ln\left(\pi+(1-\pi)\cdot P^T(t,T)\right)}{T-t}$. Although the above formulae give the value of a defaultable zero coupon bond, they can immediately be used also to value a coupon bond, since a coupon bond is equivalent to a portfolio of zero coupon bonds. We notice that equation 14 is still valid even when θ_1 is chosen to be a deterministic function of time to be calibrated to the default-free yield curve as shown in Hull and White (1990).

3.1 Quasi recovery of face value assumption and CDS valuation

So far we have maintained the tractable "recovery of Treasury" assumption. Now we introduce a more accurate assumption about the bond recovery value upon default, an assumption that is as tractable as the "recovery of Treasury" assumption and that approximates as the more accurate "recovery of face value" assumption. We call this assumption "quasi recovery of face value" (QRF). If today's date is t and T is the bond maturity date, the period $[t, T]$ is the bond residual life. We set m dates during $[t, T]$ such that $t \leq T_1 < T_2 < .. < T_m = T$ and such that $(T_k - T_{k-1})$ is constant for $k = 2, 3, ..m$. Denote with $R(t, T_{k-1}, T_k)$ the value at time $t \leq T_1$ of a claim that pays 1 at time T_k if default occurs in the time interval $]T_{k-1}, T_k]$. It follows that

$$R(t, T_{k-1}, T_k) = Z(t, T_k) \cdot E_t^k \left(\frac{1_{\tau > T_{k-1}} \cdot Z(T_{k-1}, T_k)}{Z(T_{k-1}, T_k)} \right) - D(t, T_k) \qquad (15)$$

where $E_t^k(..)$ denotes time t conditional expectation in the $Z(t, T_k)$ forward risk neutral measure, where τ is the default time, where $1_{\tau > T_{k-1}}$ is the indicator function of the survival event $\tau > T_{k-1}$ and where $Z(t, T_k) \cdot E_t^k \left(\frac{1_{\tau > T_{k-1}} \cdot Z(T_{k-1}, T_k)}{Z(T_{k-1}, T_k)} \right)$ is the present value of a defaultable claim that pays off $Z(T_{k-1}, T_k)$ at T_k. Then notice that

$$E_t^k\left(\frac{1_{\tau>T_{k-1}}\cdot Z\left(T_{k-1},T_k\right)}{Z\left(T_{k-1},T_k\right)}\right)=E_t^k\left(1_{\tau>T_{k-1}}\right)=P^k\left(t,T_{k-1}\right) \tag{16}$$

and $P^k\left(t,T_{k-1}\right)$ is the survival probability up to time T_{k-1} in the $Z\left(t,T_k\right)$ forward risk neutral measure. It follows that we can write

$$R\left(t,T_{k-1},T_k\right)=Z\left(t,T_k\right)\left(P^k\left(t,T_{k-1}\right)-P^k\left(t,T_k\right)\right). \tag{17}$$

The expression $P^k\left(t,T_{k-1}\right)-P^k\left(t,T_k\right)$ denotes the probability calculated at time t in the $Z\left(t,T_k\right)$ forward risk neutral measure that default will occur in the time interval $]T_{k-1},T_k]$. We can now determine the present value of what bond holders expect to recover upon default. At time t such present value is equal to the value of a claim that pays π at T_k if default time τ falls during the interval $]T_{k-1},T_k]$ for $k=1,2,..m$, and it is equal to

$$\pi\sum_{k=1}^m R\left(t,T_{k-1},T_k\right). \tag{18}$$

We can readily compute this expression since we have closed form solutions for $Z\left(t,T_k\right)$ and $P^k\left(t,T_k\right)$ from above. Thus this QRF assumption is as tractable as the "recovery of Treasury" assumption. Moreover as the bond residual life $[t,T]$ is partitioned in a greater number m of sub-intervals, the bond recovery value approaches the recovery value we obtain under the proper "recovery of face" assumption, which is commonly regarded as the most realistic and least tractable recovery assumption. According to the "recovery of face" assumption

π is received at the exact time of default, rather than later. We conclude that the recovery assumption that gives $\pi \sum_{k=1}^{m} R(t, T_{k-1}, T_k)$ seems a good and tractable approximation to the "recovery of face" assumption. Then the value of a defaultable fixed coupon bond with face value of 1 and which promises to pay coupons at times T_i for $i = 1, 2, ..n$ equal to $c(T_i - T_{i-1})$ is

$$C(t) = \sum_{i=1}^{n} c(T_i - T_{i-1}) D(t, T_i) + D(t, T_n) + \pi \sum_{k=1}^{m} R(t, T_{k-1}, T_k). \tag{19}$$

Notice also that we can readily derive the following closed form solution for CDS spreads

$$s_{cds} = \frac{(1-\pi) \cdot \sum_{k=1}^{m} R(t, T_{k-1}, T_k)}{\sum_{k=1}^{m} (T_k - T_{k-1}) \cdot D(t, T_k)}. \tag{20}$$

In stating this formula for a CDS spread we retain all previous assumptions, in particular the assumption about the bond recovery value to be received at times T_k. Although not necessary, we also assume, for simplicity and without much loss in accuracy, that the CDS fee payment dates are also T_k, so that each fee payment amounts to $c(T_k - T_{k-1})$.

3.2 Valuation of floating rate bonds

The above results also imply convenient closed form solutions to price defaultable floating rate bonds. Consider such a bond with face value of 1 and promising to pay coupons at times T_k for $k = 1, 2, ..n$ equal to

$$L_{k-1} \cdot (T_k - T_{k-1})$$

where L_{k-1} is the Libor rate for the period $[T_{k-1}, T_k]$. t is today's date, T_1 is the next coupon payment date and the bond maturity date is $T_n = T$. For simplicity we compute the bond value at time t net of the value of the coupon payment due at time T_1. At $t \leq T_1$ the value of the defaultable floating rate bond $C'(t)$ is

$$\begin{aligned} C'(t) &= \sum_{k=1}^{n} F(t, T_{k-1}, T_k) D(t, T_k) \\ &+ D(t, T_n) + \pi \sum_{k=1}^{n} R(t, T_{k-1}, T_k). \end{aligned} \tag{21}$$

where $F(t, T_{k-1}, T_k) = \left(\frac{Z(t,T_{k-1})}{Z(t,T_k)} - 1 \right)$ denotes the default-free forward rate at time t for the period $[T_{k-1}, T_k]$. We notice that $F(T_{k-1}, T_{k-1}, T_k) = L_{k-1} \cdot (T_k - T_{k-1})$. Again $P^k(t, T_k)$ is the survival probability in the $Z(t, T_k)$ forward neutral measure. The first line of equation 21 is the present value of the defaultable floating rate coupon payments. To clarify the first line notice that the time T_{k-1} value of a defaultable floating rate coupon that is set at time T_{k-1} and paid at time T_k is $\left(\frac{1}{Z(T_{k-1},T_k)} - 1 \right)$ provided no default occurs until T_k. Thus, since the present value at time t of a default-free floating coupon is $Z(t, T_{k-1}) - Z(t, T_k) = Z(t, T_k) \cdot F(t, T_{k-1}, T_k)$, the value of the defaultable floating coupon is

$$\begin{aligned}
\left(Z\left(t,T_{k-1}\right)-Z\left(t,T_{k}\right)\right)\cdot P^{k}\left(t,T_{k}\right) &= Z\left(t,T_{k}\right)\cdot P^{k}\left(t,T_{k}\right)\cdot F\left(t,T_{k-1},T_{k}\right) \quad (22)\\
&= D\left(t,T_{k}\right)\cdot F\left(t,T_{k-1},T_{k}\right).
\end{aligned}$$

Of course this formula assumes that default entails the entire loss of all coupon payments.

3.3 Comparative static

Without much loss in generality, in Exhibit 1 we concentrate on credit spreads on zero coupon bonds for various maturities. The base case assumes $\frac{S}{K}=3$, $m=0.05$, $\lambda=0$, $\sigma=0.3$, $\sigma_1=0.01$, $\rho=0$, $k=0.1$, $\pi=0.5$ and it assumes for simplicity that $r=x_1$. The other columns assume the same parameters as in the base case, but for the different parameter values shown in the respective column headings. As expected credit spreads rise with σ, the volatility of S. This emerges from comparing the right-most column with heading "$\sigma=0.3$" with the base case column, which assumes $\sigma=0.2$. We can interpret the results in the other columns in a similar way. To do so notice that equation 14 implies that credit spreads decrease as $\lambda\left(u\right)$ rises for $t\leq u\leq T$. We recall that $\int_t^T\lambda\left(u\right)du=\left(\frac{m}{\sigma}-\lambda_s-\frac{1}{2}\sigma\right)\left(T-t\right)-\sum_{i=1}^{3}\frac{\rho_i\sigma_i}{k_i}\left(T-t-\frac{1-e^{-k_i(T-t)}}{k_i}\right)$ and that, when $\rho_1=\rho_2=\rho_3=0$, $P^T\left(t,T\right)$ in equation 14 becomes equal to the more familiar survival probability in the risk-neutral world of equation 3.

Then as the correlation parameter ρ_1 rises and as $k_1 > 0$, $\int_t^T \lambda(u)\,du$ decreases and credit spreads rise. In other words, credit spreads rise with the degree of correlation between the default free short interest rate r and the latent default process S. This is shown in the columns "$\rho_1 = -0.5$" and "$\rho_1 = 0.5$". The sensitivity of credit spreads to σ_1 and k_1 depends on the sign of the instantaneous correlation ρ_1.

When $\rho_1 > 0$ ($\rho_1 < 0$) credit spreads increase (decrease) in σ_1. This emerges by comparing the columns headed "$\rho_1 = 0.5$" and "$\rho_1 = 0.5, \sigma_1 = 0.02$". When $\rho_1 > 0$ ($\rho_1 < 0$) credit spreads decrease (rise) as the mean reversion speed k_1 rises, i.e. as the conditional and unconditional variance of the instantaneous interest rate r decreases. This is shown in the columns headed "$\rho_1 = 0.5$" and "$\rho_1 = 0.5, k_1 = 0.5$". Generally, as the time to maturity $(T-t)$ increases, $\int_t^T \lambda(u)\,du$ becomes more sensitive to changes in the parameters ρ_1, σ_1 and k_1 and so do credit spreads.

[Exhibit 1 here]

We notice that, although we have assumed a constant default barrier K, approximate closed form solutions are still available if K follows the process $dK = \alpha(t) K dt$. In such case in the risk-neutral world $d\left(\frac{S}{K}\right) = \frac{S}{K}(\mu - \lambda_s \sigma - \alpha(t))\,dt + \frac{S}{K}\sigma dw_s$ and at most only $\int_t^T \lambda(u)\,du$ may have to be computed numerically, while equation 14 would still be valid.

4 Unexpected default and liquidity risk

So far we have assumed that default can only take place in an "expected" way as the latent process S hits the barrier K and we have omitted any bond liquidity risk considerations. Now we introduce the possibility of "unexpected" default as well as the pricing of liquidity risk. These two extensions to the above structural model are presented together since both "unexpected" default and liquidity risk seem to drive short term yield spreads of defaultable bonds. Although not necessary to retain tractable solutions, for the sake of simplicity of exposition we now assume that $r = x_1$ and the two other latent factors x_2 and x_3 will be re-interpreted below. We retain all the other assumptions as above. We also make the following two additional conjectures.

Conjecture 1 *Unexpected default.*

Default can not only occur in an "expected" way as S hits K, but also in an "unexpected" way. As in Cathcart and El-Jahel (2003), in any infinitesimal time period dt there is a probability λdt that default may unexpectedly occur. When pricing bonds or credit derivatives λ is the risk-neutral default intensity, i.e. the intensity in the risk-neutral world. Unexpected default may correspond to the discovery of substantial misgivings in the firm's accounts or other unforeseen adverse event. If the risk of unexpected default has no systematic component, such risk commands no premium and the real and the risk-neutral intensity of unexpected default are the same. We assume that $\lambda = x_2$,which implies that λ may turn negative and is correlated with S. The possibility of negative λ

seems tolerable and the correlation between λ and S is likely to be negative if the probability of unexpected default tends to rise S decreases. As we assume $\theta_2 \neq 0$, λ seems unlikely to turn negative and, even when λ does turn negative, overall credit spreads are unlikely to. Whether the default event is expected or unexpected, the recovery value of the bond is a fraction π of the bond face value, with $0 \leq \pi \leq 1$. For now we make the "recovery of Treasury assumption", i.e. we assume that π is received at time T, the bond contractual maturity date.

Conjecture 2 *Liquidity risk.*

It is well documented by now, see e.g. Perradin and Taylor (2003), that bond prices are also affected by liquidity risk, i.e. the risk for an investor of having to sell the bond at a discount if the need to immediately sell the bond should suddenly arise. To price liquidity risk we assume that during any infinitesimal period dt there is a risk-neutral probability ldt of an investor suddenly needing to sell the bond for a discounted price equal to $(1 - q)$, where q is a constant such that $q \geq 0$ and expressed the discount as a fraction of the bond market value. We assume that $lq = x_3$ and that $\theta_3 \neq 0$. Again this implies that l may turn negative and that l is correlated with S, r and λ. A negative l is not so worrying since a negative liquidity premium seems possible when the bond to be valued is not very liquid.

Under these assumptions, in order to take unexpected default and liquidity risk into account, the value of a defaultable and not perfectly liquid zero coupon bond becomes

$$D(S,\mathbf{x},t) = Z(r = x_1, lq = x_3, t)\cdot\pi + (1-\pi)\cdot Z(r = x_1, \lambda = x_2, lq = x_3, t)\cdot P^T(t,T) \tag{23}$$

where $Z(r = x_1, lq = x_3, t)$ is the same as $Z(\mathbf{x}, t)$ when $\mathbf{x} = (r, 0, lq)'$ and $Z(r = x_1, \lambda = x_2, lq = x_3, t)$ is the same as $Z(\mathbf{x}, t)$ when $\mathbf{x} = (r, \lambda, lq)'$. $Z(r = x_1, lq = x_3, t)$ is the value of a default-free zero coupon bond that is exposed to liquidity risk. $Z(r = x_1, \lambda = x_2, lq = x_3, t)$ is the value of a defaultable zero coupon bond that is exposed to both liquidity risk and "unexpected" default, but not to "expected" default risk. $Z(r = x_1, \lambda = x_2, lq = x_3, t)\cdot P^T(t,T)$ is the value of a defaultable zero coupon bond that is exposed to liquidity risk, "unexpected" default risk and "expected" default risk and that recovers nothing in case of default. The last formula for $D(S,\mathbf{x},t)$ reflects the fact that bond holders recover π at time T in case of default, whether default is expected or unexpected, and it suggests that we need to disentangle unexpected default risk from liquidity risk. One single stochastic factor set equal to $(lq + \lambda)$ cannot capture both "unexpected" default risk and "liquidity risk" at once.

4.0.1 QRF assumption and no need to disentangle unexpected default and liquidity risk

We now consider that, as we make the QRF assumption instead of the "recovery of Treasury assumption", it is no longer necessary to disentangle "unexpected" default and liquidity risk. Thus we now assume that π is to be received at time

T_k as assumed above according to the QRF assumption. As the number m of time intervals used to compute the bond recovery value rises, we can employ the following accurate approximation

$$\begin{aligned} R\left(t, T_{k-1}, T_k\right) &= Z\left(t, T_k\right)\left(P^k\left(t, T_{k-1}\right) - P^k\left(t, T_k\right)\right) \\ &\simeq D\left(t, T_{k-1}\right) - D\left(t, T_k\right). \end{aligned} \tag{24}$$

This approximation makes it not necessary to disentangle "unexpected" default risk from liquidity risk, which seems an interesting simplification for pricing defaultable bonds. In other words, in order to take "unexpected" default and liquidity risk into account, we just need to employ an instantaneous discount rate equal to $r + (lq + \lambda)$ and we need not disentangle unexpected default risk from liquidity risk. One single stochastic factor set equal to $(lq + \lambda)$ can capture both "unexpected" default risk and liquidity risk at once. This consideration applies to the pricing of bonds, but it does not so much apply to the pricing of credit default swaps, as Longstaff-Neis-Mittal (2005) highlighted how credit default swap spreads are not driven by liquidity risk.

5 Conclusions

This paper has presented an extended structural credit risk model that provides closed form solutions to price fixed and floating rate bonds and credit default swaps. In its most general formulation the model has "extended" previous structural credit risk models as follows. The default-free term structure is described by a multi-factor Gaussian rather than by a more restrictive single factor model. The latent factors that drive the default-free term structure are correlated with the default process. The default process is latent and is not the firm's assets value. Default may be expected or unexpected. Unexpected default and liquidity risks are correlated with the other factors driving "expected" default and the default-free yield curve. A tractable and accurate recovery assumption is proposed. To price defaultable bonds, it is not necessary to disentangle unexpected default from liquidity risk, both of which drive short term credit spreads.

6 Appendix: derivation of formula

Substituting the solution $D(S,\mathbf{x},t) = Z(\mathbf{x},t)\cdot\left(\pi + (1-\pi)\cdot P^T(t,T)\right)$ into 8, we find that $P^k(t,T_k)$ satisfies

$$\frac{\partial P^T(t,T)}{\partial t} + \frac{\partial^2 P^T(t,T)}{\partial S^2}\frac{1}{2}\sigma^2 S^2 + \frac{\partial P^T(t,T)}{\partial S} S\left(m - \lambda_s\sigma - \sum_{i=1}^3 \rho_i\sigma\sigma_i\frac{1-e^{-k_i(T-t)}}{k_i}\right) = 0 \tag{25}$$

subject to $\lim_{S\to\infty} P^T(t,T) \to 1$, $\lim_{S\to K} P^T(t,T) \to 0$, $P^T(T,T) = 1$. $P^T(t,T)$ is the survival probability in the $Z(\mathbf{x},t)$ forward risk-neutral world. To solve this PDE define $X = \frac{1}{\sigma}\ln\left(\frac{S}{K}\right)$. Then using Ito's lemma we obtain

$$dX = \lambda(t)\,dt + dw \tag{26}$$

with

$$\lambda\left(t\right)=\left(\frac{m}{\sigma}-\lambda_{s}-\sum_{i=1}^{3}\rho\sigma_{i}\frac{1-e^{-k_{i}(T-t)}}{k_{i}}-\frac{1}{2}\sigma\right). \tag{27}$$

Let S_T denote S at at time T and S_t denote S at time t. X_t and X_T have similar meaning. The probability at time t that, given S_t, $S_T > K$ is denoted by

$$P\left[S_T > K, S_t\right] = P\left[X_T > 0, X_t\right].$$

Now we define $\inf X_{t,T} = \min\left(X_u, t \le u \le T\right)$ and $P\left[X_T \ge 0, \inf X_{t,T} \le 0, X_t\right]$ as the probability that $X_T > 0$ and $\inf X_{t,T} \le 0$ given X_t. Then

$$P\left[X_T \ge 0, \inf X_{t,T} > 0, X_t\right] = P\left[X_T \ge 0, X_t\right] - P\left[X_T \ge 0, \inf X_{t,T} \le 0, X_t\right].$$

If $\lambda\left(t\right) = 0$, dX is a Brownian motion and by the *reflection principle*

$$P\left[X_T \ge 0, \inf X_{t,T} > 0, X_t\right] = P\left[X_T > 0, X_t\right] - P\left[X_T < 0, X_t\right].$$

This is the probability of $S_T > K$ and $S_t > K$ for $t \le u \le T$ assuming that $\lambda\left(t\right) = 0$. Similarly we obtain the probability density

$$p\left[X_T, \inf X_{t,T} > 0, X_t\right] = n\left(\frac{X_T - X_t}{\sqrt{T-t}}\right) - n\left(\frac{-X_T - X_t}{\sqrt{T-t}}\right) \tag{28}$$
$$= \frac{1}{\sqrt{2\pi\left(T-t\right)}} e^{-\frac{(X_T - X_t)^2}{2(T-t)}} - \frac{1}{\sqrt{2\pi\left(T-t\right)}} e^{-\frac{(X_T + X_t)^2}{2(T-t)}}.$$

When $\lambda\left(t\right) \neq 0$, we can find $P\left[X_T \geq 0, \inf X_{t,T} > 0, X_t\right]$ using the Girsanov theorem, according to which, if dw is a standard Brownian motion under the $p\left[X_T, \inf X_{t,T} > 0, X_t\right]$ measure, then $dw - \lambda\left(t\right) dt$ is a Brownian motion under the $p^*\left[X_T, \inf X_{t,T} > 0, X_t\right]$ measure such that

$$p^*\left[X_T, \inf X_{t,T} > 0, X_t\right] = p\left[X_T, \inf X_{t,T} > 0, X_t\right] \cdot e^{\int_t^T \lambda(u) dw_u - \frac{1}{2}\int_t^T \lambda(u)^2 du} \tag{29}$$
$$= \frac{1}{\sqrt{2\pi\left(T-t\right)}} \left(e^{-\frac{(X_T - X_t)^2}{2(T-t)}} - e^{-\frac{(X_T + X_t)^2}{2(T-t)}}\right) \cdot e^{\int_t^T \lambda(u) dX_u - \frac{1}{2}\int_t^T \lambda(u)^2 du}$$
$$= \frac{e^{-\frac{\left(X_T - X_t - \int_t^T \lambda(u) du\right)^2}{2(T-t)}}}{\sqrt{2\pi\left(T-t\right)}} - e^{\frac{-2X_t X_T}{T-t}} \frac{e^{-\frac{\left(X_T - X_t - \int_t^T \lambda(u) du\right)^2}{2(T-t)}}}{\sqrt{2\pi\left(T-t\right)}}.$$

where in the last line we have made use of the fact that $\left(X_T + X_t\right)^2 = \left(X_T - X_t + 2X_t\right)^2 = \left(X_T - X_t\right)^2 + 4X_t X_T$. It also follows that

$$P^*\left[X_T \geq 0, \inf X_{0,T} > 0, X_0\right] = \int_0^\infty p_T^*\left[X_T, \inf X_{0,T} > 0, X_0\right] dX_T \tag{30}$$
$$= \frac{\int_0^\infty e^{-\frac{\left(X_T - X_t - \int_t^T \lambda(u) du\right)^2}{2(T-t)}} dX_T}{\sqrt{2\pi\left(T-t\right)}} - \frac{\int_0^\infty e^{\frac{-2X_t X_T}{T-t}} e^{-\frac{\left(X_T - X_t - \int_t^T \lambda(u) du\right)^2}{2(T-t)}} dX_T}{\sqrt{2\pi\left(T-t\right)}}.$$

We notice that

$$\frac{\int_0^\infty e^{-\frac{\left(X_T - X_t - \int_t^T \lambda(u)du\right)^2}{2(T-t)}} dX_T}{\sqrt{2\pi\left(T-t\right)}} = N\left(\frac{X_t + \int_t^T \lambda\left(u\right)du}{\sqrt{T-t}}\right)$$

and

$$\frac{\int_0^\infty e^{\frac{-2X_tX_T}{T-t}} e^{-\frac{\left(X_T - X_t - \int_t^T \lambda(u)du\right)^2}{2(T-t)}} dX_T}{\sqrt{2\pi\left(T-t\right)}} = e^{-\frac{2X_t}{T-t}\int_t^T \lambda(u)du} N\left(\frac{\int_t^T \lambda\left(u\right)du - X_t}{\sqrt{T-t}}\right).$$

Thus

$$P^*\left[X_T \geq 0, \inf X_{t,T} > 0, X_t\right] = N\left(\frac{X_t + \int_t^T \lambda\left(u\right)du}{\sqrt{T-t}}\right) - e^{-\frac{2X_t}{T-t}\int_t^T \lambda(u)du} N\left(\frac{\int_t^T \lambda\left(u\right)du - X_t}{\sqrt{T-t}}\right).$$

$P^*\left[X_T \geq 0, \inf X_{t,T} > 0, X_t\right]$ is the same as $P^T\left(t,T\right)$ where

$$P^T\left(t,T\right) = N\left(\frac{\ln\left(\frac{S_t}{K}\right) + \sigma\int_t^T \lambda\left(u\right)du}{\sigma\sqrt{T-t}}\right) - \left(\frac{S_t}{K}\right)^{-\frac{2}{\sigma}\frac{\int_t^T \lambda(u)du}{T-t}} N\left(\frac{\ln\left(\frac{K}{S_t}\right) + \sigma\int_t^T \lambda\left(u\right)du}{\sigma\sqrt{T-t}}\right) \tag{31}$$

and where $\int_t^T \lambda\left(u\right)du = \left(\frac{m}{\sigma} - \lambda_s - \frac{1}{2}\sigma\right)\left(T-t\right) - \sum_{i=1}^3 \rho_i\sigma_i\frac{1-e^{-k_i(T-t)}}{k_i}$. This gives the result shown in the text.

QED

References

[1] Babbs S.H. and Nowman K B., 1999, "Kalman Filtering of Generalized Vasicek Term Structure Models", Journal of Financial and Quantitative Analysis 34, 1, 115-130.

[2] Bakshi G., Madan D. and Zhang F.X., 2006, "Investigating the Role of Systematic and Firm-Specific Factors in Default Risk: Lessons from Empirically Evaluating Credit Risk Models" , The Journal of Business, vol. 79, 1955–1987.

[3] Black F. and Scholes M., 1973, "The pricing of options and corporate liabilities", Journal of political economy 637-659.

[4] Briys E. and de Varenne F., 1997, Valuing risky fixed rate debt: an extension, Journal of financial and quantitative analysis 32, n.2, 239-248.

[5] Cathcart L. and El-Jahel Lina, 1998, Valuation of defaultable bonds, Journal of fixed income, 65-79.

[6] Cathcart L. and El-Jahel Lina, 2003, Semianalytical pricing of defaultable bonds in a signaling jump-default model, Journal of computational finance 6, n.3.

[7] Cox J. and Ingersoll J.E.jr and Ross S.A., 1985, A theory of the term structure of interest rates, Econometrica 53, n.2, 384-408.

[8] Dai Q. and Singleton K, 2002, "Expectation puzzles, time-varying risk premia, and affine models of the term structure", Journal of Financial Economics 63, 415-441.

[9] Driessen J., 2005, "Is Default Event Risk Priced in Corporate Bonds?", Review of Financial Studies 18, n.1, 165-195.

[10] Duffie D. and Kan R., 1996, "A yield-factor model of interest rates", Mathematical Finance 6, n.4, 379–406.

[11] Duffie D. and Lando D., 2001, Term Structures of Credit Spreads with Incomplete Accounting Information, Econometrica, Vol. 69, pp. 633-664.

[12] Dufresne P. and Goldstein R., 2001, Do credit spreads reflect stationary leverage rations?, Journal of Finance 56, 1929 - 1957.

[13] Elton E., Gruber M., Agrawal D. and Mann C., 2001, Explaining the rate spread on corporate bonds, Journal of Finance 56, 247 - 277.

[14] Ericsson J. and Renault O., 2001, Liquidity and credit risk, EFA 2003.

[15] Ericsson J. and Reneby J., 1998, A framework for pricing corporate securities, SSE/EFI working paper series in economics and finance n. 314.

[16] Ericsson J. and Reneby J., 1998, On the tradeability of firm's assets, Working paper 89 SSE.

[17] Francois P. and Hubner G., 2004, "Credit derivatives with multiple debt issues", Journal of Banking and Finance 28, 997-1021.

[18] Hull J. and White A., 1990, "Pricing interest rate derivative securities", Review of Financial Studies 3, 573-592.

[19] Kim J., Ramaswamy K. and Sundaresan S., 1993, "Does default risk in coupons affect the valuation of corporate bonds?: A contingent claims model", Financial Management 117-131.

[20] Langetieg T., 1980, "A multivariate model of the term structure", Journal of Finance 35, n.1, 71-97.

[21] Leland H., 1994a, "Corporate debt value, bond covenants and optimal capital structure", Journal of finance 49, n.4, 1213-1252.

[22] Longstaff F., Neis S. and Mittal E., 2005, "Corporate yield spreads; default risk or liquidity? New evidence from the credit default swap market", Journal of Finance 60, n.5, 2213-2253.

[23] Longstaff F. and Schwartz E., 1995, A simple approach to valuing risky fixed and floating rate debt, Journal of finance 50, n.3, 789-819.

[24] Merton R., 1974, On the pricing of corporate debt: the risk structure of interest rates, Journal of finance 29, 449-470.

[25] Perraudin W. and Taylor A., 2003, "Liquidity and bond market spreads", presented at European Finance Association 2003.

[26] Saa-Requejo J. and Santa-Clara P., 1999, Bond pricing with default risk, Working paper UCLA.

[27] Schobel L., 1999, A note on the valuation of risky corporate bonds, OR Spectrum, 35-47.

[28] Vasicek O., 1977, "An equilibrium characterisation of the term structure", Journal of Financial Economics 5, 177-188.

[29] Zhou C., 2001, A jump-diffusion approach to modelling credit risk and valuing defaultable securities, Journal of banking and finance 25, 2015-2040.

EXHIBIT 1

(T-t) in years	Base case	ρ_1=-0.5	ρ_1=0.5	ρ_1=0.5, σ_1=0.02	ρ_1=0.5, k_1=0.5	σ=0.3
1	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%
2	0.04%	0.04%	0.04%	0.04%	0.04%	0.57%
3	0.21%	0.21%	0.22%	0.23%	0.22%	1.03%
4	0.37%	0.36%	0.39%	0.40%	0.38%	1.17%
5	0.47%	0.45%	0.49%	0.50%	0.48%	1.18%
6	0.51%	0.49%	0.53%	0.56%	0.52%	1.14%
7	0.53%	0.51%	0.55%	0.58%	0.54%	1.09%
8	0.53%	0.50%	0.55%	0.58%	0.54%	1.04%
9	0.52%	0.49%	0.55%	0.57%	0.53%	1.00%
10	0.51%	0.48%	0.53%	0.56%	0.52%	0.96%
11	0.49%	0.47%	0.52%	0.55%	0.50%	0.92%
12	0.48%	0.45%	0.51%	0.53%	0.49%	0.89%
13	0.46%	0.44%	0.49%	0.52%	0.47%	0.86%
14	0.45%	0.42%	0.48%	0.51%	0.46%	0.83%
15	0.44%	0.41%	0.46%	0.49%	0.45%	0.81%
16	0.43%	0.40%	0.45%	0.48%	0.44%	0.79%
17	0.42%	0.39%	0.44%	0.47%	0.42%	0.76%
18	0.41%	0.38%	0.43%	0.46%	0.42%	0.74%
19	0.40%	0.37%	0.42%	0.45%	0.41%	0.73%
20	0.39%	0.37%	0.41%	0.44%	0.40%	0.71%