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Horizontal Equity and Progression when Equivalence Scales are not Constant

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Abstract

Household needs must be taken into account when designing an equitable income tax. If the equivalence scale is income dependent it is not transparent how to achieve equity. In this paper we explore the question of horizontal equity and the implications (vertical equity), when the equivalence scale depends on income level. In particular an 'equal progression among equals' criterion is articulated and shown to be achievable along with horizontal equity under specified conditions.

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> *"... the most interesting and important issues involving the application of equivalence scales to tax equity questions are intimately bound up with the variation of equivalence scales with the level of income"*
> (Seneca and Taussig, 1971, p. 255).

1. Introduction[1]

Ever since the days of Vickrey (1947) it has been well-appreciated that household needs must be taken into account in designing an equitable income tax. Various devices have become familiar: exemptions, deductions, tax credits, and the use of income splitting. Some of these approaches are consistent with the equivalence scale methodology. Thus the exemption can be thought of as an absolute equivalence scale rendering taxable incomes equivalent by fixed subtractions (Lambert and Yitzhaki (1997)). Similarly income splitting as e.g. in the French Quotient Familial corresponds to deflating incomes by fixed relative factors for tax purposes and then aggregating liabilities of the equivalent adults (Ebert (1997) and Lambert (2001)).

This is all well and good, but recent theoretical research (Donaldson and Pendakur (1999)) demonstrates that income dependent equivalence scales provide less restrictive household demands and can, under certain conditions, be uniquely estimated from demand data.[2] Empiricists have long been interested in income dependent equivalence scales. As Seneca and Taussig (1971) noted some time ago (see above) tax equity must take such income dependency into account.

In this paper we explore the question of horizontal equity in the income tax and the implications for progression (vertical equity) when the equivalence scale depends on income level. We identify three different principles which might be used to secure horizontal equity (namely the 'equal tax treatment of equals'). We go on to examine whether 'equal progression among equals' obtains under each of these principles. The analysis uncovers some interesting properties of constant equivalence scales in the process. It emerges that both classical horizontal equity and the 'equal progression among equals' criterion can be met using one of our three equity principles in conjunction with one of Donaldson and Pendakur's relative scales (exhibiting constant income elasticity).

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2 See also Habib (1979) and Conniffe (1992).

The structure of the paper is as follows. In section 2 we establish the notation and definition and some examples of (income dependent) equivalence scales. Section 3 sets out the three equity principles we advocate and explores the relationships between these. In section 4 the consequences for income tax progression are determined for each of these principles. We identify the conditions for equal progression among equals at this stage. Section 5 sums up the paper and provides conclusions and suggestions for further research.

2. Preliminaries

Let $x > 0$ be household gross income. Furthermore let $t(x)$, assumed differentiable, be the household's tax liability and let $v(x) = x - t(x)$ be disposable income. We assume that $t(x) > 0$ and $v(x) > 0$, i.e. we focus on tax paying households. Define as $\varepsilon(f(x), x)$ the elasticity of a differentiable function $f(x)$ with respect to *x*. Residual and liability progression, respectively, of the tax schedule $t(x)$ are $RP(x) = \varepsilon(v(x), x)$ and $LP(x) = \varepsilon(t(x), x)$.

For simplicity we consider two household types, namely single person households (s) and couples (c). The tax and net income schedules will be $t_s(x)$ and $v_s(x)$, and $t_c(x)$ and $v_c(x)$, respectively. The progression measures $RP(x)$ and $LP(x)$ will also carry suffixes *s* and *c* in what follows. We assume that singles are the reference type and will denote by $S(x)$ the equivalent income function which expresses the living standard of a couple with income *x* where $S(x)$ is both continuous and strictly increasing in *x*.[3] Later we shall apply this measure to pretax income *x* and to posttax income $v_c(x)$. For the poorest couples equivalent incomes may be zero or negative. Such couples could be expected to receive low income support from the government rather than pay taxes. In everything which follows we shall assume all equivalent incomes to be positive, thereby focussing on tax payers only.

The general form of relative equivalence scale uses a deflator $m(x)$ so that $S(x) = x/m(x)$, whilst the general form for an absolute equivalence scale, $a(x)$ say, satisfies $S(x) = x - a(x)$. Assuming couples to be needier than single adults we have $m(x) > 1$ and $a(x) > 0$ for all *x*.[4] The familiar case of a constant relative equivalence scale is $m(x) = m$ for all *x*, but as we have said in the introduction this need not be. Donaldson and Pendakur (1999) have proposed $m(x) = x(\beta/x)^{\alpha}$ as an improved specification (see their equation (4.14)), where α and $\beta > 0$ are constants. The constant absolute scale is given by $a(x) = a$ for all *x*. Donaldson and Pendakur (equation (3.7)) suggest $a(x) = \left[(\alpha - 1)x + \beta\right]/\alpha$ as an improvement in this context where $\alpha > 0$ and β are constants.

3 For more on this function see Donaldson and Pendakur (1999) and Ebert (2000).

4 Had we chosen couples as reference type and applied these scales to single persons' incomes then we would have $m(x) < 1$ and $a(x) < 0$.

3. Equity principles

If the tax schedule for singles is $t_s(x)$ and the net income schedule $v_s(x)$, what should be the associated schedules for couples in order to secure equity? The principle of classical horizontal equity calls for the equal tax treatment of equals. For us equals shall be those with the same (gross) living standard. Let a single have gross income x_s and a couple x_c.

Let us consider first the use of relative scales. The pretax living standards of the single person and the couple are equal for the relative scale $m(x)$ if $x_s = x_c / m(x_c)$. Evaluating post tax incomes in the same way, one principle of horizontal equity, which we might call naïve horizontal equity for the relative equivalence scale (NHE-R) would postulate

$$x_s = \frac{x_c}{m(x_c)} \Rightarrow v_s(x_s) = \frac{v_c(x_c)}{m(x_c)} \tag{1}$$

equivalently

$$v_c(x) = m(x) v_s\left(\frac{x}{m(x)}\right) \tag{2}$$

or

$$t_c(x) = m(x) t_s\left(\frac{x}{m(x)}\right). \tag{3}$$

One interpretation for these results is that the equivalence scale $m(x)$ splits the couple into $m(x)$ equivalent adults and taxes each according to the given schedule for singles (see Ebert (1997), Ebert and Moyes (2000) for the case of constant equivalence scales $m(x) = m$). That is, pretax circumstances, namely gross income *x*, determine the number of equivalent adults the tax system will judge a two person household to contain, each to be taxed as a single person. Tax practitioners are becoming familiar with this procedure; but a more careful consideration reveals that NHE-R is not a pure horizontal equity criterion. For horizontal equity per se (HE) the condition must be

$$x_s = S(x_c) \Rightarrow v_s(x_s) = S(v_c(x_c)) \tag{4}$$

which taking $S(x) = x/m(x)$ implies that

$$v_c(x) = m(v_c(x))\, v_s\left(\frac{x}{m(x)}\right). \tag{5}$$

No longer can the interpretation involving equivalent adults be said to hold. Rearranging (5) we find

$$t_c(x) = m(x)\, t_s\left(\frac{x}{m(x)}\right) + \left(m(x) - m(v_c(x))\right) v_s\left(\frac{x}{m(x)}\right), \tag{6}$$

the second term being a correction factor which arises whenever the equivalence scale is not constant. This means that the appropriate deflator for posttax incomes differs from the one applied in the naïve case (recall (3)). Moreover the equitable tax is now defined implicitly (by (5) or (6)) rather than explicitly (as in (3)). We shall compare both criteria in what follows.

Turning to absolute scales $a(x)$ naïve horizontal equity (NHE-A) would demand

$$x_s = x_c - a(x_c) \;\Rightarrow\; v_s(x_s) = v_c(x_c) - a(x_c) \tag{7}$$

equivalently

$$v_c(x) = v_s(x - a(x)) + a(x) \tag{8}$$

or

$$t_c(x) = t_s(x - a(x)). \tag{9}$$

Condition (9) rationalizes the use of income related deductions, again familiar to the tax designer. But NHE-A is not a pure horizontal equity criterion either; returning to the criterion HE as in (4) and setting $S(x) = x - a(x)$ we find

$$v_c(x) = v_s(x - a(x)) + a(v_c(x)) \tag{10}$$

and, equivalently,

$$t_c(x) = t_s(x - a(x)) + \left(a(x) - a(v_c(x))\right) \tag{11}$$

in which the last term represents the necessary correction for nonconstancy of the scale.

This discussion has brought up three possible equity criteria, namely NHE-R, HE, and NHE-A. How are these related? First in the case of constant scales we observe the following:

Proposition 1

(i) If $m(x)=m$ *for all x, then the criteria NHE-R and HE both reduce to*

$$v_c(x)=m\,v_s\left(\frac{x}{m}\right) \text{ and } t_c(x)=m\;t_s\left(\frac{x}{m}\right),$$

i.e. both criteria are equivalent.

(ii) If $a(x)=a$ *for all x, then the criteria NHE-A and HE both reduce to*

$$v_c(x)=v_s(x-a) \text{ and } t_c(x)=t_s(x-a),$$

i.e. both criteria are equivalent.

The constant scales of either kind reduce, respectively, to income tax splitting and to the use of lump-sum tax exemptions. For nonconstancy of scales the criteria will in general differ.

Allowing that relative and absolute equivalence scales may depend on income we can interpret any equivalent income function $S(x)$ in terms of both relative and absolute scales, namely we can set

$$S(x)=\frac{x}{m(x)} \text{ and } S(x)=x-a(x) \tag{12}$$

by appropriate choice of the functions $m(x)$ and $a(x)$:

$$\begin{aligned}\frac{x}{m(x)}=x-a(x) &\Leftrightarrow\; a(x)=\left(1-\frac{1}{m(x)}\right)x\\ &\Leftrightarrow\; m(x)=\frac{x}{x-a(x)}\end{aligned} \tag{13}$$

For the constant relative scale $m(x)=m$, the corresponding absolute scale is income related: $a(x)=(1-1/m)x$; for the constant absolute scale $a(x)=a$, the relative scale is $m(x)=x/(x-a)$. Donaldson and Pendakur (1999) who exploit this connection specify the absolute version of their relative scale $m(x)=x(\beta/x)^{\alpha}$ as $a(x)=x-(x/\beta)^{\alpha}$ and the relative version of their absolute scale $a(x)=\left[(\alpha-1)x+\beta\right]/\alpha$ as $m(x)=\alpha\,x/(x+\beta)$. This latter scale has also been proposed by Conniffe (1992). The equivalent income function presented in Ebert (2000),

$$S(x) = \ln\left[1 + \alpha\left(e^{\delta x} - 1\right)\right] / \delta \tag{14}$$

gives rise to relative and absolute scales

$$m(x) = \frac{x}{\frac{1}{\delta}\ln\left(1 + \alpha\left(e^{\delta x} - 1\right)\right)} \quad \text{and} \quad a(x) = x - \frac{1}{\delta}\ln\left(1 + \alpha\left(e^{\delta x} - 1\right)\right). \tag{15}$$

Can the same equivalent income function $S(x)$, when interpreted in these two different ways (i.e. in terms of a relative and absolute scale), generate two different equity principles? The answer is 'no' for the principle HE, but 'yes' for the naïve principles so long as there is any tax at all.

Proposition 2

When $S(x) = \dfrac{x}{m(x)} = x - a(x)$

(i) conditions (5), (6), (10), and (11) for HE are all equivalent.

(ii) NHE-R and NHE-A are equivalent if and only if $v_s(x) = x$ *and* $t_s(x) = 0$ *for all x.*[5]

Hence the three criteria necessarily differ.

What is the tax designer to do? If faced with a constant relative scale ($a(x)$ varying with income) she should select NHE-R which is both transparent (in terms of income tax splitting) and equivalent to HE. If faced with a constant absolute scale ($m(x)$ varying) she should select NHE-A which is transparent (in terms of exemptions) and equivalent to HE. For all other equivalence scales ($m(x)$ and $a(x)$ varying) there is a choice between three distinct principles: HE, NHE-R, and NHE-A. As we shall shortly see, the consequences for the degree of progression which singles and couples face may be very different.

[5] The proof of this and all other mathematical assertions made in this paper may be found in the appendix (which is not intended for publication).

4. Consequences for progression

Let us begin by positing an equivalent income function $S(x)$. Given any particular tax schedule for singles $t_s(x)$, with residual and liability progression measures $RP_s(x)$ and $LP_s(x)$, what are the consequences for the degree of progression faced by the couples when each of the equity principles NHE-R, NHE-A, and HE are applied?

When $S(x) = x/m(x)$ and NHE-R is invoked then

$$\left(RP_c(x)-1\right) = \left(RP_s\left(\frac{x}{m(x)}\right)-1\right)\left(1-\varepsilon\left(m(x),x\right)\right) \tag{16}$$

and

$$\left(LP_c(x)-1\right) = \left(LP_s\left(\frac{x}{m(x)}\right)-1\right)\left(1-\varepsilon\left(m(x),x\right)\right) \tag{17}$$

can be shown.

It is quite immediate that, if singles face constant residual progression α, namely $v_s(x) = k\,x^{\alpha}$ for some *k*, then couples will not necessarily face constant progression. From (2) $v_c(x) = k\left[m(x)\right]^{1-\alpha} x^{\alpha}$ so that $RP_c(x) = \alpha + (1-\alpha)\varepsilon\left(m(x),x\right)$. The following general results can be inferred.[6]

Proposition 3

Suppose that $S(x) = x/m(x)$ and that NHE-R is satisfied. Then

(i) *if* $\varepsilon\left(m(x),x\right) = 0$: $\qquad RP_c(x) = RP_s\left(\frac{x}{m(x)}\right)$

and $\quad LP_c(x) = LP_s\left(\frac{x}{m(x)}\right)$

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[6] Notice under the maintained assumptions $\varepsilon\left(m(x),x\right) < 1$ for all *x* (since $x/m(x)$ has to be strictly increasing in *x*).

(ii) *if* $\varepsilon\left(m(x),x\right)<0$: $\quad RP_c(x) \gtreqqless RP_s\left(\frac{x}{m(x)}\right) \Leftrightarrow RP_s\left(\frac{x}{m(x)}\right) \gtreqqless 1$

and $\quad LP_c(x) \gtreqqless LP_s\left(\frac{x}{m(x)}\right) \Leftrightarrow LP_s\left(\frac{x}{m(x)}\right) \gtreqqless 1$

(iii) *if* $\varepsilon\left(m(x),x\right)>0$: $\quad RP_c(x) \lesseqqgtr RP_s\left(\frac{x}{m(x)}\right) \Leftrightarrow RP_s\left(\frac{x}{m(x)}\right) \gtreqqless 1$

and $\quad LP_c(x) \lesseqqgtr LP_s\left(\frac{x}{m(x)}\right) \Leftrightarrow LP_s\left(\frac{x}{m(x)}\right) \gtreqqless 1.$

For the constant relative scale ((i) of Proposition 3) the income splitting device ensures that couples face the same progression as singles with the same living standard. For a nonconstant equivalence scale which decreases with income (Seneca and Taussig (1971) display such scales) the progression faced by couples under NHE-R is unambiguously higher at each living standard than for singles (provided that the given $t_s(x)$ is progressive). For equivalence scales which are increasing with income (Donaldson and Pendakur (1999) exhibit such scales) couples face unambiguously less progression than singles at their living standard.

Now suppose that NHE-A is invoked, given the same equivalent income function $S(x)$, now written as $x-a(x)$, and single tax schedule $t_s(x)$. We can show

$$LP_c(x)=LP_s\left(x-a(x)\right)\varepsilon\left(x-a(x),x\right) \tag{18}$$

in respect of liability progression, but no general useful expression for residual progression obtains. The resulting proposition is less comprehensive in this case, but still informative.

Proposition 4

Suppose that $S(x)=x-a(x)$ *and that NHE-A is satisfied.*

(i) $\quad LP_c(x) \gtreqqless LP_s\left(x-a(x)\right) \Leftrightarrow \varepsilon\left(a(x),x\right) \lesseqqgtr 1$

(ii) $\quad$ *If* $a(x)=\left(1-\frac{1}{m}\right)x$ $\left(\Leftrightarrow m(x)=m\right)$ *for all x then*

$$\left[RP_c(x) \gtreqqless RP_s\left(x-a(x)\right) \Leftrightarrow RP_s\left(x-a(x)\right) \lesseqqgtr 1\right].$$

(iii) $\quad$ *If* $a(x)=a$ *for all x, then*

$RP_c(x) \leq RP_s(x-a)$ *for all x.*

As part (i) of this proposition shows the progression comparison between singles and couples at any given living standard can go either way; $\varepsilon(a(x),x)$ could be bigger or less than 1 (cf. Donaldson and Pendakur (1999)). In two particularly interesting cases, those of constant relative and absolute equivalence scales, we can go further. For the constant relative scale (under NHE-A) the couples face unambiguously less progression than the singles. For the constant absolute scale (the most natural situation in which to apply NHE-A) couples face more progression. Since HE is equivalent to NHE-A in this case, evidently HE cannot be a progression preserving principle.

Suppose now that HE is invoked and that $S(x)$ is interpreted as $x/m(x)$. The following may be deduced from (5).

$$RP_c(x) = RP_s\left(\frac{x}{m(x)}\right)\frac{1-\varepsilon(m(x),x)}{1-\varepsilon(m(v_c(x)),v_c(x))} \tag{19}$$

Proposition 5

Suppose that $S(x) = x/m(x)$ *and that HE is satisfied. Then*

$$RP_c(x) \gtreqless RP_s\left(\frac{x}{m(x)}\right) \Leftrightarrow \varepsilon(m(x),x) \lesseqgtr \varepsilon(m(v_c(x)),v_c(x))$$

A particular implication is that for an equivalence scale $m(x)$ whose elasticity increases (decreases) with income, progression for couples is higher (lower) than for corresponding singles. For a nonconstant scale with constant income elasticity (such as the scale $m(x) = x(\beta/x)^{\alpha}$ of Donaldson and Pendakur) the principle HE is thus progression preserving.

It proves impossible to infer any meaningful or useful properties in terms of liability progression for the principle HE when we interpret $S(x)$ as either $x/m(x)$ or $x-a(x)$. However a final manipulation yields the following connection between liability and residual progression for the principle HE and $S(x) = x/m(x)$.

$$\frac{LP_c(x)-1}{LP_s\left(\frac{x}{m(x)}\right)-1}=\frac{m(v_c(x))t_s\left(\frac{x}{m(x)}\right)}{\left(m(x)-m(v_c(x))\right)\frac{x}{m(x)}+m(v_c(x))t_s\left(\frac{x}{m(x)}\right)}\;\frac{1-RP_c(x)}{1-RP_s\left(\frac{x}{m(x)}\right)} \tag{20}$$

From this we can conclude as follows:

Proposition 6

Suppose that $S(x)=x/m(x)$ *and that HE is satisfied. Then*

(i) *If* $m(x)$ *is decreasing in x, then*

$$RP_c(x)<RP_s\left(\frac{x}{m(x)}\right)\;\Rightarrow\;LP_c(x)>LP_s\left(\frac{x}{m(x)}\right).$$

(ii) *If* $m(x)$ *is increasing in x, then*

$$RP_c(x)>RP_s\left(\frac{x}{m(x)}\right)\;\Rightarrow\;LP_c(x)<LP_s\left(\frac{x}{m(x)}\right).$$

In both cases the effect on liability progression is in the same direction as the effect on residual progression.

5. Summary and conclusions

In this paper we have explored the question of tax design and the implications for progression when the equivalence scale is income dependent. Three different principles of horizontal equity have been exposed and their consequences for tax design have been investigated. In particular we found that horizontal equity does not generally ensure that couples face the same degree of progression as singles. Our analysis has also covered constant relative and absolute scales as special cases. The findings in these cases are important in their own right since most empirical analyses up to now have used constant scales.

Table 1 summarizes these findings.

a) $m(x) = m$

	RP	LP
$NHE-R$ HE	$RP_c = RP_s$	$LP_c = LP_s$
$NHE-A$	$RP_c \gtreqless RP_s$ $\Leftrightarrow RP_s \lesseqgtr 1$	$LP_c = LP_s$

b) $a(x) = a$

	RP	LP
$NHE-A$ HE	$RP_c \le RP_s$	$LP_c \ge LP_s$
$NHE-R$	$RP_c \gtreqless RP_s$ $\Leftrightarrow RP_s \gtreqless 1$	$LP_c \gtreqless LP_s$ $\Leftrightarrow LP_s \gtreqless 1$

Table 1: Horizontal equity and progression for constant equivalence scales.

Only in the most familiar case of income splitting (i.e. constant relative scale and NHE-R which is equivalent to HE in this scenario) do couples face the same degree of progression as singles. However, we have also shown that the principle HE is progression preserving in the case of the income dependent, isoelastic relative scale, which has been advocated on other grounds by Donaldson and Pendakur (1999).

Comparing the three principles NHE-R, NHE-A, and HE we have seen that the naïve versions are transparent and lead to taxes whose progression properties are more easily delineated. On the other hand the pure principle HE though less tractable is formally more correct. Transparency is important for the policy makers, but so is horizontal equity. The naïve principles offer transparency. HE guarantees classical horizontal equity. If we would add progression preservation as a new ‘equity’ principle then, we have shown, this can be achieved with the iso-elastic relative scale and HE. This includes income splitting under NHE-R as a special case.

Future research should look more closely at the progression preservation criterion and also seek to extend the analysis to benefits and indeed to combined tax and benefit systems (see Ebert and Lambert (1999) on this).

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Appendix

General rules (see e.g. Berck and Sydsaeter (1993))

Assume that $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ and that *f* and *g* are differentiable.

(E1) $$\varepsilon\left(f(x)g(x),x\right) = \varepsilon\left(f(x),x\right) + \varepsilon\left(g(x),x\right)$$

(E2) $$\varepsilon\left(f(x)/g(x),x\right) = \varepsilon\left(f(x),x\right) - \varepsilon\left(g(x),x\right)$$

(E3) $$\varepsilon\left(f(x)+g(x),x\right) = \frac{f(x)}{f(x)+g(x)}\varepsilon\left(f(x),x\right) + \frac{g(x)}{f(x)+g(x)}\varepsilon\left(g(x),x\right)$$

(E4) $$\varepsilon\left(f(x)-g(x),x\right) = \frac{f(x)}{f(x)-g(x)}\varepsilon\left(f(x),x\right) - \frac{g(x)}{f(x)-g(x)}\varepsilon\left(g(x),x\right)$$

if $f(x) - g(x) \neq 0$

(E5) $$\varepsilon\left(f(g(x)),x\right) = \varepsilon\left(f(g(x)),g(x)\right)\varepsilon\left(g(x),x\right)$$

Proof of Proposition 2

(i) Suppose that HE is satisfied.

$$\text{HE} \Leftrightarrow v_c(x) = m\left(v_c(x)\right)v_s\left(\frac{x}{m(x)}\right)$$

$$\Leftrightarrow v_c(x) = \frac{v_c(x)}{v_c(x) - a\left(v_c(x)\right)}v_s\left(\frac{x}{x/\left(x-a(x)\right)}\right)$$

$$\Leftrightarrow v_s\left(x - a(x)\right) = v_c(x) - a\left(v_c(x)\right)$$

so that (5) and (10) are equivalent.

(ii) Suppose that NHE-R $\equiv$ NHE-A

Then

$$v_c(x) = m(x)v_s\left(\frac{x}{m(x)}\right) \equiv \text{NHE-R}$$

as in (2) for all *x* and all feasible $m(x)$.

Now suppose that

$$m(x)=\frac{x}{x-a(x)}$$

Then

$$v_c(x)=\frac{x}{x-a(x)}v_s\left(\frac{x}{x/(x-a(x))}\right)$$

$$=\frac{x}{x-a(x)}v_s(x-a(x))$$

On the other hand

$v_c(x)=v_s(x-a(x))+a(x)$ = NHE-A as in (18).

Thus

$$\frac{x}{x-a(x)}v_s(x-a(x))=v_s(x-a(x))+a(x)$$

$$\Leftrightarrow v_s(x-a(x))\,\frac{a(x)}{x-a(x)}=a(x)$$

$\Leftrightarrow v_s(x-a(x))=x-a(x)$ for all x and all feasible $a(x)$

$\Leftrightarrow v_s(x)\equiv x$ for all x.

Proof of (16)

$v_c(x)=m(x)v_s\left(\frac{x}{m(x)}\right)$ by (2). Now

$$RP_c(x)=\varepsilon(v_c(x),x)=\varepsilon\left(m(x)v_s\left(\frac{x}{m(x)}\right),x\right)$$

$$=\varepsilon(m(x),x)+\varepsilon\left(v_s\left(\frac{x}{m(x)}\right),x\right) \qquad \text{by (E1)}$$

$$=\varepsilon(m(x),x)+\varepsilon\left(v_s\left(\frac{x}{m(x)}\right),\frac{x}{m(x)}\right)\varepsilon\left(\frac{x}{m(x)},x\right) \qquad \text{by (E5)}$$

$$=\varepsilon(m(x),x)+\varepsilon\left(v_s\left(\frac{x}{m(x)}\right),\frac{x}{m(x)}\right)\left[\varepsilon(x,x)-\varepsilon(m(x),x)\right] \qquad \text{by (E2)}$$

Thus

$$RP_c(x) = \varepsilon(m(x), x) + RP_s\left(\frac{x}{m(x)}\right)\left(1 - \varepsilon(m(x), x)\right)$$

$$RP_c(x) - 1 = \left(\varepsilon(m(x), x) - 1\right) + RP_s\left(\frac{x}{m(x)}\right)\left(1 - \varepsilon(m(x), x)\right)$$

$$= \left(RP_s\left(\frac{x}{m(x)}\right) - 1\right)\left(1 - \varepsilon(m(x), x)\right)$$

Proof of (17)

Starting from

$$t_c(x) = m(x)\, t_s\left(\frac{x}{m(x)}\right)$$

the proof of (17) is analogous to the proof of (16)

Proof of (18)

$$t_c(x) = t_s\left(x - a(x)\right)$$

$$LP_c(x) = \varepsilon\left(t_c(x), x\right) = \varepsilon\left(t_s\left(x - a(x)\right), x\right)$$

$$= \varepsilon\left(t_s\left(x - a(x)\right), x - a(x)\right)\varepsilon\left(x - a(x), x\right) \qquad \text{by (E5)}$$

This reduces directly to (18).

Proof of Proposition 4

$$v_c(x) = v_s\left(x - a(x)\right) + a(x) = \text{NHE-A}$$

$$RP_c(x) = \varepsilon\left(v_c(x), x\right) = \varepsilon\left(v_s\left(x - a(x)\right) + a(x), x\right)$$

$$= \frac{v_s\left(x - a(x)\right)}{v_s\left(x - a(x)\right) + a(x)}\varepsilon\left(v_s\left(x - a(x)\right), x\right) + \frac{a(x)}{v_s\left(x - a(x)\right) + a(x)}\varepsilon\left(a(x), x\right) \text{by (E3)}$$

$$= \frac{v_s}{v_s + a}\varepsilon\left(v_s\left(x - a(x)\right), x - a(x)\right)\varepsilon\left(x - a(x), x\right) + \frac{a}{v_s + a}\varepsilon\left(a(x), x\right) \qquad \text{by (E5)}$$

$$= \frac{v_s\left(x - a(x)\right) RP_s\left(x - a(x)\right)\left[\dfrac{x - a(x)\varepsilon\left(a(x), x\right)}{x - a(x)}\right] + a(x)\varepsilon\left(a(x), x\right)}{v_s\left(x - a(x)\right) + a(x)}$$

by (E5)

Hence (i) is obvious.

For (ii), note that if $\varepsilon\left(x-a(x),x\right)=1=\varepsilon\left(a(x),x\right)$ then

$$RP_c(x)=\frac{v_s\left(x-a(x)\right)RP_s\left(x-a(x)\right)+a(x)}{v_s\left(x-a(x)\right)+a(x)}$$

which is equivalent to

$$RP_c(x)v_s\left(x-a(x)\right)+RP_c(x)a(x)=v_s\left(x-a(x)\right)RP_s\left(x-a(x)\right)+a(x)$$

or

$$v_s\left(x-a(x)\right)\left(RP_c(x)-RP_s\left(x-a(x)\right)\right)=a(x)\left(1-RP_c(x)\right).$$

For (iii) note that $a(x)=a\Leftrightarrow\varepsilon\left(a(x),x\right)=0$. Then

$$RP_c(x)=\frac{v_s(x-a)}{v_s(x-a)+a}\cdot\frac{x}{x-a}RP_s(x-a)$$

$$\frac{v_s(x-a)}{v_s(x-a)+a}\,\frac{x}{x-a}\leq 1\Leftrightarrow xv_s(x-a)\leq(x-a)\left(v_s(x-a)+a\right)$$

$$\Leftrightarrow xv_s(x-a)\leq xv_s(x-a)-av_s(x-a)+a(x-a)$$

$$\Leftrightarrow v_s(x-a)\leq x-a$$

Thus

$$RP_c(x)\leq RP_s(x-a)$$

Proof of (19)

$$v_c(x) = m(v_c(x))\, v_s\left(\frac{x}{m(x)}\right) \equiv (1)$$

$$RP_c(x) = \varepsilon(v_c(x), x) = \varepsilon\left(m(v_c(x))\, v_s\left(\frac{x}{m(x)}\right), x\right)$$

$$= \varepsilon(m(v_c(x), x)) + \varepsilon\left(v_s\left(\frac{x}{m(x)}\right), x\right) \qquad \text{by (E1)}$$

$$= \varepsilon(m(v_c(x)), v_c(x))\,\varepsilon(v_c(x), x) + \varepsilon\left(v_s\left(\frac{x}{m(x)}\right), x\right) \qquad \text{by (E5)}$$

$$= \varepsilon(m(v_c(x)), v_c(x))\, RP_c(x) + \varepsilon\left(v_s\left(\frac{x}{m(x)}\right), \frac{x}{m(x)}\right)\left[1 - \varepsilon(m(x), x)\right]$$

by (E5) and (E2)

Thus

$$RP_c(x)\left[1 - \varepsilon(m(v_c(x)), v_c(x))\right]$$
$$= RP_c\left(\frac{x}{m(x)}\right)\left[1 - \varepsilon(m(x), x)\right]$$

proving the result.

Proof of (20)

Using a result from Lambert (2001) (exercise 8.2.2, page 198) we obtain

$$LP_c(x) = \frac{x - v_c(x)\, RP_c(x)}{t_c(x)}$$

and thus

$$LP_c(x) - 1 = \frac{v_c(x)(1 - RP_c(x))}{t_c(x)}$$

and similarly

$$LP_s\left(\frac{x}{m(x)}\right)-1=\frac{v_s\left(\frac{x}{m(x)}\right)\left(1-RP_s\left(\frac{x}{m(x)}\right)\right)}{t_s\left(\frac{x}{m(x)}\right)}$$

Therefore, combining these,

$$\frac{LP_c(x)-1}{LP_s\left(\frac{x}{m(x)}\right)-1}=\frac{v_c(x)t_s\left(\frac{x}{m(x)}\right)}{t_c(x)v_s\left(\frac{x}{m(x)}\right)}\frac{1-RP_c(x)}{1-RP_s\left(\frac{x}{m(x)}\right)}$$

Now

$$\frac{v_c(x)t_s\left(\frac{x}{m(x)}\right)}{t_c(x)v_s\left(\frac{x}{m(x)}\right)}=\frac{m(v_c(x))v_s\left(\frac{x}{m(x)}\right)t_s\left(\frac{x}{m(x)}\right)}{\left[\left(m(x)-m(v_c(x))\right)\frac{x}{m(x)}+m(v_c(x))t_s\left(\frac{x}{m(x)}\right)\right]v_s\left(\frac{x}{m(x)}\right)}$$

$$=\frac{m(v_c(x))t_s\left(\frac{x}{m(x)}\right)}{\left(m(x)-m(v_c(x))\right)\frac{x}{m(x)}+m(v_c(x))t_s\left(\frac{x}{m(x)}\right)}$$

and (20) now follows.