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On Ehrhart Polynomials and Probability Calculations in Voting Theory *

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Abstract

In voting theory, analyzing how frequent is an event (*e.g.* a voting paradox) is, under some specific but widely used assumptions, equivalent to computing the exact number of integer solutions in a system of linear constraints. Recently, some algorithms for computing this number have been proposed in social choice literature by Huang and Chua [17] and by Gehrlein ([12, 14]). The purpose of this paper is threefold. Firstly, we want to do justice to Eugène Ehrhart, who, more than forty years ago, discovered the theoretical foundations of the above mentioned algorithms. Secondly, we present some efficient algorithms that have been recently developed by computer scientists, independently from voting theorists. Thirdly, we illustrate the use of these algorithms by providing some original results in voting theory.

Key words: voting rules, manipulability, polytopes, lattice points, algorithms.
JEL Classification: D70, D71

1 Introduction

Consider an election on three alternatives or candidates $\{a, b, c\}$. Assume that voters have complete linear preference rankings on these candidates. Then, there are six possible preference orders that voters might have:

$abc\ (n_1)\quad acb\ (n_2)\quad bac\ (n_3)\quad bca\ (n_4)\quad cab\ (n_5)\quad cba\ (n_6)$

Here, n_i denotes the number of voters with the associated preference ranking on candidates. For n voters, we then have $\sum_{i=1}^{6} n_i = n$, and any such combination of n_i's is referred to as a voting situation, or simply as a situation. In order to compute

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the probability that some event takes place, it is often assumed in the social choice literature that all voting situations are equally likely to occur. It is the so-called *Impartial Anonymous Culture* (IAC) condition, first explicitly introduced by Gehrlein and Fishburn [16]. Under this assumption, computing the desired probability amounts to evaluating the number of situations corresponding to the voting event under consideration. Typically, the voting events can be described by a system of linear constraints with rational coefficients. For example, the event "candidate a is the Condorcet winner[1]" corresponds to $n_1+n_2+n_3 > n_4+n_5+n_6$ and $n_1+n_2+n_5 > n_3+n_4+n_6$. As the n_i's must be integers, the problem is then to compute the number of integer solutions of a system of linear (in)equalities. Recently, Huang and Chua [17] and Gehrlein [12] have proposed in *Social Choice and Welfare* some new methods for computing the number of situations exhibiting a particular voting event. These methods are based on the fundamental result (proved in Huang and Chua [17]) that the number of integer solutions of a system of linear constraints can be represented by a polynomial in n with periodic coefficients.

We have very recently realized that this result is more than 40 years old and is due to Ehrhart [7][2]. Ehrhart's theory is very general and provides solid theoretical foundations for the IAC probability calculations. We strongly believe that the knowledge of this theory can be useful for all social choice theorists interested in probability calculations. To give a brief overview of Ehrhart's theory is the first objective of the present paper.

We have also discovered that numerous studies exist in applied mathematics and in the computer science literature that propose and analyze different ways for computing the number of integer solutions of a set of linear constraints. Some of them are based on Ehrhart's theory; others make use of new theoretical developments. The second purpose of this paper is to present the two main algorithms that can be implemented.

Finally, we will illustrate the use and the efficiency of these algorithms by providing some new and original probability representations[3].

2 Ehrhart polynomials and their computation

How can one calculate the number of points with integer coordinates in a set S described by a finite system of linear constraints with rational coefficients? Frequently, this problem appears as an important step in a wide variety of topics in pure and applied mathematics. The first fundamental contribution in this area is due to the work of the French mathematician *Eugène Ehrhart* $(1906-2000)$[4]. He considered the special case when the linear constraints depend on a single positive parameter n and showed that the number of points with integer coordinates in S can be represented by a polynomial in n with periodic coefficients. Before presenting more details on Ehrhart's theory

[1] A Condorcet winner exists as a candidate who could beat every other candidate on the basis of pairwise majority rule.

[2] It turns out that, in mathematics, a polynomial with periodic coefficients is called "Ehrhart polynomial".

[3] The reader not interested in technical aspects may directly consult this part of the paper (Section 3).

[4] For a short biography, see the tribute written by Philippe Clauss in honor of E. Ehrhart: http://icps.u-strasbourg.fr/ clauss/ehrhart.html

and recent advances in this domain, it is convenient to introduce some notations and terminology.

Let $\mathbb{R}^d$ be the Euclidean d-space of all d-tuples $\mathbf{x} = (x_1, \cdots, x_d)$ of real numbers. The integer lattice $\mathbb{Z}^d$ is the subset of $\mathbb{R}^d$ consisting of points with integer coordinates (for short, called lattice points or integer points). A rational polyhedron of dimension d is a set $P \subset \mathbb{R}^d$, that is the solution of a finite system of linear inequalities with integer coefficients [5]:

$$P = \{\mathbf{x} \in \mathbb{R}^d : \quad A\mathbf{x} \leq b\}$$

where A is a $m \times d$ integer matrix, b an integer vector with m components and m the number of independent linear inequalities. The inequality $A\mathbf{x} \leq b$ is understood component-wise. Usually, the polyhedron is bounded and called a polytope. The extremal points of a polytope are called its vertices. A lattice polytope is a polytope with integer vertices. It is clear that the problem of counting lattice points that satisfy a finite set of linear constraints with rational coefficients is now equivalent to counting lattice points inside a given rational polyhedron.

2.1 Ehrhart's theory

Let $P \subset \mathbb{R}^d$ be a d-dimensional rational polytope. For an integer parameter $n \geq 1$, define the dilation of P by n as the polyhedron $nP = \{n\mathbf{x} : \mathbf{x} \in P\}$. Geometrically, this can be interpreted as dilating P while leaving the angles and proportions fixed. Consider the function $L(P, n) = |nP \cap \mathbb{Z}^d|$ of the variable n, that describes the number of lattice points that lie inside the dilation nP. Ehrhart [7] inaugurated the systematic study of general properties of this function by proving in particular, that it can be represented by a polynomial in n when P is a lattice polytope and by a finite family of a polynomials called *quasi-polynomials* or Ehrhart polynomials, in the general case.

Definition 2.1 *A function $f : \mathbb{Z} \longrightarrow \mathbb{Q}$ is a (univariate) quasi-polynomial of period q if there exists a list of q polynomials g_i ($0 \leq i < q$) such that $f(n) = g_i(n)$ if $n \equiv i \mod q$.*

Instead of representing a quasi-polynomial by a long list of polynomials, Ehrhart [9] uses the practical concept of periodic numbers.

Definition 2.2 *A rational periodic number $U(n)$ is a function $U : \mathbb{Z} \longrightarrow \mathbb{Q}$, such that there exists a period q such that $U(n) = U(n')$ whenever $n \equiv n' \mod q$.*

The possible values of $U(n)$ are usually made explicit by a list of q rational numbers enclosed in square brackets.

Example 2.1 $U(n) = [\frac{1}{2}, \frac{3}{4}, 1]_n$ is a periodic number with period $q = 3$, $U(n) = \frac{1}{2}$ if $n \equiv 0 \mod 3$, $U(n) = \frac{3}{4}$ if $n \equiv 1 \mod 3$ and $U(n) = 1$ if $n \equiv 2 \mod 3$.

[5] P could be not full–dimensional (this is the case when the linear system describing the polyhedron contains equalities). However, without loss of generality, P can be assumed to be full-dimensional [28].

Definition 2.3 *A (univariate) quasi-polynomial f of degree d is a function $f(n) = c_d(n)n^d + \cdots + c_1(n)n + c_0(n)$ where the $c_i(n)$'s are rational periodic numbers. The period q of a quasi-polynomial is the least common multiple (lcm) of the periods of its coefficients.*

Example 2.2 $f(n) = \frac{1}{4}n^2 + [\frac{1}{2}, \frac{3}{4}, 1]_n n + [0, \frac{1}{3}]_n$ is a quasi-polynomial of degree 2 and period 6.

The fundamental result of Ehrhart can be described by the following theorem.

Theorem 2.1 (Ehrhart) *Let P be a rational polytope. The function $L(P, n)$ representing the number of integer points in the dilation nP is given by a degree-d quasi-polynomial. The coefficient of the leading term is independent of n and is equal to the Euclidean volume of P. The period of the quasi-polynomial is a divisor of the lcm of the denominators of the vertices of nP. When P is a lattice polytope, $L(P, n)$ is given by a single polynomial.*

Example 2.3 Consider the following parametric system:

$$(S_n) \quad \left\{ \begin{array}{ccccc} \boldsymbol{x_1} & + & \boldsymbol{x_2} & \leq & \boldsymbol{3n} \\ \boldsymbol{2x_1} & & & \leq & \boldsymbol{5n} \\ \boldsymbol{x_1} & & & \geq & \boldsymbol{0} \\ & & \boldsymbol{x_2} & \geq & \boldsymbol{0} \end{array} \right.$$

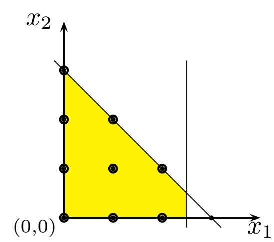

Fig. 1. Integer points in P

The number of integer solutions of S_n is the number of integer points inside the dilation nP of the polytope:

$$P = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 3, 2x_1 \leq 5\} \qquad \text{(Fig. 1)}$$

The vertices of nP are: $(0, 0), (\frac{5n}{2}, 0), (0, 3n)$ and $(\frac{5n}{2}, \frac{n}{2})$. Hence, by Theorem 2.1, $L(P, n)$ is a quasi-polynomial of degree 2, it has the general form: $\alpha n^2 + [\beta_1, \beta_2]_n n + [\gamma_1, \gamma_2]_n$.

It should be mentioned that if the above theorem is compared with the theoretical result obtained by Huang and Chua in [17], it turns out that these results are essentially identical. However, the seminal work of Ehrhart is more general and involves more information about the coefficients and the period of the quasi-polynomial representing the $L(P, n)$ function. In their paper, Huang and Chua also suggested an algorithm for computing periodic coefficients. This algorithm (further refined and improved by Gehrlein [12]) is based on the classical technique of interpolation. Currently, there exist two general methods for computing Ehrhart polynomials: Clauss's algorithm (1998) and the parameterized Barvinok's algorithm (2004). The Huang-Chua algorithm can be considered as a particular case (with a single parameter) of Clauss's method that we present now.

2.2 Interpolation method and Clauss's generalization

Ehrhart polynomials have many applications concerning computer science. Their use in this area was initiated by Clauss and Loechner [4]. They were the first to propose a method for computing the quasi-polynomials coefficients. Based on the information provided by Theorem 2.1, the algorithm counts the number of lattice points for a set of fixed values of the parameter and then calculates the quasi-polynomial through interpolation.

Example 2.4 Consider once again the system (S_n) from example 2.3. To find the unknown values of $L(P,n)$ coefficients, five independent linear equations on α, β_i and γ_i $(i = 1, 2)$ can be obtained by counting the number of lattice points in nP for fixed values of n in $[0,4]$. This initial counting ($L(P,0) = 1,\ L(P,1) = 9, L(P,2) = 27, L(P,3) = 52, L(P,4) = 88$) allows us to construct a system of linear equalities for which the solutions are the desired coefficients. Resolving this system, we obtain: $L(P,n) = \frac{35}{8}n^2 + [\frac{17}{4}, 4]_n n + [1, \frac{5}{8}]_n$.

In order to apply the above technique to a larger class of problems associated with counting lattice points, Clauss [5] extended Ehrhart's result to parameterized polytopes with any number of integer parameters.

Definition 2.4 *A rational d-dimensional parameterized polyhedron is a set of real vectors defined by parametric linear inequalities:$P_{\mathbf{p}} = \{\mathbf{x} \in \mathbb{Z}^d : A\mathbf{x} \leq C\mathbf{p} + b\}$, where A and C are integer matrices, b is an integer vector and $\mathbf{p}$ a vector of r integer parameters. When$P_{\mathbf{p}}$ is bounded for each value of $\mathbf{p}$, it will be called a parametric polytope.*[6]

Note that the coordinates of the vertices of a parametric polytope are affine functions of parameters. Each vertex exists only if $\mathbf{p}$ belongs to a subset of the parameter domain $\mathbb{N}^r$. Subsets where the vertices have stable expressions are called *validity domains* (see Example 2.7 and Figures 2 and 3, further).

Before presenting Clauss's generalization of Ehrhart's theorem, we need to extend the concept of periodic number and quasi-polynomial.

Definition 2.5 *Let* $\boldsymbol{p} = (p_1, \cdots, p_r)$ *be a r-dimensional parameter vector. A r-dimensional periodic number $U(P)$ is a function $U : \mathbb{Z}^r \longrightarrow \mathbb{Q}$ such that there exist periods $q = (q_1, \cdots, q_r) \in \mathbb{N}^r$ such that $U(\mathbf{p}) = U(\mathbf{p}')$ whenever $p_i \equiv p'_i$ mod $q_i (1 \leq i \leq r)$. The lcm of all q_i's is called the period of $U(\mathbf{p})$.*

The multidimensional periodic numbers are usually represented by a look-up table.

Example 2.5 $\big[[1, \frac{1}{2}]_{p_2}, [0, \frac{3}{2}]_{p_2}, [-1, \frac{1}{4}]_{p_2}\big]_{p_1}$ is a 2-periodic number with period $q = (3, 2)$.
$U(n,m) = (-1)^{n-m}$ is a 2-periodic number with period $q = (2, 2)$. It can be represented by $U(n,m) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{n,m} = \big[[1,-1]_m, [-1,1]_m\big]_n$

[6]Note that if $\mathbf{p} = (n, \cdots, n)$ and $b = (0, \cdots, 0)$, then $P_{\mathbf{p}} = nP$.

Note that the matrix array notation works well for 2-dimensional periodic numbers, whereas the notation with square brackets works for all dimensions.

Definition 2.6 *A multivariate quasi-polynomial is a polynomial in r variables $p_1, \cdots, p_r$ such that each coefficient is a multidimensional periodic number on a subset of $\{p_1, \cdots, p_r\}$. The period of a multivariate quasi-polynomial is the lcm of the periods of its coefficients.*

Example 2.6 Consider the quasi-polynomial

$$f(n,m) = \frac{-1}{8}n^2 - \frac{-1}{8}m^2 + \frac{3}{2}nm + \big[[\frac{11}{4}, \frac{5}{4}]_m, [\frac{5}{2}, 1]_m\big]_n n + [\frac{1}{4}, \frac{1}{2}]_m m + \big[[1, \frac{5}{8}]_m, [\frac{5}{8}, \frac{1}{4}]_m\big]_n.$$

It is a quasi-polynomial with period 2 in variables n and m. $f(n,m)$ can be expressed as a quasi-polynomial in n, with periodic coefficients being periodic linear expressions in m:

$$-\frac{1}{8}n^2 + [(\frac{3}{2}m + [\frac{11}{4}, \frac{5}{4}]_m), (\frac{3}{2}m + [\frac{5}{2}, 1]_m)]_n n$$
$$+[(-\frac{1}{8}m^2 + [\frac{1}{4}, \frac{1}{2}]_m m + [1, \frac{5}{8}]_m), (-\frac{1}{8}m^2 + [\frac{1}{4}, \frac{1}{2}]_m m + [\frac{5}{8}, \frac{1}{4}]_m)]_n.$$

Theorem 2.2 (Clauss) *The enumerator function $E(P_\mathbf{p})$ that describes the number of lattice points in a d-dimensional parametric polytope $P_\mathbf{p}$ can be represented by a finite set of multivariate quasi-polynomials of degree d in $\mathbf{p}$, each valid on a different validity domain. The period of the quasi-polynomial in a given validity domain divides the lcm of the denominators that appear in the expression defining the vertices on this domain.*

Using Theorem 2.2, Clauss and Loechner have developed a general method to count lattice points in a parametric polytope[7]. Their method is based on the knowledge of the structure of the solution; the implemented algorithm consists of the following steps:

1. Compute the validity domains and the parametric coordinates of the vertices.[8]
2. For each validity domain, since the general form of the associated quasi-polynomial is known:
   (a) Count the number of points for some initial values of the parameters;
   (b) Solve a system of linear equations of which the solutions are the quasi-polynomial coefficients.

Example 2.7 We modify the system in Example 3 by introducing a second parameter (m) and adding a supplementary constraint:

[7]It is worth noticing that Gehrlein [14] has recently proposed a method, called EUPIA2, to compute quasi-polynomials for the specific case of two parameters. The algorithm developed by Clauss and Loechner generalizes in some sense Gehrein's EUPIA2.

[8]The parametric vertices are computed by the Loechner-Wilde algorithm implemented in PolyLib [20, 21].

$$
\begin{cases}
x_1 + x_2 \leq 3n \\
2x_1 \leq 5n \\
2x_2 \leq m \\
x_1 \geq 0 \\
x_2 \geq 0
\end{cases}
$$

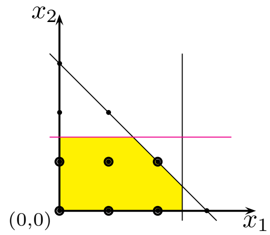

Fig. 2. Parameterized polytope (for $n = 1, m = 3$)

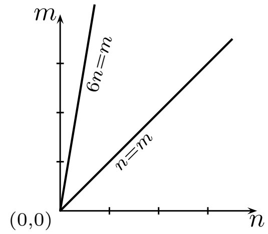

Fig. 3. Validity domains

Applying Clauss's algorithm to compute the enumerator function, we obtain:

Validity domain	**Vertices**	**Quasi-polynomial**
$-6n+m \geq 0$ $n \geq 0$	$(0,0), (\frac{5n}{2}, 0)$ $(0,3n), (\frac{5n}{2}, \frac{n}{2})$	$\frac{35}{8}n^2 + [\frac{17}{4}, 4]_n n + [1, \frac{5}{8}]_n$
$6n-m \geq 0$ $-n+m \geq 0$	$(0,0), (0, \frac{m}{2}), (\frac{5n}{2}, 0)$ $(3n-\frac{m}{2}, \frac{m}{2}), (\frac{5n}{2}, \frac{n}{2})$	$-\frac{1}{8}n^2$ $+[(\frac{3}{2}m + [\frac{11}{4}, \frac{5}{4}]_m), (\frac{3}{2}m + [\frac{5}{2}, 1]_m)]_n n$ $+[(-\frac{1}{8}m^2+[\frac{1}{4}, \frac{1}{2}]_m m+[1, \frac{5}{8}]_m), (-\frac{1}{8}m^2+[\frac{1}{4}, \frac{1}{2}]_m m$ $+[\frac{5}{8}, \frac{1}{4}]_m)]_n$
$n-m \geq 0$ $m \geq 0$	$(0,0), (0, \frac{m}{2})$ $(\frac{5n}{2}, 0), (\frac{n}{2}, \frac{m}{2})$	$(\frac{5}{4}m + [\frac{5}{2}, \frac{5}{4}]_m)n$ $+[(\frac{1}{2}m + [1, \frac{1}{2}]_m), (\frac{1}{4}m + [\frac{1}{2}, \frac{1}{4}]_m)]_n$

The above interpolation method is the very first algorithm ever developed to compute Ehrhart polynomials. However, it presents some drawbacks. The first limitation concerns the problem of *degenerate domains*: to interpolate a quasi-polynomial of degree d in r parameters with period $q = (q_1, \cdots, q_r)$, the algorithm requires $\prod_{i=1}^{r}(d+1)q_i$ initial countings. For certain validity domains, it is not always possible to find a subregion with $(d+1)q_i$ consecutive values in each dimension and then it may be impossible to get a complete set of appropriate instances for interpolation and the algorithm fails to produce a solution[9]. The second problem is related to time complexity[10]. The method used for initial countings basically enumerates all points, so if any instance contains a large number of points, the computation time rises accordingly. Moreover, if the periods are large, then the number of instances will be very large and interpolation will take an exponential time[27, 28].

2.3 Barvinok's algorithm

In 1993, Barvinok [1, 2] developed an algorithm that counts integer points inside rational polytopes. This algorithm is time-polynomial in input size when the dimension of the polytope is fixed. We will not explain Barvinok's algorithm in details but we will try to give a brief description of its main steps. Given a polyhedron P, define the multivariate generating function attached to P as:

[9]Since the implementation is based on interpolation, it searches for fixed parameter values located in an hyper rectangle. For more details, see [28].

[10]Time complexity refers to the function describing the way in which the number of steps required by an algorithm varies with the input size of the problem it is solving.

$$f(P;\mathbf{x}) = \sum_{\alpha \in P \cap \mathbb{Z}^d} \mathbf{x}^\alpha \quad where \quad \mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \quad with \quad \mathbf{x} = (x_1, \cdots, x_d)$$

Note that if P is a polytope, this formal power series is just a (Laurent) polynomial with one monomial per lattice point. Counting the number of integer points $|P \cap \mathbb{Z}^d|$ is then equivalent to evaluating $f(P;\mathbf{x})$ at $\mathbf{x} = (1, \cdots, 1)$. This allows the computation of the generating function as a reasonably short function.

Example 2.8 Consider the one-dimensional polytope $P = [0, N]$. The long polynomial $f(P;x) = 1 + x + \cdots + x^N$ can be represented by the short rational function $\frac{1-x^{N+1}}{1-x}$. Substituting $x = 1$ in this expression yields a denominator equal to zero, so we must take the limit as x approaches 1 (by L'Hospital theorem) and get, as expected, $f(P;1) = N + 1$, the number of integer points in P.

In this simple example the basic observation is that the compact expression of the generating function can be obtained by considering the two rays $K_0 = [0, \infty)$ and $K_N = (-\infty, N]$. Their generating functions are:

$$f(K_0;x) = \sum_{\alpha \geq 0} x^\alpha = \frac{1}{1-x} \text{ and } f(K_N;x) = \sum_{\alpha \leq N} x^\alpha = \frac{x^N}{1-x^{-1}}$$

Adding the two rational function right-hand sides (representing two infinite series) collapses into the rational function representing $f(P;x)$:

$$\frac{1}{1-x} + \frac{x^N}{1-x^{-1}} = \frac{1-x^{N+1}}{1-x}$$

This is a one dimensional instance of a theorem due to M. Brion [3]. In order to present this crucial theorem, we need to introduce some more concepts.

Definition 2.7 *A cone with generators $u_1, \cdots, u_t \in \mathbb{Z}^d$ is the set K defined by:* $K = \{\sum_{i=1}^t \lambda_i u_i : \quad \lambda_i \geq 0$*, for all i*$\}$.

This definition is somewhat restrictive, a cone is defined as the set of all positive combinations of its generators, so it must contain the origin. Given $v \in \mathbb{Z}^d$, we will use the notation $v + pos\{u_i : i = 1, \cdots, t\}$ to refer to the (shifted) cone defined by: $K = \{v + \sum \alpha_i, \alpha_i \geq 0\}$, which is the sum of v and the cone generated by $u_1, \cdots, u_t$, also called generators of K.

Another important class of cones that have a key role in Barvinok's algorithm is the class of unimodular cones.

Definition 2.8 *A (shifted) cone $K \subset \mathbb{R}^d$ is unimodular if its generators form a basis of $\mathbb{Z}^d$.*

Here, by basis of $\mathbb{Z}^d$, we mean a set of d linearly independent integer vectors which generate $\mathbb{Z}^d$. The significance of unimodular cone $K = v + pos\{u_i : i = 1, \cdots, t\}$ is that its fundamental half open parallelepiped $\Pi = \{\sum \alpha_i u_i : 0 \leq \alpha_i < 1\}$ contains only one lattice point $E(v, K)$, equivalently K is unimodular if and only if

$det(u_1, \cdots, u_d) = \pm 1$ (see [6]). It can also be shown ([1]) that the generating function of an unimodular cone has a simple and short form:

$$f(K;\mathbf{x}) = \frac{\mathbf{x}^{E(v,K)}}{(1-\mathbf{x}^{u_1})\cdots(1-\mathbf{x}^{u_d})} \tag{2.1}$$

Definition 2.9 *Let P be a polyhedron and $V(P)$ be the set vertex of P. The supporting cone $K(P,v)$ of P at $v \in V(P)$ is $K(P,v) = v + \{u \in \mathbb{R}^d : v + \delta u \in P$ for all sufficiently $\delta > 0\}$.*

Note that using Definition 2.9, the supporting cone of $K(P,v)$ is not always a cone itself, but it is the (possibly translated) cone defined by the facets touching vertex v. In the above example, the polytope $P = [0,N]$ has two supporting cones, $K(P,0) = [0,\infty)$ and $K(P,N) = (-\infty, N]$.

One fundamental step in Barvinok's algorithm is its ability to distribute the computation of $f(P;\mathbf{x})$ on the vertices of the polytope . This is described by the following Theorem.

Theorem 2.3 (Brion) *Let P be a rational polyhedron. Then*

$$f(P;\mathbf{x}) = \sum_{v \in V(P)} f(K(P,v);\mathbf{x}).$$

Brion's theorem allows the computation of $f(P;\mathbf{x})$ by computing the generating functions of the supporting cones of P.

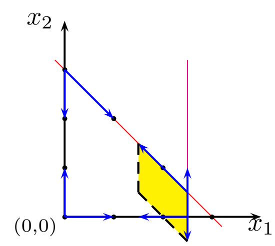

Fig.4. Generators, supporting cones and half-open parallelepiped

Example 2.9

The quadrilateral P from Example 2.3 ($n = 1$), has four supporting cones: $(0,0) + pos\{(1,0),(0,1)\}$, $(\frac{5}{2},0) + pos\{(-1,0),(0,1)\}$, $(\frac{5}{2},\frac{1}{2}) + pos\{(0,-1),(-1,1)\}$, and $(0,3) + pos\{(0,-1),(1,-1)\}\}$. It is easy to see that all these cones are unimodular (determinant $= \pm 1$), the only lattice points belonging to corresponding fundamental half-open parallelepipeds are respectively : $(0,0), (2,0), (2,1)$, and $(0,3)$ (fig. 4). Applying Brion's theorem and formula (2.1), we obtain (with $\mathbf{x} = (x,y)$):
$f(P;\mathbf{x}) = \frac{1}{(1-x)(1-y)} + \frac{x^2}{(1-x^{-1})(1-y)} + \frac{x^2 y}{(1-y^{-1})(1-x^{-1}y)} + \frac{y^3}{(1-y^{-1})(1-xy^{-1})}$.

Making the variable substitution[11] $x = (1+t)^1, y = (1+t)^2$, we obtain $f(P;\mathbf{x})$ as a univariate (Laurent) polynomial:
$g(t) = \frac{1}{(1-(1+t))(1-(1+t)^2)} + \frac{(1+t)^2}{(1-(1+t)^{-1})(1-(1+t)^2)} + \frac{(1+t)^4}{(1-(1+t)^{-2})(1-(1+t)^{-1}(1+t)^2)}$
$+ \frac{(1+t)^6}{(1-(1+t)^{-2})(1-(1+t)(1+t)^{-2})}$).

[11] The integer vector $\lambda = (1,2)$ used in this substitution is chosen such that λ is not orthogonal to any generator. See [6] for more details on how to evaluate $f(P;\mathbf{x})$ at $\mathbf{x} = (\mathbf{1}, \cdots, \mathbf{1})$.

Simplifying this expression in order to obtain only positive powers in the denominators and factorizing out $\frac{1}{t^2}$ from each term, we obtain through Taylor expansion:
$g(t) = \frac{1}{t^2}(\frac{1}{2} - \frac{1}{4}t - \frac{1}{8}t^2 + \cdots) + \frac{1}{t^2}(\frac{-1}{2} - \frac{5}{4}t - \frac{7}{8}t^2 + \cdots) + \frac{1}{t^2}(-\frac{1}{2} - \frac{11}{4}t - \frac{49}{8}t^2 + \cdots) + \frac{1}{t^2}(\frac{1}{2} + \frac{17}{4}t + \frac{127}{8}t^2 + \cdots)$.

Finally, the number of lattice points in the polytope P is given by:
$|P \cap \mathbb{Z}^2| = \lim_{\mathbf{x}\to(1,1)} f(P;\mathbf{x}) = \lim_{t\to 0} g(t) = \frac{1}{8} + \frac{-7}{8} + \frac{-49}{8} + \frac{127}{8} = \frac{72}{8} = 9$.

In this easy example, the polytope decomposition is simple since all supporting cones are unimodular. In the general case, one or more supporting cones are not necessarily unimodular. The fundamental idea of Barvinok was to decompose each cone $K \subset \mathbb{R}^d$ into a (signed) sum of unimodular cones $K_i \subset \mathbb{R}^d$: $[K] = \sum_{i\in I} \epsilon_i [K_i]$, where $[.]$ denotes the indicator function and $\epsilon_i \in \{-1, 1\}$ depending whether K_i is added or subtracted. Via this decomposition, we can write the expression: $f(K;\mathbf{x}) = \sum_{i\in I} \epsilon_i f(K_i;\mathbf{x})$. Consequently, the generating function $f(P;\mathbf{x})$ can be written as a signed sum of short rational functions.

Theorem 2.4 (Barvinok) *Let P be a rational polytope of dimension d. The multivariate generating function $f(P;\mathbf{x})$ can be written in polynomial time as:*

$$f(P;\mathbf{x}) = \sum_{i\in I} \epsilon_i \frac{\mathbf{x}^{w_i}}{\prod_{j=1}^{d}(1 - \mathbf{x}^{u_{ij}})}$$

where I is a (polynomial-size) indexing set, $\epsilon_i \in \{-1, 1\}$ and w_i, $u_{ij} \in \mathbb{Z}^d$ for all i and j.

The polynomial-time algorithm described in the above Theorem was further generalized by Barvinok and Pommersheim [2] to parametric polytopes. In 2004, De Loera et al [6] developed the programm **LattE**, a computer package for lattice point enumeration, which contains the first implementation of the technique of Barvinok for enumerating non-parametric polytopes.

Note that Barvinok's algorithm could be used to perform initial countings needed for Clauss's method in order to make these countings more efficient. However, using the extension proposed by Barvinok and Pommersheim , Ehrhart polynomials can be obtained *analytically*. This extension, implemented by Verdoolaege et al [28], takes into account the validity domains while keeping the overall structure of Barvinok's algorithm (See appendix). The first step is to compute parametric vertices and validity domains. Then, Barvinok's algorithm is applied to the fixed set of parametric vertices that belongs to each validity domain. Obviously, this parameterized version of Barvinok's algorithm needs to handle periodic numbers. To avoid the exponential behavior of the look-up tables used in the interpolation method, periodicity is represented using fractional parts, with the following notation: For a rational number x, the rational part is denoted by $\{x\}$ and is defined as follows: $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Example 2.10 The periodic number $U(n,m) = \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}_{n,m}$ can be written as: $(\{\frac{1}{2}n\} - 1)\{\frac{1}{3}m\} + (\{-\frac{1}{2}n\} + 1)$. The quasi-polynomial from Example 2.7 (for the last validity domain) can be written as:

$(\frac{5}{4}m - \frac{5}{2}\{\frac{1}{2}m\} + \frac{5}{2})n + ((\frac{-1}{2}\{\frac{1}{2}n\} + \frac{1}{2})m + (\{\frac{1}{2}m\} - 1)\{\frac{1}{2}n\} - \{\frac{1}{2}m\} + 1)$
Notice that this representation is more convenient when the period is very large.

To conclude this expository section, we mention that the proofs of Theorems and formulas given above can be found in the references already cited, particularly [1, 2, 3]. For a general background on algorithms computing Ehrhart polynomials, limits and time complexity of these algorithms, we recommend the excellent and complete report written by Verdoolaege et alii [29].

3 Applying Clauss and Barvinok algorithms to voting theory

The purpose of this section is to illustrate the use of the Clauss and the Barvinok algorithms for obtaining new probabilistic results in voting theory. We consider three-candidate elections with n voters (the notation is the same as in the introduction) and IAC is assumed. In what follows, we will make use of the well known relation:

$$D(n) = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{120}$$

which gives the total number of voting situations as a function of n, *i.e.* the number of integer solutions associated with the following system:

$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n$ and $n_i \geq 0$ for $i = 1, 2, ..., 6$.

The programs we use to implement Clauss and Barvinok algorithms have been coded by Ahmed Louichi [12]. Three issues will be examined with the help of these programs.

3.1 Problem 1: Disagreement between plurality and plurality runoff

The two voting methods that are the most commonly used in presidential elections are (simple) plurality voting and plurality runoff. Under plurality voting, each voter votes for one of the candidates and the candidate with the highest number of votes is elected. Under plurality runoff, a candidate is elected at the first stage if she obtains more than 50% of the votes; if no candidate obtains this absolute majority, then a second stage is organized in which the two candidates with the highest plurality scores at the first stage are confronted in a pairwise majority contest. It is of interest to ask the following question: What is the probability that these two methods disagree when IAC is assumed in a three-candidate election?

The two methods disagree when, for instance, a is the plurality winner, b obtains a plurality score higher than c and a majority of voters prefer b to a.[13] The system of linear (in)equalities that characterize this event is given as:

[12]The Clauss program is based on: Polylib library [22] by Chenikova (kernel), Wilde, Loechner and IRISA team. Parameterized Barvinok program is based on Latte project [25] and Verdoolaege library [26]. In order to deal with multi-precision integers, GMP [18] and NTL [24] libraries have been used. The two programs are compiled with the GNU tools under GPL licence.

[13]We ignore in this illustrative investigation the problem of tied elections.

$n_1 + n_2 > n_3 + n_4,$
$n_3 + n_4 > n_5 + n_6,$
$n_1 + n_2 + n_5 < n/2,$
$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n,$
$n_i \geq 0$ for $i = 1, 2, ..., 6.$

Using Clauss's algorithm, we obtain (after a computation time of 1 minute and 40 seconds) the following quasi-polynomial:

$$\frac{71}{414720}n^5 + [\frac{-1}{1728}, \frac{91}{82944}]_n n^4 + [\frac{-17}{5184}, \frac{-209}{41472}, \frac{-49}{5184}, \frac{47}{41472}, \frac{-49}{5184}, \frac{-209}{41472}]_n n^3$$
$$+[\frac{1}{36}, \frac{-1589}{41472}, \frac{-65}{5184}, \frac{-7}{1536}, \frac{-1}{324}, \frac{-1333}{41472}, \frac{7}{576}, \frac{-2237}{41472}, \frac{1}{324}, \frac{17}{1536}, \frac{-97}{5184}, \frac{-1981}{41472}]_n n^2$$
$$+[\frac{4}{45}, \frac{-13661}{414720}, \frac{101}{2880}, \frac{-869}{46080}, \frac{29}{270}, \frac{4259}{414720}, \frac{-59}{2880}, \frac{-59021}{414720}, \frac{13}{90}, \frac{4171}{46080}, \frac{-17}{8640}, \frac{-41101}{414720}]_n n$$
$$+[0, \frac{6223}{82944}, \frac{77}{1296}, \frac{-65}{1024}, \frac{16}{81}, \frac{13391}{82944}, \frac{-3}{16}, \frac{-11921}{82944}, \frac{20}{81}, \frac{159}{1024}, \frac{13}{1296}, \frac{-4753}{82944}]_n.$$

The program indicates that this relation is valid for $n \geq 9$ and the periodicity can be seen to be equal to 12.

The use of Barvinok's algorithm gives (after 2 seconds) the following output, with the same validity domain:

$$71/414720*n^5+(139/41472*\{(1/2*n+0)\}+-1/1728)*n^4+(1/36*\{(1/3*n+0)\}^2+-1/36*\{(1/3*n+0)\}+(61/6912*\{(1/2*n+0)\}+-17/5184))*n^3+(-1/12*\{(1/4*n+1/2)\}^3+(1/8*\{(1/2*n+0)\}+1/16)*\{(1/4*n+1/2)\}^2+(-3/32*\{(1/2*n+0)\}+1/48)*\{(1/4*n+1/2)\}+(-1/12*\{(1/3*n+0)\}^3+(1/6*\{(1/2*n+0)\}+1/4)*\{(1/3*n+0)\}^2+(-1/6*\{(1/2*n+0)\}+-1/6)*\{(1/3*n+0)\}+(-77/2304*\{(1/2*n+0)\}+7/576)))*n^2+(-1/12*\{(1/4*n+1/2)\}^4+(1/6*\{(1/2*n+0)\}+-1/2)*\{(1/4*n+1/2)\}^3+(11/16*\{(1/2*n+0)\}+11/24)*\{(1/4*n+1/2)\}^2+(-61/96*\{(1/2*n+0)\}+1/8)*\{(1/4*n+1/2)\}+(-3/16*\{(1/3*n+0)\}^4+(-1/6*\{(1/2*n+0)\}+-1/8)*\{(1/3*n+0)\}^3+(3/2*\{(1/2*n+0)\}+17/48)*\{(1/3*n+0)\}^2+(-4/3*\{(1/2*n+0)\}+-1/24)*\{(1/3*n+0)\}+(5/384*\{(1/2*n+0)\}+-59/2880)))*n+(4/15*\{(1/4*n+1/2)\}^5+(-1/6*\{(1/2*n+0)\}+-1)*\{(1/4*n+1/2)\}^4+(1*\{(1/2*n+0)\}+-1/3)*\{(1/4*n+1/2)\}^3+(29/48*\{(1/2*n+0)\}+1)*\{(1/4*n+1/2)\}^2+(-33/32*\{(1/2*n+0)\}+1/15)*\{(1/4*n+1/2)\}+(3/5*\{(1/3*n+0)\}^5+(-3/8*\{(1/2*n+0)\}+-3/2)*\{(1/3*n+0)\}^4+(-1/4*\{(1/2*n+0)\}+2/3)*\{(1/3*n+0)\}^3+(27/8*\{(1/2*n+0)\}+-1/2)*\{(1/3*n+0)\}^2+(-11/4*\{(1/2*n+0)\}+11/15)*\{(1/3*n+0)\}+(5/16*\{(1/2*n+0)\}+-3/16))).$$

It can be checked that this formulation is equivalent to the one we have obtained with Clauss's algorithm. If, for example, we consider the coefficient of n^4 in Barvinok's result, we observe that $\{(1/2*n+0)\} = 0$ if n is even and $\{(1/2*n+0)\} = 1/2$ if n is odd. Consequently, the n^4 coefficient is $-1/1728$ if n is even and $139/41472 * 1/2 - 1/1728 = 91/82944$ if n is odd, in accordance with Clauss's algorithm result.

From this quasi-polynomial, it can be deduced that for $n \equiv 9 \mod 12$, the number of voting situations for which alternative a is the plurality winner and is not the plurality runoff winner is[14]:

$$\frac{71}{414720}n^5 + \frac{91}{82944}n^4 + \frac{47}{41472}n^3 + \frac{17}{1536}n^2 + \frac{4171}{46080}n + \frac{159}{1024}.$$

[14] Clearly, this step is much easier when starting from Clauss's formulation rather than from Barvinok's result.

The eleven other polynomials can be obtained in a similar way.

Multiplying by 6 (the plurality winner may be a, b or c and, when a is the plurality winner, b or c may obtain the second rank position) and dividing by the total number of voting situations $D(n)$, we obtain the desired probability. For $n \equiv 9 \mod 12$, this probability is given as:

$$\frac{71n^5 + 455n^4 + 470n^3 + 4590n^2 + 37539n + 64385}{576(n+1)(n+2)(n+3)(n+4)(n+5)}.$$

This relation allows us to conclude that, for large electorates, the likelihood of a disagreement between plurality voting and plurality runoff is about 12%.

3.2 Problem 2: Manipulability of the Borda rule

Although there exists different ways for measuring the manipulability of alternative voting rules (see *e.g.* Pritchard and Wilson [23]), the most common approach consists in computing the proportion of voting situations at which the rule under consideration is manipulable by a single voter (*individual* manipulation) or by a coalition of voters (*collective manipulation*). Adopting this approach, Lepelley and Mbih [19] have shown, among other results, that, for large electorates, the plurality rule can be manipulated by a coalition of voters in 29% of the voting situations (7/24, to be exact), and the corresponding proportion for plurality runoff is 1/9 (11%). In a recent paper, Favardin and Lepelley [10] have used a Clauss-Huang-Chua type algorithm to compute the manipulability of a large number of voting rules. However, they failed to obtain the exact limiting coalitional manipulability of the famous Borda rule[15] : in this case, Clauss's algorithm does not work. We are going to show that the use of Barvinok's algorithm can solve the problem.

A bit of notation is needed. $V(n)$ is the proportion of voting situations at which the Borda rule is manipulable by a coalition of voters. We wish to evaluate $V(\infty)$. Let B_{ij} be the difference between the Borda score of candidate i and the Borda score of candidate j. Consider a voting situation where the Borda winner is candidate a. Favardin *et alii* [11] have shown that the Borda rule is manipulable by a coalition of voters at this situation if and only if:

(1) $B_{bc} + 2n_6 \geq n_3$ and $B_{ab} < n_3 + n_6$; or
(2) $n_3 > B_{bc} + 2n_6 \geq 0$ and $B_{ab} < B_{bc} + 3n_6$; or
(3) $B_{cb} + 2n_4 > n_5$ and $B_{ac} < n_4 + n_5$; or
(4) $n_5 \geq B_{cb} + 2n_4 \geq 0$ and $B_{ac} < B_{cb} + 3n_4$.

Let $\#(i)$ denote the number of situations that are compatible with inequalities (i), $i = 1, 2, 3, 4$. Let $\#(i, j)$ be the number of situations that are compatible with both (i) and (j). Clearly, $\#(1, 2) = 0$ and $\#(3, 4) = 0$. Noting that, for large n, all the above inequalities can be considered as strict for our purpose (the proportion of situations

[15] Under this rule, the voters are asked to rank the candidates. In three-candidate elections, the Borda rule consists in giving 2 points to a candidate for each ballot on which she is ranked first, 1 point for each on which she is ranked second and 0 point for a last rank position. The winner is the candidate with the highest number of points.

corresponding to an equality tends towards 0), and using symmetry arguments, we obtain that $V(n)$ can be computed as:

$$V(n) = \frac{3(2\#(1) + 2\#(2) - \#(1,3) - \#(2,4) - 2\#(1,4))}{D(n)}.$$

It remains to calculate $\#(1)$, $\#(2)$, $\#(1,3)$, $\#(2,4)$ and $\#(1,4)$.

Given the definition of $B_{i,j}$, the system of inequalities associated with (1) can easily be written as (the third and fourth inequalities mean that a is the Borda winner):

$$\begin{aligned}
&n_1 - n_2 + n_3 + n_4 - 2n_5 + n_6 \geq 0,\\
&-n_1 - 2n_2 + 2n_3 + 2n_4 - n_5 + 2n_6 \geq 0,\\
&n_1 + 2n_2 - n_3 - 2n_4 + n_5 - n_6 > 0,\\
&2n_1 + n_2 + n_3 - n_4 - n_5 - 2n_6 > 0,\\
&n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n,\\
&n_i \geq 0 \text{ for } i = 1, 2, ..., 6.
\end{aligned}$$

Confronted by this set of inequalities, Clauss's algorithm is inoperative. Barvinok's algorithm, on the other hand, provides an output after a calculation time of 3 seconds. This output is quite huge and more than 10 pages would be necessary to exhibit it. But all that we need for computing the limiting value of $V(n)$ is the coefficient of n^5, which is independent from n (*cf* Theorem 2.1). This coefficient is $43871/61236000$.

We can obtain, in a similar way, the coefficients of the leading term of the quasi-polynomials corresponding to (respectively) $\#(2)$, $\#(1,3)$, $\#(2,4)$ and $\#(1,4)$. Finally, the desired result is given as:

$$V(\infty) = 3\frac{2\frac{43871}{61236000} + 2\frac{473}{5832000} - \frac{234989}{1714608000} - \frac{2059}{122472000} - 2\frac{1237}{54432000}}{\frac{1}{120}} = \frac{132953}{264600}.$$

It is worth noticing that this exact result (50.247%) is very close to the approximation given in Favardin *et alii* [11] (50.25%).

3.3 Problem 3: Manipulability, plurality rule and single-peakedness

An interesting feature of the Clauss and Barvinok algorithms is that they enable the obtaining of quasi-polynomials as a function of more than one parameter. For example, it becomes possible to derive probability representations that depend not only on n, the number of voters, but also on another parameter which captures some given attribute of preference profiles. This is precisely what we wish to illustrate with this third problem.

We follow here an idea developped by Gehrlein [14, 13, 15]. Suppose we want to study the coalitional manipulability of the plurality rule in three-alternative elections. Lepelley and Mbih [19] provided a representation for the proportion of situations at which plurality is manipulable by a coalition of voters. It is possible to go further by investigating what happens when some degree of consistency of individual preferences is introduced. The notion of single peakedness had been proposed by Black in order to reflect this coherence of preferences in voting situations. In three-alternative elections, preferences are single peaked when some candidate is never ranked in the last place in the preference rankings of the voters. Assuming that preferences are "perfectly" single peaked is an hypothesis that can be considered as too radical in many circumstances. It is probably more interesting and more realistic to consider a *tendency* towards single peakedness. For that purpose, we will use the parameter k that measures the minimum number of times that some candidate is ranked last in the preferences of voters:

$$k = Min(n_4 + n_6, n_2 + n_5, n_1 + n_3).$$

The number k serves as a simple measure of the proximity of a voting situation to being perfectly single peaked. When $k = 0$, the associated situation has perfectly single peaked preferences, and taking k close to its maximum value ($n/3$) reflects a situation that is far from perfect single peakedness. We know from Gehrlein [14] that, given n and k, $0 \leq k \leq n/3$, the total number of situations is given as:

$$D(n,k) = \frac{(k+1)(n-3k)((n+1)(n+5)-3k(2+k))}{2}$$

Instead of directly computing the number of situations at which plurality is manipulable, we will compute the number of situation where no manipulation can occur; such situations will be said to be *stable*. Once again, we will ignore tied elections: it means that the plurality winner is supposed to be unique and the candidate obtaining the minimum number of last place in the preference rankings is also unique. Assume that candidate a is the plurality winner. Three cases must be considered, according to the identity of the candidate who obtains the minimum number of last positions. In each of these cases, we must have:

$n_1 + n_2 - n_3 - n_4 - n_6 > 0,$
$n_1 + n_2 - n_4 - n_5 - n_6 > 0,$
$n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n,$
$n_i \geq 0$ for $i = 1, 2, ..., 6.$ (3.1)

The two first inequalities come from Lepelley and Mbih and characterize the stable situations for which a is the plurality winner. In addition to (3.1), we must also have:

$n_4 + n_6 < n_2 + n_5$, $n_4 + n_6 < n_1 + n_3$ and $n_4 + n_6 = k$ (3.2)
if a obtains the minimum number of last positions;
$n_2 + n_5 < n_4 + n_6$, $n_2 + n_5 < n_1 + n_3$ and $n_2 + n_5 = k$ (3.3)
if b obtains the minimum number of last positions;
$n_1 + n_3 < n_2 + n_5$, $n_1 + n_3 < n_4 + n_6$ and $n_1 + n_3 = k$ (3.4)
if c obtains the minimum number of last positions.

It can be checked by symmetry arguments that the number of situations compatible with ((3.1) and (3.3)) is equal to the number of situations compatible with ((3.1) and (3.4)). Consequently, the proportion of stable situations given n and k can be computed as:

$$P(n,k) = \frac{3\ (\#((3.1) \text{ and } (3.2))\ +\ 2\#((3.1) \text{ and } (3.3)))}{D(n,k)}.$$

We have calculated $\#((3.1)$ and $(3.2))$ and $\#((3.1)$ and $(3.3))$ by using Barvinok's algorithm. The computation time was about 2 seconds for each set of inequalities. We give hereafter the results we have derived from the output for the particular case where the parameters n and k are even multiples of three (similar representations could be obtained for all the other cases). Three validity domains must be distinguished. For each of these domains, the representation for $P(n,k)$ is as follows:

Domain 1: $0 \le k \le \frac{n-4}{5}$

$$\frac{124n^3(k+1)-6n^2(73k^2-40k-104)-12n(35k^3+396k^2+418k+84)+k(1771k^3+7984k^2+7500k+1584)}{144(k+1)(n-3k)(n^2+6n-3k^2-6k+5)}$$

Domain 2: $\frac{n-2}{5} \le k \le \frac{n-4}{4}$

$$\frac{-n^4+12n^3(4k+3)-4n^2(24k^2-72k-83)-32n(20k^3+102k^2+1000k+21)+32k(51k^3+204k^2+220k+69)}{144(k+1)(n-3k)(n^2+6n-3k^2-6k+5)}$$

Domain 3: $\frac{n-2}{4} \le k \le \frac{n-2}{3}$

$$\frac{n^4-9n^3k+3n^2(12k^2+5k+14)-2n(36k^3+36k^2+11k+6)+6k(9k^3+12k^2+2k+3)}{144(k+1)(n-3k)(n^2+6n-3k^2-6k+5)}$$

Computer enumeration was used to verify these representations for small values of the parameters. Table 1 lists $P(300,k)$ values for $k = 0, 6, 12, \cdots, 96$.

Table 1
Impact of a tendency towards single peakedness
on the frequency of stable situations under plurality rule:
(3 candidates, 300 voters)

k	$P(300,k)$	k	$P(300,k)$
0	0.858	54	0.693
6	0.849	60	0.658
12	0.837	66	0.617
18	0.824	72	0.574
24	0.808	78	0.538
30	0.791	84	0.511
36	0.771	90	0.491
42	0.749	96	0.459
48	0.723		

The results show that introducing some degree of homogeneity in individual preferences clearly increases stability. However, it can be observed that, even in presence of perfect single peakedness, the possibility of manipulation by coalition of voters remains significant: for $k = 0$ and large n, the manipulability measure is still equal to $1 - \frac{124}{144} = 5/36$ (about 14%).

4 Concluding remark

The use of the Clauss and Barvinok algorithms greatly facilitates the derivation of probability representations for voting outcomes. Barvinok's method appears to be particularly efficient and should be able to solve most of the problems of a probabilistic nature that we could considerer in voting theory for three candidate elections. The main limit of these algorithms is related to the number of variables and parameters that they can take into account. The maximum number that the Clauss and Barvinok methods can deal with seems to be about 20. Consequently, it is not possible to analyze four candidate elections, where the total number of variables (possible preference rankings) is 24. We hope that further developments of these algorithms will enable the overcoming of this difficulty.

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Appendix

Algorithms (From [28, 29])

Algorithm 1: Barvinok's algorithm

1. For each vertex v_i of P
   (a) Determine supporting cone, $K(P, v_i)$
   (b) Let $K = K(P, v_i) - v_i$
   (c) Decompose K into unimodular cones K_j such that: $[K] = \sum_j \epsilon_j [K_j]$
   (d) For each K_j, determine $f(K_j; x)$
   (e) $f(K(P, v_i); x) = \sum_j \epsilon_j x^{E(v_i, K_j)} f(K_j; x)$, $\epsilon_j \in \{-1, 1\}$ and $E(v_i, K_j)$ is the unique lattice point belonging to the fundamental half-open parallelepiped corresponding to the translated cone $K_j + v_i$
2. $f(P; x) = \sum_{v_i \in D} f(K(P, v_i); x)$
3. evaluate $f(P; 1)$

Algorithm 2: Parameterized Barvinok

1. For each (parametric) vertex $v_i(\mathbf{p})$ of P
   (a) Determine supporting cone, $K(P, v_i(\mathbf{p}))$
   (b) Let $K = K(P, v_i(\mathbf{p})) - v_i(\mathbf{p})$
   (c) Decompose K into unimodular cones: $[K] = \sum_j \epsilon_j [K_j]$
   (d) For each K_j, determine $f(K_j; x)$
   (e) $f(K(P, v_i(\mathbf{p})); x) = \sum_j \epsilon_j x^{E(v_i(\mathbf{p}), K_j)} f(K_j; x)$
2. For each validity domain D of P
   (a) $f(P; x) = \sum_{v_i(\mathbf{p}) \in D} f(K(P, v_i(\mathbf{p})); x)$
   (b) evaluate $f(P; 1)$