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Abstract. *In this note we study the existence of Nash equilibria in nonsymmetric finite congestion games, complementing the results obtained by Milchtaich on monotone-decreasing congestion games. More specifically, we examine the case of two resources and we propose a simple method describing all Nash equilibria in this kind of congestion games. Additionally, we give a new and short proof establishing the existence of a Nash equilibrium in this type of games without invoking the potential function or the finite improvement property.*

Keywords: Singleton congestion games, Nash equilibria, Potential function, Finite improvement property.

JEL classification: C72

1 Introduction

In recent years, many economists have been interested in the general class of congestion games introduced by Rosenthal [8]. In these games, a set of players compete for a set of resources, and the payoff of each resource depends only on the number of players using it. The utility a player derives from a combination of resources is the sum of the payoffs associated with each resource included in his choice. A key game-theoretic property of these games is that they always have at least one pure strategy Nash equilibrium. This result follows from the existence of a potential function (Rosenthal). However, Konishi, Le Breton and Weber [4], Quint and Shubik [7], and Milchtaich [5] consider that congestion games do not admit (in general) a potential function, but are likely to admit a Nash equilibrium in pure strategies. A slightly different formulation of congestion games was introduced by Milchtaich [5] under the name of congestion games with player-specific payoff

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functions. Each player has individual nonincreasing payoff functions and is allowed to choose any resource but must choose exactly one. Milchtaich showed that such a game possesses at least one Nash equilibrium without invoking a potential function but by using the finite improvement property. His proof implicitly contains an efficient algorithm for computing an equilibrium. Additionally, he shows that players iteratively playing best responses in such games do not necessarily reach a Nash equilibrium, that is, the best response dynamics may cycle. However, he shows that from every state of such a game there exists a polynomially long sequence of best responses to a Nash equilibrium. Ieong et *al.* [3] generalized this result to the largest class of singleton congestion games (where the payoff functions are not required to be monotone). They also showed that even optimal equilibria (Nash equilibria that maximize the sum of players' utilities) can be found in a polynomial time. Holzman and Law-Yone [2] and Voorneveld et *al.* [10] investigated the set of strong Nash equilibria [1] in monotone singleton congestion games. It turns out that this set coincides with the set of Nash equilibria and with the set of profiles which maximize the potential. Variants of (monotone) singleton congestion games have been studied in terms of time convergence of the best-reply dynamics to a Nash equilibrium (Even-Dar et *al.* [1]) and in terms of the existence of an alternative concept of solution (Rozenfeld and Tennenholtz, [9]).

A substantial literature has been devoted to particular subclasses and extensions of congestion games. Most of the studies focus on the problem of finding and computing efficiently only one Nash equilibrium, leaving open the question of identifying all Nash equilibria. However, the characterization of the set of all equilibria, beyond its theoretical interest, can be very useful when we have to choose between these equilibria on the basis of performance criteria such as social optimality, or to explore intrinsic proprieties of the game such as the price of anarchy[2]. In this paper, we address this question for a simple subclass of congestion games which lie in the intersection between Rosenthal's and Milchtaich's model. We refer to games in this class as monotone singleton congestion games. Our approach yields a new and short proof establishing the existence of a Nash equilibrium in this kind of congestion games and shows how to compute all equilibria using a simple and direct formula. The rest of this paper is organized as follows: section 3 provides congestion games: definitions and notations, section 3 destabilizes the result and section 4 concludes the paper.

2 Congestion games: definitions and notations

Formally, a *game* (in strategic form) is defined by a tuple $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is a set of n players, S_i a finite set of strategies available to player i and $u_i : S = S_1 \times \ldots \times S_n \to \mathbb{R}$ is the utility function of player i. The set S is the strategy space of the game, and its elements are the (strategy) profiles. For a profile $\sigma = (\sigma_i)_{i\in N}$ on S, we will use the notation σ_{-i} to stand for the same profile with i's strategy excluded, so that (σ_{-i}, σ_i) forms a complete profile of strategies.

[1] A strong Nash equilibrium is a profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies.

[2] When utilities are replaced by costs, the price of anarchy of a game is the ratio of the social cost in the worst Nash equilibrium to the minimum social cost possible.

A (pure) Nash equilibrium of the game Γ is a profile σ^* such that each σ_i^* is a best-reply strategy: For each player $i \in N$, $u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$, for all $\sigma_i \in S_i$. Thus, no player can benefit from unilaterally deviating from his strategy.

In a *standard congestion game* [8] we are given a finite set $R = \{1, \ldots, m\}$ of m resources. A player's strategy is to choose a subset of resources among a family of allowed subsets: $S_i \subseteq 2^R$, for all $i \in N$. A payoff function $d_r : \{1, \ldots, m\} \to \mathbb{R}$ is associated with each resource $r \in R$, depending only on the number of players using this resource. For a profile σ and a resource r, the congestion on resource r (i.e. the number of players using r) is defined by $n_r(\sigma) = |\{i \in N : r \in \sigma_i\}|$. The vector $(n_1(\sigma), \ldots, n_m(\sigma))$ is the congestion vector corresponding to σ. The utility of player i from playing strategy σ_i in profile σ is given by $u_i(\sigma) = \sum_{r \in \sigma_i} d_r(n_r(\sigma))$.

There is a well known extension of congestion games, namely the monotone nonsymmetric *singleton congestion games* (singleton congestion games for short) which can be seen as the intersection between Rosenthal's and Milchtaich's model. A game in this class is defined by a tuple $\Gamma(N, R, (d_r)_{r \in R})$, where N is a set of n players, R is a set of m resources/strategies (a player's strategy consists of any single resource in R) and d_r is a nonincreasing payoff function associated with resource r. The utility of player i for a profile σ is simply given by $u_i(\sigma) = d_{\sigma_i}(n_{\sigma_i}(\sigma))$. We note that these games are nonsymmetric : Players are restricted to choose only one strategy, but they each have their own utility function. Since the utility of an anonymous player derived from selecting a single resource depends only on the number of the players doing the same choice, the common utility function is simply a mapping: $u : R \times \{1, \ldots, n\} \to \mathbb{R}$, $(r, k) \mapsto u(r, k)$, where u decreases with k.

In the remainder of this section we develop a technique which attempts to simplify the analysis of such games by moving to the ordinal representation of preferences. Indeed, in the case of singleton congestion games, we can, without affecting the set of Nash equilibria, replace the values of the payment functions by their ranks in a preference ordering representing the common utility function. More formally, a singleton congestion game will be represented by a tuple $\Gamma(N, R, \precsim)$ where N is a set of n players, R a set of m resources and $\precsim$ a weak ordering on $R \times \{1, \ldots, n\}$. In this ordinal context, a strategy profile σ^* is a Nash equilibrium of the game Γ if $\sigma^* \succsim (\sigma_i, \sigma_{-i}^*)$ for all σ_i in R. We also note that, since players are anonymous, all strategy profiles that differ only by a permutation of players can be identified by the corresponding congestion vector. We refer to a congestion vector $\sigma^* = (n_1, \ldots, n_m)$ as a Nash equilibrium if, for all r, r' in R with $r \neq r'$, we have $(r, n_r) \succsim (r', n_{r'} + 1)$. Thus, no player can benefit from joining a group of players sharing a different resource.

In this paper, we are interested to introduce variants of the above notion in the simple case of two resources in singleton congestion games which are suitable for establishing at least one Nash equilibrium.

Let $G(N, R, (\precsim_i)_{i \in N})$ be a singleton congestion game and $R = \{a, b\}$ a set of two alternatives. To develop our approach, we need the following notation: For a player $i \in N$, we note $(a, 0) \succsim_i (b, n+1)$ (or $0 \cdot a \succsim_i (n+1) \cdot b$ by adopting the simplified notation) when $(a, 1) \precsim_i (b, n)$. Similarly, we note $(b, 0) \succsim_i (a, n+1)$ (or $0 \cdot b \succsim_i (n+1) \cdot a$) when $(b, 1) \precsim_i (a, n)$. For all $i \in N$, we define the following integers:

$$p_i = \max \{p \in \{0, 1, \ldots, n\} : (a, p) \succsim_i (b, n+1-p)\}$$

$$q_i = \max \{q \in \{0, 1, \ldots, n\} : (b, q) \succsim_i (a, n+1-q)\}$$

The entire p_i denotes the maximum size of a group choosing the alternative a in a given strategy profile, in which the player i can belong. Beyond this size, the player i will choose the resource b. Indeed, by definition we have $p_i \cdot a \succsim_i (n+1-p_i) \cdot b$ and $(p_i+1) \cdot a \precsim_i (n-p_i) \cdot b$. The entire q_i is interpreted in the same way; we replace a by b. We note that when ex-aequo cases are used, we have: $p_i + q_i \geq n$, for all $i \in N$. It is therefore possible, for some players i, to have $p_i + q_i > n$. This point is important because in this case there exists the possibility to have more than one congestion vector corresponding to a Nash equilibrium. However, when the preferences orders are strict we have $p_i + q_i = n$, for all $i \in N$.

Using the list of integers p_i and q_i ($i \in N$), we define two other integers that will serve to identify the congestion vector that may correspond to a Nash equilibrium of the game:

$$n(a) = \max \{p \in \{0, 1, \ldots, n\} :\mid \{i \in N : p_i \geq p\} \mid \geq p\}$$
$$n(b) = \max \{q \in \{0, 1, \ldots, n\} :\mid \{i \in N : q_i \geq q\} \mid \geq q\}$$

We point out that $n(a)$ (resp. $n(b)$) represents the maximum size of a group of players that can choose the resource a (resp. b) without any member of this group having interest in deviating from his strategy. When the preference orders include ex-aequo possibilities, we have the inequality $n(a)+n(b) \geq n$ and the corresponding congestion vector is $v = (\alpha, \beta)$, where $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$. In the case of strict orders, we necessarily have $n(a) + n(b) = n$, with the corresponding congestion vector being $v = (n(a), n(b))$.

In order to describe all Nash equilibria, we introduce the three following sets that allow us to identify the alternatives that correspond to each player:

- When the preference orders present ex-aequo cases:

  $$A(G, v) = \{i \in N : p_i \geq \alpha \text{ and } q_i < \beta\}, B(G, v) = \{i \in N : p_i < \alpha \text{ and } q_i \geq \beta\}$$
  $$C(G, v) = \{i \in N : p_i \geq \alpha \text{ and } q_i \geq \beta\}$$

  N is the disjoint union of these three sets and that each of these sets may be empty. We do not examine the case in which $p_i < \alpha$ and $q_i < \beta$, as $p_i + q_i \geq n$ and $\alpha + \beta = n$.

- If the preference orders are strict:

  $$A(G) = \{i \in N : p_i > n(a)\}, B(G) = \{i \in N : p_i < n(a)\}$$
  $$C(G) = \{i \in N : p_i = n(a)\}$$

  Here, N is the disjoint union of these three sets, each of which may be empty and $|C(G)| \succeq na - |A(G)|$.

The following section allows to establish the existence of at least one Nash equilibrium and to determine a complete list of all equilibria of a given singleton congestion game.

3 The result

Many of the methods proposed in the literature until now attempt to find an equilibrium for a corresponding class of games. The main drawback is that most of these approaches, such as Rosenthal's and Milchtaich's ones, give only one equilibrium, ignoring the general structure of the set of all Nash equilibria. Our aim in this paper is to improve the study of singleton congestion games by providing a general method to describe *all* Nash equilibria and to establish a comprehensive list of all of them. We investigate the special case of two resources and our approach is such that we are not making use of the potential function or the FIP invoked by Rosenthal and Milchtaich respectively.

Theorem 1 . *Let $R = \{a, b\}$ and $G(N, R, (\prec)_{i \in N})$ be a singleton congestion game where all preference orderings are strict.*

1. *G admits at least one Nash equilibrium. All equilibria correspond to the same congestion vector : $v = (n(a), n(b))$.*
2. *Each Nash equilibrium of G, $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$, is characterized by a unique subset D (possibly empty) of $C(G)$, of cardinal $n(a) - |A(G)|$, such that: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G) \cup D$ and $\sigma_i^* = b$ if $i \in B(G) \cup (C \setminus D)$.*
3. *The game admits exactly $C_{|C(G)|}^{n(a)-|A(G)|}$ Nash equilibria. In particular, if $n(a) = |A(G)|$ the game admits a single Nash equilibrium.*

Theorem 2 . *Let $R = \{a, b\}$ and $G(N, R, (\precsim)_{i \in N})$ be a singleton congestion game where the preference orders present ex-aequo possibilities.*

1. *Each congestion vector $v = (\alpha, \beta)$ such that $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$, corresponds to (at least) one Nash equilibrium of G.*
2. *Each of the Nash equilibrium of G corresponding to the vector v, $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$, is characterized by a unique subset D (possibly empty) $C(G, v)$, of cardinal $\alpha - |A(G, v)|$, to ensure that: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G, v) \cup D$ and $\sigma_i^* = b$ if $i \in B(G, v) \cup (C(G, v) \setminus D)$.*

Let illustrate our main results by the following two examples.

Example 1 . *Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $R = \{a, b\}$. Suppose that the players' preferences are given by the following weak ordering:*

$$J_1: \quad 8b \prec 7b \prec 8a \prec 6b \prec 5b \prec 4b \prec 7a \prec 6a \prec 5a \prec 3b \prec 4a \prec 3a \prec 2b \prec 2a \prec b \prec a$$
$$J_2: \quad 8a \prec 7a \prec 6a \prec 8b \prec 7b \prec 6b \prec 5b \prec 5b \prec 5a \prec 4a \prec 3a \prec 2a \prec a \prec 3b \prec 2b \prec b$$
$$J_3: \quad 8b \prec 7b \prec 6b \prec 5b \prec 4b \prec 3b \prec 8a \prec 7a \prec 6a \prec 2b \prec 5a \prec 4a \prec 3a \prec b \prec 2a \prec a$$
$$J_4: \quad 8b \prec 7b \prec 6b \prec 5b \prec 4b \prec 3b \prec 2b \prec b \prec 8a \prec 7a \prec 6a \prec 5a \prec 4a \prec 3a \prec 2a \prec a$$
$$J_5: \quad 8a \prec 7a \prec 8b \prec 7b \prec 6a \prec 6b \prec 5b \prec 4b \prec 3b \prec 5a \prec 4a \prec 3a \prec 2a \prec a \prec 2b \prec b$$
$$J_6: \quad 8b \prec 7b \prec 6b \prec 5b \prec 4b \prec 3b \prec 2b \prec 8a \prec 7a \prec 6a \prec 5a \prec 4a \prec b \prec 3a \prec 2a \prec a$$
$$J_7: \quad 8a \prec 7a \prec 6a \prec 8b \prec 5a \prec 7b \prec 4a \prec 3a \prec 6b \prec 5b \prec 4b \prec 3b \prec 2a \prec a \prec 2b \prec b$$
$$J_8: \quad 8b \prec 8a \prec 7a \prec 7b \prec 6b \prec 5b \prec 4b \prec 6a \prec 3b \prec 5a \prec 2b \prec 4a \prec 3a \prec b \prec 2a \prec a$$

We have omitted the indices of players in the order of preferences. For each player i, we search the integer p_i which is the greatest p such that $na \succ_i (n+1-p)b$. We obtain $(n - p_i)b \succ_i (p+1)a$.

$$
\begin{array}{llll}
p_1 = 5: & 5a \succ_1 4b & and & 3b \succ_1 6a \\
p_2 = 5: & 5a \succ_2 4b & and & 3b \succ_2 6a \\
p_3 = 6: & 6a \succ_3 3b & and & 2b \succ_3 7a \\
p_4 = 8: & 8a \succ_4 b & and & 0b \succ_4 9a \\
p_5 = 5: & 5a \succ_5 4b & and & 3b \succ_5 6a \\
p_6 = 7: & 7a \succ_5 2b & and & b \succ_5 8a \\
p_7 = 2: & 2a \succ_5 7b & and & 6b \succ_5 3a \\
p_8 = 5: & 5a \succ_5 4b & and & 3b \succ_5 6a
\end{array}
$$

So, we can verify that $n(a) = 5$ *and* $n(b) = 3$. *The only congestion vector corresponding to a Nash equilibrium is the vector* $(5a, 3b)$. *Furthermore, we have:* $A(G) = \{3, 4, 6\}$, $B(G) = \{7\}$ *and* $C(G) = \{1, 2, 5, 8\}$. *By theorem 1, we know that there are exactly* $C_4^2 = 6$ *different Nash equilibria. All these equilibria are common:* $\sigma^* = (a)$ *if* $i \in A(G)$ *and* $\sigma^* = (b)$ *if* $i \in B(G)$. *Each of these equilibria is characterized by a subset* D *of* $C(G)$ *with* $|D| = 2$ *and* $\sigma_i^* = a$ *if* $i \in D$. *The list of the Nash equilibria of this game is:*

$$
(a,a,a,a,b,a,b,b),\ (a,b,a,a,a,a,b,b),\ (a,b,a,a,b,a,b,a),
$$
$$
(b,a,a,a,a,a,b,b),\ (b,a,a,a,b,a,b,a),\ (b,b,a,a,a,a,b,a).
$$

Example 2 . *Let* $N = \{1, 2, 3, 4, 5\}$ *and* $R = \{a, b\}$. *Suppose that the players' preferences are given by the following weak ordering:*

$$
\begin{array}{ll}
J_1: & 5a \prec 5b \prec 4b \prec 4a \prec 3b \sim 3a \sim 2a \prec 2b \sim a \prec b \\
J_2: & 5b \sim 4b \sim 5a \sim 4a \sim 3b \sim 3a \sim 2a \sim 2b \sim a \sim b \\
J_3: & 5a \prec 5b \prec 4b \prec 4a \sim 3b \sim 3a \sim 2b \prec 2a \prec a \prec b \\
J_4: & 5b \prec 5a \prec 4b \prec 4a \sim 3b \sim 3a \sim 2a \prec 2b \prec a \prec b \\
J_5: & 5b \sim 4b \sim 5a \sim 4a \sim 3b \sim 3a \sim 2a \sim 2b \sim a \sim b
\end{array}
$$

It is easy to see that:
$p_1 = 3,\ q_1 = 3,\ p_2 = 5,\ q_2 = 5,\ p_3 = 4,\ q_3 = 3,\ p_4 = 4,\ q_4 = 3,\ p_5 = 5,\ q_5 = 5.$
We have $n(a) = 4$ *and* $n(b) = 3$. *By theorem 2, the possible congestion vectors are:* $v_1 = (4a, b)$, $v_2 = (3a, 2b)$, $v_3 = (2a, 3b)$.
Since $v_1 = (4a, b)$, *we have:* $A(G, v_1) = \emptyset$, $B(G, v_1) = \{1\}$ *and* $C(G, v_1) = \{2, 3, 4, 5\}$. *Thus, there exists a unique equilibrium corresponding to* v_1, *which is the profile* (b, a, a, a, a).
Similarly, $v_2 = (3a, 2b)$. *We have:* $A(G, v_2) = \emptyset$, $B(G, v_2) = \emptyset$ *and* $C(G, v_3) = \{1, 2, 3, 4, 5\}$. *The Nash equilibria corresponding to* v_2 *are:*

$$
(b,b,a,a,a),\ (b,a,b,a,a),\ (b,a,a,b,a),\ (b,a,a,a,b),\ (a,b,a,a,b),
$$
$$
(a,a,b,a,b),\ (a,a,a,b,b),\ (a,b,a,b,a),\ (a,b,b,a,a),\ (a,a,b,b,a).
$$

Finally for $v_3 = (2a, 3b)$ *we have:* $A(G, v_3) = \emptyset$, $B(G, v_3) = \emptyset$ *and* $C(G, v_3) = \{1, 2, 3, 4, 5\}$. *The Nash equilibria corresponding to* v_3 *are:*

$$
(b,b,b,a,a),\ (b,b,a,b,a),\ (b,b,a,a,b),\ (b,a,a,b,b),\ (a,b,b,b,a),
$$
$$
(a,a,b,b,b),\ (b,a,b,b,a),\ (b,a,b,a,b),\ (a,b,b,a,b),\ (a,b,a,b,b).
$$

Proof 1 . *1) By definition of* $n(a)$, *there are at least* $n(a)$ *players* $i \in N$ *such that* $p_i \geq n(a)$. *Therefore, we choose* $n(a)$ *players satisfying this condition including all players for whom* $p_i > n(a)$. *Note* A *the set of these players. For all players*

who are in $B = N\backslash A$, we must have $p_i \leq n(a)$ and therefore $q_i \geq n(b)$. It is easy, returning to the definition of p_i and q_i, to verify that the profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ defined by $\sigma_i^* = a$ if $i \in A$ and $\sigma_i^* = b$ if $i \in B$ is a Nash equilibrium. Let σ^* be a Nash equilibrium of G and let (α, β) be the congestion vector associated with σ^*. Suppose that $\alpha > n(a)$. As σ^* is a Nash equilibrium, there exist α players such that $p_i \geq \alpha$, which contradicts the maximality of $n(a)$. We must therefore have $\alpha \leq n(a)$. Similarly, we show that $\beta \leq n(b)$. As $\alpha + \beta = n$ and $n(a) + n(b) = n$, we necessarily have $\alpha = n(a)$ and $\beta = n(b)$.
2) Let D be a subset (possibly empty) of $C(G)$, of cardinal $n(a) - |A(G)|$. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be the strategy profile defined by: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G) \cup D$ and $\sigma_i^* = b$ if $i \in B(G) \cup (C(G) \setminus D)$. The profile σ^* is a Nash equilibrium. Indeed, let $i \in A(G) \cup D$. By definition of $A(G)$ and D, we have $p_i \geq n(a)$. By definition of p_i and the assumption of monotonicity, we get: $n(a) \cdot a \succsim_i (n(b)+1) \cdot b$. Similarly, we show that for all i in $B(G) \cup (C(G) \setminus D$, $n(b) \cdot b \succsim_i (n(a) + 1) \cdot a$. Reciprocally, let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium of G. It is known from (1) that the congestion vector associated with σ^* is $(n(a), n(b))$. We must have $\sigma_i^* = a$ if $i \in A(G)$ and $\sigma_i^* = b$ if $i \in B(G)$. We just have to consider $D = \{i \in N : \sigma_i^* = a \text{ and } i \notin A(G)\}$.
3) The result is obtained by a simple calculation from (2). □

Proof 2 . *It suffices to prove (2), because (1) is obtained as a consequence of (2). Let $v = (\alpha, \beta)$ be a congestion vector such that $\alpha \leq n(a)$, $\beta \leq n(b)$ and $\alpha + \beta = n$. Let D be a subset (possibly empty) of $C(G, v)$, of cardinal $\alpha - |A(G, v)|$. Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a strategy profile such that: For all $i \in N$, $\sigma_i^* = a$ if $i \in A(G, v) \cup D$ and $\sigma_i^* = b$ if $i \in B(G, v) \cup (C(G, v) \setminus D)$. σ^* is a Nash equilibrium. Indeed, let $i \in A(G, v) \cup D$. By definition of $A(G, v)$ and of D, we have $p_i \geq \alpha$. By definition of p_i and by the assumption of monotonicity, we obtain: $\alpha \cdot a \succsim_i (\beta + 1) \cdot b$. Similarly, we show that for all i in $B(G, v) \cup (C(G, v) \setminus D$, $\beta \cdot b \succsim_i (\alpha + 1) \cdot a$. Reciprocally, let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium of G and let $v = (\alpha, \beta)$ be the congestion vector associated with this equilibrium. We have $\alpha \leq n(a)$, otherwise there exist α players i with $p_i \geq \alpha > n(a)$. This is impossible by definition of $n(a)$. Similarly, we show that $\beta \leq n(b)$. By definition of a congestion vector, we also have $\alpha + \beta = n$. As σ^* is a Nash equilibrium, for any $i \in N$, we must have: $\sigma_i^* = a$ if $i \in A(G, v)$ and $\sigma_i^* = b$ if $i \in B(G, v)$. We just need to consider $D = \{i \in N : \sigma_i^* = a \text{ et } i \notin A(G, v)\}$ and to note that the case $p_i < \alpha$ and $q_i < \beta$ is not possible.* □

4 Concluding remarks

Contrary to the studies done in the past, which provide only one Nash equilibrium in a specific class of games, in this paper, we have presented a method for describing the general structure of all Nash equilibria and identifying all of them in nonsymmetric singleton congestion games. Our approach is valid for the case of two resources and we consider that is complete. The ordinal representation of preferences allowed us to simplify the analysis of such games and to easily find a method for describing all Nash equilibria without using either the potential function or the finite improvement path invoked by Rosenthal and Milchtaich. It is also important to underline that our analysis can be used to obtain optimal Nash equilibria. As a future research, it remains an interesting open question to extend our approach to the general case ($R \succeq 3$) of nonsymmetric congestion games with player-specific payoff functions.

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