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College Choice Mechanism: The Respect of the Vagueness of Choices

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College Choice Mechanism: The Respect of the Vagueness of Choices

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Abstract

Taking as a starting point the theory of matching applied in the case of a problem of college admissions, where one is interested only to strict preference profiles for students and colleges, a part of the literature has been oriented towards profiles of priorities for colleges. In this paper we will assume that students have also their own priorities to which is associated some 'fuzzy'. This vagueness designates the preference of an individual (resp. college) for a college relative to parameters that characterize the latter one (resp. individual). Thus, we talk about fuzzy priorities. Our purpose is to analyze this aspect and to propose a real-life mechanism which will take into consideration the fuzzy priority profiles of both students and colleges, in order to achieve the best possible matching that is stable, strategy-proof, Pareto efficient and fair.

KEYWORDS: Education, Priorities, Preferences, Fuzzy, Algorithm, Matching.
JEL Codes: C78, D80, I20, I31.

1 Introduction

Matching theory, beyond its theoretical interest, is in the heart of the economic reality.

> *'Matching is one of the important functions of markets. Who gets which job, which school places, who marries whom, these help shape lives and careers'.* (Roth, 2008)

Gale and Shapley (1962) were the first to introduce *matching* models based on *marriage* and *college admission problems*. Their required objective was to assign a type of agent to another: in their case, the question was to make correspond a man to a woman or a student at a college. They proposed a class of *two-sided matching* models for studying such processes, since in such cases there are always two disjoint sets where each agent of a

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set is associated to another agent of the other set, via a bijective correspondence. They introduced the *Deferred Acceptance Algorithm* (DAA) which achieves this idea, considering that each agent has strict preferences over the members of the opposite side, and proved that the matching that emerges is *stable*. This means that any agent must be matched to another in such a way that they do not form a blocking pair. In other words, no individual prefers other alternative than the one he is matched to. Roth and Sotomayor (1989, 1990), Knuth (1976), Roth (1982, 1985a,b)[1] developed a large literature on the above dimensions, trying in particular to tackle the problems of assignment between employees and employers. Our article is directed towards the field of education, towards some 'college admission perspectives': each student makes a proposal at a college that may, or not, accept it. The problem being that, in such a situation, there is a *quota of admission* for each college concerned. Generally, a student carries out only one proposal for only one college which, in turn, can accept several individuals, until its quota is achieved. We refer to this kind of mechanisms as *'many-to-one'*. For simplicity, in what follows, we suppose a quota equal to the unit, what leads us to a mechanism *'one-to-one'*.

Following the work of Gale and Shapley (1962), Balinski and Sönmez (1999) introduced a *student placement problem* by considering this time that students have preferences over colleges and that colleges have priorities determined by local laws, and so on. Technically, priorities are mathematical arguments similar to preferences. Being subjected to such kind of constraints of selection, colleges become passive. This assumption allows us to focus on the welfare loss among students and not over colleges. However, their model is based on two specific priorities, namely, the individual skills and the individual results obtained during examinations. Abdulkadiroğlu and Sönmez (2003) had the idea of going over a student placement problem, applied, this time, in the case of schools (and not of colleges), considering a wider set of priorities. Their idea gave rise to what is called a *school choice problem*. In a school choice problem, students (including their respective families) have the opportunity to choose the public school they prefer, while the latter one has strict priorities. These priorities can be the obligation to admit students living in a specific geographical zone, the obligation to admit students with at least one member of their family being already to the school concerned etc. The central issue in a school choice problem is the design of a rigorous and specific *student assignment mechanism*, which is a procedure that selects a matching for each school choice problem.

We point up that a mechanism is valid if only certain ethical properties are respected: the truth (condition of *strategy-proofness*), the optimality (condition of *Pareto efficiency*) and fairness (condition of *elimination of justified envy*). We will say that a mechanism *Pareto dominates* another one if not only is at least as preferable as any other mechanism for any student, but also strictly better for some of them. A mechanism is *Pareto optimal* when it is not Pareto dominated. A mechanism is *strategy-proof* if any strategy that represents the true preferences of an individual is a dominant strategy. Finally, a mechanism *eliminates justified envy* if it always selects a matching that eliminates justified envy. This means that no individual prefers the college that another one is assigned to and also he has not the priority by the college concerned. The elimination of justified envy in a school choice problem is equivalent to the notion of stability in a college admissions problem.

[1]The idea of students' assignment developed by Gale and Shapley (1962) has its origin the article of Mullin (1950) (see also Mullin and Stalnaker [1951], Stalnaker [1953], Darley [1959], or McJoynt and Crosby [1957] and so on). Roth (1984) was the first economist to point out this aspect.

Some mechanisms applied in Boston, New York, etc. have actually significant defects. They violate the strategy-proofness, the Pareto efficiency and do not eliminate justified envy. Abdulkadiroğlu and Sönmez (2003) were the first to focus on these deficiencies and have therefore attempted to approach the student placement problem from a mechanism design perspective. They proposed the *Student Optimal Stable Mechanism* (SOSM) that selects a student optimal stable matching for each school choice problem[2]. Although strategy-proof, such a mechanism is not Pareto efficient[3]. In parallel, Abdulkadiroğlu and Sönmez (2003) developed the *Top Trading Cycles Mechanism* (TTCM). The latter, although Pareto efficient and strategy-proof (Ma [1994]), is not fair[4][5]. Kesten (2006, 2010), proposed two alternative mechanisms to the SOSM and the TTCM: the *Efficiency Adjusted Deferred Acceptance Mechanism* (EADAM) and the *Equitable Trading Top Cycles Mechanism* (ETTCM). He actually proved that the EADAM is fair and Pareto efficient but not strategy-proof and that the ETTCM is fair, Pareto efficient and also strategy-proof. These works have a great importance, particularly the two proposals made by Kesten.

However, we note that all these algorithms are based on strict preferences. Nevertheless, in real life is far to be always the case! In fact, strict preferences mean that an individual may choose a college to another one in a strict way, but a college ranked 'relatively' better, may be 'almost' or 'nearly' desirable as another one, without meaning that the individual is indifferent. In other words, some students can be quite undecided about their choice of a college. Thus, individual choices may be characterized by some 'vagueness' associated to some of their attributes. And what better way than using the concept of *fuzzy* preferences to set clear this point of view! Such preferences were originally introduced in the theory of fuzzy sets by Zadeh (1965), through his early attempts to formalize. They are based on certain multi-valued logics known as perpetuated via certain philosophical questionings. Barrett Pattanaik and Salles (1986), in turn, applied this concept to the social choice under an Arrovian prospect.

Nonetheless, the concept of preferences (and fuzzy preferences) used particularly in social choice theory, is wide! The problem being that when using preferences one tends to disregard the notion of priorities (and priorities neglect the notion of preferences). In this article, preferences can be thought of as the combination of priorities and tastes.

In fact, students may have, *by choice*, *priorities* over colleges, which are necessary and sufficient alternatives to their welfare. If, for example, a student chooses a college for the teaching program then such an option corresponds to the individual priorities. In the same time, students have *tastes* or, in other means, *preferences on relative attributes (or advantages) that characterize a college* (PACC), with respect to another one, which may not be primordial to individuals' welfare (even if they participate in its improvement). It is possible that an individual desires more a college than another for its social and commercial environment, for example. This last point calls for individual tastes (PACC).

[2] This leads us to say that there is a close link with the standard theory of Gale and Shapley (1962).

[3] The algorithm introduced by Gale and Shapley (1962) is not strategy-proof (Dubins and Freedman [1981] and Roth [1982]).

[4] Note that such a result goes against the theory of Alcalde and Barberà (1994) who tell us that there is no mechanism that is in the same time Pareto efficient, individually rational and strategy-proof.

[5] The notion of fairness plays an important role in this article; it is still very present in other contributions to the theory of matching such as Masarani and Gokturk (1989), Özkal-Sanver (2004), Klaus (2009), etc.

And because a college relatively no prioritary still has some advantages with respect to another one, the relatively no prioritary option becomes relatively 'less' or 'largely less', and so on, prioritary. The difference between the advantages of a prioritary college and a 'less' prioritary one implies the 'vagueness' of individual choices!

In parallel, colleges, from their part, may accept students that respect, *by obligation*, *priorities* considered by the law (being subjected to laws and other internal regulations, colleges are regarded as passive). Nevertheless, they may be undecided about their real choices. Next to college priorities, they may indeed have *preferences on relative attributes that characterize a student* (PACS), with respect to another one; for example, a student that is serious and works hard (these characteristics are not necessarily taken into account by the laws!). We underline that, college priorities are seen as a 'constrain', while individual priorities denote above all the essential needs of an individual. As an example, consider priorities ranked as follows: 'live in the area geographically close to the college' and 'have good test scores'. If an individual less talented lives in the defined area, while the talented one lives on the border of this zone, only the individual respecting the first priority will be accepted (assuming a quota equal to the unit). However, the college faced to such a situation would better appreciate the other person! So, students and colleges may be quite undecided on their actual choices.

There are certainly many ways of representing an agent's fuzzy choices in a concrete way. We are based on the distinction between tastes and priorities on an institutional *and* individual level. So, privileging priorities and taking into consideration the associated tastes, we talk about *fuzzy priorities.*

In this work, we introduce a *college choice problem* by considering that both students and colleges have fuzzy priorities. However, what matters is the design of a mechanism that will be able to select a matching for each college choice problem. The latter matching needs to be the best possible for all agents! Before trying to do so, we ask the following question: does the access to an alternative relatively less acceptable, necessarily mean an important welfare loss, compared to the level of welfare that the individual would have had if he had been admitted by the college that he ranks first in his profile (called the top-ranking of the individual's profile)? No, if the individual has a high fuzzy between the alternative with which he is matched and his top-ranked option!

Our aim will be to analyze the latter situation by attempting to, particularly, reduce the envy among students for which the fuzzy is low, in favor of students who really tend to 'hesitate' between some of the options that are, of course, acceptable (high fuzziness). We will therefore establish a procedure, based on the SOSM and called *Extended*-SOSM, where the intervention of a *social planner* will seek to improve the fate of envious students with weak fuzzy on their choices, relatively to the least envious for whom the fuzzy is high. In other words, we will try to obtain the best possible matching for all students, taking into account their possible indecision.

Afterwards, we will develop an algorithm that will take into consideration the above mentioned procedure and the fuzzy profiles of individuals and institutions. The purpose of the algorithm, called *College Choice Mechanism* (CCM), will be to select the best possible matching for a college choice problem where agents have vagueness in their choices, i.e. their choices are elaborated in a more or less undecided way. Compared to the mechanisms previously stated, the CCM respects almost all ethical properties and is easy to apply in real-life. To simplify the analysis, we suppose a quota of selection equal to the unit.

This paper is organized as follows: after presenting some basic definitions and mechanisms already existing in the literature (Section 2), we introduce the problem of indecision in the theory of matching (Section 3). The introduction of this new hypothesis allow us to develop a procedure which takes into account the fuzzy priorities of individuals and their ranking to enable the most envious to achieve a better situation and this, considering the vagueness existing in the selection of students by colleges (Section 4). This leads us to the presentation of a concrete real-life mechanism, based on the procedure mentioned above, what will be the subject of the fifth section. The last section concludes.

2 From a School Choice Onwards

The mechanisms proposed in the literature are based on a school choice problem. Such a problem is defined as a 5-tuple $(N, X, \mathbf{Q}, \Gamma_N, \Delta_X)$ where $N = \{1, ..., n\}$ is a set of n individuals, X the set of possible schools x, and $\mathbf{Q}$ a vector of quotas $Q(x)$, $\forall x \in X$, with $Q(x)$ being the maximum number of seats available in a school x. We note $\Gamma_N := \{i \in N : \mathscr{P}(i)\}$, where $\mathscr{P}(i) := \{x, y, ..., m(i) \in X(i) : xP(i)yP(i)...P(i)m(i)P(i)\varnothing\}$, with $\varnothing$, the situation: 'I prefer not to go to a school rather than go to a particular one' ($m(i)$ being the last acceptable alternative for i). It is for this reason that we note, $X(i)$, the set of *acceptable* alternatives for the individual i, with $X(i) \subseteq X$. In the same way, we define $N(x)$ as the set of acceptable individuals for a school x, with $N(x) \subseteq N$, $Q(x) \leq |N(x)|$. The binary preference relation P, here considered, means '*strictly preferred to*'. In other words, when we have $xP(i)y$, $\forall x, y \in X(i)$, we say that a school x is strictly preferred by i to a school y. Finally, $\Delta_X := \{\mathcal{P}_x\}_{x \in X}$, where $\mathcal{P}_x := \{i, j, ..., m(x) \in N(x) : i\mathcal{P}(x)j\mathcal{P}(x), ..., \mathcal{P}(x)m(x)\mathcal{P}(x)\varnothing\}$ is a ranking of priorities such that $i\mathcal{P}(x)j$ means that i has the priority over j for school x.

Let $\mathscr{S} = N \cup X$, where $\mathscr{S}(\rho)$ represents the set of all possible correspondences, defined by:

$$\mathscr{S}(\rho) = \begin{cases} i & \text{if } \rho = x \\ x & \text{if } \rho = i \end{cases}$$

for all $x \in X$, $i \in N$.

Definition 1. A matching is a function $\mu : \mathscr{S} \to \mathscr{S}$, such that $\forall \rho \in \mathscr{S}$, $\mu(\rho) \in \mathscr{S}(\rho)$ and $\forall i \in \mathscr{S}$, $\forall\ x \in X(i)$, $\mu(i) = \{x\} \Leftrightarrow \mu(x) = \{i\}$, with $|\mu^{-1}(x)| \leq Q(x)$.

We consider $\mathscr{M}(\mathscr{S})$, the set of all μ for $\mathscr{S}$.

Definition 2. A μ-problem is a triplet $\pi \equiv (\mathscr{S}, \Gamma_N, \Delta_X)$, for all $i \in N$ and all alternative $x \in X$. We denote Π the set of all problems π.

A very closely related problem to the school choice one, is the well-known college admissions problem described by Gale and Shapley (1962). The difference between the two problems is that in the college admissions one, schools are active and have preferences over students whereas in a school choice problem, schools are passive and viewed as objects to be consumed. The central concept in college admissions is stability.

Definition 3. A matching μ is stable for a μ-problem π if there is no pair (i, x) that blocks μ (where i and x may block in the same time, or, i may block and no x, or the inverse).

Definition 4. A pair (i,x) blocks $\mu \in \mathscr{M}(\mathscr{S})$ if there exists an $\widetilde{x} \in X(i)$ or $\widetilde{i} \in N(x)$, such that $\widetilde{x}P(i)\mu(i)$ or $\widetilde{i}P(x)\mu(x)$, $\forall i \in N(\mu(i))$, $\forall x \in X(\mu(x))$.

The notion of stability, imposed in the entire literature of two-sided matching models, is relative to the concept of elimination of justified envy (which is more oriented towards the notion of fairness) in the context of priority-based allocations: there is no form of justified envy if for $\mu(j) = \{x\}$, $\forall j \in N(x)$, $\nexists\ i \in N\backslash\{j\}$, such that $xP(i)\mu(i)$, where $i\mathcal{P}(x)j$, $\forall x \in X$ (condition (**No-EJ**)).

A matching μ is *individually rational* if for all $i \in N$, $\mu(i)P(i)\varnothing$ (condition (**IR**)).

A matching μ is *non wasteful* if, $\exists i \in N(\mu(i))$ such that $xP(i)\mu(i)$ while $Q(x) = |\{j \in N(x)\backslash\{i\} : \mu(j) = \{x\}\}|$, $\forall x \in X(k)$, $k = i, j$ (condition (**NW**)).

Let S be the strategic choices of individuals such that $S = (s^*_i, s^*_{-i})$, where s^*_i is the true choice of the individual i and s^*_{-i}, the real choice of any individual except i. The set $S' = (s_i, s^*_{-i})$ differs from S only concerning the choice of i, which in this case, reflects a manipulated choice. The strategic choice s^*_i of i determines the outcome $g(i, s^*_i, s^*_{-i})$ of all matching μ, representing the correspondence between an individual and a school. A μ-problem is *strategy-proof* if no student can achieve a better outcome when he manipulates his true preferences. In other words, for all $i \in N$ and strategy $S = (s^*_i, s^*_{-i})$, $S' = (s_i, s^*_{-i})$ there does not exist $g'(i, s_i, s^*_{-i})P(i)g(i, s^*_i, s^*_{-i})$, with $g \neq g'$, where $g(S) = \mu$ and $g'(S') = \eta$.

Finally, a matching μ *Pareto dominates* η if $\mu(i)R(i)\eta(i)$ for all $i \in N$, and $\mu(j)P(j)\eta(j)$ for some $j \in N\backslash\{i\}$. We say that a matching μ is *Pareto efficient* if there is no matching η, $\mu \neq \eta$, such that η *Pareto dominates* μ (condition (**PE**)).

Given a school choice problem, the aim is to construct a mechanism that selects a matching for the former one and that respects the above stated ethical properties. We will present some of the mechanisms that already exist in the literature, namely the *Boston Mechanism*, the *Gale and Shapley's Student Optimal Stable Mechanism* (SOSM), the *Top Trading Cycles* (TTC), the *Efficiency-Adjusted Deferred Acceptance Mechanism* (EADAM) and the *Efficiency Top Trading Cycles Mechanism* (ETTCM).

Mechanism 1 (Boston Mechanism).

1. Students have preferences on schools. Such preferences are ordered in individual profiles $\mathscr{P}(i)$, $\forall i \in N$.

2. Each school x, $\forall x \in X$ has priorities $\mathcal{P}_x$.

3. Each individual proposes to his top-ranked college that will reject or accept him. *Individuals accepted by an alternative, can never be rejected* by any other person making a proposal to the latter, even if they happen to be relatively more preferred: when individuals rejected propose to their next 'best' option, they may obtain only the alternatives for which places have not yet been assigned (i.e. when the quota for each alternative is not reached).

4. The process stops when every individual is affected to an option, for quotas achieved.

In the mechanism of Boston even if a student has a high priority at a school x, unless he lists it as his top choice, he loses his priority to students who have listed x as their top choices. It is a mechanism neither stable nor strategy-proof. The algorithm introduced by Gale and Shapley (1962), allows us to consider the latter aspect. It is relative to the *Student Optimal Stable Mechanism* (SOSM) introduced by Abdulkadiroğlu and Sönmez (2003).

Mechanism 2 (**Student Optimal Stable Mechanism (SOSM)**).

1. Students have preferences on schools. Such preferences are ordered in individual profiles $\mathscr{P}(i)$, $\forall i \in N$.

2. Each school x, $\forall x \in X$ has priorities $\mathcal{P}_x$[6].

3. Each individual propose to his top-ranked college that will reject or accept him.

4. Individuals rejected, will then propose to their next 'best' option of their profile. The schools will once again be able to accept or refuse them. Since the individual previously rejected, is accepted, *any individual initially accepted by that same option, may be in turn rejected* if he is relatively less preferred to the individual making a new proposal (when the quota for each alternative is neither unitary nor reached). And so on.

5. The process stops when every individual is affected to an option.

However, the SOSM does not satisfy the Pareto efficiency (Abdulkadiroğlu and Sönmez [2003]) . The *Top Trading Cycles* (Shapley and Scarf [1974]) has been introduced to solve this problem. A mechanism which was rehabilitated by Abdulkadiroğlu and Sönmez (2003) in the case of a *school choice problem.* It is well known as the *Top Trading Cycles Mechanism* (TTCM).

Mechanism 3 (**Top Trading Cycles Mechanism (TTCM)**).

1. Students have preferences on schools. Such preferences are ordered in individual profiles $\mathscr{P}(i)$, $\forall i \in N$.

2. Each school x, $\forall x \in X$ has priorities $\mathcal{P}_x$.

3. A loop is the logical path (designated as an exchange process or in other words a trade)[7] that associates a school with a student who happens to be his highest priority and the latter to a school that happens to be her best choice, and so on, until that individual is again associated at that same school: in this case, we say that the loop is closed. A cycle exists when a loop is closed (i.e. when one has something like: school x prefers the individual 1, but the individual 1 prefers a school y which prefers individual 3, [...], school z prefers individual k but the individual k prefers school x). This step consists to identify p-cycle(s) (with $p \geq 1$) from the set of alternatives considered. Any individual identified in these cycles is then accepted by the 'best' option for which the first proposal was performed.

[6] In the case of the DAA, preferences of colleges are considered instead of priorities of schools.

[7] This is indeed an exchange process in the sense that the student who has the highest priority for the school is assigned a place (for a given quota). Afterwards, he is free to exchange this place with any individual who considers this school to have the highest priority.

4. We determine p-cycle(s) considering this time the alternatives for which the quota is not reached (*no individual initially accepted* (via its previous proposal) *can not be rejected*). And so on.

5. The process stops when every individual is affected to an option, for quotas achieved.

Although this mechanism has the advantage of satisfying the Pareto efficiency, it violates the condition of elimination of justified envy. Kesten (2006, 2010) provides another formulation of the TTCM of Abdulkadiroğlu and Sönmez (2003), in order to simplify it: he assumes that for a given quota, all the seats are given from the start, from the school to the student who has the highest priority. Thus, this student will be ready to 'exchange' his seats with another one who considers this school as his best option. We will therefore have, once again, a loop that will appear. Once each student associated with the school that has the priority, the remaining seats are 'put into play'. And so on, until the quota of each school is reached. Kesten, based on this variant of the concept, develops another mechanism which, this time, is more fair:

> '[...] *unlike the TTCM of Abdulkadiroğlu and Sönmez (2003), instead of giving all the trading power to those students with the highest priority for a school, we distribute the trading rights of seats for each school among those who are entitled one seat at that school and allow them to trade in such a way so as situations of justified envy are avoided as much as possible*'. (Kesten 2005)[8]

Mechanism 4 (**Efficiency Top Trading Cycles Mechanism (ETTCM)**).

1. Students have preferences on schools. Such preferences are ordered in individual profiles $\mathscr{P}(i)$, $\forall i \in N$.

2. Each school x, $\forall x \in X$ has priorities $\mathcal{P}_x$.

3. For each school, all available seats are assigned to students one by one following their priority order to form student-seat pairs. Each student-seat pair (i, x) points to the student-seat pair (i', x') such that (**a.**) school x' is the best choice of student i and, (**b.**) student i' is the student with the highest priority for school x among the students who are assigned a seat from school x'. If there is already a student-seat pair at which student i is already assigned one seat from his best choice school, then all student-seat pairs containing him point to that student-seat pair. There is at least one cycle. In each cycle, corresponding trades are performed, i.e. if a student-seat pair (i, x) is pointing to the pair (i', x') in a cycle, then student i is placed to school x' and he is removed as well as the seat student i' is assigned. It is possible that the student-seat pairs containing the same student, say student i, appear in the same or in different cycles. In such a case, student i is placed to his best choice and the extra seats of that school (for which the student-seat pairs containing him are pointing to in those other cycles) remain to be inherited. If a student is removed and there are student-seat pairs containing him which do not

[8] "Student Placement to Public Schools in the US: Two New Solutions," mimeo, University of Rochester, 2005.

participate in a cycle, then the seats assigned to him in those student-seat pairs also remain to be inherited. The seats that remain to be inherited at the end of the step are not necessarily inherited at the very next step by the remaining students.

4. At every step k, $k \geq 2$, for each school x such that (**a.**) there are seats of school x which remained to be inherited from previous steps , and (**b.**) no student who was assigned a seat of school x at a previous step is left, its seats which remained to be inherited from previous steps are assigned to the remaining students one by one following their priority order to form new student-school pairs. Again, student-seat pairs point to each other in the way described in the previous step. Corresponding trades are carried out in each cycle and some seats remain, once again, to be inherited. The procedure continues in a similar way.

5. The algorithm stops when all students are assigned a seat to a school.

For both TTCM and ETTCM, it is necessary to suppose that students have access to the information concerning the profiles of the other candidates. Such an assumption is rather difficult to consider in reality. Besides this, Kesten (2006, 2010) proposes a second mechanism that begins with the DAA of Gale and Shapley.

Mechanism 5 (**Efficiency-Adjusted Deferred Acceptance Mechanism (EADAM)**).

1. Students have preferences over schools. Such preferences are ordered in individual profiles $\mathscr{P}(i)$, $\forall i \in N$.

2. Each school x, $\forall x \in X$ has priorities $\mathcal{P}_x$.

3. Each individual proposes to his top-ranked school that will reject or accept him. The rejected individuals propose to another 'best' option. Gale and Shapley's mechanism stops when all individuals are matched to an alternative of their respective profile.

4. A cycle is designated as a logical path between the top-ranking of an individual and the option with which he is paired. When there is a cycle from which certain individuals can not reach the top-ranking of the individual considered through the cycle, he is designated as an 'interrupter'. The top-ranking option is therefore eliminated by his profile (the order of the other preferences remaining unchanged), since any possibility of achieving this alternative is actually rejected. Thus, we go back to Step 1.

5. The process stops when no individual is considered as an 'interrupter' for the others and every individual is matched to an option.

This algorithm was recently introduced by Kesten (2010). Note that this algorithm has a perspective enough 'immoral' in the sense that the individual who has ambitions is denied for the simple reason that he has not had the opportunity to be matched! The aim of such an approach is to to annihilate 'artificially' any form of envy towards the 'forbidden' alternative, which is quite criticizable. This algorithm implicitly raises some constraint on the individual's freedom, even if it suggests, explicitly, the consent of the individual to 'delete' an option from his profile. However, an individual, because he is too 'good', and so on, can sometimes be quite 'naive' (or 'myopic') with regard to such actions. Eliminate his goal(s) telling that they will lead to nothing leave us perplexed. Especially, if we ask him directly to abandon such ambitions!

Let us take an example: suppose that individual i knows that his competitors have a greater chance of success than him, given his social position. Suppose also that a school decides to refuse i because of his foreign origin, or because he has not studied at a prestigious school. In this case, understanding that he has no chance of success, he eliminates that school from his profile of preferences, which moreover rejected him. Yet, his naivety is misleading since he was rejected for circumstantial reasons and not to his own responsibility. Speaking of consent is therefore delicate in this kind of problems.

In parallel with the school choice problem, identified through its mechanism, we propose a new problem called the *college choice problem*, which will also lead us, thereafter, to a mechanism. The later one will be constructed first and foremost through a procedure.

3 The College Choice Problem

Going further the *student placement problem* of Balinski and Sönmez (1999) and the *school choice problem* presented by Abdulkadiroğlu and Sönmez (2003), we introduce a *college choice problem.*

3.1 Definitions and Basic Properties

In a college choice problem, **colleges** have *fixed* priorities imposed by regulations and laws. Such priorities are called *colleges' priorities.* However, colleges may have tastes (PACS) given the priority list of candidates, already fixed. It is to every student that has the priority that is associated a degree of intensity, noted d, of the taste concerned.

More precisely, the intensity accorded to a student by a college is representative of all the advantages of the student relatively to any other one (this may be a great level of effort, grades on exams, and so on). Say that student i has the priority relatively to j implies that i is strictly better than j in the eyes of the college. However, j may also have some advantages (admittedly insufficient as to those of i) and, consequently, such an individual cannot be strictly considered as detestable with respect to i. We will say that i has a 'little more' or 'far more', and so on, the priority over j. That is why we talk about *fuzzy colleges' priorities.*

In what follows, we note d^+ the degree of intensity of the option which is relatively more prioritary and d^- the degree of intensity of the one that is relatively no prioritary.

Definition 5. A binary relation of fuzzy colleges' priorities is a function $f : \mathscr{H} \times \mathscr{H} \to \mathcal{P}$, such that for all $i, j \in N(x)$ and for all $d^+, d^- \in [0;1]$, $(i, d^+)\mathcal{P}(x)(j, d^-)$, for all $x \in X$, with $d^+ \geq d^-$, $\mathscr{H} = N \times [0;1]$ and $\mathcal{P}$ the relation 'strict priority over'.

Thus, the relation $(i, d^+ = a)\mathcal{P}(x)(j, d^- = b)$, $\forall i, j \in N(x)$, $\forall x \in X$, stipulates that, even if a commission decided that only i had the priority over j, college x could easily find j relatively a little less desirable and i not 'strictly' desirable (with $a > b$). Therefore, we note that when $a = 1$ and $b = 0$, this means that i is strictly desirable and j strictly detestable (relatively speaking). This would suggest the consideration of strict and not fuzzy priorities.

On the other hand, **students** have *non-fixed* priorities (they are freely and individually constructed priorities) that reflect the real needs of every student on his academic success. These are called *individual priorities*. Alongside these priorities, the students may have tastes (PACC) over the colleges, now *fixed*. At each college, classified by priority, is assigned a degree of intensity, defined in the same manner as above. In this sense, we say that college x is a 'little more' or 'much more' prioritary relatively to y if we consider that x has more advantages than y, even if y has also its own advantages (certainly, relatively insufficient). For example, one can think of a low entry cost, a better quality of education and so on. This time, we talk about *fuzzy individual priorities*.

Definition 6. A binary relation of fuzzy individual priorities is a function $f' : \mathscr{H}' \times \mathscr{H}' \to \mathcal{P}$, such that for all $x, y \in X(i)$ and for all $d^+, d^- \in [0;1]$, $(x, d^+)\mathcal{P}(i)(y, d^-)$, for all $i \in N$, with $d^+ \geq d^-$, $\mathscr{H}' = X \times [0;1]$.

The relation $(x, d^+ = c)\mathcal{P}(i)(y, d^- = f)$, $\forall x, y \in X(i)$, $\forall i \in N$, stipulates that i gives priority to the option x compared to the option y. This is, the alternative x is not strictly desirable to y for i (when $c = 1$ and $f = 0$ we talk about, once again, strict priorities). Just like above where $a > b$, we have $c > f$. In other words, as students must select *options for what they are worth and not for what they are*, they can actually 'weaken' their priorities through some characteristics that may give advantage to the relatively lower priority option.

Properties to be respected over fuzzy priorities: For all $i \in N$ and $x, y, z \in X(i)$, let F' be the set of all functions $f' : \mathscr{H}' \times \mathscr{H}' \to \mathcal{P}$, such that $\mathcal{P}$ respects the following conditions:

- *F-Asymmetry*: If $[(x, d)\mathcal{P}(i)(y, d')]$, then we cannot have $[(d' = d^+) \wedge (d = d^-)]$, with $d^+ \geq d^-$; and if the relation $[(x, d)\mathcal{P}(i)(y, d')]$ is considered, then the contrary is not possible.
  If an alternative has a relatively higher priority, then it is relatively more desirable; and an alternative relatively more desirable cannot be less prioritary.

- *F-Irreflexivity*: $\neg(x, d^+)\mathcal{P}(i)(x, d^-)$, with $d^+ \geq d^-$.
  This condition stipulates that an individual cannot choose an option to itself.

- *F-Transitivity*:
  (a.) If $(x, d^+)\mathcal{P}(i)(y, d^-)$ and $(y, d^+)\mathcal{P}(i)(z, d^-)$, implies $(x, d^+)\mathcal{P}(i)(z, d^-)$, with $d^+(i, x) = d^+(i, y)$ or $d^+(i, x) \neq d^+(i, y)$ and $d^-(i, y) = d^-(i, z)$ or $d^-(i, y) \neq d^-(i, z)$
  (b.) If $d^+(i, x) > d^-(i, y)$ and $d^+(i, y) > d^-(i, z)$, implies $d^+(i, x) > d^-(i, y) > d^-(i, z)$, where $d^+(i, x)$ denotes the degree of intensity associated to the option x, by i.

Respectively, $\mathcal{P}$ has to satisfy the above conditions for F, that is the set of all functions $f : \mathscr{H} \times \mathscr{H} \to \mathcal{P}$.

Concurrently to a school choice problem, any college choice problem will be defined as a 5-tuple $(N, X, \mathbf{Q}, \Delta_N^F, \Delta_X^F)$, where $\Delta_N^F := \{\mathcal{P}_i\}_{i \in N}$, with $\mathcal{P}_i := \{x, y \in X(i) : (x, d^+)\mathcal{P}(i)$

$(y, d^-)\}$ and $\Delta_X^F := \{\mathcal{P}_x\}_{x \in X}$, with $\mathcal{P}_x := \{i, j \in N(x) : (i, d^+)\mathcal{P}(x)(j, d^-)\}$. The outcome of a college choice problem will be a matching between students and colleges, that will take into consideration their respective fuzziness. Henceforth, let us define the notion of a *fuzzy matching*.

Definition 7. A fuzzy matching is a function 'one-to-one' $\mu : \mathscr{S}_F \to \mathscr{S}_F$, such that $\forall \rho \in \mathscr{S}_F$, $\mu(\rho) \in \mathscr{S}_F(\rho)$ and $\forall i \in N(x)$, $\forall\ x \in X(i)$, $\mu(i) = \{x\} \Leftrightarrow \mu(x) = \{i\}$, where,

$$\mathscr{S}_F(\rho) = \begin{cases} i & \text{if } \rho = x \\ (x, d) & \text{if } \rho = i \end{cases}$$

$\forall d \in [0; 1]$ and $Q(x) = 1$.

We consider $\mathscr{M}_F(\mathscr{S}_F)$, the set of all μ for $\mathscr{S}_F$ and Π, the set of all problems π, for each of which a *μ-problem* is indicated by the triple $\pi \equiv (\mathscr{S}_F, \Delta_N^F, \Delta_X^F)$.

Remark 1. *In what follows, a matching μ will be determined via a College Choice Mechanism and will be denoted as μ^{CCM}.*

3.2 Conditions of Robustness

Every college choice problem has to respect certain conditions: elimination of justified envy, individual rationality, non-wastefulness, strategy-proofness and Pareto optimality.

On a purely formal basis, we have to present the stability condition by introducing this time the vagueness of priorities previously determined.

Definition 8. A pair (i, x) blocks $\mu \in \mathscr{M}_F(\mathscr{S}_F)$ if there exists an $\widetilde{x} \in X(i)$ or $\tilde{i} \in N(x)$, such that $(\widetilde{x}, d^+)P(i)(\mu(i), d^-)$, $\forall i \in N$ or $(\widetilde{i}, d^+)P(x)(\mu(x), d^-)$, $\forall x \in X$, and $\forall d^+, d^- \in [0; 1]$, with $d^+ \gg d^-$ in both cases.

The notion of a blocking pair implies the non-satisfaction of the respective decisional criteria of any $x \in X$ and/or any $i \in N$. In other words, either the individual likes better another alternative that seems more in line with his expectations, or the college is interested in an individual other than the one designated by the correspondence, under the pretext that the former has better dotations than the individual being matched.

Definition 9. A fuzzy matching μ is F-stable (Fuzzy Stability) for a μ-problem π if there is no pair (i, x) that blocks μ[9].

In our context, the natural counterpart of the F-stability is the *elimination of justified envy*. If for $\mu(j) = \{x\}$, $\forall j \in N(x)$, $\nexists\ i \in N \backslash \{j\}$, such that $(x, d^+)\mathcal{P}(i)(\mu(i), d^-)$, and $(i, d^+)\mathcal{P}(x)(j, d^-)$, $\forall d^+, d^- \in [0; 1]$, $\forall x \in X(i)$, there exists elimination of justified envy (condition (**No-EJ**)*).

A matching μ is *individually rational* if for all $i \in N$, $(\mu(i), d^+)\mathcal{P}(i)(\varnothing, d^-)$ (condition (**IR**)*).

[9]When $d^+ = 1$ and $d^- = 0$, if there is no pair (i, x) that blocks a matching, we go back to the general notion of stability, defined in Section 2. Mathematically, if there is no $\widetilde{x} \in X(i)$ and/or an $\tilde{i} \in N(x)$ that implies $(\widetilde{x}, 1)\ P(i)(\mu(i), 0)$, and/or $(\tilde{i}, 1)P(x)(\mu(x), 0)$, $\forall i \in N$ and/or $\forall x \in X$, such that no pair $(i, \widetilde{x})$, $(\tilde{i}, x)$ or $(\tilde{i}, \widetilde{x})$ *blocks* $\mu \in \mathscr{M}_F(\mathscr{S}_F)$, there is stability.

A matching μ is *non wasteful* if, $\exists i \in N(\mu(i))$ such that $(x, d^+)\mathcal{P}(i)(\mu(i), d^-)$ while $Q(x) = |\{j \in N(x)\backslash\{i\} : \mu(j) = \{x\}\}|$, $\forall x \in X(k)$, $k = i, j$ (condition $(\mathbf{NW})^*$).

Furthermore, we add a condition of fairness for the college choice problem. To do so, we need the following definition.

Definition 10. The loss of intensity, $LI(i)$, $\forall i \in N$, of a binary relation $(x, d^+)\mathcal{P}(i)(y, d^-)$, $\forall x, y \in X(i)$, is proportional to $d^+(i, x) - d^-(i, y)$, and it is equivalent to the notion of the vagueness of choices.

Definition 11. A fuzzy matching η is fair (**F**), for a μ-*problem* π if, $\forall i, j \in N$, $(\mu(j), d^+)\mathcal{P}(i)$ $(\mu(i), d^-)$ and $(\mu(j), d^+)\mathcal{P}(j)(\mu(i), d^-)$, with $\mu(i) \neq \mu(j)$, we have $\eta(i) = \mu(j)$ and $\eta(j) = \mu(i)$ if $LI(i) \gg LI(j)$ where $d^+(j, \mu(i)) \rightleftarrows d^-(j, \mu(j))$ and if $LI(\eta(i)) > LI(\eta(j))$ where $d^+(\mu(k), \eta(\mu(k))) \not\rightleftarrows d^-(\mu(k), k)$, with $\eta(\mu(k)) = k'$, for $k = i, j$ and, $k' = i$ if $k = j$ and $k' = j$ if $k = i$.

We get a stricter version of the previous definition by considering that change the matching allows to reach a more acceptable situation, for every individual concerned (only the social planner carries out the 'exchange'). So, we have the following definition:

Definition 12. A fuzzy matching η is strictly fair (**SF**) for a μ-problem π if, $\forall i, j \in N$, $(\mu(j), d^+)$ $\mathcal{P}(i)(\mu(i), d^-)$ and $(\mu(i), d^+)\mathcal{P}(j)(\mu(j), d^-)$, with $\mu(i) \neq \mu(j)$, we have $\eta(i) = \mu(j)$ and $\eta(j) = \mu(i)$ for all $LI(i) > LI(j)$ or $LI(i) < LI(j)$ where $LI(k) = d^+(\eta(k)) - d^-(\mu(k))$, with $\eta(k) \neq \mu(k)$, for $k = i, j$ and, $LI(\eta(i)) > LI(\eta(j))$ where $d^+(\mu(k), \eta(\mu(k))) \not\rightleftarrows d^-(\mu(k), k)$, with $\eta(\mu(k)) = k'$, for $k' = i$ if $k = j$ and $k' = j$ if $k = i$.

Moreover, if we have, generally speaking, $\mu(i) \vdash^P \eta(i)$, for all $i \in N$, we say that the matching μ is better for i, in the sens of Pareto, than it was the previous one, where the notation $\vdash^P$ indicates the 'dominance' relation. In our setting, we note $\vdash^P_F$, the relation of 'fuzzy dominance' within the sense of Pareto.

A matching μ *Pareto dominates* another matching μ' (i.e. $\mu \vdash^P_F \mu'$) if for,

$$[\mu(i) \wedge \mu'(i)] \vee [(\neg\mu'(i)) \wedge \mu(i)] \in \mathscr{M}^{\mathbf{SF}}_F,\ \forall i \in N,$$

where $\mathscr{M}^{\mathbf{SF}}_F$ being the set of all possible fuzzy matching satisfying the condition of *strict fairness*, with $\mathscr{M}^{\mathbf{SF}}_F \subset \mathscr{M}_F(\mathscr{S}_F)$, we have,

$$(\mu(i), 1)\mathcal{P}(i)(\mu'(i), 0), \text{ for some } i \in N, \text{ and,}$$

$$(\mu(i), d^+)\mathcal{P}(i)(\mu'(i), d^-),\ \forall i \in N,\ \forall d^+, d^- \in [0; 1].$$

Consequently, we say that a fuzzy matching μ is $\vdash^P_F$-*Pareto-efficient* (**PE**) when there exists no other matching η such that $\eta \vdash^P_F \mu$.

4 Procedure Extended-Sosm

Before establishing the mechanism which will be able to select a matching for a college choice problem in a very precise way, we provide a procedure, called *Extended*-SOSM (hereafter mentioned *ext*-SOSM), which will be applied during a phase of the former one. Using the procedure *ext*-SOSM, we can obtain the best possible correspondance between students and colleges. Our process is an extension of the one given by Abdulkadiroğlu and Sönmez (2003). More specifically, we begin by applying the following version of the SOSM of Abdulkadiroğlu and Sönmez, to obtain an initial matching μ^{SOSM}: we assume that both students and colleges have, foremost, priorities. Then, the SOSM is applied.

Initial Step for the Procedure (SOSM) : Each individual $k = i, j, ..., n$, makes a proposal to the college that has for him the highest priority (independently of the fuzzy associated to each priority!). The college accepts him or not with respect to its quota (here equal to one). If the college accepts the student, the mechanism stops. Otherwise, the individual rejected, proposes to his second best alternative. And so on, until the moment where every individual k of N is matched to an option that has the priority for him. We will denote such a matching, μ^{SOSM}.

Afterwards, we try to improve the later one in order to satisfy more people when they seem to be quite undecided on their actual choices. The above mentioned improvement gives rise to a fuzzy (extended) matching, denoted μ^{CCM}.

We can summarize our aim in the following question: would not be better to give someone the advantage of access to a college, knowing that the person already assigned to it may be particularly 'hesitant' about the choice of such an alternative?

Procedure *ext*-SOSM :

- **Step 1 :**
  Condition ℓ (**necessity**) : A *social planner* takes into consideration the fuzzy attached to the priorities of the student and the college[10]: in fact, he realizes that the individual i having made a proposal at the beginning, designating the college ranked in $(\mu^{\text{SOSM}}(i) + \alpha_i - \gamma_i)$[11], with $\alpha_i, \gamma_i \in \mathbb{N}_{++}$, and $\gamma_i \in [0; \alpha_i[$, desires the latter much more than the individual j, that, nevertheless, the college accepted (strong loss of intensity noted LI). In other words, if $\exists i \in N(\mu^{\text{SOSM}}(i))$ and $j \in N(\mu^{\text{SOSM}}(j))$, $\forall i, j \in N$, for whom $LI(i) \gg LI(j)$, with $LI(i) = d^+(i, \mu^{\text{SOSM}}(i) + \alpha_i - \gamma_i) - d^-(i, \mu^{\text{SOSM}}(i))$ and $LI(j) = d^+(j, \mu^{\text{SOSM}}(j) + \alpha_j - \gamma_j) - d^-(j, \mu^{\text{SOSM}}(j))$, where $\alpha_k, \gamma_k \in \mathbb{N}_{++}$, $k = i, j$, $\alpha_j = \alpha_i$ or $\alpha_j \neq \alpha_i$ and $\gamma_j = \gamma_i$ or $\gamma_j \neq \gamma_i$, or $LI(j) = d^+(j, \mu^{\text{SOSM}}(j)) - d^-(j, \mu^{\text{SOSM}}(j) - \beta_j)$, with $\beta_j, \delta_j \in \mathbb{N}_{++}$, $\beta_j \in [1; \delta_j]$, when $(\mu^{\text{SOSM}}(i) + \alpha_i - \gamma_i) \equiv \mu^{\text{SOSM}}(j)$, then i is 'admissible'. The individual j matched to the alternative $(\mu^{\text{SOSM}}(i) + \alpha_i - \gamma_i)$ must make a proposal to the college $\mu^{\text{SOSM}}(i)$ that belongs to the profile of his acceptable options and for which he has a low loss of intensity $LI(j)$ $(LI(j) \to \epsilon)$.

  If the condition ℓ is verified, we suppose, implicitly, a preliminary matching denoted μ^{σ}, where,

[10] To promote the impartiality of the social planner and prevent any judgment on behalf of colleges and students, the way with which he proceeds during the procedure is unknown by the agents.

[11] The college ranked in $(\mu^{\text{SOSM}}(i) + \alpha_i)$ into $\mathcal{P}_i \in \Delta_N^F$, is considered as the 'top-ranking' of i. The college ranked in $(\mu^{\text{SOSM}}(i) - \delta_i)$ into $\mathcal{P}_i \in \Delta_N^F$, is considered as the 'bottom-ranking' of i. We note that, $\forall i \in N$, $\alpha_i = |\{z \in X : (z, d^+)\mathcal{P}(i)(\mu^{\text{SOSM}}(i), d^-), \forall d^+, d^- \in [0;1]\}|$ and $\delta_i = |\{z \in X : (\mu^{\text{SOSM}}(i), d^+)\mathcal{P}(i)(z, d^-), \forall d^+, d^- \in [0;1]\}|$.

$$\begin{cases} \mu^{\sigma}(i) = (\mu^{\text{SOSM}}(i) + \alpha_i - \gamma_i) \\ \mu^{\sigma}(j) = (\mu^{\text{SOSM}}(j) + \alpha_j - \gamma_j) \vee (\mu^{\text{SOSM}}(j) + \beta_j) \end{cases}$$

In the same way, we will use the notation μ^{σ} to represent the preliminary matching of colleges.

Condition $\ell + 1$ (**sufficiency**) : Although the college $\mu^{\text{SOSM}}(j)$ has accepted j for the priorities that characterize it, if the *social planner* considers that the individual i, desiring to be admitted, is almost as desirable for this college as the individual j initially accepted, he will be preliminarily 'admissible'. More specifically, we must have $LI(\mu^{\sigma}(k)) \to \epsilon$[12], when $\mu^{\sigma}(k) = (\mu^{\text{SOSM}}(k) + \alpha_k - \gamma_k)$, $k = i, j$, with $LI(\mu^{\sigma}(k)) = d^{+}(\mu^{\sigma}(k), \mu^{\text{SOSM}}(\mu^{\text{SOSM}}(k))) - d^{-}(\mu^{\sigma}(k), \mu^{\sigma}(\mu^{\sigma}(k)))$ or $LI(\mu^{\sigma}(j)) \to \epsilon$, when $\mu^{\sigma}(j) = (\mu^{\text{SOSM}}(j) - \beta_j)$, with $LI(\mu^{\sigma}(j)) = d^{+}(\mu^{\sigma}(j), \mu^{\text{SOSM}}(\mu^{\text{SOSM}}(i))) - d^{-}(\mu^{\sigma}(j), \mu^{\sigma}(\mu^{\sigma}(j)))$.

If both conditions are satisfied, we will have a new matching denoted μ^{CCM}, such that $\mu^{\sigma} = \mu^{\text{CCM}}$.

- **Step 2 :** *When one of the two conditions is violated*, $\mu^{\text{CCM}} = \mu^{\text{SOSM}}$. In this case, the social planner will check if another 'exchange' is *possible* for individual i. This means that he will look if there exists another individual $j' \in N \backslash \{i, j\}$ for whom $\mu^{\text{SOSM}}(j')$ is a better matching than $\mu^{\text{SOSM}}(i)$ for i (we go back to Step 2)[13]. And so on. The process stops when all possibilities of improvement of individuals' situations towards the top-ranking or towards any 'less aggravating' alternative for the individual (alternatives with low loss of intensity) were considered by the social planner (i.e. while all matching μ^{CCM} are determined). However, when we have to decide between different possibilities of matching, we should take the one which maximizes the number of individuals for whom μ^{CCM} is better than μ^{SOSM}.

The final objective (of the procedure) will be: $\forall i, j \in N$, we will opt more for the possibility A than B, if

$$|N_A| \geq |N_B|$$

with,

$$N_{A \vee B} = \{i : \ (\mu^{\text{CCM}}(i), d^{+}) \mathcal{P}(i) (\mu^{\text{SOSM}}(i), d^{-})\}.$$

Remark 2. *See Appendix for special cases of the Procedure ext-SOSM.*

Remark 3. *If* $0 < LI(i) \leq \epsilon$, $\forall i \in N$, *this is close to the notion of the indifference. In this case, the exchange by the social planner is realized automatically. However, we note that the indifference relative to the alternatives ranked according to their respective priority cannot be supposed (Roth and Sotomayor [1990]).*

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[12]The characterization of ϵ is variable. Indeed, such a parameter depends on the degree of exigency of universities while they choose students, but also of the environment (number of available universities, etc.). This implies that ϵ may be equal to 0,1 either 0,11 or 0,111, and so on. Here, we consider $\epsilon = 0,1$, in order to simplify our analysis.

[13]See Appendix for special cases.

To illustrate the procedure outlined above, two examples will be introduced. Before doing so, we have to point up that the profiles of fuzzy priorities will be presented as rows. The alternative with the highest priority will be the one that is in the head of the ranking. Recall that the tastes formed must not go against the linear ranking of priorities. In other words, if we consider for example, for three distinct colleges $y, z, \omega \in X(i)$, the profile $\mathcal{P}_i$ below,

$\mathcal{P}_i$: $(\mathbf{y}, d^+ = a)\mathcal{P}(i)(z, d^- = b), (y, d^+ = c)\mathcal{P}(i)(\omega, d^- = e)$
$(\mathbf{z}, d^+ = f)\mathcal{P}(i)(\omega, d^- = p)$
$(\omega, d^+ = q)\mathcal{P}(i)(\varnothing, d^- = r)$

we must have $a < b$, $c < e$, $f < p$, $q < r$, and $(a - b) \leq (c - e)$. Each row of the profile summarizes certain information regarding the relationship of priority/taste with any other alternative of the profile (other than the succeeding option of the line). In the case of the preceding explanatory profile, the first line of $\mathcal{P}_i$, shows the link between y and z but also between y and any other option (here it is ω)!

The interest to have mentioned the relation $(\omega, d^+ = q)\mathcal{P}(i)(\varnothing, d^- = r)$, with $\omega \equiv m(i)$, is that it allows us to see how important the last option may be in a profile. Generally in the literature, one supports the idea that being paired with the last acceptable alternative is very serious (especially in terms of loss of welfare). However, this is not verified here if the loss of intensity LI between the latter option and $\varnothing$ is very low. In the case where the individual still gives a clear emphasis on the last acceptable option to his profile (high loss of intensity relatively to $\varnothing$), then having such an option would seem less serious than we usually claim. Note that it is not necessarily useful to mention the link between, for example, a top-ranking (i.e. y) and $\varnothing$ since y would obviously have a degree $d^+ \approx 1$ while the degree attached to $\varnothing$ would be $d^- \approx 0$.

Using all the above information, we go through some examples to make the procedure *ext*-SOSM better understood.

Example 1. Consider $N = \{1, 2, 3\}$, $X = \{y, z, \omega\}$ (with $N = N(x)$, $\forall x \in X$ and $X = X(k)$, $k = 1, 2, 3$), and the following respective profiles :
$\mathcal{P}_1$: $(y, d^+ = 0,85)\mathcal{P}(1)(\omega, d^- = 0,8), (y, d^+ = 0,85)\mathcal{P}(1)(z, d^- = 0,75)$
$(\omega, d^+ = 0,76)\mathcal{P}(1)(z, d^- = 0,18)$
$(z, d^+ = 0,13)\mathcal{P}(1)(\varnothing, d^- = 0,08)$

$\mathcal{P}_2$: $(z, d^+ = 0,8)\mathcal{P}(2)(y, d^- = 0,5), (z, d^+ = 0,7)\mathcal{P}(2)(\omega, d^- = 0,25)$
$(y, d^+ = 0,6)\mathcal{P}(2)(\omega, d^- = 0,34)$
$(\omega, d^+ = 0,55)\mathcal{P}(2)(\varnothing, d^- = 0,02)$

$\mathcal{P}_3$: $(y, d^+ = 0,9)\mathcal{P}(3)(z, d^- = 0,45), (y, d^+ = 0,92)\mathcal{P}(3)(\omega, d^- = 0,3)$
$(z, d^+ = 0,5)\mathcal{P}(3)(\omega, d^- = 0,44)$
$(\omega, d^+ = 0,65)\mathcal{P}(3)(\varnothing, d^- = 0,07)$

$\mathcal{P}_y$: $(1, d^+ = 0,8)\mathcal{P}(y)(2, d^- = 0,77), (1, d^+ = 0,7)\mathcal{P}(y)(3, d^- = 0,61)$
$(2, d^+ = 0,5)\mathcal{P}(y)(3, d^- = 0,4)$
$(3, d^+ = 0,8)\mathcal{P}(y)(\varnothing, d^- = 0,03)$

$\mathcal{P}_z$: $(1, d^+ = 0,9)\mathcal{P}(z)(3, d^- = 0,56), (1, d^+ = 0,5)\mathcal{P}(z)(2, d^- = 0,4)$
$(3, d^+ = 0,65)\mathcal{P}(z)(2, d^- = 0,58)$
$(2, d^+ = 0,89)\mathcal{P}(z)(\varnothing, d^- = 0,04)$

$\mathcal{P}_\omega$: $(2, d^+ = 0,81)\mathcal{P}(\omega)(3, d^- = 0,74), (2, d^+ = 0,81)\mathcal{P}(\omega)(1, d^- = 0,72)$
$(3, d^+ = 0,6)\mathcal{P}(\omega)(1, d^- = 0,52)$
$(1, d^+ = 0,73)\mathcal{P}(\omega)(\varnothing, d^- = 0,03)$

Initial Step of the Procedure (SOSM) : Initially, all individuals proposent to their top-ranked option. This is, individuals 1 and 3 make a proposition to college y, while individual 2 proposes to the college z. Individual 2 is accepted by z. Only individual 1 is selected by y and 3 is rejected. So, 3 proposes to his second best option i.e. to z, that accepts him. Thus, 2 is rejected. Individual 2 now proposes to his second best alternative, which is y. However, college y prefers student 1, so 2 is once again rejected. Student 2 proposes to his last acceptable option, to ω, that accepts him. The SOSM algorithm gives us $\mu^{\text{SOSM}}(1) = \{y\}$, $\mu^{\text{SOSM}}(2) = \{\omega\}$ and $\mu^{\text{SOSM}}(3) = \{z\}$, that will be marked in boxes in what follows.

Procedure *ext*-SOSM :

First possibility : We remark that the individual 3 has not achieved his top-ranked option. It is for this reason that it would be possible to enable him to achieve this option, namely y, which is currently matched to individual 1, via step 1. The question is whether such an exchange is possible. For this, we have to check if the conditions ℓ and $\ell + 1$ of the procedure *Extended*-SOSM are respected; we find that, indeed, it is the case: the loss of intensity of 3 is greater than the one of individual 1 $(0,1 < 0,45)$. The condition ℓ is verified. Moreover, the loss of intensity of the college y, likely to accept 3 compared to 1, is rather weak, in an absolute way, since equals to 0,09.

However, with regard to the college z, we observe that it is not really laid out to 'accept the exchange' of 3 against 1 since the loss of intensity is rather strong (0,34): this stipulates simply that z prefers a little more 3 than 1 compared to y which prefers slightly better 1 compared to 3. The condition $\ell + 1$ being violated, the exchange cannot be carried out. By the procedure *ext*-SOSM we know that when a condition is violated we get $\mu^{\text{CCM}} = \mu^{\text{SOSM}}$, that is, $\mu^{\text{SOSM}}(1) = \{y\}$, $\mu^{\text{SOSM}}(2) = \{\omega\}$ and $\mu^{\text{SOSM}}(3) = \{z\}$.

$\mathcal{P}_1$: $(\boxed{y}, d^{+} = 0,85)\mathcal{P}(1)(\omega, d^{-} = 0,8), (y, d^{+} = 0,85)\mathcal{P}(1)(z, d^{-} = 0,75)$
$(\omega, d^{+} = 0,76)\mathcal{P}(1)(z, d^{-} = 0,18)$
$(\textcircled{z}, d^{+} = 0,13)\mathcal{P}(1)(\varnothing, d^{-} = 0,08)$

$\mathcal{P}_2$: $(z, d^{+} = 0,8)\mathcal{P}(2)(y, d^{-} = 0,5), (z, d^{+} = 0,7)\mathcal{P}(2)(\omega, d^{-} = 0,25)$
$(y, d^{+} = 0,6)\mathcal{P}(2)(\omega, d^{-} = 0,34)$
$(\boxed{\omega}, d^{+} = 0,55)\mathcal{P}(2)(\varnothing, d^{-} = 0,02)$

$\mathcal{P}_3$: $(\textcircled{y}, d^{+} = 0,9)\mathcal{P}(3)(z, d^{-} = 0,45), (y, d^{+} = 0,92)\mathcal{P}(3)(\omega, d^{-} = 0,3)$
$(\boxed{z}, d^{+} = 0,5)\mathcal{P}(3)(\omega, d^{-} = 0,44)$
$(\omega, d^{+} = 0,65)\mathcal{P}(3)(\varnothing, d^{-} = 0,07)$

$\mathcal{P}_y$: $(1, d^{+} = 0,8)\mathcal{P}(y)(2, d^{-} = 0,77), (1, d^{+} = 0,7)\mathcal{P}(y)(3, d^{-} = 0,61)$
$(2, d^{+} = 0,5)\mathcal{P}(y)(3, d^{-} = 0,4)$
$(3, d^{+} = 0,8)\mathcal{P}(y)(\varnothing, d^{-} = 0,03)$

$\mathcal{P}_z$: $(1, d^{+} = 0,9)\mathcal{P}(z)(3, d^{-} = 0,56), (1, d^{+} = 0,5)\mathcal{P}(z)(2, d^{-} = 0,4)$
$(3, d^{+} = 0,65)\mathcal{P}(z)(2, d^{-} = 0,58)$
$(2, d^{+} = 0,89)\mathcal{P}(z)(\varnothing, d^{-} = 0,04)$

$\mathcal{P}_\omega$: $(2, d^{+} = 0,81)\mathcal{P}(\omega)(3, d^{-} = 0,74), (2, d^{+} = 0,81)\mathcal{P}(\omega)(1, d^{-} = 0,72)$
$(3, d^{+} = 0,6)\mathcal{P}(\omega)(1, d^{-} = 0,52)$
$(1, d^{+} = 0,73)\mathcal{P}(\omega)(\varnothing, d^{-} = 0,03)$

Second possibility : In parallel to the first possibility of exchange, this one consists in wondering whether it would not be better for individual 2 to achieve the college z and 3 to achieve ω. The social planner finds that such an exchange is possible since, the loss of intensity of individual 3 between the options z and ω is weak (0,06), and the one of individual 2 is relatively high (0,45). So, 2 may find it very beneficial to 'go up' towards his top-ranking (i.e. z) and as $0,06 < 0,45$ the social planner allows the exchange.

Now that 3 is paired to ω and 2 to his top-ranking z, the question is whether it would be possible for 3 to be matched with the option y while 1 with the alternative ω. Such an exchange is possible! Indeed, the loss of intensity of 1 between the options y and ω is more or less low, compared to 3 $(0,05 < 0,62)$. Moreover, the condition $\ell + 1$ remains valid since y has a loss of intensity of 0,09 and the one of ω equals to 0,08 (for 1 compared to 3).

Consequently, individual 2 keeps the option z from the first exchange and individuals 1 and 3 get options ω and y, respectively, by the second exchange. Thus, $\mu^{\mathrm{CCM}}(1) = \{\omega\}$, $\mu^{\mathrm{CCM}}(2) = \{z\}$ and $\mu^{\mathrm{CCM}}(3) = \{y\}$.

$\mathcal{P}_1$: $(\boxed{y}, d^+ = 0,85)\mathcal{P}(1)(\omega, d^- = 0,8), (y, d^+ = 0,85)\mathcal{P}(1)(z, d^- = 0,75)$
$(\textcircled{$\omega$}, d^+ = 0,76)\mathcal{P}(1)(z, d^- = 0,18)$
$(z, d^+ = 0,13)\mathcal{P}(1)(\varnothing, d^- = 0,08)$

$\mathcal{P}_2$: $(\textcircled{z}, d^+ = 0,8)\mathcal{P}(2)(y, d^- = 0,5), (z, d^+ = 0,7)\mathcal{P}(2)(\omega, d^- = 0,25)$
$(y, d^+ = 0,6)\mathcal{P}(2)(\omega, d^- = 0,34)$
$(\boxed{\omega}, d^+ = 0,55)\mathcal{P}(2)(\varnothing, d^- = 0,02)$

$\mathcal{P}_3$: $(\textcircled{y}, d^+ = 0,9)\mathcal{P}(3)(z, d^- = 0,45), (y, d^+ = 0,92)\mathcal{P}(3)(\omega, d^- = 0,3)$
$(\boxed{z}, d^+ = 0,5)\mathcal{P}(3)(\omega, d^- = 0,44)$
$(\textcircled{$\omega$}, d^+ = 0,65)\mathcal{P}(3)(\varnothing, d^- = 0,07)$

$\mathcal{P}_y$: $(1, d^+ = 0,8)\mathcal{P}(y)(2, d^- = 0,77), (1, d^+ = 0,7)\mathcal{P}(y)(3, d^- = 0,61)$
$(2, d^+ = 0,5)\mathcal{P}(y)(3, d^- = 0,4)$
$(3, d^+ = 0,8)\mathcal{P}(y)(\varnothing, d^- = 0,03)$

$\mathcal{P}_z$: $(1, d^+ = 0,9)\mathcal{P}(z)(3, d^- = 0,56), (1, d^+ = 0,5)\mathcal{P}(z)(2, d^- = 0,4)$
$(3, d^+ = 0,65)\mathcal{P}(z)(2, d^- = 0,58)$
$(2, d^+ = 0,89)\mathcal{P}(z)(\varnothing, d^- = 0,04)$

$\mathcal{P}_\omega$: $(2, d^+ = 0,81)\mathcal{P}(\omega)(3, d^- = 0,74), (2, d^+ = 0,81)\mathcal{P}(\omega)(1, d^- = 0,72)$
$(3, d^+ = 0,6)\mathcal{P}(\omega)(1, d^- = 0,52)$
$(1, d^+ = 0,73)\mathcal{P}(\omega)(\varnothing, d^- = 0,03)$

Third possibility : Hereafter, we examine if individual 2 can have access to the college y, what would obviously mean that individual 1 goes down to the option initially held by 2, namely the option ω. Once again, conditions ℓ and $\ell + 1$ are satisfied and the exchange can be done.

Thus, this leads us to continue this possibility by allowing this time to 2 to reach his top-ranking (since it is henceforth paired with y) by exchanging his new option with the one that 3 is paired. This exchange is possible: 2 may find it very beneficial to be paired to z since such an option is very preferable for him (compared to y) and 3 as well. Let us note that when two individuals have both interest to 'go up' towards a better alternative (i.e. equity in access), then we do not even have to look at the condition ℓ! The condition $\ell + 1$ is also respected.

Once again, we obtain $\mu^{\text{CCM}}(1) = \{\omega\}$, $\mu^{\text{CCM}}(2) = \{z\}$ and $\mu^{\text{CCM}}(3) = \{y\}$.

$\mathcal{P}_1$: $(\boxed{y}, d^+ = 0,85)\mathcal{P}(1)(\omega, d^- = 0,8), (y, d^+ = 0,85)\mathcal{P}(1)(z, d^- = 0,75)$
$(\textcircled{\omega}, d^+ = 0,76)\mathcal{P}(1)(z, d^- = 0,18)$
$(z, d^+ = 0,13)\mathcal{P}(1)(\varnothing, d^- = 0,08)$

$\mathcal{P}_2$: $(\textcircled{z}, d^+ = 0,8)\mathcal{P}(2)(y, d^- = 0,5), (z, d^+ = 0,7)\mathcal{P}(2)(\omega, d^- = 0,25)$
$(\textcircled{y}, d^+ = 0,6)\mathcal{P}(2)(\omega, d^- = 0,34)$
$(\boxed{\omega}, d^+ = 0,55)\mathcal{P}(2)(\varnothing, d^- = 0,02)$

$\mathcal{P}_3$: $(\textcircled{y}, d^+ = 0,9)\mathcal{P}(3)(z, d^- = 0,45), (y, d^+ = 0,92)\mathcal{P}(3)(\omega, d^- = 0,3)$
$(\boxed{z}, d^+ = 0,5)\mathcal{P}(3)(\omega, d^- = 0,44)$
$(\omega, d^+ = 0,65)\mathcal{P}(3)(\varnothing, d^- = 0,07)$

$\mathcal{P}_y$: $(1, d^+ = 0,8)\mathcal{P}(y)(2, d^- = 0,77), (1, d^+ = 0,7)\mathcal{P}(y)(3, d^- = 0,61)$
$(2, d^+ = 0,5)\mathcal{P}(y)(3, d^- = 0,4)$
$(3, d^+ = 0,8)\mathcal{P}(y)(\varnothing, d^- = 0,03)$

$\mathcal{P}_z$: $(1, d^+ = 0,9)\mathcal{P}(z)(3, d^- = 0,56), (1, d^+ = 0,5)\mathcal{P}(z)(2, d^- = 0,4)$
$(3, d^+ = 0,65)\mathcal{P}(z)(2, d^- = 0,58)$
$(2, d^+ = 0,89)\mathcal{P}(z)(\varnothing, d^- = 0,04)$

$\mathcal{P}_\omega$: $(2, d^+ = 0,81)\mathcal{P}(\omega)(3, d^- = 0,74), (2, d^+ = 0,81)\mathcal{P}(\omega)(1, d^- = 0,72)$
$(3, d^+ = 0,6)\mathcal{P}(\omega)(1, d^- = 0,52)$
$(1, d^+ = 0,73)\mathcal{P}(\omega)(\varnothing, d^- = 0,03)$

We just mention that the exchange between 1 and 3 is impossible since, pairing on the one hand, 1 with z and on the other hand, 3 with ω, would mean that individuals 1 and 3 would have had everything to lose (non access to an option that would be more acceptable to them).

Conclusion: The second and third possibility gives us the same result, which is better than the one given by the algorithm of Gale and Shapley. In other words, two individuals obtain a better option than the one initially given by the deferred acceptance algorithm. ■

Example 2. Consider $N = \{1, 2, 3\}$, $X = \{y, z, \omega\}$ (with $N = N(x)$, $\forall x \in X$ and $X = X(k)$, $k = 1, 2, 3$), and the following respective profiles :
$\mathcal{P}_1$: $(\omega, d^+ = 0,7)\mathcal{P}(1)(z, d^- = 0,5), (\omega, d^+ = 0,7)\mathcal{P}(1)(y, d^- = 0,5)$
$(z, d^+ = 0,6)\mathcal{P}(1)(y, d^- = 0,4)$
$(y, d^+ = 0,3)\mathcal{P}(1)(\varnothing, d^- = 0,11)$

$\mathcal{P}_2$: $(z, d^+ = 0,9)\mathcal{P}(2)(\omega, d^- = 0,8), (z, d^+ = 0,86)\mathcal{P}(2)(y, d^- = 0,7)$
$(\omega, d^+ = 0,8)\mathcal{P}(2)(y, d^- = 0,5)$
$(y, d^+ = 0,5)\mathcal{P}(2)(\varnothing, d^- = 0,004)$

$\mathcal{P}_3$: $(z, d^+ = 0,8)\mathcal{P}(3)(y, d^- = 0,75), (z, d^+ = 0,8)\mathcal{P}(3)(\omega, d^- = 0,71)$
$(y, d^+ = 0,8)\mathcal{P}(3)(\omega, d^- = 0,43)$
$(\omega, d^+ = 0,8)\mathcal{P}(3)(\varnothing, d^- = 0,02)$

$\mathcal{P}_y$: $(1, d^+ = 0,9)\mathcal{P}(y)(2, d^- = 0,84), (1, d^+ = 0,9)\mathcal{P}(y)(3, d^- = 0,8)$
$(2, d^+ = 0,8)\mathcal{P}(y)(3, d^- = 0,21)$
$(3, d^+ = 0,3)\mathcal{P}(y)(\varnothing, d^- = 0,15)$

$\mathcal{P}_z$: $(3, d^+ = 0,72)\mathcal{P}(z)(2, d^- = 0,66), (3, d^+ = 0,77)\mathcal{P}(z)(1, d^- = 0,68)$
$(2, d^+ = 0,6)\mathcal{P}(z)(1, d^- = 0,3)$
$(1, d^+ = 0,52)\mathcal{P}(z)(\varnothing, d^- = 0,06)$

$\mathcal{P}_\omega$: $(2, d^+ = 0,63)\mathcal{P}(\omega)(3, d^- = 0,55), (2, d^+ = 0,65)\mathcal{P}(\omega)(1, d^- = 0,56)$
$(3, d^+ = 0,79)\mathcal{P}(\omega)(1, d^- = 0,2)$
$(1, d^+ = 0,25)\mathcal{P}(\omega)(\varnothing, d^- = 0,12)$

Initial Step of the Procedure (SOSM) : Individual 1 proposes to ω, while individuals 2 and 3 propose to z. Individual 1 will be accepted and only individual 2 will be rejected by z since individual 3 has the priority. Thus, individual 2 proposes to his second best option of his profile, namely, the option ω. And since he has the priority compared to 1, individual 2 is matched with ω while 1 is rejected. Consequently, 1 proposes to the option z. He is once again rejected as 3, already paired with this option, has the priority. So, 1 proposes to his last acceptable alternative, that is to y and he is accepted. The SOSM-matching realized is $\mu^{\text{SOSM}}(1) = \{y\}$, $\mu^{\text{SOSM}}(2) = \{\omega\}$ and $\mu^{\text{SOSM}}(3) = \{z\}$. Once again, the above matching will be marked into boxes in what follows.

Procedure *ext*-SOSM :

First possibility : Let us ask the following question: Is it possible to let individual 1 to achieve ω and individual 2 to get y? We see that the condition ℓ is not verified: the loss of intensity of 1 is very low compared to 2 $(0,2 < 0,4)$. Such a possibility of exchange cannot be realized. In such a case $\mu^{\text{CCM}} = \mu^{\text{SOSM}}$. So, the best matching continues to be the one given above, $\mu^{\text{SOSM}}(1) = \{y\}$, $\mu^{\text{SOSM}}(2) = \{\omega\}$ and $\mu^{\text{SOSM}}(3) = \{z\}$.

Second possibility : This time, we will try to give to 1 the option z, by giving to 3 the option y. The conditions ℓ (low loss of intensity for 3 equal to 0,05 and $0,05 < 0,2$), and $\ell + 1$ (low loss of intensity for y and z concerning the acceptability of individuals 3 and 1, respectively) are satisfied. The exchange is realizable.

Now that 1 has the option z, is it possible to give him the option ω in order to ascend 1 and 2 simultaneously towards their respective top-ranking? The answer is obvious, especially since the condition $\ell + 1$ is valid in this case: the loss of intensity of ω for 1 relatively to 2 is weak (0,09). In addition, 2 has the priority compared to 1 given its position in the profile of the college ω. Thus, we obtain $\mu^{\text{CCM}}(1) = \{\omega\}$, $\mu^{\text{CCM}}(2) = \{z\}$ and $\mu^{\text{CCM}}(3) = \{y\}$.

$\mathcal{P}_1$: $(\textcircled{\omega}, d^+ = 0,7)\mathcal{P}(1)(z, d^- = 0,5), (\omega, d^+ = 0,7)\mathcal{P}(1)(y, d^- = 0,5)$
$(\textcircled{z}, d^+ = 0,6)\mathcal{P}(1)(y, d^- = 0,4)$
$(\boxed{y}, d^+ = 0,3)\mathcal{P}(1)(\varnothing, d^- = 0,11)$

$\mathcal{P}_2$: $(\textcircled{z}, d^+ = 0,9)\mathcal{P}(2)(\omega, d^- = 0,8), (z, d^+ = 0,86)\mathcal{P}(2)(y, d^- = 0,7)$
$(\boxed{\omega}, d^+ = 0,8)\mathcal{P}(2)(y, d^- = 0,5)$
$(y, d^+ = 0,5)\mathcal{P}(2)(\varnothing, d^- = 0,004)$

$\mathcal{P}_3$: $(\boxed{z}, d^+ = 0,8)\mathcal{P}(3)(y, d^- = 0,75), (z, d^+ = 0,8)\mathcal{P}(3)(\omega, d^- = 0,71)$
$(\textcircled{y}, d^+ = 0,8)\mathcal{P}(3)(\omega, d^- = 0,43)$
$(\omega, d^+ = 0,8)\mathcal{P}(3)(\varnothing, d^- = 0,02)$

$\mathcal{P}_y$: $(1, d^+ = 0,9)\mathcal{P}(y)(2, d^- = 0,84), (1, d^+ = 0,9)\mathcal{P}(y)(3, d^- = 0,8)$
$(2, d^+ = 0,8)\mathcal{P}(y)(3, d^- = 0,21)$
$(3, d^+ = 0,3)\mathcal{P}(y)(\varnothing, d^- = 0,15)$

$\mathcal{P}_z$: $(3, d^+ = 0,72)\mathcal{P}(z)(2, d^- = 0,66), (3, d^+ = 0,77)\mathcal{P}(z)(1, d^- = 0,68)$
$(2, d^+ = 0,6)\mathcal{P}(z)(1, d^- = 0,3)$
$(1, d^+ = 0,52)\mathcal{P}(z)(\varnothing, d^- = 0,06)$

$\mathcal{P}_\omega$: $(2, d^+ = 0,63)\mathcal{P}(\omega)(3, d^- = 0,55), (2, d^+ = 0,65)\mathcal{P}(\omega)(1, d^- = 0,56)$
$(3, d^+ = 0,79)\mathcal{P}(\omega)(1, d^- = 0,2)$
$(1, d^+ = 0,25)\mathcal{P}(\omega)(\varnothing, d^- = 0,12)$

Third possibility : The exchange between individuals 2 and 3 is now considered by the social planner. He verifies if it is possible for 2 to achieve z, and 3 to reach ω (even if ω is the last acceptable option in the profile of individual 3). Such an exchange is realizable: indeed, the loss of intensity of 3 between z and ω is rather low in an absolute way (0,09) and rather weak in a relative one ($0,09 < 0,1$). Moreover, the condition $\ell + 1$ is verified (z having a loss of intensity equal to 0,06 and the one of ω equals to 0,08).

Thereafter, the social planner could have considered the possibility of exchange between 2 and 1, for the respective options z and y. Nevertheless, even if the condition ℓ does not seem violated ($0.2 > 0.16$), the condition $\ell + 1$ as for it, it is violated. Indeed, college z is not ready to accept individual 1 instead of individual 2 (0,3). Consequently, this exchange considered by the social planner cannot be applied. By the procedure we know that when a condition is violated we get $\mu^{\text{CCM}} = \mu^{\text{SOSM}}$, that is, $\mu^{\text{SOSM}}(1) = \{y\}$, $\mu^{\text{SOSM}}(2) = \{\omega\}$ and $\mu^{\text{SOSM}}(3) = \{z\}$.

$\mathcal{P}_1$: $(\omega, d^+ = 0,7)\mathcal{P}(1)(z, d^- = 0,5), (\omega, d^+ = 0,7)\mathcal{P}(1)(y, d^- = 0,5)$
$(z, d^+ = 0,6)\mathcal{P}(1)(y, d^- = 0,4)$
$(\boxed{y}, d^+ = 0,3)\mathcal{P}(1)(\varnothing, d^- = 0,11)$

$\mathcal{P}_2$: $(\text{\textcircled{z}}, d^+ = 0,9)\mathcal{P}(2)(\omega, d^- = 0,8), (z, d^+ = 0,86)\mathcal{P}(2)(y, d^- = 0,7)$
$(\boxed{\omega}, d^+ = 0,8)\mathcal{P}(2)(y, d^- = 0,5)$
$(y, d^+ = 0,5)\mathcal{P}(2)(\varnothing, d^- = 0,004)$

$\mathcal{P}_3$: $(\boxed{z}, d^+ = 0,8)\mathcal{P}(3)(y, d^- = 0,75), (z, d^+ = 0,8)\mathcal{P}(3)(\omega, d^- = 0,71)$
$(y, d^+ = 0,8)\mathcal{P}(3)(\omega, d^- = 0,43)$
$(\text{\textcircled{$\omega$}}, d^+ = 0,8)\mathcal{P}(3)(\varnothing, d^- = 0,02)$

$\mathcal{P}_y$: $(1, d^+ = 0,9)\mathcal{P}(y)(2, d^- = 0,84), (1, d^+ = 0,9)\mathcal{P}(y)(3, d^- = 0,8)$
$(2, d^+ = 0,8)\mathcal{P}(y)(3, d^- = 0,21)$
$(3, d^+ = 0,3)\mathcal{P}(y)(\varnothing, d^- = 0,15)$

$\mathcal{P}_z$: $(3, d^+ = 0,72)\mathcal{P}(z)(2, d^- = 0,66), (3, d^+ = 0,77)\mathcal{P}(z)(1, d^- = 0,68)$
$(2, d^+ = 0,6)\mathcal{P}(z)(1, d^- = 0,3)$
$(1, d^+ = 0,52)\mathcal{P}(z)(\varnothing, d^- = 0,06)$

$\mathcal{P}_\omega$: $(2, d^+ = 0,63)\mathcal{P}(\omega)(3, d^- = 0,55), (2, d^+ = 0,65)\mathcal{P}(\omega)(1, d^- = 0,56)$
$(3, d^+ = 0,79)\mathcal{P}(\omega)(1, d^- = 0,2)$
$(1, d^+ = 0,25)\mathcal{P}(\omega)(\varnothing, d^- = 0,12)$

Conclusion: We keep the second possibility that improves the initial matching of Gale and Shapley. ■

The procedure described in this section allows us to obtain an acceptable matching for a given population of individuals (that minimizes the collective desire as good as possible). We just note that there are more possibilities of exchange when the quota $Q(x)$, $\forall x \in X$, or $|N|$ is big. At this point, let us mark the following proposition.

Proposition 1. *For a number of exchanges lower than the number of students, there does not exist a cycle: no individual who obtains a better alternative through a possible exchange terminates with his initial matching, via the procedure.*

Proof of Proposition 1. The proof is trivial. Indeed, if for example $|N| = 3$, each possibility must contain at most two exchanges. Otherwise, we will obtain the initial matching! This fact can be seen in the examples stated above. □

Henceforth, it is possible to present a real-life mechanism that will handle student admissions to colleges when their priorities initially formulated are fuzzy.

5 Towards a New Algorithm

It is now important to establish a rigorous mechanism that will be able to select the best possible matching for a college choice problem. The mechanism that we present, aims to highlight the concept of uncertainty (fuzzy) of agents' choices and therefore, the procedure previously presented and illustrated. It will be entitled: 'The College Choice Mechanism'.

Mechanism 6 (**The College Choice Mechanism (CCM)**).

VAGUE I : CONSTRUCTION OF THE FUZZY PROFILES OF AGENTS

THE FUZZY PRIORITIES OF STUDENTS:

I.1. Students perform a ranking of *strict priorities* over colleges, according to the disciplinary programs or other needs (all of them correlated with the objective: *'succeed academically and be able to succeed'*) like registration fees, geographical distance, and so on, that colleges propose. Since base the choices on issues individually important means to ignore some basic relative advantages associated directly with the option itself, students are supposed to ignore the identity of colleges (and most of its attributes) to which they make their proposal.

The above mentioned ranking is directly transmitted to a social planner who will create and provide them a list of colleges respecting in the best way their respective strict priorities.

I.2. The social planner sends to each student the list of colleges that correspond to their respective strict priorities. Henceforth, students have knowledge of the identity (and attributes) of the colleges being offered to them. Having knowledge of 'superflux' attributes (i.e. other than their respective priorities), this time, they will have tastes (PACC)! They should not go against the priorities identified individually and freely (since the strict priority profile of each student was provided by the social planner). All they can do is to say that a college that has the priority may be finally a 'little less' or 'slightly less' prioritary to another. Therefore, it appears a *variable* degree of intensity of tastes.

Consequently, a student with strict priorities, given his (PACC), will have a fuzzy choice, variable from one option to another. Any individual priority associated with any variation of the degree of intensity of taste for this priority will be called *fuzzy priority.* Once the latter constructed, students send the list, i.e. their *fuzzy priority profile*, back to the social planner.

THE FUZZY PRIORITIES OF COLLEGES:

I.3. The social planner is aware of the *fuzzy priority profiles* of students, from the *strict priority profiles* previously realized. He now sends a *signal* of the 'potentially possible candidates' that colleges can accept (regardless the classification of colleges in a student's profile). Thus, if for example, individual's i strict priority profile is $\mathcal{P}(i) : y, z, \omega$, then the social planner will create a signal to colleges y, z, ω. So, each college creates a profile of strict priorities on these 'possible candidates' according to the above signal and depending on whether students' attributes correspond to those required by the law and/or any other internal institutional regulation imposed. These 'possible candidates' will be designated as 'possibly acceptable candidates'. The profile of strict priorities respective to each college is then forwarded to the social planner.

I.4. Henceforth, the social planner asks colleges to provide their preferences (PACS) on 'students-candidates' based on other attributes apart from, on the one hand, those required by the law and/or any other internal institutional regulation and, on the other hand, their identity (in order to avoid any attempt of discrimination); this, *by respecting the classified priorities* which they previously formulated. Thus, variable degrees of intensity appear, according to the classified prioritary students: a student 'strictly prioritary' may ultimately be designated as 'a little more' or 'much more' prioritary relatively to another one. These profiles, along with those of students, are called *fuzzy priority profiles* for colleges. These profiles will be considered by the social planner in Vague II.

As for students, colleges have also fuzzy priorities. Therefore, we have a ranking of fuzzy priorities as well for students as for colleges.

VAGUE II : THE RESULTED FUZZY MATCHING

Initially, the SOSM is applied *only* over strict priority profiles as well for students as for colleges. Hence, the social planner considers the first priority of a student and verifies whether he is accepted or rejected, for a given quota, by a college. If the social planner sees that the student is rejected, he will then try to assign him to his second priority. And so on. The process stops when the social planner assigns each student at a college taking into consideration the strict priority profiles of each of them. At the end, the procedure *Extended*-SOSM is applied[14].

Examples 1 and 2 may represent an illustration of the above algorithm. Indeed, one can assume that the options matched are known.

6 Vigorousness of the Mechanism

In what follows, we prove that the solution obtained by the College Choice Mechanism is robust! In other means, will we verify that the conditions previously stated (Section 3) are respected.

Theorem 1. *A matching μ^{CCM} satisfies the conditions (**No-EJ**)*, (**IR**)* and (**NW**)* for a college choice problem if an only if is stable for its associated college admissions problem.*

Proof of Theorem 1. We demonstrate our theorem going through the following lemmas.

Lemma 1. *The college admission problem is associated to a fuzzy college admission problem.*

[14]The students know neither the (PACS), nor the process of 'exchange' carried out by the social planner. They do not know either the choices carried out by the others.

Proof of Lemma 1. In fact, the college admission problem (CAP) can be identified via strict preferences of the form $xP(i)y$, for all $i \in N$, and $iP(x)j$, for all $x \in X$. This is equivalent to say that $(x,1)P(i)(y,0)$, $\forall i \in N$, and $(i,1)P(x)(j,0)$, $\forall x \in X$, which implies that $LI^{CAP}(x) = 1$ and $LI^{CAP}(i) = 1$. In a fuzzy college admission problem (FCAP), we talk about fuzzy preferences of the form $(x,d^{+})P(i)(y,d^{-})$ for all $i \in N$, and $(i,d^{+})P(x)(j,d^{-})$ for all $x \in X$, with $d^{+}d^{-} \in [0;1]$, which leads us to say that $LI^{FCAP}(x) < 1$ and $LI^{FCAP}(i) < 1$.

Consequently, $LI^{FCAP}(x) < LI^{CAP}(x)$ and $LI^{FCAP}(i) < LI^{CAP}(i)$. Thus, $FCAP \subseteq CAP$. More specifically, regardless of the intensity associated to students (for the colleges) and to colleges (for students), the ranking of the alternatives in a FCAP remains the same in a CAP. So, $FCAP \subseteq CAP$, which also leads us to think that the condition of the F-stability for a FCAP is a weakening of the condition of stability related to the CAP. □

Furthermore, note that,

Lemma 2 (Balinski and Sönmez (1999)). *A matching satisfies (**IR**), fairness and (**NW**) for a student placement problem, if and only if, it is stable for its associated college admissions problem.*

As $FCAP \subseteq CAP$, the conditions concerning a student placement problem and related to the notion of stability (Lemma 2), are also related to the F-stability.

Lemma 3. *The fuzzy college admission problem is associated to the college choice problem.*

Proof of Lemma 3. To every college choice problem (CCP) is associated a fuzzy college admission problem (FCAP) considering, foremost, the priority relation $\mathcal{P}(i)$ and $\mathcal{P}(x)$, $\forall i \in N$, $\forall x \in X$, based on the concept of 'necessity' to which will be associated the degree of intensity, based on the variability of tastes; preferences are thought of as the combination of necessities and tastes. In more concrete terms, the difference from a FCAP is that the CCP, on the one hand, gives more importance to the priorities and on the other hand, is directed towards the equivalence of the F-stability, i.e. the condition of (**No-EJ**)*. □

Using all the above information, it is easy to see that $CCP \subseteq CAP$. In other means, the necessary conditions for a CAP (Lemma 2) are valid even for a CCP. In our context, (**IR**), fairness and (**NW**) are the natural counterpart of (**IR**)*, (**No-EJ**)*, and (**NW**)*. □

Corollary 1. *It always exists an F-stable matching for each **CCM**.*

This point is obvious as the fuzzy priorities are neither more nor less than a weakening of their 'strict' side. Going back to the concept of strategy-proofness, we will say (with respect to the definition stated in Section 2) that for all $i \in N$ and every strategy $S = (s_i^{*}, s_{-i}^{*})$, $S' = (s_i, s_{-i}^{*})$ there does not exist $(g'(i, s_i, s_{-i}^{*}), d^{+})\mathcal{P}(i)(g(i, s_i^{*}, s_{-i}^{*}), d^{-})$, with $g \neq g'$, where $g(S) = \mu$ and $g'(S') = \eta$. This leads us to the following proposition:

Proposition 2. *A **CCM** is strategy-proof.*

Proof of Proposition 2. Students may lie either on their priorities or on their tastes. We will explain that these two cases, noted (1) and (2) respectively, are not possible through the **CCM** considered.
(1) Let $\mathbf{s}_i = s_i \vee s_i^{*}$. Moreover, suppose that $(x, d^{+})\mathcal{P}(i)(y, d^{-})$, $\forall i \in N$.

If $\mathbf{s}_i = s_i^*$, we have $(x, d^+)\mathcal{P}(i)(y, d^-)$, $\forall x, y \in X(i)$. Suppose that the individual i decides to play another strategy aiming to lie, $\mathbf{s}_i = s_i$. In this case, $s_i = \neg s_i^* \Rightarrow \neg(x, d^+)\mathcal{P}(i)(y, d^-) \Rightarrow (y, d^+)\mathcal{P}(i)(x, d^-)$. The problem in this case is that our mechanism will have as a purpose to assign in the best way i to the option y ($\mu^{\text{CCM}}(i) = \{y\}$). However, y is not the option with the highest priority for i because we supposed that $(x, d^+)\mathcal{P}(i)(y, d^-)$.
(2) Students as well as colleges are unaware of the way that the procedure *ext*-SOSM progresses, so it is difficult to lie through a strategy, especially as only the social planner takes decisions. Moreover, establish a strategy on tastes (PACC) that may characterize an option, or an environment associated with the option, and so on, is complex as no student knows the attributes which form the basis of the degree of intensity of other students and colleges. Thus, in a problem of multidimensionality, unpredictability and heterogeneity of individual behavior, it is difficult for each student to determine a reliable strategy of lie.

Lying on the (PACC) and/or the priorities is not a way to guarantee the access to a souhaitable alternative, which, additionally, is supposed to correspond on the former ones. □

The question of the relationship between optimality and fairness must also be considered.

Lemma 4. *The (**PE**) is not always satisfied through **CCM**.*

Proof of Lemma 4. By (**PE**), $\nexists \eta$ such that $(\eta(i), 1)\mathcal{P}(i)(\mu^{\text{CCM}}(i), 0)$, for some $i \in N$, and $(\eta(i), d^+)\mathcal{P}(i)(\mu^{\text{CCM}}(i), d^-)$, for all $i \in N$. Moreover, note that each $\mu^{\text{CCM}} \vdash_F^P \mu^{\text{SOSM}}$. In fact, the CCM tries to find the best possibility that will give a matching better than the one of given by the SOSM (Vague II). Thus, there can be no other option that Pareto dominates the one which comes from the possibility chosen through the procedure. However, we must distinguish two separate cases (see the Procedure *ext*-SOSM): for all $i \in N$, $(\mu^{\text{CCM}}(i), d^+)\mathcal{P}(i)(\mu^{\text{SOSM}}(i), d^-)$, $d^+, d^- \in [0; 1]$, and for some $i \in N$, for whom $LI(i) = d^+(\mu^{\text{CCM}}(i)) - d^-(\mu^{\text{SOSM}}(i)) > 0$, whereas for some $j \in N$, we can have either $LI(j) = d^+(\mu^{\text{CCM}}(j)) - d^-(\mu^{\text{SOSM}}(j)) > 0$ (**Case 1**), or $LI(j) = d^+(\mu^{\text{CCM}}(j)) - d^-(\mu^{\text{SOSM}}(j)) \to -\epsilon$, where ϵ a very small real number (**Case 2**), with $\mu^{\text{CCM}}(i) = \mu^{\text{SOSM}}(j)$ and $\mu^{\text{CCM}}(j) = \mu^{\text{SOSM}}(i)$.

We underline that, unlike the first case, the second does not verify the (**PE**) because this condition suggests that $(\mu^{\text{CCM}}(i), d^+)\mathcal{P}(i)(\mu^{\text{SOSM}}(i), d^-)$, for all $i \in N$. Nethertheless, this relation is not verified because we have supposed that individual j obtains a matching $\mu^{\text{CCM}}(j) = \mu^{\text{SOSM}}(j) - \beta_j$. Consequently, we have a contradiction. □

The only way that the (**PE**) is respected as best as possible in Case 2 is to introduce a 'weaker' version. In this objective, we introduce a new condition which is the condition of *Justified Pareto Non-Dominance.*

We say that a matching μ^{CCM} *Pareto dominates in a justified way* another matching μ^{SOSM} ($\mu^{\text{CCM}} \vdash_F^{JP} \mu^{\text{SOSM}}$) if for,

$$[\mu^{\text{CCM}}(i) \wedge \mu^{\text{SOSM}}(i)] \vee [(\neg \mu^{\text{SOSM}}(i)) \wedge \mu^{\text{CCM}}(i)] \in \mathscr{M}_F^*, \ \forall i \in N,$$

where $\mathscr{M}_F^* = (\mathscr{M}_F^{\mathbf{F}} \vee \mathscr{M}_F^{\mathbf{SF}})$, with $\mu^{\text{SOSM}}, \mu^{\text{CCM}} \in \mathscr{M}_F(\mathscr{S}_F)$, we have,

$$(\mu^{\text{CCM}}(i), 1)\mathcal{P}(i)(\mu^{\text{SOSM}}(i), 0), \text{ for some } i \in N, \text{ and,}$$

$$(\mu^{\text{CCM}}(i), d^{+})\mathcal{P}(i)(\mu^{\text{SOSM}}(i), d^{-}), \ \forall i \in N, \ \forall d^{+}, d^{-} \in [0;1].$$

The above condition makes it possible to establish the link between the (**PE**) and the procedure *Extended*-SOSM when the latter deals with various cases, like the one of the non-satisfaction of the (**PE**). The condition that results is the *Justified Pareto Efficiency condition.*

Therefore, we say that a matching is *Justified* $\vdash_F^P$*-Pareto Efficient* (**JPE**) if there exists no other matching $\eta \neq \mu^{\text{CCM}}$, such that $\eta \vdash_F^{JP} \mu^{\text{CCM}}$ or $\eta \vdash_F^{P} \mu^{\text{CCM}}$.

Thus, it is possible for us to come to the following Lemma.

Lemma 5. *A **CCM** always satisfies the (**JPE**).*

Proof of Lemma 5. Through the developed procedure, we can say that the **CCM** always tries to find a possibility of improvement of individual situations (via the matching associated with students). Thus, every matching $\mu^{\text{CCM}} \neq \mu^{\text{SOSM}}$ is now possible (under consideration of optimality) since, considering the condition of fairness (relatively to the cases stated in the proof of Lemma 4), while there exists $LI(j) = d^{+}(\mu^{\text{SOSM}}) - d^{-}(\mu^{\text{CCM}}) \to \epsilon$, and $LI(i) = d^{+}(\mu^{\text{CCM}}) - d^{-}(\mu^{\text{SOSM}}) \gg 0$, for some $i, j \in N$, a loss of welfare for j stays justified, by (**JPE**)! Therefore, the **CCM** always verifies (**JPE**). □

Corollary 2. *(**JPE**) always implies the non-violation of the condition (**F**) and (**SF**), and reciprocally.*

Theorem 2. *A **CCM** that eliminates justified envy satisfies the (**PE**) when* $LI(i) = d^{+}(\mu^{SOSM}(i)) - d^{-}(\mu^{CCM}(i)) \to \kappa$ *is individually justified,* $\forall i \in N$*, if for* $\mu^{CCM}(i) = (\mu^{SOSM}(i) - \beta_i)$ *with* $\beta_i \in \mathbb{N}_{++}$, $\beta_i \in [1; \delta_i]$, $\delta_i \in \mathbb{N}_{++}$*, we have* $\kappa = \epsilon$*, or if for* $\mu^{CCM}(i) = (\mu^{SOSM}(i) + \alpha_i - \gamma_i)$ *with* $\alpha_i, \gamma_i \in \mathbb{N}_{++}$, $\gamma_i \in [0; \alpha_i[$*, we have* $\kappa = -\infty$ *.*

Proof of Theorem 2. Lemmas 4 and 5 justify the existence of the condition (**PE**) (or, (**JPE**)), depending on the situation. □

7 Conclusion

The purpose of this paper was to develop an equitable system of 'distribution' of colleges' seats to students through the traditional theory of matching. We have introduced a realistic challenge which was to confront the individual priorities with the associated tastes. Doing so was quite difficult since considering this hypothesis, we mean that the real choice of a college may finally be enough vague! In fact, even if the colleges have to respect regulations such as quotas of entry (for example a quota assigned to each type of individuals (with disabilities, etc.)[15] or the obtention of minimum scores during examinations and so on, they may also have tastes (PACS) concerning some candidates who, however, may (or may not) have been rejected by the educational regulations imposed.

[15] See especially Abdulkadiroğlu (2005a), Hafalir (2011), regarding the policy of affirmative action, or Ehlers (2010), regarding the issue of a school choice regulated through quotas imposed for specific types of individuals (disabilities, etc.).

Table 1. Summary

	Pareto Efficiency	Justified Pareto Efficiency	Non-Justified Envy	Strategy-Proofness
CCM	NO[c]	YES	YES	YES
Boston	NO[a]	-	NO	NO
SOSM	NO	-	YES	YES
TTCM	YES	-	NO	YES
ETTCM	YES	-	NO	YES
EADAM	YES	-	NO[b]	NO

a. When individual preferences are revealed, the Pareto efficiency is not violated.
b. This is not always verified (see Kesten [2006]).
c. No in (**Case 2**), and yes in (**Case 1**) (see demonstration of Lemma 4); it is Pareto efficient only if a justification is considered for the (**Case 2**) (see Lemma 5), which implies a 'YES' for the Justified Pareto Efficiency condition.

Conversely, students have sometimes a tendency to select colleges in order to maximize their welfare even if it is possible to consider that sometimes the attributes associated with their choices (PACC) can move away from their real needs (priorities). For these reasons it was important to distinguish individual and institutional tastes/priorities! The problem is that it is difficult to consider all of them. For this reason we included in our analysis the concept of 'vagueness', via the fuzzy priorities.

Relatively to certain algorithms such as the ETTCM, we tried to preserve even the concept of fairness, but also to introduce a simple mechanism that can be easily applied in real-life. Consequently, the College Choice Mechanism not only relies on assumptions visible in reality but also respects conditions considered as 'essential' in the literature.

Moreover, our mechanism may very well be generalized to a school choice problem. To conclude, we underline the fact that the cardinal analysis is not a feasible tool in our work, mainly because it does not consider the vagueness of students' choices and colleges' selections. Indeed, the utility function of von Neumann-Morgenstern simply analyzes the 'dominance' of some alternatives over others, via the utility that is assigned to each of them. They only allow us to say that a difference of utilities is greater than another, but they do not allow us to say 'slightly' or 'much'. Particularly, fuzzy priorities, contrary to utilities, enable us to represent numerically a subjective value over the binary relation between two distinct options. In this direction, if x has largely the priority compared to y, for a given individual, it is that y is largely less prioritary; and any level of utility does not enable us to consider this information. The use of an integer which tends towards the unit (for the option x) and of an integer that tends towards zero (for the option y) makes it possible to consider the above situation.

8 Appendix

Special cases of the Procedure Extended-Sosm

Here we are interested in the case where two individuals try to obtain their respective best ranked option, while a third one is already assigned to one of the latter alternatives. From this point onwards, consider an individual $j' \in N\backslash\{k\}$, $k = i, j$, and

$(LI(i)|\mu^{\text{SOSM}}(j')) \equiv (LI(j)|\mu^{\text{SOSM}}(j'))$ (or $(LI(i)|\mu^{\text{SOSM}}(j')) \approx (LI(j)|\mu^{\text{SOSM}}(j'))$), with $(LI(k)|\mu^{\text{SOSM}}(j')) = d^{+}(k, \mu^{\text{SOSM}}(j')) - d^{-}(k, \mu^{\text{SOSM}}(k))$, then:

Case (a) : Consider $(\mu^{\text{SOSM}}(i)+\alpha_i) = (\mu^{\text{SOSM}}(j)+\alpha_j)$, $\alpha_i, \alpha_j \in \mathbb{N}_{++}$, $\alpha_i = \alpha_j$ or $\alpha_i \neq \alpha_j$, and $(LI(j')|\mu^{\text{SOSM}}(i)) \approx (LI(j')|\mu^{\text{SOSM}}(j))$, where, $(LI(j')|\mu^{\text{SOSM}}(k)) = d^{+}(j', \mu^{\text{SOSM}}(k)) - d^{-}(j', \mu^{\text{SOSM}}(j'))$, when $\mu^{\text{CCM}}(k) = (\mu^{\text{SOSM}}(j')+\alpha_{j'}-\gamma_{j'})$, $\gamma_{j'} \in \mathbb{N}_{+}$, $\gamma_{j'} \in [0; \alpha_{j'}[$, or, $(LI(j')|$ $\mu^{\text{SOSM}}(k)) = d^{+}(j', \mu^{\text{SOSM}}(j')) - d^{-}(j', \mu^{\text{CCM}}(k))$, when $\mu^{\text{CCM}}(k) = \mu^{\text{SOSM}}(j') - \beta_{j'}$, $\beta_{j'} \in \mathbb{N}_{+}$, $\beta_{j'} \in [1; \delta_{j'}]$, with $\alpha_{j'} \in \mathbb{N}_{++}$, $k = i, j$. Moreover, suppose that $(\mu^{\text{SOSM}}(i) + \alpha_i) = \mu^{\text{SOSM}}(j')$.

(a.1) : If $d^{+}(i, \mu^{\text{SOSM}}(j')) > d^{+}(j, \mu^{\text{SOSM}}(j'))$, we get $\mu^{\text{CCM}}(j) \neq \mu^{\text{CCM}}(i)$ where,

$$\begin{cases} \mu^{\text{CCM}}(i) = (\mu^{\text{SOSM}}(i) + \alpha_i) \\ \mu^{\text{CCM}}(j) \neq (\mu^{\text{SOSM}}(j) + \alpha_j) \end{cases}$$

for $Q(\mu^{\text{SOSM}}(i) + \alpha_i) = 1$.

(a.2) : While $d^{+}(i, \mu^{\text{SOSM}}(j')) \equiv d^{+}(j, \mu^{\text{SOSM}}(j'))$ (or, $d^{+}(i, \mu^{\text{SOSM}}(j')) \approx d^{+}(j, \mu^{\text{SOSM}}(j'))$), and if $d^{+}(\mu^{\text{SOSM}}(j'), i) > d^{+}(\mu^{\text{SOSM}}(j'), j)$, then,

$$\begin{cases} \mu^{\text{CCM}}(i) = (\mu^{\text{SOSM}}(i) + \alpha_i) \\ \mu^{\text{CCM}}(j) \neq (\mu^{\text{SOSM}}(j) + \alpha_j) \end{cases}$$

for $Q(\mu^{\text{SOSM}}(i) + \alpha_i) = 1$.

(a.3) : If $d^{+}(i, \mu^{\text{SOSM}}(j')) \equiv d^{+}(j, \mu^{\text{SOSM}}(j'))$ and $d^{+}(\mu^{\text{SOSM}}(j'), i) \equiv d^{+}(\mu^{\text{SOSM}}(j'), j)$, then the *social planner* makes his decision by lottery.

Case (b) : Suppose this time that $(\mu^{\text{SOSM}}(i)+\alpha_i) = (\mu^{\text{SOSM}}(j)+\alpha_j)$, $\alpha_i, \alpha_j \in \mathbb{N}_{++}$, $\alpha_i = \alpha_j$ or $\alpha_i \neq \alpha_j$, and $(LI(j')|\mu^{\text{SOSM}}(i)) \neq (LI(j')|\mu^{\text{SOSM}}(j))$. In this case, if $(LI(i)|\mu^{\text{SOSM}}(j')) \equiv (LI(j)|\mu^{\text{SOSM}}(j'))$ (or $(LI(j')|\mu^{\text{SOSM}}(i)) \approx (LI(j')|\mu^{\text{SOSM}}(j))$) and if $(LI(j')|\mu^{\text{SOSM}}(i)) > (LI(j')|\mu^{\text{SOSM}}(j))$, with $(LI(j')|\mu^{\text{SOSM}}(k)) = d^{+}(j', \mu^{\text{SOSM}}(k)) - d^{-}(j', \mu^{\text{SOSM}}(j'))$, $k = i, j$, *when*,

(b.1) : $\mu^{\text{SOSM}}(i) = (\mu^{\text{SOSM}}(j') + \alpha_{j'} - \gamma^{i}_{j'})$ and $\mu^{\text{CCM}}(j) = (\mu^{\text{SOSM}}(j') + \alpha_{j'} - \gamma^{i}_{j'})$, with $\gamma^{i}_{j'} < \gamma^{j}_{j'}$ or with $\gamma^{i}_{j'} = \gamma^{j}_{j'}$, which implies that $d^{+}(j', \mu^{\text{SOSM}}(i)) - \tilde{d}^{-}(j', \mu^{\text{SOSM}}(j')) > d^{+}(j', \mu^{\text{SOSM}}(j)) - \tilde{d}^{-}(j', \mu^{\text{SOSM}}(j'))$, with $\tilde{d}$ being a benchmark degree, or,

(b.2) : $\mu^{\text{SOSM}}(i) = \mu^{\text{SOSM}}(j') - \beta^{i}_{j'}$ and $\mu^{\text{SOSM}}(j) = \mu^{\text{SOSM}}(j') - \beta^{j}_{j'}$, with $\beta^{i}_{j'} < \beta^{j}_{j'}$, which implies $\tilde{d}^{+}(j', \mu^{\text{SOSM}}(j')) - d^{-}(j', \mu^{\text{SOSM}}(i)) < \tilde{d}^{+}(j', \mu^{\text{SOSM}}(j')) - d^{-}(j', \mu^{\text{SOSM}}(j))$, or,

(b.3) : $\mu^{\text{SOSM}}(i) = (\mu^{\text{SOSM}}(j') + \alpha_{j'} - \gamma_{j'})$ and $\mu^{\text{SOSM}}(j) = \mu^{\text{SOSM}}(j') - \beta_{j'}$,
then we get,

$$\begin{cases} \mu^{\text{CCM}}(i) = (\mu^{\text{SOSM}}(j')) \\ \mu^{\text{CCM}}(j) \neq (\mu^{\text{SOSM}}(j')). \end{cases}$$

Case (c) : Now, consider $(\mu^{\text{SOSM}}(i)+\alpha_i) \neq (\mu^{\text{SOSM}}(j)+\alpha_j)$, $\alpha_i, \alpha_j \in \mathbb{N}_{++}$, $\alpha_i = \alpha_j$ or $\alpha_i \neq \alpha_j$, and $(LI(j')|\mu^{\text{SOSM}}(i)) \approx (LI(j')|\mu^{\text{SOSM}}(j))$. Here, either $\mu^{\text{SOSM}}(j') \neq (\mu^{\text{SOSM}}(k)+\alpha_k)$, for $k = i \vee j$, or $\mu^{\text{SOSM}}(j') \neq (\mu^{\text{SOSM}}(k)+\alpha_k)$, for $k = i, j$. Under this latter situation, there is no problem. Each individual acts independently towards an option $(\mu^{\text{SOSM}}(k)+\alpha_k-\gamma_k)$, with $\gamma_k \in \mathbb{N}_{+}$, $\gamma_k \in [0; \alpha_k[$, $k = i, j$, if there exist at least two other individuals $j'', j''' \in N \setminus \{i, j, j'\}$ for whom $\mu^{\text{SOSM}}(j'') = (\mu^{\text{SOSM}}(i)+\alpha_i-\gamma_i)$ and $\mu^{\text{SOSM}}(j''') = (\mu^{\text{SOSM}}(j)+\alpha_j-\gamma_j)$ and when it is possible to have

$$\begin{cases} \mu^\sigma(i) = (\mu^{\text{SOSM}}(j'')) \\ \mu^\sigma(j) \neq (\mu^{\text{SOSM}}(j'''))\text{.} \end{cases}$$

We go back to the *Extended*-SOSM. And so on.

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