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Working Paper

A formula for Nash equilibria in monotone singleton congestion games

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A formula for Nash equilibria in monotone singleton congestion games

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Abstract. *This paper provides a simple formula describing all Nash equilibria in symmetric monotone singleton congestion games. Our approach also yields a new and short proof establishing the existence of a Nash equilibrium in this kind of congestion games without invoking the potential function or the finite improvement property.*

Keywords: Singleton congestion games, Nash equilibria, Potential function, Finite improvement property.

JEL classification: C72

1 Introduction

Congestion games provide a natural framework for a wide range of economics and computer science applications such as resource allocation, routing and network design problems. Rosenthal [8], who was the first to consider this class of noncooperative games, showed by a potential function argument, that they possess a pure-strategy Nash equilibrium. In fact, Nash dynamics, where players iteratively improve their utilities, always converge to an equilibrium after a finite number of steps. In Rosenthal's model, a player's strategy consists of a subset of a common set of resources. The payoff received for selecting a particular resource depends only on the total number of players sharing this resource. The utility a player derives from a combination of resources is the sum of the payoffs associated with each resource included in his choice. A slightly different formulation of congestion games was introduced by Milchtaich [5] under the name of congestion games with player-specific payoff functions. Each player has individual non increasing payoff functions and is allowed to choose any resource but must choose exactly one. Milchtaich

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showed that each game in this class admits at least one Nash equilibrium that can be rehashed as a terminal point of a particular improvement dynamic.

A substantial literature has been devoted to particular subclasses and extensions of congestion games. Most of the studies focus on the problem of finding and computing efficiently only one Nash equilibrium, leaving open the question of identifying all Nash equilibria. However, the characterization of the set of all equilibria, beyond its theoretical interest, can be very useful when we have to choose between these equilibria on the basis of performance criteria such as social optimality, or to explore intrinsic proprieties of the game such as the price of anarchy[1]. In this paper, we address this question for a simple subclass of congestion games which lie in the intersection between Rosenthal's and Milchtaich's model. We refer to games in this class as monotone singleton congestion games. Our approach yields a new and short proof establishing the existence of a Nash equilibrium in this kind of congestion games and shows how to compute all equilibria using a simple and direct formula. The rest of this paper is organized as follows: In section 2, we introduce notations and definitions, section 3 reviews related work, section 3 destabilizes the result and section 4 concludes the paper.

2 Definitions and notations

A game (in strategic form) is defined by a tuple $\Gamma = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where $N = \{1, 2, \dots, n\}$ is a set of n players, S_i a finite set of strategies available to player i and $u_i : S = S_1 \times \dots \times S_n \to \mathbb{R}$ is the utility function for player i. The set S is the strategy space of the game, and its elements are the (strategy) profiles. For a profile $\sigma = (\sigma_i)_{i\in N}$ on S, we will use the notation σ_{-i} to stand for the same profile with i's strategy excluded, so that (σ_{-i}, σ_i) forms a complete profile of strategies. A (pure) Nash equilibrium of the game Γ is a profile σ^* such that each σ_i^* is a best-reply strategy: for each player $i \in N$, $u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$, for all $\sigma_i \in S_i$. Thus, no player can benefit from unilaterally deviating from his strategy.

In a (standard) congestion game [8] we are given a finite set $R = \{1, \dots, m\}$ of m resources. A player's strategy is to choose a subset of resources among a family of allowed subsets: $S_i \subseteq 2^R$, for all $i \in N$. A payoff function $d_r : \{1, \dots, m\} \to \mathbb{R}$ is associated with each resource $r \in R$, depending only on the number of players using this resource. For a profile σ and a resource r, the congestion on resource r (i.e. the number of players using r) is defined by $n_r(\sigma) = |\{i \in N : r \in \sigma_i\}|$. The vector $(n_1(\sigma), \dots, n_m(\sigma))$ is the congestion vector corresponding to σ. The utility of player i from playing strategy σ_i in profile σ is given by $u_i(\sigma) = \sum_{r\in\sigma_i} d_r(n_r(\sigma))$. Rosenthal [8] shows that every congestion game possesses at least one Nash equilibrium by considering the exact potential function $P : S \to \mathbb{N}$ with $P(\sigma) = \sum_{r\in R}^{m} \sum_{j=1}^{n_r(\sigma)} d_r(j)$ [2], $\forall\sigma \in S$. A consequence of the existence of an exact potential function is the finite improvement property

[1] When utilities are replaced by costs, the price of anarchy of a game is the ratio of the social cost in the worst Nash equilibrium to the minimum social cost possible.

[2] Rosenthal's potential function shows that congestion games are potential. Monderer and Shapley (1996) proved that every potential game can be represented in a form of a congestion game. Thus, classes of potential games and congestion games coincide. Hence, congestion games are essentially the only class of games for which one can show the

(FIP) (Monderer and Shapley [6]): Any sequence of strategy-profiles in which each strategy-profile differs from the preceding one in only one coordinate and the unique deviator in each step strictly increases his utility (such a sequence is called an improvement path), is finite. Obviously, any maximal improvement path, an improvement path that cannot be extended, terminates by a Nash equilibrium. A slightly different formulation of congestion games was introduced by Milchtaich [5] under the name of congestion games with player-specific payoff functions [3]. Each player i has individual payoff functions $(d^i_r)_{r\in R}$: The payoff function associated with each resource is not common but player-specific. In this sense, theses games are more general than Rosenthal's model. However, this generalization is accompanied by two limiting (restrictive) assumptions. The first restriction is that each player is allowed to choose any resource from R but must choose only one. The second restriction is that, for each player i and each resource r, the specific payoff function d^i_r is a monotonically non increasing function along with the number of players selecting r. Milchtaich shows that games in this class do not generally satisfy the FIP (thus they are not potential games anymore) but that they always possess Nash equilibria. In fact, he showed that the best-replay dynamics may cycle (i.e. the improvement paths in which players iteratively shift to the best-replay strategy do not necessarily lead to an equilibrium). Nevertheless, he also proved that there is always at least one best-replay improvement path that connects an arbitrary initial profile to a Nash equilibrium.

In this study, we are interested in the simple class of monotone symmetric singleton congestion games (singleton congestion games for short) which can be seen as the intersection between Rosenthal's and Milchtaich's model. A game in this class is defined by a tuple $\Gamma(N, R, (d_r)_{r\in R})$, where N is a set of n players, R is a set of m resources/strategies (a player's strategy consists of any single resource in R) and d_r is a non increasing payoff function associated with resource r. The utility of player i for a profile σ is simply given by $u_i(\sigma) = d_{\sigma_i}(n_{\sigma_i}(\sigma))$. Note that these games are symmetric : players share the same strategy set ($S_i = R$, for all $i \in N$) and the same utility function. Since the utility of an anonymous player derived from selecting a single resource depends only on the number of the players doing the same choice, the common utility function is simply a mapping: $u : R \times \{1, \dots, n\} \to \mathbb{R}$, $(r, k) \mapsto u(r, k)$, where u decreases with k.

3 Related work

Since singleton congestion games are a special case of standard congestion games (with the restrictions cited above), the existence of a Nash equilibrium is guaranteed by Rosenthal's potential function. This class of games has been initially studied by Milchtaich [5] as the symmetric case of his model. Without invoking a potential function, he showed that, unlike general (nonsymmetric) congestion games with specific-payoff functions, monotone singleton congestion games possess the finite improvement property. It follows from this proof that best-replay dynamics always converge to an equilibrium in a polynomial number of steps. Ieong et *al.* [3]

existence of pure equilibria with an exact potential function.

[3]This class of games was also investigated, independently and under different names by Quint and Shubik [7] and by Konishi et *al.* [4]).

generalized this result to the largest class of singleton congestion games (where the payoff functions are not required to be monotone). They also showed that even optimal equilibria (Nash equilibria that maximize the sum of players utilities) can be found in a polynomial time. Holzman and Law-Yone [2] and Voorneveld et *al.* [10] investigated the set of strong Nash equilibria [4] in monotone singleton congestion games. It turns out that this set coincides with the set of Nash equilibria and with the set of profiles which maximize the potential. Variants of (monotone) singleton congestion games have been studied in terms of time convergence of the best-reply dynamics to a Nash equilibrium (Even-Dar et *al.* [1]) and in terms of the existence of alternative concept of solution (Rozenfeld and Tennenholtz, [9]).

4 The result

The main drawback of Rosenthal and Milchtaich methods is that they give only one equilibrium, ignoring the general structure of the set of all Nash equilibria. In what follows, we propose to improve the study of singleton congestion games by providing a simple formula describing all these equilibria. In order to state our result, we need first to simplify the analysis by moving to the ordinal representation of preferences. Indeed, in the case of singleton congestion games, we can, without affecting the set of Nash equilibria, replace the values of the payment functions by their ranks in a preference ordering representing the common utility function. More formally, a singleton congestion game will be represented by a tuple $\Gamma(N, R, \precsim)$ where N is a set of n players, R a set of m resources and $\precsim$ a weak ordering on $R \times \{1, \ldots, n\}$. In this ordinal context, a strategy profile σ^* is a Nash equilibrium of the game Γ if $\sigma^* \succsim (\sigma_i, \sigma^*_{-i})$ for all σ_i in R. We also note that, since players are anonymous, all strategy profiles that differ only by a permutation of players can be identified with the corresponding congestion vector. We refer to a congestion vector $\sigma^* = (n_1, \ldots, n_m)$ as a Nash equilibrium if, for all r, r' in R with $r \neq r'$, we have $(r, n_r) \succsim (r', n_{r'} + 1)$. Thus, no player can benefit from joining a group of players sharing a different resource.

Our result is based on the following notion

Definition 1 *Let $\precsim$ be a (weak) ordering on $R \times \{1, \ldots, n\}$. An n-sequence derived from $\precsim$ is a subset T of $R \times \{1, \ldots, n\}$ such that:*

- $|T| = n$.
- $((r, k) \in T$ *and* $(r', k') \notin T) \Rightarrow (r, k) \succsim (r', k')$.
- $(r, k) \in T \Rightarrow ((r, k') \in T, \forall k' < k)$.

Thus, an $n-$sequence is simply a set of the most preferred n elements of $R \times \{1, \ldots, n\}$. To illustrate this definition, let's consider the two following situations.

[4] A strong Nash equilibrium is a profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players are expected to maintain their equilibrium strategies.

Example 1

- *Let* $N = \{1, 2, 3, 4, 5\}$ *and* $R = \{a, b, c\}$*. For simplicity, we will denote the couple* (r, k) *by* rk*. Suppose that the common ordinal utility function is given by the following strictly decreasing ordering:*

$$5c \prec 4c \prec 5a \prec 5b \prec 4b \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec \underbrace{3c \prec a \prec 2c \prec c \prec b}.$$

By definition 1, the unique 5-sequence is $T = \{3c, a, 2c, c, b\}$*.*

- *Let* $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *and* $R = \{a, b, c, d\}$*. Suppose that the players' preferences are given by following weak ordering:*
$8c \sim 8b \prec 8a \sim 8d \prec 7c \sim 7b \sim 6c \prec 7d \sim 5c \sim 4c \prec 3c \sim 6b \sim 6d \prec 5d \sim 5b \sim 4b \sim 7a \prec 5a \sim 4d \sim 6a \prec 4a \sim 3a \sim 3b \sim 2b \sim 3d \prec 2a \sim b \sim 2c \prec c \sim 2d \prec a \sim d$.
We have exactly three 8-sequences: $T_1 = \{3d, 2a, b, 2c, c, 2d, a, d\}$, $T_2 = \{2b, 2a, b, 2c, c, 2d, a, d\}$ *and* $T_3 = \{3a, 2a, b, 2c, c, 2d, a, d\}$*.*

We can now formulate our result.

Theorem 1 *. Let* $\Gamma(N, R, \precsim)$ *be a monotone symmetric singleton congestion game, with* $|N| = n$ *et* $|R| = m$*. Then,*

1. *There is a unique Nash equilibrium per* n*-sequence. Let* T *be an* n*-sequence of* $\precsim$*. The corresponding Nash equilibrium is defined by:* $\sigma = ((1, \alpha_1), \ldots, (m, \alpha_m))$*, where* α_j *is the greater integer* p *satisfying* $(r_j, p) \in T$*.*
2. *When the players' preferences are expressed by a strictly decreasing ordering, the game* Γ *admits exactly one Nash equilibrium.*
3. *The number of Nash equilibria of the game* Γ *equals the number of all* n*-sequences extracted from* $\precsim$*.*

Proof. Since the second and the third point are simple consequences of the first assertion, we have just the following statement.Let T be an n-sequence and let $\sigma^* = ((1, \alpha_1), \ldots, (m, \alpha_m))$ be the $m-$ components vector defined by: $\alpha_r = max\{p : (r, p) \in T\}$. By definition of T and σ^*, we have $\sum_{r=1}^{m} \alpha_r = n$. Indeed, the sequence T consists exclusively of the following terms:

$$(1, \alpha_1), \ldots, (1, 1), (2, \alpha_2), \ldots, (2, 1), \ldots, (m, \alpha_m), \ldots, (m, 1).$$

Therefore, σ^* is a congestion vector. Furthermore, for all r, r' in R, $(r, \alpha_r) \succsim (r', \alpha_{r'} + 1)$ because $(r, \alpha_r) \in T$ and $(r', \alpha_{r'} + 1) \notin T$. Hence, σ^* is a Nash equilibrium. Reciprocally, let $\sigma^* = ((1, \alpha_1), \ldots, (m, \alpha_m))$ be a Nash equilibrium. It is easy to see that $T = \{(1, \alpha_1), \ldots, (1, 1), \ldots, (m, \alpha_m), \ldots, (m, 1)\}$ is an n-sequence. In fact, as σ^* is a congestion vector, we have $\sum_{r=1}^{m} \alpha_r = n$ and so $|T| = n$. On the other hand, by definition of T, $(r, k) \in T \Rightarrow ((r, k') \in T, \forall k' < k)$. Finally, let $(r, k) \in T$ and $(r', k') \notin T$. By definition of T, we have $k \leq \alpha_r$ and $k' \geq \alpha_{r'} + 1$. Since σ^* is a Nash equilibrium, we have $(r, \alpha_r) \succeq (r', \alpha_{r'} + 1)$. By the monotonicity hypothesis, we obtain $(r,k) \succsim (r,\alpha_r) \succsim (r',\alpha_{r'}+1) \succsim (r',k')$. $\square$

To illustrate the above theorem, we continue with the previous example to show how we can easily characterize all Nash equilibria.

Example 2 *Reconsider the two cases of the first example. Applying our theorem, we can easily find a Nash equilibrium for each n-sequence.*

- *Let* $N = \{1,2,3,4,5\}$ *and* $R = \{a,b,c\}$*. Considering the players' ordinal utility :* $5c \prec 4c \prec 5a \prec 5b \prec 4b \prec 4a \prec 3a \prec 3b \prec 2b \prec 2a \prec \underbrace{3c \prec a \prec 2c \prec c \prec b}$*, we obtain* $T = \{b,c,2c,a,3c\}$*. Selecting the greatest integer corresponding to each resource, we identify the unique Nash equilibrium:* $\sigma^* = (a,b,3c)$*.*
- *Let* $N = \{1,2,3,4,5,6,7,8\}$ *and* $R = \{a,b,c,d\}$ *and the weak ordering :* $8c \sim 8b \prec 8a \sim 8d \prec 7c \sim 7b \sim 6c \prec 7d \sim 5c \sim 4c \prec 3c \sim 6b \sim 6d \prec 5d \sim 5b \sim 4b \sim 7a \prec 5a \sim 4d \sim 6a \prec 4a \sim 3a \sim 3b \sim 2b \sim 3d \prec 2a \sim b \sim 2c \prec c \sim 2d \prec a \sim d$*.*
  Here we find one Nash equilibrium per n-sequence:

  $$\text{For } T_1 = \{d,a,2d,c,2c,b,2a,3d\},\ \sigma_1^* = (2a,b,2c,3d);$$
  $$\text{For } T_2 = \{d,a,2d,c,2c,b,2a,2b\},\ \sigma_2^* = (2a,2b,2c,2d);$$
  $$\text{For } T_3 = \{d,a,2d,c,2c,b,2a,3a\},\ \sigma_3^* = (3a,b,2c,2d).$$

  Hence, the are exactly three Nash equilibria in this game.

Note that for the nonsymmetric case, Theorem 1 does not work. The following example illustrates this fact.

Example 3 *Let* $N = \{1,2,3\}$ *and* $R = \{a,b,c\}$*. We consider the following players' ordinal utility:*

$$3a \prec_1 3b \prec_1 2 \prec_1 3c \prec_1 2b \prec_1 a \prec_1 b \prec_1 2c \prec_1 c.$$

$$3c \prec_2 2c \prec_2 3b \prec_2 c \prec_2 2b \prec_2 3a \prec_2 b \prec_2 2a \prec_2 a.$$

$$3c \prec_3 3a \prec_3 2a \prec_3 2c \prec_3 3b \prec_3 c \prec_3 2b \prec_3 b \prec_3 a.$$

The concept of an $n-$sequence does not apply in this case because we have three different utility functions. For player 1, we have the 3-sequence $b \prec_1 2c \prec_1 c$, for player 2: $b \prec_2 2a \prec_2 a$ and for player 3: $2b \prec_3 b \prec_3 a$. Applying Theorem 1 to these sequences, we obtain the following three congestion vectors: $(b,2c)$, $(2a,b)$ and $(a,2b)$. None of these three congestion vectors is appropriate to the three players simultaneously. We can then think about taking the last term of each of the three utility functions. In this way, the strategy profile is (c,a,a). But one can easily check that this profile does not correspond to a Nash equilibrium. Nevertheless, there is a Nash equilibrium (c,a,b).

5 Concluding remarks

In this paper we have proposed a new approach which allows to find all Nash equilibria of a given singleton congestion game. While we do not address the question of the computational complexity, we believe that our formula can contribute to the algorithmic analysis of this class of congestion games. For example, it can help to improve the time complexity of computing optimal Nash equilibria or calculating the price of anarchy. As a future research, we hope to extend our approach to the general case of nonsymmetric congestion games with player-specific payoff functions.

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