Coin Sizes and Payments in Commodity Money Systems[*]

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ABSTRACT _____

Contemporaries, and economic historians, have noted several features of medieval and early modern European monetary systems that are hard to analyze using models of centralized exchange. For example, contemporaries complained of recurrent shortages of small change and argued that an abundance/dearth of money had real effects on exchange. To confront these facts, we build a random matching monetary model with two indivisible coins with different intrinsic values. The model shows that small change shortages can exist in the sense that adding small coins to an economy with only large coins is welfare improving. This effect is amplified by increases in trading opportunities. Further, changes in the quantity of monetary metals affect the real economy and the amount of exchange as well as the optimal denomination size. Finally, the model shows that replacing full-bodied small coins with tokens is not necessarily welfare improving.

1. Introduction

Modern monetary economies rely on fiat money, largely a 20th century innovation. Prior to that time, commodity monies, in a variety of guises, were used to facilitate exchange. This paper takes the view that there is an important distinction between fiat money and commodity money that has not been given sufficient attention. Specifically, with commodity money there is a necessary link between the physical characteristics (e.g., weight and size) of a unit of money and its value. However, with fiat money — because it is intrinsically useless and inconvertible, — there is no such link.

This link between physical characteristics and value created real challenges for monetary authorities operating under a commodity money regime. For one, coins could not be too small or light, because the smaller or lighter the coin, the more likely it was to be physically lost or lost through wear and tear. In addition, coins could not be too large or heavy, because the heavier or larger the coin, the more inconvenient it was to transport or exchange. Further, restrictions on minting technology meant that coins had to be produced in distinct sizes; that is, coins had to be indivisible. That monetary authorities operating under commodity money standards were not always able to meet these denominational challenges is shown by the recurring complaints about shortages of small change that disproportionately hurt the poor.

In this paper, we build a model of a commodity money system that explicitly takes account of these constraints on the monetary authority. We show that introducing and modeling these constraints helps in understanding commodity money systems in general and the monetary policies of medieval Europe and the complaints of contemporaries about that system in particular. Thus, we agree with Mayhew (2004) who states that "any study of the money supply [in medieval Europe] needs to take account not only of the total face value of the currency, but also of the metals and denominations of which it is composed"(p. 82).

Specifically, we set up an economy in which there is an absence of double coincidence of wants and decentralized exchange among anonymous agents. In this economy, a medium of exchange is essential in the sense that it allows the economy to achieve allocations that could not be achieved without it. The role of media of exchange is filled by indivisible coins of different sizes and metals. The indivisibility of coins means that the existence of multiple coins of different sizes (different "denominations") can improve allocations in the sense of delivering higher ex ante welfare than can be achieved with only a single size coin.

Several previous studies have built explicit models of commodity money systems with multiple types of coins. Sargent and Wallace (1983) build an overlapping generations model with two storable goods, gold and silver, which can be transformed into and out of a perishable good at a some cost. They determine the conditions under which there are monetary equilibria (equilibria in which at least one storable good has value). Sargent and Velde (2002) build a model with a large coin, which they call a "dollar," and a small coin, which they call a "penny." Denomination issues are introduced by assuming that one of the two goods in their model can only be purchased with pennies. In other words, their model has a "penny-in-advance" constraint in addition to the usual budget constraint. Velde and Weber (2000) build a model with gold and silver coins in which agents get direct utility from the uncoined stocks of these metals that they hold.

These models are subject to several criticisms about their abilities to analyze the problems with commodity money systems. The Sargent and Velde (2002) and Velde and

Weber (2000) models are subject to the standard criticism of all cash-in-advance models: that the market incompleteness that gives rise to the need for a medium of exchange is simply assumed. It does not arise from fundamentals such as preferences or technologies. And all three models are subject to the criticism that even though coins exist in these models, they are perfectly divisible. Therefore, these models seem inadequate to address denomination issues, which are essentially issues of indivisibility. Finally, they are stand-in agent models, so that distributional effects of different coin denominational structures cannot be analyzed in them.

Other studies of commodity money systems do not use the cash-in-advance framework but rather use a decentralized bilateral matching framework similar to that in this paper. These are the papers by Velde, Weber, and Wright (1999) and Bignon and Dutu (2007). However, these papers cannot fully analyze the issues with commodity money systems because they impose a unit upper bound on money holdings and only permit exchanges of coins for goods. They do not permit exchanges of coins for goods plus coins. Thus, the denomination structure of the coinage, in the sense of the value of coins that can be offered for purchases, is extremely restricted.

In our model, we expand the denomination structure of the coinage in several ways. First, our model has two different coins with different intrinsic values. In this way, we capture the historical reality that for most of the last millennium there were bimetallic monetary systems in the West. We generate a demand for large value coins by introducing a cost to carrying coins that is monotonically increasing (we assume linear) in the number of coins that an agent carries.

Second, the model permits exchanges of coins for goods plus coins; that is, change-giving is permitted in our model. This gives our model another attractive feature. In it, the quantity of small coins affects both buyers and sellers. That is, in our model, the terms of trade between buyers and sellers, and in some cases the ability to trade, depends upon the coin portfolios of both, not just the portfolio of buyers as was the case in, for example, the Sargent and Velde (2002) analysis. This dependence of trade on the portfolios of both buyers and sellers arises from the indivisibility of coins, which we regard as a major contributor to the problems with commodity money systems.

Finally, building on the work of Lee, Wallace, and Zhu (2005), we allow agents to hold multiple numbers of each of the two coins. The result is that in the equilibria of our model, there is a distribution of poor agents who hold few coins and rich agents who hold many coins or coins with the higher intrinsic value. This heterogeneity is important in the analysis for reasons we discuss below.

The paper proceeds as follows. In section 2, we discuss some stylized facts about commodity money regimes that we want our model to confront. We draw heavily upon the experience of England for these facts. In section 3, we present our model. In section 4, we present the implications of the model and relate them to the stylized facts. The final section concludes.

2. Commodity Money Systems

In medieval and early modern Europe, the money stock was comprised of coins of silver and (from the 14th century) gold. Two features of the monetary system of this period that have attracted the attention of economic historians are the impact of changes in the

quantity of money in aggregate and the quantity of small denomination coins. In each case there are discussions of both scarcity and abundance of coins and the impact of the quantities of coins on output and welfare.

A. The quantity of money

Writ very large, the records speak of an abundance of silver in the 13th century and a dearth of silver and gold in the mid-15th century. The mint output data in Figure 1 support this. Metcalf (1977) speaks of England being "awash" in bullion from 1180 to 1300. Mayhew (2004) estimates that while annual per capita real income in England was essentially unchanged (at 43d.) between 1086 and 1300, the real money stock per capita doubled from 4d. to 8d.[1]
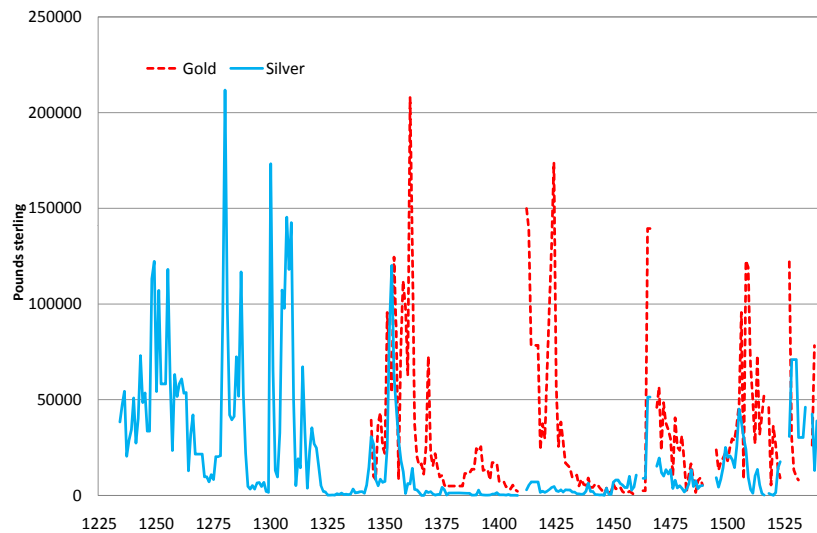
Source: Challis (1992)

Figure 1: Mint output in medieval/early modern England

In contrast to the descriptions of plentiful money stocks in the 13th century, the 15th century, and especially the 1440s and 1450s, are portrayed as a period of scant money. The existing silver mines in Europe had been played out, and no new mines opened until the 1460s. Spufford (1989) notes that many European mints closed for 20 years in the middle of the 14th century, and concludes that the "silver-famine of the late 14th and 15th centuries

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[1]By the 13th century, coins were used throughout England not only for large transactions such as taxes, rents, and wages, but also for small purchases, such as beer and bread. Metcalf (1977), for example, comments that "throughout the thirteenth century the use of coinage was in fact pervasive, not just in towns and ports and among the merchant class, but in villages throughout the land" (p. 13). Similarly, Mayhew (1999) notes that "money had already become a central consideration in the lives of even the poorest by the beginning of the thirteenth century" (p. 16).

appears more severe than anything that had taken place since the seventh century."(p. 340) Figure 1 shows the low level of English mint output from 1380 to 1460, excluding the recoinages of 1412 - 14 and 1421. Notice too that the mint coined about 100 tonnes of silver in the recoinage of 1278 - 80, but only 2 tonnes of silver and gold coin equivalent in value to 70 tonnes of silver during the recoinage of 1412-14.[2] Challis (1992) emphasizes that this low English mint output must be seen in the context of "the great bullion famine which afflicted North West Europe in general at this time."(p. 190)

If we take as exogenous the broad changes in the quantity of monetary metals, the interesting question is, what would have been the consequences? The quantity theory answer is clear: an exogenous change in the quantity of money will, perhaps after a transitory short-term impact on output, raise prices and leave output unchanged. Yet, historians have seen a different outcome. Mayhew (1999)'s picture of the 13th century is the clearest: "Put simply, buying and selling became easier when there was more coin about"(p. 13). Put alternatively, an increase in the money stock led to an increase in output and welfare as well as prices. Our model generates such a result.

B. The scarcity of small change

From the time of Charlemagne to approximately 1200, the silver penny was the only coin minted in Western Europe, although the silver content of the penny varied across countries and even cities. Sargent and Velde (2002) note the contrast between Venice, where the penny contained .05 g of silver, and England, where it contained 1.3 g of silver (p. 45). The English penny was both small in size (a US dime weighs about 2.3 g) and large for retail payments. Bread cost a farthing (a quarter pence) a loaf, with the size of the loaf being set by the Assizes in each city depending on the price of corn. Daily wages for a labourer were about 1d. per day. Not surprisingly, historians such as Britnell (2004) considered the penny "an inconveniently large unit for retail trade"(p. 24).

The need for lower denomination coins led to the introduction of halfpennies and farthings beginning in 1279. Britnell (2004) concludes that by 1300, "the monetary system could suit the needs of small households better than a hundred years before"(p. 24).[3]

Yet, permission to strike small denomination coins did not necessarily imply that they were produced. Data on mint output by denomination are available only for the period 1280 - 1351 (see Figure 2), but these show the generally low level of small denomination coin production. Regulations varied slightly over time, but broadly speaking, the mint workers had no incentive to produce halfpennies and farthings, which required double or quadruple the effort.

The primary mechanism for obtaining small denomination coins appears to have been complaining of the injury done by the lack of small change sufficiently aggressively that a quota was imposed on the mint workers. For example, Ruding (1840) describes a petition

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[2]Sussman (1998) has pointed out the weakness of the case of those who argue that the bullion famine and the accompanying price decline reflected a drain of specie to the East. He argues that in the early 15th century, the cause of low mint output was lack of demand; however, he is agnostic about the low mint output of the mid-15th century.

[3]The groat — containing four times as much silver as the penny — was also created under the mint indenture of 1279, but does not seem to have been minted until the mid-14th century when gold coins were also introduced. The smallest gold coin was worth approximately 80 pence.

from the late 1370s stating that

> certain weights for bread, and measures for beer, such as gallon, pottle, and quart, were ordained by statute, and that they the said Commons had no small money to pay for the smaller measures, which was greatly injurious to them. (p. 237)

Ruding (1840) also mentions similar small change shortages in 1380, 1393, 1402, and 1421.
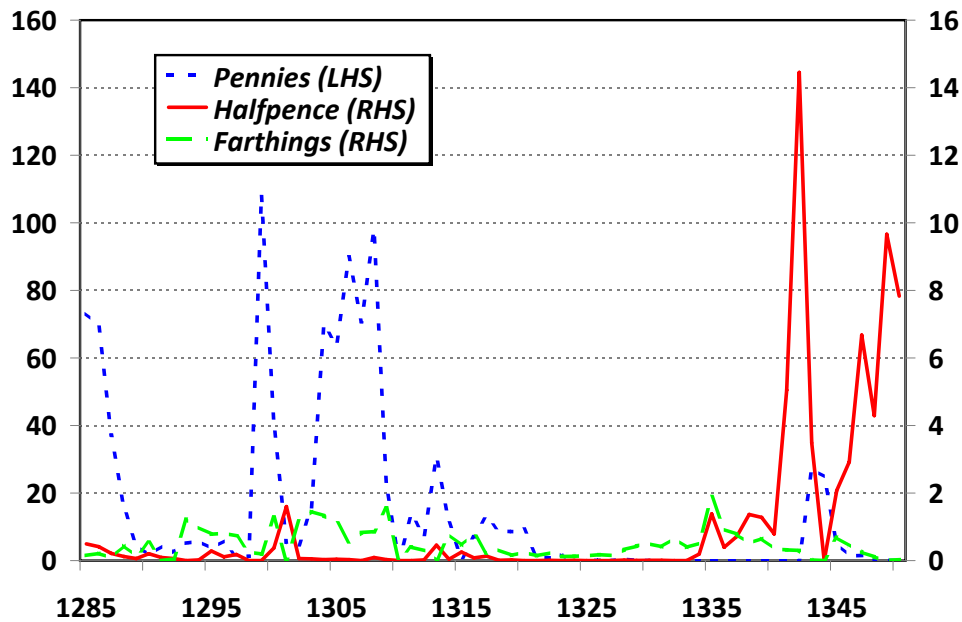
Source: Challis (1992)

Figure 2: London mint: Silver coin output (thousands of pounds)

One of the common complaints encountered in the literature is that of a scarcity of small change that affected both buyers and sellers. These complaints were almost always accompanied by the claim that such shortages disproportionately hurt the poor — for example, this excerpt from a petition to the king asking for issues of halfpence and farthings in February 1444 - 45:

> Men traveling over countries, for part of their expenses of necessity must depart our sovereign lord's coin, that is to wit, a penny in two pieces, or else forego all the same penny, for the payment of a half penny; and also the poor common retailers of victuals, and of other needful things, for default of such coin of half pennies and farthings, oftentimes may not sell their said victuals and things, and many of our said sovereign lord's poor liege people, which would buy such victuals and other small things necessary, may not buy them, for default of half pennies and farthings not had on the part of the buyer nor on the part of the seller; which scarcity of half pennies and farthings, has fallen, and daily yet does,

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because that for their great weight, and the fineness of allay, they be daily tried and molten, and put into other use, unto the increase of winning of them that do so. (from Ruding (1840) p. 275)

This petition was granted, and the king ordered the mint to coin halfpennies and farthings at 33 shillings per pound rather than 30 shillings per pound for two years. The increased revenue was shared between seignorage, the mint workers, and the bullion sellers, and the legal tender of the coins was restricted to small transactions. (Mayhew (1992),p. 110; Craig (1953),p. 87).

In 1560, after the Great Debasement, Queen Elizabeth "restored" the silver coinage to its traditional sterling fineness, but she did not restore the weight to its 1540 level. The penny now weighed 0.5 g, and the mint, concluding that halfpennies and farthings would be too small to handle, instead produced coins valued at $\frac{3}{2}$d. and $\frac{3}{4}$d. which would permit payments of a farthing (Craig (1953),p. 123). The scarcity of small denomination coins continued to cause complaints. In 1599 the mint workers were given 1d. (financed by a reduction in seignorage) to produce 1.5% of silver output in silver denomination coins. This effort seems to have been too little to quell complaints and too little to pay for the costs; in the 1660s the money was withheld from the mint for lack of performance, and the mint was required to produce £125 in change "for the more convenient change of our people" (Craig (1953), p. 159).

The challenge of minting small denomination coins continued until the mint authorized private copper coins in the late 18th century (see Sargent and Velde (2002) and Redish (2000)).

C. Token coins

By "token coins", we mean coins either that are silver but contain less metal than the penny or that are made of a base metal such as copper or lead. Such coins seemed to come into frequent use during the medieval period. The literature is full of references of the attempts of the English sovereigns to prevent the importation and circulation of foreign token or billon coins. However, some private English tokens were permitted to be used at least once. As Ruding (1840) states, "At an early period of his [Henry VIII's] reign, or about the conclusion of his father's, private tokens were used to supply the want of silver coins" (p. 301). We would like our model to deliver the implication that in an economy without an intrinsically valuable small denomination money, the existence of tokens — that is, coins with no intrinsic value — can lead to higher welfare.

We will also use our model to examine a claim of Elizabeth I. In justifying an edict to reduce the values of base monies, she proclaimed:

First of all it is known that the honour and reputation of the singular wealth that this realm was wont to have above all other realms, was partly in that it had no current monies but gold and silver, whereas contrary all other countries . . . have had, and still have, certain base monies now of late days, by turning of fine monies into base, much decayed and daily grown into infamy and reproach, and therefore is thought necessary to be recovered. (from Ruding (1840) p. 334)

To examine whether Elizabeth's claim is correct, we will use our model to determine the welfare effects of replacing a given supply of token money with the same quantity, in terms

of numbers of coins, of intrinsically valuable small denomination money. In other words, we will examine the effects of replacing a given supply of token coins with the equivalent number of intrinsically valuable small denomination coins.

3. The Model

A. Environment

The model has discrete time and an infinite number of periods. There are two non-storable and perfectly divisible goods: a special consumption good and a general consumption good.

In addition, there are two metals (durable commodities) — silver and gold — in the economy. There are m_s ounces of silver and m_g ounces of gold in existence. Each ounce of these metals gives off one unit of the general consumption good at the beginning of each period.[4] These metals are divisible, but not infinitely so. We will refer to the objects into which these metals are divided as *coins*. Thus, there can be many silver and many gold coins in the economy, but not infinitely many. In other words, coins are indivisible monies, and the supply of each type of coin must be finite. The assumption that metals generate dividends is our way of modeling commodity money.

The monetary authority in this environment chooses how many ounces of a metal to put into a coin of that metal. We let b_s be the ounces of silver that it puts in a silver coin and b_g be the number of ounces of gold that it puts in a gold coin. Thus, a silver coin yields a dividend of b_s units of the general consumption good per coin to the holder at the beginning of a period, and a gold coin yields a dividend of b_g units of the general consumption good per coin to the holder at the beginning of a period. The total supplies of the two types of coins are $M_s = m_s/b_s$ and $M_g = m_g/b_g$, respectively.

As was the case with coins throughout most of the time during which commodity monies were used, these gold and silver coins do not have denominations. They are simply amounts of the two metals that have been turned into coins with some type of standardized markings that allow one type of coin to be easily differentiated from a different type of coin. To capture the fact that historically silver coins were less valuable than gold coins, we assume that for technological reasons $b_s < b_g$, silver coins must be less valuable than gold coins in the sense of yielding a lower dividend per coin.[5]

Letting s and g be an agent's holdings of silver and gold coins, respectively, an agent's portfolio of coin holdings is
$$y = \{(s, g) : s \in \mathbb{N}, g \in \mathbb{N}\}.$$
Let $\mathbf{Y} = \mathbb{N} \bigotimes \mathbb{N}$ be the set of all possible portfolios.

There is a $[0, 1]$ continuum of infinitely lived agents in the model. Let q denote the quantity of special good. We assume the agent's preferences are

$$u(q) - q + sb_s + gb_g - \gamma(s + g)$$

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[4]Instead of viewing silver and gold as metals, they could be viewed as two different kinds of Lucas trees.

[5]It is not critical to the analysis that the coins be of different metals. Both coins could be gold or both could be silver. What is important is that the two coins have different intrinsic values (different b_j).

with $u(0) = 0$, $u' > 0$, $u'' < 0$, and $u'(0) = \infty$. The disutility of special good production is assumed to be linear without loss of generality. The term $sb_s + gb_g$ is the utility the agent gets from general goods received by holding coins, and γ is the utility cost, also in terms of general goods, that the agent suffers for each coin held coming into a period.[6]

At the beginning of each period, an agent has a probability $\frac{1}{2}$ of being a consumer but not a producer and the same probability of being a producer but not a consumer. This assumption rules out double coincidence matches, and therefore gives rise to the essentiality of a medium of exchange.

After agents' types (consumer or producer) are revealed, a fraction $\theta \in (0, 1]$ of agents are matched bilaterally. In a match, the coin portfolio of both agents is known. However, past trading histories are private information and agents are anonymous. These assumptions rule out gift-giving equilibria and the use of credit. Thus, trading can only occur through the use of media of exchange, which is the role that the gold and silver coins can play.

B. Consumer choices

We assume that in a single coincidence pairwise meeting, the potential consumer gets to make a take-it-or-leave-it (TIOLI) offer to the potential seller. This offer will be the triple (q, p_s, p_g), where $q \in \Re_+$ is the quantity of production demanded, $p_s \in \mathbb{Z}$ is the quantity of silver coins offered, and $p_g \in \mathbb{Z}$ is the quantity of gold coins offered. Offers with $p_s < 0$ or $p_g < 0$ can be thought of as the seller being asked to make change.

Let $w(y) : \mathbb{N} \bigotimes \mathbb{N} \to \Re_+$ be the expected value of an agent's beginning of period portfolio of coin holdings. The set of feasible TIOLI offers is a combination of special good output and coins that is a feasible coin offer and satisfies the condition that the seller be no worse off than not trading. Denoting this set by $\Gamma(y, \tilde{y}, w)$,

$$\Gamma(y, \tilde{y}) = \{\sigma : q \in \Re_+, -\tilde{s} \leq p_s \leq s, -\tilde{g} \leq p_g \leq g,$$
$$-q + \beta w(\tilde{s} + p_s, \tilde{g} + p_g) \geq \beta w(\tilde{y})\},$$

where $\tilde{y}$ denotes the seller's portfolio. The arguments of σ are (q, p_s, p_g).

C. Equilibrium

We will consider only steady state equilibria in this section. The three components needed are the value functions (Bellman equations), the asset transition equations, and the market clearing conditions. We proceed to describe each in turn.

Value functions

The steady state value functions are

$$w(y) = \frac{\theta}{2}\sum_{\tilde{y}}\pi(\tilde{y})\max_{(q,p_s,p_g)\in\Gamma}[u(q) + \beta w(s - p_s, g - p_g)] + (1 - \frac{\theta}{2})\beta w(y)$$
$$+ gb_g + sb_s - \gamma(s + g),$$

where $\pi(\tilde{y})$ is the fraction of agents holding $\tilde{y}$. The first term on the right-hand side is the expected payoff from being a buyer in a single coincidence meeting, which occurs with probability $\frac{\theta}{2}$. The second term is the expected payoff either from being the seller in a single coincidence meeting or from not being in a meeting. The final terms are the net utility from holding silver and gold coins.

Asset holdings

Define $\lambda(k, k'; y, \tilde{y})$ to be the probability that a buyer with portfolio y in a meeting with a seller with portfolio $\tilde{y}$ leaves the meeting with k silver coins and k' gold coins. That is,

$$\lambda(k, k'; y, \tilde{y}) = \begin{cases} 1 & \text{if } k = s - p_s(y, \tilde{y}) \text{ and } k' = g - p_g(y, \tilde{y}) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then the asset transitions are

$$\pi_{t+1}(k, k') = \frac{\theta}{2}\left\{\sum_{y,\tilde{y}}\pi_t(y)\pi_t(\tilde{y})[\lambda(k, k'; \cdot) + \lambda(s + \tilde{s} - k, g + \tilde{g} - k'; \cdot)]\right\}$$

$$+ (1 - \theta)\pi_t(k, k').$$

The first term on the right-hand side is the fraction of single coincidence meetings in which the buyer leaves with k silver coins and k' gold coins. The second term is the fraction of such meetings in which the seller leaves with k silver coins and k' gold coins. The final term is the probability that no meeting occurs, in which case no coins change hands.

Of course, asset holdings must also satisfy $\sum_y \pi(y) = 1$.

Market clearing

The market clearing conditions are that the stocks of gold and silver coins must be held. That is,

$$\sum_y s\pi(y) = M_s$$

$$\sum_y g\pi(y) = M_g.$$

Definition 1. *Steady state equilibrium:* *A steady state equilibrium is a (w, π, Γ) that satisfies the value functions, the asset transition equations, $\sum_y \pi(y) = 1$, and market clearing.*

4. Results

We are unable to prove the existence of steady state equilibria or to obtain analytic results for our model. Therefore, we rely instead on computed equilibria for numerical examples to obtain our results. Specifically, we assume $u(q) = \sqrt{(q)}, \beta = 0.9, \theta = \frac{2}{3}, m_g = 0.05$, and $m_s = 0.03$. For most of our examples we will assume a gold coin size of $b_g = 0.05$, which implies a per capita supply of gold coins $M_g = 1$. Because we are interested in small change, we will consider various values for b_s. Note that because we assume the quantity of silver is fixed, changing the size of the silver coin also changes the number of coins in existence. Further, even though the theoretical model assumes no upper bound on coin holdings, we impose upper bounds $S = 40$ on silver coin holdings and $G = 2$ on gold coin holdings.[7] Throughout the analysis, our welfare criterion is ex ante welfare, computed as $\bar{w} = \sum_y \pi(y)w(y)$.
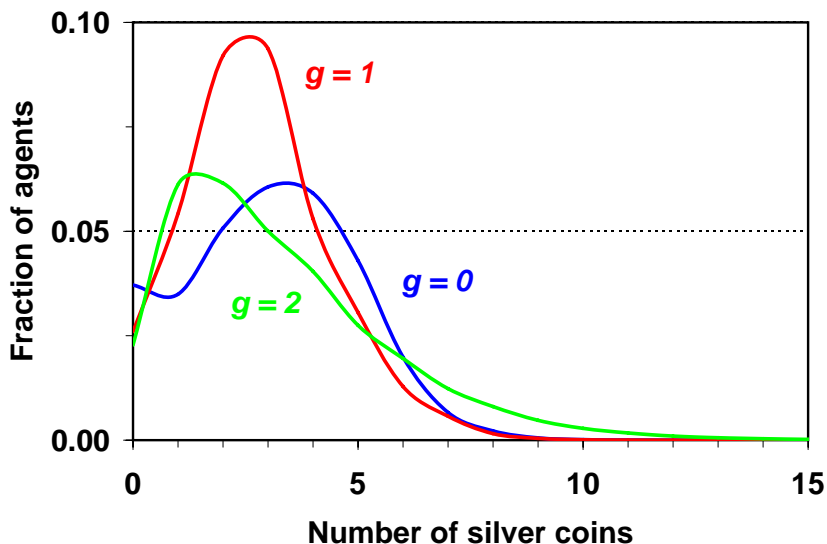

Figure 3: Distribution of coin holdings

The distribution of portfolios in a sample steady state where $b_s = .01$ and $\gamma = .001$ is shown in Figure 3. Approximately 4% of the agents hold zero coins of either type, while the most widely held portfolio is three silver coins and one gold coin, which is held by 9.5% of the agents. The ex ante welfare $\bar{w}$ is 1.50, which represents a loss relative to the welfare a planner would achieve, which is 1.63.[8] Only a small part of this loss $(\frac{\gamma}{1-\beta}(M_g + M_s) = 0.04)$ arises from the costs of carrying coins. By far the largest part is due to the fact that the output produced in the individual matches differs from the efficient quantity, q^*. Although

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[7]For several cases we also computed equilibria assuming $S = 30$ and $G = 3$. These equilibria did not differ much from the equilibria reported in the text.

[8]We assume that the planner would have the efficient quantity q^* produced in every meeting and that no

we find that the amount produced is too small in most cases, there are matches in which a quantity greater than q^* is produced.

A. Small change shortages

Definition 2. *Small change shortage:* *A small change shortage occurs in an economy when adding small (silver) coins to an economy with only large (gold) coins is welfare improving.*

Here we show that small change shortages could occur. We also examine whether there is an optimal (in terms of ex ante welfare) ratio between the sizes of the coins and whether changes in the trading opportunities available to agents in an economy (think: development of organized markets) affect the optimal size of silver coins. Further, we examine how the poor are affected by the introduction of small coins, and how their welfare changes as the size of small coins changes.

Our model's results concerning whether the introduction of small coins improves welfare and whether there is an optimal ratio between large and small coins are given in Figure 4. There we plot as a function of the size of a silver coin the ex ante welfare of the agents in an economy with both gold and silver coins over that if only gold coins (or only pennies in the case of England) were in existence.[9]

The figure shows that small change shortages can occur under a commodity money system. The addition of silver coins to gold coins (or halfpennies or farthings to pennies in the case of England) improves ex ante welfare.[10] Both lines in the figure lie above zero for at least some sizes of small coins.

Figure 4 also shows that if silver coins exist in the economy, decreasing the size of the silver coin can improve welfare. Intuitively, this occurs because the smaller the silver coin, the more precisely the buyer can adjust the TIOLI offer made to the seller. However, the figure also shows that this is only the case when the silver coins are not too small. After some point, ex ante welfare falls as the size of the silver coin is decreased. (The dotted lines mark the silver coin size that yields maximum ex ante welfare for each of the γ's.) Further, the fact that the lines intersect the x-axis for $b_s > 0$ shows that after some point silver coins can become too small. This occurs when there are so many of them in existence that the cost of holding them exceeds their intrinsic value plus their value in trade. In such a case, agents would prefer to throw them away rather than hold them. Not surprisingly, the minimum size for a small coin such that agents are willing to hold it increases as the cost of holding coins increases, as shown by the fact that the dashed line intersects the x-axis to the right of the solid one.[11]

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coins would be held. Under these assumptions, the planner's welfare is

$$w* = \frac{1}{1-\beta}\{\frac{\theta}{2}[u(q*) - q*] + m_s + m_g\}.$$

[9]For the gold coins only case, we assume that agents still get per capita dividends of m_s, so that the change in welfare is due strictly to the addition of silver coins, not to the addition of silver per se.

[10]We also checked whether there could be "large denomination shortages" in the sense that adding gold coins to an economy with only silver coins would improve welfare and found that it did.

[11]For $\gamma = 0.01$, the dashed line intersects the x-axis for $b_s < 0.01$. Thus, in this case it is welfare improving
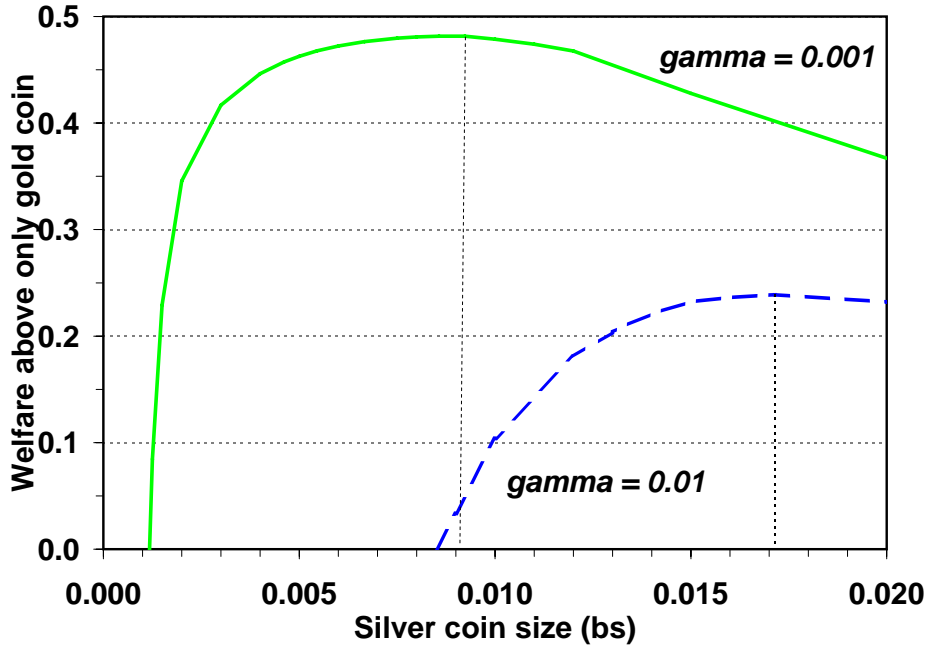
Figure 4: Welfare from the addition of small coins of various sizes for different γ

Because ex ante welfare is not always decreasing in coin size, there is an optimal size ratio between the two coins. The figure shows that this ratio decreases as the cost of holding a coin increases.

Figure 5 shows the fraction of single coincidence matches by the type of payment made in the match.[12] The figure shows that the largest fraction of matches are ones in which trade is done solely with silver coins. This fraction is approximately $\frac{3}{4}$ of all matches except when silver coins became large, when it drops to about $\frac{3}{5}$. The next largest fraction of matches are ones in which silver coins are given as change for gold coins. This is almost $\frac{1}{4}$ of all matches when silver coins are small, and this fraction also falls as silver coins are made larger. The figure also shows that matches in which trade is done solely with gold coins are infrequent and only happen when silver coins are large. There are no matches in which both gold and silver coins are offered in trade. There is an increasing fraction of matches in which even though trades are possible, no trade takes place (the dark area at the top of the figure). Finally, there is an increasing fraction of matches (the white area at the top of the figure) in which trade cannot take place because potential buyers have no coins.

In the model, θ is the probability of a match, so changing θ can be interpreted as

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to have silver coins in addition to gold coins even if the cost of holding a silver coin is greater than its intrinsic value. This is due to the fact that silver coins are useful in trade. However, this is not true when $\gamma = 0.001$. There the intrinsic value of the silver coin must be greater than the cost of holding it in order for the addition of silver coins to be welfare improving.

[12]To compute these fractions we calculated $\sum_{y,\tilde{y}} \pi(y)\pi(\tilde{y})I(y,\tilde{y})$ where $I(y,\tilde{y})$ is an indicator function that takes on the value of 1 if trade of a particular type occurs in a match between an agent holding y and an agent holding $(\tilde{y})$.
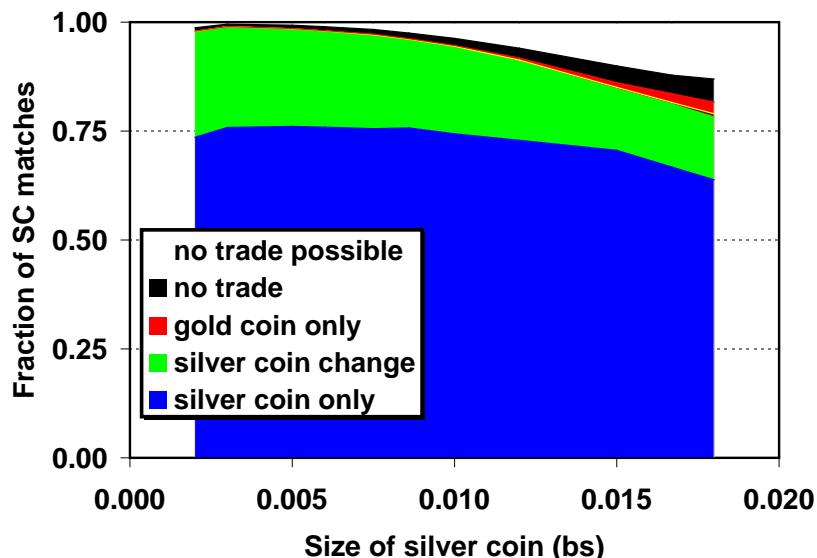
Figure 5: Fractions of matches by type of trade and size of silver coin, $\gamma = 0.001$

changing the trading opportunities available to agents in the economy. We now examine how changes in trading opportunities available to agents in an economy affect the optimal size of silver coins. To do so, we compute ex ante welfare for $\theta = \{\frac{1}{5}, \frac{1}{3}, \frac{2}{3}\}$ for various sizes of silver coins, holding the size of the gold coin fixed. Throughout we assume that $\gamma = 0.001$.

The results are shown in Figure 6. When $\theta = \frac{1}{5}$, that is, when an agent has a one-in-five chance of having a single coincidence match in any period, the ex ante welfare maximizing size of the silver coin is $b_s = 0.015$, which, given that we have assumed a gold coin size of $b_g = 0.05$, gives an optimal ratio of $3\frac{1}{3} : 1$.[13] When $\theta = \frac{1}{3}$, the figure shows that the optimal size of the silver coin falls to $b_s = 0.011$, or an optimal ratio of approximately $4\frac{1}{2} : 1$. And when $\theta = \frac{2}{3}$, the case studied above, the optimal coin size is $b_s = 0.0092$, for an optimal coinage ratio of approximately $5\frac{1}{2} : 1$. Thus, our model indicates that decreasing the size of silver coins would be an optimal response to an increase in a country's trading opportunities.

Intuitively, the smaller the silver coin, the more finely a potential buyer is able to calibrate an offer of wealth (in the form of coins) for goods to a potential seller. The finer the wealth offer that a buyer can make, the less likely it is that the buyer will have to take a smaller quantity of goods from the seller in exchange for a given wealth transfer or have to give up additional wealth to get the desired quantity.

Obviously, buyers would like to have silver coins be infinitely small. However, the benefits of making finer offers have to be traded off against the costs of carrying more coins. When a buyer has a high probability (the $\theta = \frac{2}{3}$ case) of being in potential trade matches,

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[13]We have not made any attempt to try to calibrate our model to any actual coin ratios of the period.
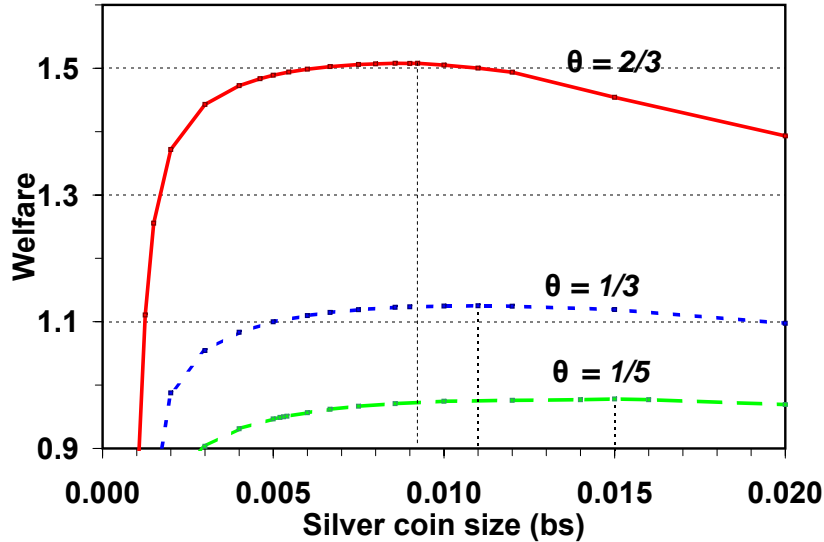
Figure 6: Ex ante welfare as a function of silver coin size for various θ, $\gamma = 0.001$

the benefits of finer offers is high and small silver coins are preferred. But these benefits fall as trade occurs less often, so larger silver coins are preferred because of the lower carrying costs.

Lastly, in this section we examine the claim that the shortage of small denomination coins falls hardest on the poor. We do this in two ways. First, we look at how the welfare of an agent with a given wealth in terms of silver coins changes as the size of the silver coin changes. That is, we calculate how $w(y|\hat{w})$, where $\hat{w} = sb_s$ is some value of the agent's silver coin wealth, varies as the size of the silver coin varies. Because we are concerned with poor agents, we consider only agents who hold no gold coins, which is why gold coin holdings do not appear in $\hat{w}$.[14] Second, since the pattern of trade affects the distribution of agents' coin holdings, we examine how this distribution changes as the size of the silver coin varies.

The effects of changes in small silver coin size on the welfare of agents with small amounts of silver coin wealth are shown in Table 1. The entries in the left column of the table are the metallic content of an agent's silver coin holdings. The entries in the second column are an agent's welfare for a silver coin size of $b_s = 0.01$, which from Figure 4 is close to optimal size of the silver coin. The entries in the third column are for a larger silver coin size $b_s = 0.015$. The entries in the fourth column are for an even larger silver coin size $b_s = 0.02$. Note that the implied coin holdings for a given metallic content of silver coin holdings differ by column. For example, for a metallic content of $\hat{w} = sb_s = 0.06$, an agent is holding 6 coins when $b_s = 0.01$, 4 coins when the metallic content is $b_s = 0.015$, and 3 coins

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[14]Consistent with Mayhew (2004)'s comment cited in the introduction, we find that for similar values of total coin wealth, $sb_s + gb_g$, $w(y)$ is different for different y.

Silver coin	Welfare		
wealth ($\hat{w} = sb_s$)	$b_s = 0.01$	$b_s = 0.015$	$b_s = 0.02$
0	0.00	0.00	0.00
0.01	0.35		
0.02	0.63		0.53
0.03	0.86	0.83	
0.04	1.06		0.98
0.05	1.23		
0.06	1.39	1.43	1.37
0.07	1.53		
0.08	1.67		1.70
0.09	1.80	1.89	
0.1	1.92		2.00

Table 1: Welfare by metallic content of agents' wealth holdings for various coin sizes

when the metallic content is $b_s = 0.02$. This is why there are gaps in the table.

Table 1 shows that small change shortages have a greater effect on the poor. The welfare of agents with small amounts of silver coin wealth (≤ 0.05) increases as the silver coin becomes smaller. After that level of wealth, however, this is no longer always the case, and as an agent's silver coin wealth becomes large enough, the agent would prefer to have larger size silver coins.

Next, we examine how changing the size of the silver coin affects the distribution of wealth. This is shown in Figure 7, which plots the cumulative distribution of agents by their wealth in terms of silver coins (recall that $g = 0$). The figure shows that the fraction of agents with little such wealth falls sharply as the size of the silver coin is decreased. Specifically, when $b_s = 0.02$, approximately 14% of agents have no wealth at all and approximately 25 % have wealth in terms of silver coins of 0.02 or less. When $b_s = 0.015$, the percentages fall to roughly 10 and 22, respectively. And when the size of the small coin falls to $b_s = 0.01$, the percentages are approximately 4 and 12, respectively. Note that the fraction of agents with larger amounts of wealth in silver coins is approximately the same over the three coin sizes and is invariant to coin size. This further shows that variations in the size of small coins have the largest effects on the poor.

B. Quantity of money
**Definition 3. *Monetary neutrality:* ** *Let $\bar{w}(m)$ be ex ante welfare in an economy with metal stocks $m = (m_s, m_g)$, and let $\tilde{w}(m)$ be that value net of the present discounted value of the dividends from metal stocks, that is, $\tilde{w}(m) = \bar{w}(m) - \frac{1}{(1-\beta)}(m_s + m_g)$. Metal stocks (the quantity of money) are neutral if, ceteris paribus, $\tilde{w}(m) = \tilde{w}(\mu m)$ for all μ.*

Consider two economies, one with μ times more of both metals than the other. The quantity of money is neutral if ex ante welfare in the economy with more money equals that in the lower money stock economy, net of the present discounted value of the difference in
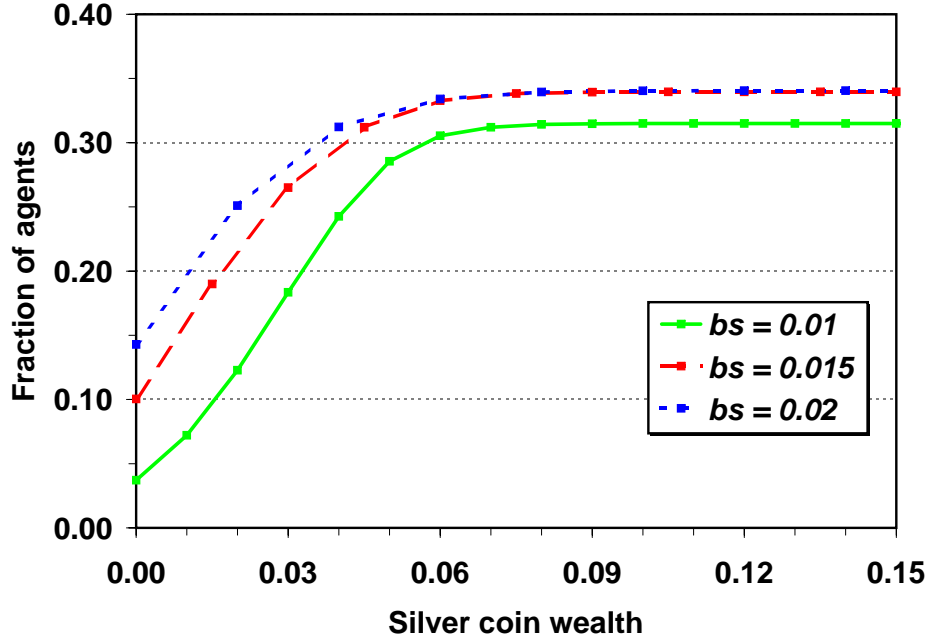
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Figure 7: Fractions of agents with $g = 0$ and various levels of silver coin wealth for various sizes of silver coins, $\gamma = 0.001$

dividends from the different money stocks.

To determine whether or not commodity money is neutral we consider three economies $m = (0.015, 0.025), m' = (0.02, 0.033)$, and $m'' = (0.03, 0.05)$. That is, we compare economies with $\mu = \frac{4}{3}$ and $\mu = 2$ to the original. In Figure 8 we plot $\tilde{w}(m)$ for various sizes of silver coins for each of these economies. If money were neutral, the lines would lie on top of each other. Obviously they do not. Our model implies that commodity money is not neutral.

Varying the stock of monetary metal could affect ex ante welfare through three channels:

1. There is the direct positive wealth effect from having more monetary metals generating more dividends ($\frac{\mu-1}{(1-\beta)}(m_s + m_g)$), which is incorporated into the definition of neutrality.
2. If coin sizes remain unchanged, there is the direct negative cost of carrying more coins, $-\gamma \frac{\mu-1}{(1-\beta)}(M_s + M_g)$.
3. There may be an effect on the distribution of coin holdings and the quantity of output transacted in matches. We will refer to this as the transactions effect.

It is the possibility of these last two effects that can lead to monetary nonneutrality.

As an example, consider the economies m and m''. Table 2 shows that the direct welfare gain from the increased dividends arising from the increased money stock is 0.4. We compute the total welfare gain (given $b_s = .005$) to be larger, 0.424. The difference arises because even though there is a loss due to the costs of carrying the additional coins, it is more than offset by the transactions effect. The proportion of agents that do not trade (because the potential buyer has no coins or cannot make a welfare improving trade) falls from 7.4%
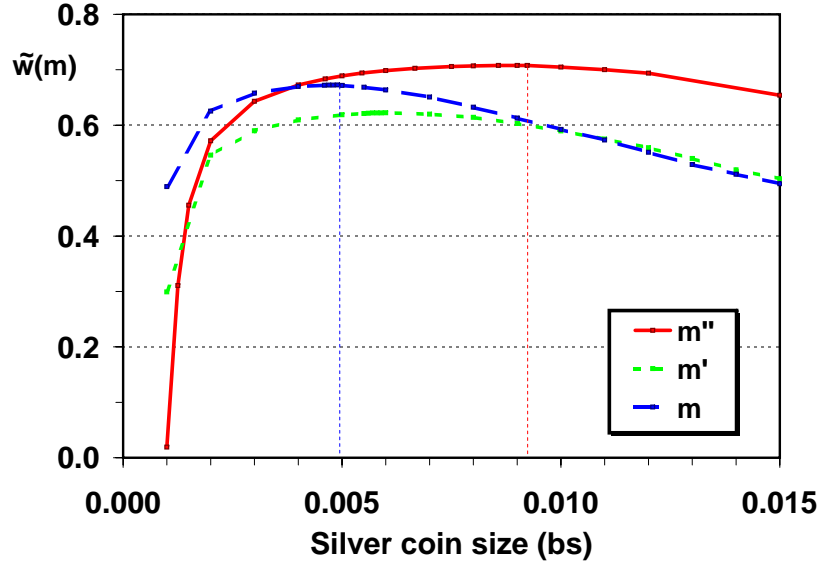
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Figure 8: Ex ante welfare as a function of silver coin size for various levels of metallic wealth, $\gamma = 0.001$

Table 2: Welfare impact of increasing monetary stocks

	Equilibrium		
	$\mu = 1$	$\mu = 2$	$\mu = 2$
		$b_s = .005$	$b_s = .0092$
m_s, m_g	.015,.025	0.03,.05	0.03,.05
Silver coin size b_s	.005	.005	.0092
Number of coins M_s, M_g	3, 0.5	6, 1	3.25, 1
Total welfare $(\bar{w})$	1.072	1.496	1.518
Welfare from metal $\frac{1}{(1-\beta)}(m_s + m_g)$	0.4	0.8	0.8
Welfare net of metal value $(\tilde{w})$	0.672	0.696	0.718
Welfare loss from carrying $\frac{1}{(1-\beta)}\gamma(M_s + M_g)$	(.035)	(.07)	(.0425)
Welfare from transactions	0.707	0.766	0.760
Percentage of agents that do not trade	7.4	1.5	4.4

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to 1.5% capturing Mayhew (1999)s description of the impact of more monetary metal in the 13th century: "buying and selling became easier when there was more coin about."

Of course, there is no reason why the size of coins must remain unchanged when the stock of monetary metals changes. In fact, it may be optimal for the size of coins to be changed in response to save on carrying costs or to facilitate more transactions. Figure 8 shows that it is in fact optimal to increase the size of silver coins in economies with proportionally more quantities of the two metals. Specifically, in the economy with m, the optimal size of the silver coin is $b_s = 0.005$. However, in the economy with m'', the optimal size of the silver coin is $b_s = 0.0092$. Table 2 shows that the increase in coin size causes a reduction in the welfare from transactions, but this is more than offset by the decrease in the loss from carrying coins.

C. Token coins

In this section, we examine whether ex ante welfare is higher with token coins or with full-bodied coins as claimed in the earlier quote from Elizabeth I. For this, we compute ex ante welfare for various quantities of tokens and silver coins, where we define a token to be a coin with no intrinsic value. To eliminate any wealth effects, we assume the same m_s in both and that it is distributed lump sum to all the agents in the economy with tokens. That is, in terms of the value function, we assume that $b_s = 0$ and add m_s. Further, we assume that silver coins and tokens have the same carrying cost. Our justification is that such coins were "liable to be soon worn, and easily lost" (Ruding (1840), p. 281).
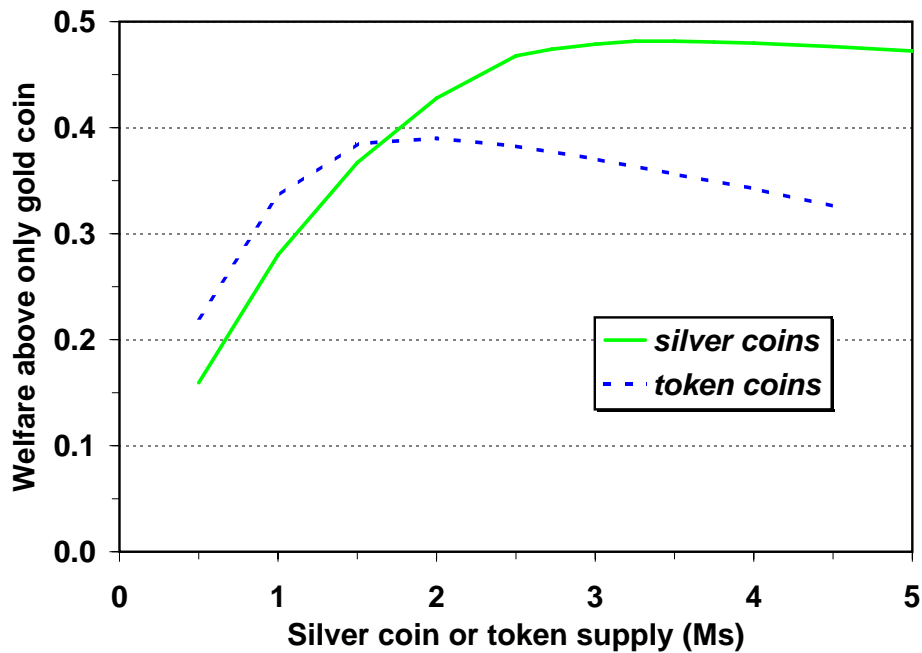

Figure 9: Ex ante welfare for various quantities of silver coins and tokens, $\gamma = 0.001$

The results are shown in Figure 9. We find that ex ante welfare is higher with token coins than with full-bodied silver coins when the quantity of silver coins or tokens is small.

However, the opposite is true when the quantity of silver coins or tokens is large.

Because we have eliminated wealth effects and carrying cost effects by assumption, the differences in ex ante welfare must be due to transactions effects — effects on the quantity of output transacted in matches and the distribution of coin holdings. Define the average quantity traded in matches to be $\bar{q} = \sum_y \sum_{\tilde{y}} q(y, \tilde{y})$, where $q(y, \tilde{y})$ is the quantity traded in $y, \tilde{y}$ matches. We find that for any M_s, $\bar{q}$ is always higher for full-bodied silver coins than tokens. A buyer with silver coins is able to extract more output from the seller because a more intrinsically valuable object is being offered. We also find that $\bar{q}$ is decreasing as M_s increases. For silver coins this is explained by the fact that there is less metal in the coin as M_s increases. For token coins, the relative scarcity of tokens when M_s is small makes them more valuable. Further we find that the difference between $\bar{q}$ for silver coins and $\bar{q}$ for tokens is decreasing as M_s increases.

These results might seem to suggest that welfare should always be higher with silver coins than with tokens, because utility is increasing in q. However, that neglects the fact that because the cost of production is linear in q, sellers are worse off as q increases. The surplus in a trade, $u(q) - q$ is maximized at q^*, and is decreasing in q when $q > q^*$ and increasing in q when $q < q^*$. At low levels of M_s, $\bar{q}$ in both the silver and token economies exceeds q^*, so that the higher average output per trade in the economy with silver coins generates a lower ex ante welfare. Conversely at high levels of M_s, where $\bar{q}$ in both the silver coin and token economies is lower than q^*, the higher average output per trade in the economy with silver coins generates a higher ex ante welfare.

5. Conclusion

Contemporaries, and economic historians, have noted several features of medieval and early modern European monetary systems that are hard to analyze using models of centralized exchange. For example, contemporaries complained of recurrent shortages of small change that affected the ability of both buyers and sellers to trade. These shortages were seen as disproportionately hurting the poor. It has also been argued that an abundance/dearth of money had real effects on exchange.

To confront these facts, we build a random matching monetary model with two indivisible coins with different intrinsic values. The model shows that small change shortages can exist in the sense that adding small coins to an economy with only large coins is welfare improving, This effect is amplified by increases in trading opportunities. Further, changes in the quantity of monetary metals affect the real economy and the amount of exchange as well as the optimal denomination size. Finally, the model shows that replacing full-bodied small coins with tokens is not necessarily welfare improving.

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