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ABSTRACT

We define the median stable matching for two-sided matching markets with side payments and prove constructively that it exists.

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1. Introduction

We study one-to-one matching problems with endogenous side transfers as in Shapley and Shubik (1971), Crawford and Knoer (1981) and Kelso and Crawford (1982).[1] In this problem, there are two sides of the market, firms and workers. Each agent on one side of the market has cardinal preferences over the agents on the other side. A matching in our setup not only assigns who is going to be matched with whom but also the monetary transfers to be made.

An intuitive solution concept in this setup is the *core*, which coincides with the so-called *stable matchings* (Shapley and Shubik (1971))- matchings that give non-negative utilities to all agents and that do not have any blocking pairs who could be better off through matching with each other than the current one.

There are several classic results about the core of matching markets both with transfers (for example, the labor market where wages are set endogenously) and without transfers (such as the marriage market). The core is known to be non-empty. Moreover, there are two matchings in the core which are the best outcome for one side of the market and also the worst for the other (Shapley and Shubik (1971) and Gale and Shapley (1962)). These matchings are called the *firm-optimal stable matching* and the *worker-optimal stable matching*.

The two extreme matchings of the core are often viewed both as a metaphor of what happens in a market and a recipe for market design (Roth (2008)). However, choosing an outcome that is most favorable for one side seems somewhat arbitrary both as a modeling choice and as a market design choice. This motivated research aimed at identifying a matching in the core that represents a fair compromise between the two sides of the market. For one-to-one matching markets without wages, such a matching is proposed in Teo and

[1] Crawford and Knoer (1981) and Kelso and Crawford (1982) study more general matching models with transfers.

Sethuraman (1998).

Teo and Sethuraman established a remarkable result for one-to-one matching markets: There exists a stable matching μ such that for any agent a, μ gives the median match of all matches in the core for a. This result was later extended to one-to-many matching markets in Sethuraman, Teo, and Qian (2006)- both papers use linear programming tools. Fleiner (2003) and Klaus and Klijn (2006) show the existence of such a stable matching independently using only the lattice structure of the core.[2] The same technique is used in this paper. The results in the aforementioned literature are limited to markets where agents form matches without a possibility of making transfers.

When side transfers are endogenous, we define an assignment (see Definition 3) which we call the *median stable matching* (borrowing the term from the literature on matching without side transfers). To define it, consider all the stable matchings in the core when the side transfers are discretized. For any agent, order all the possible matches (with possible multiplicities) in these stable matchings and take the median one. Then the median stable matching is such that all agents are matched to their median matches. Typically, there are multiple matchings in the core that give the median match to any worker. However, it is not obvious that there exists a stable matching that gives the median matches to all the workers at the same time. Theorem 1 establishes the existence. Moreover, the same stable matching also gives the median matches to all the firms. In some sense the median stable matching is a compromise solution for matching markets in which no side of the market can dictate the outcome.

It is harder to define the median stable matching when transfers are continuous as generically there will be a continuum of stable matchings. However, we show that there is an appropriate Lebesgue measure[3] which

[2]The lattice structure of stable matchings was first presented in Knuth (1976) which attributes it to John Conway.

[3]Lebesgue measure can be used to measure lower dimensional sets in higher dimensions,

can be used to define the median stable matching. Using this measure we can find the median utility levels for all agents and then show that they arise as a core outcome (Theorem 2). Furthermore, under some conditions we also show that the median utility levels are just the average of the utility levels in the firm-optimal stable matching and the worker-optimal stable matching.

Having defined the median stable matching both in the discrete case and the continuous case, we show that the median stable matching in the discrete case converges to the one in the continuous case (Theorem 3). This result shows that our definitions are consistent in some sense.

We also show that the discrete core converges to the continuous core (Theorem 4) using similar techniques. Although this result may feel intuitively obvious, it has not been established in the previous literature. In fact, most of the papers only consider the discrete case or the continuous case. Two exceptions are Crawford and Knoer (1981) and Kelso and Crawford (1982). However, both papers first establish the non-emptiness of the discrete core using a deferred-acceptance algorithm and then show the non-emptiness of the continuous core using the result in the discrete case. None of these papers shows convergence of the core or convergence of the two extreme stable matchings.

The nature of the median stable matching in a market with side transfers is very different from that of the market without side transfers. First note that in a market without side transfers, the number of possible matchings is finite in contrast to the matching with continuous side transfers where there is a continuum of stable matchings. Second in a market without side transfers firms may get matched to different workers in two stable matchings whereas in a market with side transfers generically agents get matched to the same agents but at different wages. Therefore, the median stable matching in the case without side transfers is about the median partners whereas the median stable matching with side transfers is about the median wages.

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see section 5 for details.

It is worth pointing out that the choice between worker-optimal and firm-optimal stable matchings can be a contentious issue when a centralized matching mechanism is used in practice. The best-known centralized matching market without endogenous side transfers is the national residence matching program (NRMP). Hospitals have fixed wages for residents which cannot be adjusted for different candidates. Every year NRMP matches tens of thousands of medical residents to hospitals in the USA. NRMP used to implement an algorithm based on the hospital-optimal matching until 1998 after which an algorithm based on the student-optimal one has been implemented.[4]

NRMP takes wages as exogenously given and then find a stable matching. Even if the student-optimal matching is used, it is not clear that students benefit compared to a stable matching with flexible wages. Antitrust charges were filed against the NRMP arguing that bilateral negotiations would benefit students.[5] A paper by Bulow and Levin (2006) gives some merit to the argument by showing that even if we use the student-optimal outcome the wages might be compressed.[6]

It is understood that not endogenizing wages in a centralized matching mechanism may lead to inefficiency. However, as far as we know, there are no centralized matching markets that allow for endogenous side transfers. In a recent paper Crawford (2008) proposes such a centralized procedure for NRMP. His proposal considers implementing both of the extreme matchings of the core but not the other ones. The median stable matching described in this paper may be considered as a plausible alternative that treats both sides of the market in a symmetric way.

The rest of the paper is organized as follows. In section 2, we introduce the matching problem with side transfers and define the notions of feasibility,

[4]For details of the history of the NRMP see Roth (2008).

[5]The lawsuit was dismissed on the grounds of antitrust exemption.

[6]Niederle (2007) and Kojima (2007) provide counterarguments to the plaintiff's claim if other features of the NRMP are also taken into account.

optimality and stability. We give an example in section 3 to demonstrate some of our results. In section 4, we restrict transfers to be discrete and define the median stable matching and show that it exists. In section 5, we let the transfers be continuous, define the median stable utility imputation, show that it exists and provide some conditions under which it is just the average of the firm-optimal and the worker-optimal utility imputations. In section 6, we prove that the median stable imputation in the discrete case converges to that of the continuous case as the bid increment goes to zero. We provide the proofs of Theorems 2, 3 and 4 in the Appendices.

2. Model

There are two disjoint sets of agents, the set of firms, $F = \{f_1, f_2, \dots, f_m\}$ and the set of workers, $W = \{w_1, w_2, \dots, w_n\}$ where the number of firms m can be different from the number of workers n. Let $A = F \cup W$ be the set of all agents. Each firm $f \in F$ has a quasi-linear utility over $(w, t) \in W \cup \{f\} \times \mathbb{R}$ which is $u_f(w) + t$. Similarly, each worker $w \in W$ has a quasi-linear utility over $(f, t) \in F \cup \{w\} \times \mathbb{R}$ which is $u_w(f) + t$. Without loss of generality, assume that for each agent $a \in A$ the utility of being unmatched is zero, $u_a(a) = 0$. A utility function profile U is a list of utility functions of all the agents, $U = (u_{f_1}, \dots, u_{f_m}, u_{w_1}, \dots, u_{w_n})$. A matching problem with side transfers is a triple $\langle F, W, U \rangle$.

Definition 1: *A (discrete)* ***feasible matching*** $c = \langle \mu, t \rangle$ *consists of a* ***matching function*** μ *and a* ***monetary transfer function*** t *satisfying the following conditions:*

1. $\mu : A \to A$ *is a bijection such that* $\mu(f) \in W \cup \{f\}$ *for all* $f \in F$ *and* $\mu(w) \in F \cup \{w\}$ *for all* $w \in W$
2. $\mu^2(a) = a$ *for all* $a \in A$
3. $t : A \to \mathbb{R}(\Delta\mathbb{Z})$ *such that* $\sum t(f_i) + \sum t(w_j) \leq 0$.

Conditions 1 and 2 are about the feasibility of the matching function μ. Condition 1 states that each agent a is either single or matched to an agent a' from the other side of the market and condition 2 guarantees that if it is the latter case then agent a' is also matched to agent a. On the other hand, condition 3 is about the feasibility of the transfer function. It basically states that the sum of the transfers for all of the agents should not be positive. In the discrete case, the transfers are restricted to be integer multiples of some positive real number Δ.

A feasible matching $\langle \mu, t \rangle$ is called ***optimal*** if the sum of the utilities of all agents are maximized at $\langle \mu, t \rangle$. For an optimal matching condition 3 always binds. If $\langle \mu, t \rangle$ is an optimal matching then $\langle \mu, t' \rangle$ such that $\sum t'(f_i) + \sum t'(w_j) = 0$ is also an optimal matching. In this case, μ is also called an optimal matching without specifying any transfer function t for which condition 3 binds. Usually there is a unique optimal matching μ, otherwise a small adjustment in the utilities gives uniqueness.

Given a matching $c = \langle \mu, t \rangle$, we let $v(a \mid c) \equiv u_a(\mu(a)) + t(a)$ denote the ***utility imputation*** of agent $a \in A$ from the matching. We also use $v(a \mid c(a))$ to denote the same utility imputation where $c(a) = \langle \mu(a), t(a) \rangle$ is the match of agent a from matching c. If the matching is clear from the context then we use $v(a)$ instead, without specifying the matching.

Definition 2: *A (discrete) feasible matching $\langle \mu, t \rangle$ is (discrete)* ***stable*** *if*

1. $v(a) \geq 0$ *for all* $a \in A$
2. *For all* $(f, w) \in F \times W$ *there does not exist any* $x \in \mathbb{R}(\Delta\mathbb{Z})$ *such that* $v(f) \leq u_f(w) - x$ *and* $v(w) \leq u_w(f) + x$ *with one of these inequalities being strict.*

Condition 1 is about the individual rationality of the matching. It states that no agent should be better off by staying alone and receiving zero payment in a stable matching. Whereas condition 2 guarantees that no firm-worker pair can block the matching. If it fails for a pair (f, w) and a transfer x then these agents can both be made as well off and one of them better off by matching together and the firm paying the worker x. For the continuous case condition 2 is equivalent to:

$$v(f) + v(w) \geq u_f(w) + u_w(f) for\ all (f, w) \in F \times W. \tag{1}$$

3. An Example

In this section we give an example to illustrate and motivate our results.

There are two firms f_1 and f_2 and two workers w_1 and w_2. The agents have the following utility functions: $u_{f_1}(w_1) = 2, u_{f_1}(w_2) = 1; u_{f_2}(w_1) = 2, u_{f_2}(w_2) = 0; u_{w_1}(f_1) = 2, u_{w_1}(f_2) = 0$ and $u_{w_2}(f_1) = 1, u_{w_2}(f_2) = 3$. We can write the **production function** $\phi(f, w) \equiv u_f(w) + u_w(f)$ related to these utility functions as:

$\phi(\cdot,\cdot)$	w_1	w_2
f_1	4	2
f_2	2	3

Let us first characterize the continuous core.[7] There is a unique optimal matching which maximizes the sum of utilities of the agents in which f_1 gets matched to w_1 and f_2 gets matched to w_2. Note that matchings in the core must be optimal as the utilities of all agents can be improved with an appropriate transfer function and the optimal matching.

The sum of the transfers for a matched pair must be zero. This is easy to see. By stability the sum of the transfers for a matched pair is non-negative. On the other hand, by feasibility the sum of all transfers must be non-positive. Hence the sum of the transfers must be zero for a matched

[7]Matchings that are in the "core" and stable matchings coincide. See Shapley and Shubik (1971) when transfers are continuous and Crawford and Knoer (1981) for the discrete one. In this paper, we use these two concepts interchangeably.

pair. Therefore, individual rationality constraints can be written as:

$$4 \geq v(f_1) \geq 0 \textit{ and } 3 \geq v(f_2) \geq 0. \tag{2}$$

There are two possible blocking pairs (f_1, w_2) and (f_2, w_1). Non-blocking conditions for these pairs require:

$$v(f_1) + v(w_2) \geq 2 \textit{ and } v(f_2) + v(w_1) \geq 2. \tag{3}$$

Since the sum of the transfers must add up to zero for a matched pair in a stable matching we have $v(w_2) = 3 - v(f_2)$ and $v(w_1) = 4 - v(f_1)$. Therefore, (3) can be rewritten as:

$$v(f_1) - v(f_2) \geq -1 \textit{ and } v(f_2) - v(f_1) \geq -2. \tag{4}$$

[**INSERT FIGURE 1 HERE**]

Hence, the continuous core is characterized by the inequalities given in (2) and (4) which is shown in Figure 1. The median utility imputation in this example is constructed as follows. First we find the vertical line which divides the core into two sets of equal area which is $v(f_1) = 2$. Then we find the horizontal line which divides the core into two sets of equal area which is $v(f_2) = 3/2$. The intersection of these two lines is $(2, 3/2)$ which gives the median utility levels for the firms. The corresponding median utility levels for w_1 and w_2 are 2 and 3/2, respectively. Hence the median utility imputation is $(2, 3/2, 2, 3/2)$.

Let us characterize the discrete core when $\Delta = 3/4$ (denote it by $Core(3/4)$). Note that for this choice of Δ no agent is indifferent between two different feasible matchings.

We claim that any stable matching is optimal. Assume otherwise. Then f_1 must be matched to w_2 and f_2 must be matched to w_1. The social surplus is then four. Since the optimal social surplus is seven, we can give everyone more than what they are getting in a feasible way in an optimal matching. We get a contradiction.

Now, consider the Cartesian product of feasible utility imputations for firms f_1 and f_2 in the optimal matching. First note that all the points of the Cartesian product inside the continuous core are still in the core since the discrete core restricts the feasible blocking pairs. Other than these nineteen points which are denoted by nodes, there are six more points which satisfy the individual rationality constraints (denoted by crosses in the figure). One easily verifies that none of these six points are stable.

Hence, there are nineteen points in $Core(3/4)$. By calculating the tenth utility level for each agent we get the median stable utility imputation which is $(2, 3/2, 2, 3/2)$ that coincides with the continuous median utility imputation. Note that there are multiple points in the core that give median utility to f_1 and there are many such points for f_2. It is not obvious that there exists a stable matching that gives median utilities to both f_1 and f_2. Moreover, the same stable matching also gives median utilities to the workers. This example illustrates the claims of Theorems 1 and 2.

4. DISCRETE CASE

In this section we suppose that the transfer function can only take integer multiples of a positive real number Δ. Formally, for any $a \in A$, $t(a) \in \Delta\mathbb{Z}$.

Given a matching problem $\langle F, W, U \rangle$ with Δ, we know that the set of stable matchings is finite and non-empty since there exist a worker-optimal stable matching and a firm-optimal stable matching. For a given agent a we want to be able to rank any two different choices of a without using any tie-breaking rules. Hence, we make the following assumption:

ASSUMPTION 1: *Let u and Δ be such that if c and c' are two matchings such that $c(a) \neq c'(a)$ for some agent $a \in A$ then $v(a \mid c) \neq v(a \mid c')$.*

This assumption is satisfied for example if u_a is one-to-one and Δ is not an integer fraction of $u_a(\mu(a)) - u_a(\mu'(a))$ where $\mu(a)$ and $\mu'(a)$ are two different partners of a (either a or an agent from the other side of the market) for all agents a.

Let $\{c_1, c_2, \ldots, c_k\}$ be the set of all stable matchings. For every agent a consider the set $\{c_1(a), c_2(a), \ldots, c_k(a)\}$ with possible multiplicities and order them such that agent a's utility is non-increasing and denote the i-th element in this non-increasing sequence by $c^i(a)$. Equivalently,

$$\{c^1(a), c^2(a), \ldots, c^k(a)\} = \{c_1(a), c_2(a), \ldots, c_k(a)\}$$

and $v(a \mid c^1(a)) \geq v(a \mid c^2(a)) \geq \ldots \geq v(a \mid c^k(a))$.

For any $i \in \{1, 2, \ldots, k\}$ consider the following vector:

$$x_i \equiv (c^i(f_1), \ldots, c^i(f_m), c^{k+1-i}(w_1), \ldots, c^{k+1-i}(w_n)).$$

By definition, x_i takes the i-th best match for the firms and the i-th worst match for the workers. We next show that x_i is a stable matching. To be more explicit, we construct a stable matching using all the stable matchings and then show that this coincides with x_i.

Theorem 1: *For all $i \in \{1, 2, \ldots, k\}$, x_i is a stable matching.*

We can now define what the median stable matching is.

Definition 3: *If k is odd then let $x_{\frac{k+1}{2}}$ be the median stable matching, otherwise let $x_{\frac{k}{2}}$ be the median stable matching. Denote the utility imputation of the median stable matching by $v_{median}(\Delta)$.*

Note that when k is even we could as well choose $x_{\frac{k}{2}+1}$ to be the median stable matching. Our results hold under both choices.

Corollary 1: *Median stable matching exists.*

Before we proceed with the proof of Theorem 1, we note in Lemma 1 that the set of stable matchings is closed with respect to two orders which we define below.

With a slight abuse of notation we let $min_a S$ and $max_a S$ denote the least preferred and the most preferred choices respectively for an agent a from any finite set S of partner-transfer pairs. This is well-defined and unique by Assumption 1.

Given two stable matchings c and c' we let $c \wedge_W c'$ be the vector such that for all workers w, $(c \wedge_W c')(w) = min_w\{c(w), c'(w)\}$ and for all firms f, $(c \wedge_W c')(f) = max_f\{c(f), c'(f)\}$. Similarly, let $c \vee_W c'$ be the vector such that for all workers w, $(c \vee_W c')(w) = max_w\{c(w), c'(w)\}$, and for all firms f, $(c \vee_W c')(f) = min_f\{c(f), c'(f)\}$. Let $c \vee_F c' \equiv c \wedge_W c'$ and $c \wedge_F c' \equiv c \vee_W c'$.

LEMMA 1: *Let c and c' be two stable matchings. Then $c \wedge_W c'$ and $c \vee_W c'$ are also stable matchings.*[8]

Now, we are ready to prove our first theorem. Note here that our proof parallels the proof of Theorem 3.2 in Klaus and Klijn (2006).

Proof of Theorem 1. Consider all combinations of i stable matchings $d_1, d_2, \ldots, d_i$ where $1 \leq i \leq k$. For each such combination construct $d_1 \wedge_F d_2 \wedge_F \ldots \wedge_F d_i$. By Lemma 1, we get a stable matching in this way. There are $l = \binom{k}{i}$ of these stable matchings, denote them by $e_1, e_2, \ldots, e_l$. Now let $\beta_i = e_1 \vee_F e_2 \vee_F \ldots \vee_F e_l$. By Lemma 1, β_i is also a stable matching. We claim that $\beta_i = x_i$.

First, let us show that $\beta_i(f) = x_i(f)$ for all firms $f \in F$. By definition of e_j we get that $v(f \mid c^i(f)) \geq v(f \mid e_j)$ for all j which implies $v(f \mid c^i(f)) \geq v(f \mid \beta_i)$ by definition of β_i. Now if we let $\{d_1, d_2, \ldots, d_i\}$ to be a set of stable matchings such that $\{d_1(f), d_2(f), \ldots, d_i(f)\} = \{c^1(f), c^2(f), \ldots, c^i(f)\}$ then we get that $v(f \mid c^i(f)) = v(f \mid e_j)$ for $e_j = d_1 \wedge_F d_2 \wedge_F \ldots \wedge_F d_i$. Since β_i if formed by taking the best option for all firms among e's we get that $c^i(f) = \beta_i(f)$ implying $x_i(f) = \beta_i(f)$.

[8] Similar results to Lemma 1 are well-known in the matching literature. However, we were not able to find a direct reference for the assignment game with discrete transfers. There is a related result in Hatfield and Milgrom (2005) which has a more general model but the operation they use is different than ours. However, our lemma is straightforward using their Theorem 3(a) and the idea in the proof of Theorem 4. Therefore, we do not give a proof here.

Second, we show that $\beta_i(w) = x_i(w)$. Since $\wedge_F = \vee_W$ and $\vee_F = \wedge_W$, we have:

$d_1 \wedge_F d_2 \wedge_F \ldots \wedge_F d_i = d_1 \vee_W d_2 \vee_W \ldots \vee_W d_i$ and $\beta_i = e_1 \wedge_W e_2 \wedge_W \ldots \wedge_W e_l$.

These inequalities and definition of e_j imply that $v(w \mid e_j) \geq v(w \mid c^{k+1-i}(w))$ for all j. Now by taking $\{d_1, d_2, \ldots, d_i\}$ to be a set of stable matchings such that $\{d_1(w), d_2(w), \ldots, d_i(w)\} = \{c^{k+1-i}(w), c^{k+2-i}(w), \ldots, c^k(w)\}$ we get that $e_j(w) = c^{k+1-i}(w)$ for some j. Hence we have established $\beta_i(w) = c^{k+1-i}(w)$. Therefore $\beta_i(w) = x_i(w)$ by definition of x_i. ∎

Hence, the median stable matching is well-defined.

5. CONTINUOUS CASE

In the previous section we showed that the median stable matching is well-defined. Now, let us define the median stable imputation for the continuous case. Since we allow continuous transfers, our setup is the same as Shapley and Shubik (1971). Then it is clear that every stable matching must be optimal which follows immediately from inequality 1. The next lemma shows that any stable utility imputation can be achieved via any optimal matching μ.

LEMMA 2: (Shapley and Shubik) *If v is a utility imputation of a stable matching then it can be achieved via any optimal matching μ with an appropriate transfer function t.*

From now on, we fix an optimal matching $\tilde{\mu}$. For any stable matching $\langle \mu, t \rangle$ we only consider the corresponding transfer function $\tilde{t}$ for $\tilde{\mu}$ such that $v(a \mid \langle \mu, t \rangle) = v(a \mid \langle \tilde{\mu}, \tilde{t} \rangle)$ for all agents $a \in A$. Therefore, given any stable matching we can concentrate only on the utility imputation which we call the stable utility imputation.

Note that for a stable matching $\langle \mu, t \rangle$, the sum of the transfers should add up to zero. However, the feasibility allows not only transfers between firm-worker pairs but also between firms or between workers. Therefore it is not clear *a priori* that the sum of the transfers for a matched firm-worker pair should be zero. The next lemma shows that the only transfers made in a stable matching are between matched firm-worker pairs.

Lemma 3: (Roth and Sotomayor (1990), Lemma 8.5) *Let $\langle \mu, t \rangle$ be a stable matching. Then*

(i) $t(f) + t(w) = 0$ *for matched pairs* $(f, w) \in F \times W$

(ii) $t(a) = 0$ *for all unmatched agents* $a \in F \cup W$.

Given a matching problem with continuous transfers $\langle F, W, U \rangle$ we denote the set of stable utility imputations in $(m+n)$-dimensional Euclidean space by P. If we restrict our attention to only a subset A' of k agents then we denote the set of stable utility imputations in k-dimensional Euclidean space by $P_{A'}$.

If P is a single point then there exists a unique stable imputation and it can naturally be defined to be the median stable imputation. For what follows, we assume that there exist at least two stable imputations. Since the transfers are continuous, there will be an uncountable number of stable imputations. Therefore, a measure on P must be defined to be able to define the median stable imputation.

Theorem 2: *Suppose the dimension of the core is equal to the number of matched pairs r. And let $\hat{F}$ and $\hat{W}$ be the set of matched firms and workers, respectively. Then there exists a unique stable utility imputation v_{median} such that $\lambda_r(\{v \in P_{\hat{F}} | v(f) \geq v_{median}(f)\}) = \lambda_r(\{v \in P_{\hat{F}} | v(f) \leq v_{median}(f)\})$ for every $f \in \hat{F}$ and $\lambda_r(\{v \in P_{\hat{W}} | v(w) \geq v_{median}(w)\}) = \lambda_r(\{v \in P_{\hat{W}} | v(w) \leq v_{median}(w)\})$ for every $w \in \hat{W}$.*[9]

[9] Here λ_r denotes the Lebesgue measure on $\mathbb{R}^r$. Similarly, for every $0 < \alpha < 1$, there exists a unique stable utility imputation v_α such that $\alpha \cdot \lambda_r(\{v \in P_{\hat{F}} | v(f) \geq v_\alpha(f)\}) = (1-\alpha) \cdot \lambda_r(\{v \in P_{\hat{F}} | v(f) \leq v_\alpha(f)\})$ for all $f \in \hat{F}$ and $(1-\alpha) \cdot \lambda_r(\{v \in P_{\hat{W}} | v(w) \geq v_\alpha(w)\}) = \alpha \cdot \lambda_r(\{v \in P_{\hat{W}} | v(w) \leq v_\alpha(w)\})$ for all $w \in \hat{W}$. The theorem is stated for $\alpha = 1/2$.

Obviously, the dimension of the core cannot exceed the number of matched pairs. Unless there are degeneracies (arithmetic relations between the utilities), the dimension of the core is equal to the number of matched pairs as noted by Shapley and Shubik (1971).

The main idea of the proof is finding hyperplanes of the form $v(a) = c$ which divide the core into two sets of equal λ_r-measure. The rest of the proof shows that the intersection of these hyperplanes is in the core. See Appendix A for the proof and also how this theorem can be extended to the case when the dimension of the core is smaller than the number of matched pairs.

5.1. Median Stable Matching for Monotone Supermodular Production Functions

We have established the existence of v_{median} when the side transfers are continuous. However, the construction of v_{median} needs full characterization of the core which can be done by finding all the extreme points in P (see Núñez and Rafels (2003) on the characterization of the extreme points).

Instead of taking this approach, we give some conditions on the utility functions of the agents such that v_{median} coincides with the average of the firm-preferred utility imputation v_F and the worker-preferred utility imputation v_W which we call the ***mean stable utility imputation*** and denote it by v_{mean}.

PROPOSITION 1: *The mean stable utility imputation is a core utility imputation.*

Proof. Since the set of stable utility imputations is characterized by a set of linear inequalities, it is a convex set. Therefore, v_{mean} is a stable utility imputation. ∎

The mean stable utility imputation is also a potential compromise solution along with the median stable utility imputation. We formulate some conditions on the utility functions for which these two imputations coincide.

Now, we define the sufficient conditions for ϕ to satisfy $v_{median} = v_{mean}$.

DEFINITION 4: *ϕ satisfies **supermodularity** if for all $i \geq i'$ and $j \geq j'$ we have:*

$$\phi(w_i, f_j) + \phi(w_{i'}, f_{j'}) \geq \phi(w_i, f_{j'}) + \phi(w_{i'}, f_j).$$

DEFINITION 5: *ϕ satisfies **monotonicity** if $\phi(w_i, f_j) \geq \phi(w_i, f_{j'})$ and $\phi(w_i, f_j) \geq \phi(w_{i'}, f_j)$ for all $i \geq i'$ and $j \geq j'$.*

Note that multiplicative production functions routinely used in the matching literature are particular examples of monotone supermodular functions.

PROPOSITION 2: *If ϕ satisfies monotonicity and supermodularity then $v_{median} \equiv v_{mean}$.*

For simplicity we assume that the number of firms and workers are equal. If there were excessive number of workers or firms, the first inequality for $v(w_1)$ in the characterization below would have different boundaries but everything else would be the same and our result goes through.

Take an optimal matching. Consider its assortative counterpart where the highest indexed matched firm gets matched to the highest indexed matched worker, the second highest indexed matched firm gets matched to the second highest indexed matched worker and so on... Because of supermodularity the assortative matching must also be optimal. By monotonicity, if there are r coupled pairs in an optimal matching then the assortative matching of the highest r indexed firms and workers must also be optimal. Consider this optimal matching. Now, we can also assume for simplicity that $\phi(w_1, f_1) \geq 0$. Otherwise we could exclude pairs (w_i, f_i) with negative production functions.

LEMMA 4: *Suppose ϕ satisfies monotonicity and supermodularity, then* *the following inequalities characterize the set of stable utility imputations:*

$$
\begin{gathered}
0 \leq v(w_1) \leq \phi(w_1, f_1) \\
\phi(w_2, f_1) - \phi(w_1, f_1) \leq v(w_2) - v(w_1) \leq \phi(w_2, f_2) - \phi(w_1, f_2) \\
\vdots \\
\phi(w_r, f_{r-1}) - \phi(w_{r-1}, f_{r-1}) \leq v(w_r) - v(w_{r-1}) \leq \phi(w_r, f_r) - \phi(w_{r-1}, f_r).
\end{gathered}
$$

Proof. Let us first show that these inequalities are necessary.

By individual rationality $v(w_1) \geq 0$, $v(f_1) \geq 0$ and $v(f_1) + v(w_1) = \phi(w_1, f_1)$ by Lemma 3, so we get that $0 \leq v(w_1) \leq \phi(w_1, f_1)$.

Consider the pair (w_{i-1}, f_i). They must not be a blocking pair. Hence, $(\phi(w_i, f_i) - v(w_i)) + v(w_{i-1}) = v(f_i) + v(w_{i-1}) \geq \phi(w_{i-1}, f_i)$. Therefore, $v(w_i) - v(w_{i-1}) \leq \phi(w_i, f_i) - \phi(w_{i-1}, f_i)$. This gives the right hand side of the i-th inequality.

Similarly, consider the pair (w_i, f_{i-1}). They don't form a blocking coalition, so $(\phi(w_{i-1}, f_{i-1}) - v(w_{i-1})) + v(w_i) = v(f_{i-1}) + v(w_i) \geq \phi(w_i, f_{i-1})$. Hence, $v(w_i) - v(w_{i-1}) \geq \phi(w_i, f_{i-1}) - \phi(w_{i-1}, f_{i-1})$. This is the left hand side of the i-th inequality.

We have shown necessity, next we show sufficiency. We need to show two things. First, we need to show that for any vector of utilities for firms which satisfy all the inequalities we get an individually rational matching. Second, we need to show that for these utility vectors there are no blocking pairs.

To show individual rationality, let us add up the first i inequalities to get:

$$\sum_{j=1}^{i-1} (\phi(w_{j+1}, f_j) - \phi(w_j, f_j)) \leq v(w_i) \leq \sum_{j=1}^{i} \phi(w_j, f_j) - \sum_{j=1}^{i-1} \phi(w_j, f_{j+1}).$$

The left hand side of this inequality is non-negative since each term in the summation is non-negative by monotonicity. Now, let us rewrite the right hand side as $\phi(w_i, f_i) + \sum_{j=1}^{i-1} (\phi(w_j, f_j) - \phi(w_j, f_{j+1}))$ which is less than $\phi(w_i, f_i)$ since monotonicity implies that every term in the summation is non-positive by feasibility. Hence, we get $0 \leq v(w_i) \leq \phi(w_i, f_i)$. This also implies $0 \leq v(f_i) \leq \phi(w_i, f_i)$. Thus individual rationality is implied by this set of inequalities.

Now, let us show that (w_{i+k}, f_i) and (w_i, f_{i+k}) are not blocking pairs for all $i, k \in \mathbb{N}$ such that $i + k \leq n$.

If we add up the inequalities from the $(i+1)$-th to the $(i+k)$-th we get the following inequalities:

(5)
$$\sum_{j=i}^{i+k-1} (\phi(w_{j+1}, f_j) - \phi(w_j, f_j)) \leq v(w_{i+k}) - v(w_i) \leq \sum_{j=i}^{i+k-1} (\phi(w_{j+1}, f_{j+1}) - \phi(w_j, f_{j+1})).$$

Now, note that we have the following inequality by supermodularity:

$$\phi(w_{i+k}, f_i) + \sum_{j=i+1}^{i+k-1} \phi(w_j, f_j) \leq \sum_{j=i}^{i+k-1} \phi(w_{j+1}, f_j).$$

If we rewrite this inequality we get:

$$\phi(w_{i+k}, f_i) - \phi(w_i, f_i) \leq \sum_{j=i}^{i+k-1} (\phi(w_{j+1}, f_j) - \phi(w_j, f_j)).$$

Combining the last inequality with the left hand side of equation 5 we get:

$$\phi(w_{i+k}, f_i) - \phi(w_i, f_i) \leq v(w_{i+k}) - v(w_i).$$

This proves that (w_{i+k}, f_i) is not a blocking pair.

Similarly, by supermodularity we get the following inequality:

$$\sum_{j=i}^{i+k-1} \phi(w_j, f_{j+1}) \geq \sum_{j=i}^{i+k-2} \phi(w_{j+1}, f_{j+1}) + \phi(w_i, f_{i+k}).$$

By rewriting this we get the following:

$$\phi(w_{i+k}, f_{i+k}) - \phi(w_i, f_{i+k}) \geq \sum_{j=i}^{i+k-1} (\phi(w_{j+1}, f_{j+1}) - \phi(w_j, f_{j+1})).$$

With the last inequality and the right hand side of equation 5 we get:

$$\phi(w_{i+k}, f_{i+k}) - \phi(w_i, f_{i+k}) \geq v(w_{i+k}) - v(w_i).$$

This proves that (w_i, f_{i+k}) is not a blocking pair. ∎

Hence, under supermodularity and monotonicity, it is sufficient to consider individual rationality constraints of w_1 and f_1, and only the non-blocking constraints for pairs with adjacent numbers. Using this characterization of the set of stable utility imputations we give a proof of Proposition 2.

Proof of Proposition 2. Consider $P_{\hat{F}}$. By Lemma 4, $P_{\hat{F}}$ is an r-dimensional parallelotope.[10] Therefore, there exists a point v_O which is the center of symmetry.

Since $v_F|_{\hat{F}}$ and $v_W|_{\hat{F}}$ are two opposite corners of this parallelotope, we get that $v_O = (v_F|_{\hat{F}} + v_W|_{\hat{F}})/2 = v_{mean}|_{\hat{F}}$.[11]

Since v_O is the center of symmetry we also have that $\lambda_r(\{v \in P_{\hat{F}} | v(w) \geq v_O(w)\}) = \lambda_r(\{v \in P_{\hat{F}} | v(w) \leq v_O(w)\})$. Hence $v_O = v_{median}|_{\hat{F}}$ which implies that $v_{mean}|_{\hat{F}} = v_{median}|_{\hat{F}}$.

Analogously, working with $P_{\hat{W}}$ we get that $v_{mean}|_{\hat{W}} = v_{median}|_{\hat{W}}$. We conclude that $v_{mean} = v_{median}$. ∎

6. Convergence

We have defined the median utility imputations for the discrete and continuous cases. In this section, we show that the median stable utility imputation in the discrete case converges to the median utility imputation in the continuous case.

Let $Core$ be the set of stable utility imputations in the continuous case and $Core(\Delta)$ be the corresponding one in the discrete case with Δ increment. In this case also let $Feas(\Delta)$ be the set of feasible utility imputations.

[10] An r-dimensional parallelotope is the generalization of a parallelogram to r-dimensional Euclidean space. It has $2r$ faces such that each face has a parallel face opposite of it. The diagonals of a parallelotope intersect at a point which is called the *center of symmetry*. Any hyperplane passing through the center of symmetry divides the parallelotope into two sets of equal λ_r measure.

[11] Here, $v|_{A'}$ for a vector of utilities v and a set of agents A' denotes a restricted vector consisting of utilities of agents in A'.

ASSUMPTION (Full-dimension): *The dimension of Core is equal to the number of matched pairs.*

This assumption ensures that for small enough Δ there are stable utility imputations which are in $Core$.

THEOREM 3: *If the full-dimension assumption is satisfied then* $v_{median}(\Delta) \rightarrow v_{median}$ *as* $\Delta \rightarrow 0$.

This result makes the connection between the continuous and the discrete median stable utility imputations and shows that our definitions are consistent in some sense.

A similar convergence result also holds between the discrete core and the continuous core. Before proceeding to the result, we note that to have a convergence result we need an underlying distance function. Since the core is a set of points in the Euclidean space, we need to define the distance between two sets in the Euclidean space. We use the following distance function.

Let S and S' be two sets in a Euclidean space with the underlying distance function d. Then the ***Hausdorff distance*** of S and S' is defined as follows:

$$d_H(S,S') = max\{\sup_{s\in S} \inf_{s'\in S'} d(s,s'), \sup_{s'\in S'} \inf_{s\in S} d(s,s')\}.$$

THEOREM 4: $Core(\Delta) \rightarrow Core$ *with respect to* d_H *as* $\Delta \rightarrow 0$ *under the full-dimension assumption.*

7. CONCLUSION

Two-sided matching literature primarily considers two points in the core: the worker-optimal stable matching and the firm-optimal stable matching. These two extreme points are often viewed both as a metaphor of what happens in the labor market and a recipe for market design. They provide useful insight into the workings of the labor market by choosing an outcome that is most favorable for one side of the market. However, they seem somewhat arbitrary both as a modeling choice and as a market design choice. We consider two-sided matching markets with wages, such as the labor market and point out another stable outcome which has the property of being the median outcome for all agents in the market.

Appendix A: Proof of Theorem 2

We now give a couple of mathematical definitions that we need for the rest of the section.[12]

A ***hyperplane*** in $\mathbb{R}^n$ is any affine subspace with dimension $n-1$. In other words, a hyperplane in $\mathbb{R}^n$ is given by $\{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | a_1x_1 + a_2x_2 + \ldots + a_nx_n = b\}$ for some $a \in \mathbb{R}^n - \{0\}$ and $b \in \mathbb{R}$. For example, in $\mathbb{R}^2$ a line is a hyperplane, in $\mathbb{R}^3$ a plane is a hyperplane. Two hyperplanes are ***parallel*** if they do not intersect.

A ***polytope*** $Q \subseteq \mathbb{R}^n$ is a set characterized by the intersection of a finite number of linear inequalities. Note that the core is a polytope.

The ***dimension of a polytope*** Q in $\mathbb{R}^n$ is k if Q is locally homeomorphic to $\mathbb{R}^k$. For example, a circle has dimension 1, whereas a filled square has dimension 2.

A k-dimensional polytope in $\mathbb{R}^n$ possesses a k-dimensional density and by integrating the polytope with respect to this density a measure. For example, a circle still has a perimeter in higher dimensions or a square still has an associated area. Therefore, for any polytope Q of dimension k, we can use the k-dimensional Lebesgue measure λ_k to measure Q.

We proceed with the proof.

Proof of Theorem 2. We start with showing that there exists a unique choice $v_{median}(a)$ for each agent a. To complete the proof we need to argue that

[12]These definitions are standard in mathematics and can be found in many textbooks, see Lee (2000) for topological definitions and Duistermaat and Kolk (2004) for measuring lower dimensional objects in higher dimensions.

v_{median} defined in this way is in P. We prove this in a couple of steps. First we show that v_{median} is such that for every pair (f, w) that is matched in the optimal matching we get that $v_{median}(f) + v_{median}(w) = u_f(w) + u_w(f)$. Second we show that $v_{median}|_{\hat{F}} \in P_{\hat{F}}$ (analogously $v_{median}|_{\hat{W}} \in P_{\hat{W}}$). This completes the proof.

To show v_{median} is well-defined: For all single agents a, $v_{median}(a) = 0$. Consider $P_{\hat{F}}$. Let r be the dimension of $P_{\hat{F}}$. Now define functions $h_f(v_f) \equiv \lambda_r(\{x \in P_{\hat{F}} | x_f \geq v_f\})$ and $g_f(v_f) \equiv \lambda_r(\{x \in P_{\hat{F}} | x_f \leq v_f\})$ for every $f \in \hat{F}$. They are both monotone functions, so they can only have a countable number of jumps. Since $\lambda_r(\{x \in P_{\hat{F}} | x_f = v_f\}) = 0$ for any v_f, there cannot be any jumps, so both h and g are continuous. Moreover, for all v_f we have that $h_f(v_f) + g_f(v_f) = \lambda_r(P_{\hat{F}})$ which is a non-zero constant. If $v_f > sup(\{x_f \mid x \in P_{\hat{F}}\})$ then $h_f(v_f) = 0$ and $g_f(v_f) = \lambda_r(P_{\hat{F}})$ and if $v_a < inf(\{x_f \mid x \in P_{\hat{F}}\})$ then $h_f(v_f) = \lambda_r(P_{\hat{F}})$ and $g_f(v_f) = 0$. Hence, by continuity there exists a $\tilde{v}_f$ such that $h_f(\tilde{v}_f) = g_f(\tilde{v}_f) = \lambda_r(P_{\hat{F}})/2$. Since P is an r-dimensional polytope there cannot exist more than one. Let $\tilde{v}_f = v_{median}(f)$. We define $v_{median}(w)$ for all $w \in \hat{W}$ analogously.

To show $v_{median}(f) + v_{median}(w) = u_f(w) + u_w(f)$ for a pair (f, w) which is matched in an optimal stable matching: By Lemma 3, we get that $v(f) + v(w) = u_f(w) + u_w(f)$ for any stable utility imputation v. Hence, $h_f(v(f)) = g_w(v(w))$ and $h_w(v(w)) = g_f(v(f))$. Therefore, $v_{median}(f) + v_{median}(w) = u_f(w) + u_w(f)$ by construction.

To show $v_{median}|_{\hat{F}} \in P_{\hat{F}}$: Assume for contradiction that it is not in $P_{\hat{F}}$. By the separating hyperplane theorem, there exists a hyperplane H through $v_{median}|_{\hat{F}}$ such that $P_{\hat{F}}$ lies entirely in one side of this hyperplane. Since $P_{\hat{F}}$ is a polytope we can take H to be parallel to a face of $P_{\hat{F}}$. Moreover, H cannot be a hyperplane parallel to $v_f = 0$ for any $f \in \hat{F}$ since $v_{median}(f)$ is the median utility level for f. Since all the faces of H of are either parallel to $v_f = 0$ or $v_f - v_{f'} = 0$, it must be the latter one. Now, consider the following hyperplanes: $v_f = v_{median}(f)$ and $v_{f'} = v_{median}(f')$. They partition the

Euclidean space into four quadrants. Let A, B, C and D be the λ_r measures of $P_{\hat{F}}$ within these four quadrants in the usual order. Since $v_f = v_{median}(f)$ and $v_{f'} = v_{median}(f')$ divide $P_{\hat{F}}$ into two polytopes of equal measure then we must have that $A = C$ and $B = D$ for the opposite quadrants. If $A = 0$ then we get that $P_{\hat{F}}$ is divided into two sets within two opposite quadrants which cannot be true since $P_{\hat{F}}$ is a convex r-dimensional set. Hence $A \neq 0$ and similarly $B \neq 0$. Now, H is such that there exists at least one quadrant in either side of H with a positive measure of H within. This gives a contradiction to the fact that $P_{\hat{F}}$ lies completely on one side of H. ∎

For the rest of the appendix, we similarly construct v_{median} even when the dimension of the core is not equal to the number of matched pairs. To do this, we use the characterization of the core given in Núñez and Rafels (2008).[13]

LEMMA 5: (Núñez and Rafels) *There exists a partition of firms* $\{F_0, F_1, \ldots, F_d\}$ *such that* F_0 *includes all the firms which get fixed utility in all stable matchings and for any* $f, \tilde{f} \in F_i$ *for* $i = 0, 1, \ldots, d$ $v(f|\langle\mu, t\rangle) - v(\tilde{f}|\langle\mu, t\rangle)$ *is fixed for all stable matchings* $\langle\mu, t\rangle$*. Moreover, if you take a representative firm* $\tilde{f}_i \in F_i$ *for* $i = 1, 2, \ldots, d$ *then* $P_{\tilde{F}}$*, where* $\tilde{F} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_d\}$*, has dimension* d*.*

We explain how $v_{median}(f)$ is constructed for firm f. For worker w, $v_{median}(w)$ is 0 if w is unmatched in an optimal matching, otherwise it is $u_f(w) + u_w(f) - v_{median}(f)$ where f is the firm that w is matched to.

If $f \in F_0$ then $v_{median}(f)$ is the fixed utility that f is getting. Otherwise, construct a set of representative firms including f: that is pick a firm from each $f_i \in F_i$ where $i = 1, 2, \ldots, d$ such that $f_j = f$ for some $j \in \{1, 2, \ldots, d\}$. Let $\tilde{F} = \{f_1, f_2, \ldots, f_d\}$. Consider $P_{\tilde{F}}$. Find hyperplanes of the form $v(f_i) = c$ that divide $P_{\tilde{F}}$ into two sets of equal measure like in the proof. The intersection of these hyperplanes lies within $P_{\tilde{F}}$ just as before.

[13]The first and second parts of Lemma 5 are Corollary 8 and Theorem 10 of their paper.

Now use the first part of Lemma 5 to show that the choice of representative firms do not change the median utility level of f.

Appendix B: Proofs of Theorems 3 and 4

Before we proceed to the proof of Theorem 3, we prove a couple of lemmas. As a by-product we get that the discrete core converges to the continuous core stated as Theorem 4 which is of independent interest from the rest of the paper.

First note that if Δ is small enough then any discrete stable matching is optimal.

Each single agent has a utility level of zero and thus only affects the lower bound of the utility levels of the agents on the other side of the market. Therefore, we can restrict $Core$, $Core(\Delta)$ and $Feas(\Delta)$ to the space of utility imputations of the matched agents. Moreover, since the sum of the transfers are zero for matched agents by Lemma 3, we can further restrict these sets to the space of utility imputations for the matched firms.[14] Note that in this space $Core$ has full-dimension by the assumption.

Lemma 6: *Let $v \in Feas(\Delta) \cap Core$ then $v \in Core(\Delta)$.*

Proof. If a utility imputation is in $Core$ and still feasible for some increment Δ then it is also in $Core(\Delta)$ as the discrete core considers blocking pairs with only restricted transfers. ∎

$Core$ is a polytope characterized by the following two types of hyperplanes. One type is of the form $c_1 \geq v(f) \geq c_2$, and the other is of the form $d_1 \geq v(f) - v(f') \geq d_2$. Let C_Δ be a polytope with faces parallel to these

[14] Lemma 3 states this result for the continuous case. Similarly, the same result holds for the discrete case. This can be proven as follows. By stability the transfer of a single agent or the sum of transfers for a matched pair cannot be negative. However, the sum of all transfers must be non-positive by feasibility. Therefore we conclude that the transfer of a single agent and the sum of transfers for a matched pair must be zero.

hyperplanes defined as follows. The faces of $Core$ of the first type are still the faces of C_Δ whereas the second type faces are shifted and can be written as $d_1 + 2\Delta \geq v(f) - v(f') \geq d_2 - 2\Delta$. The next lemma shows that $Core(\Delta)$ has to lie inside this polytope.

LEMMA 7: *For any $\Delta > 0$ we have $Core(\Delta) \subseteq C_\Delta$.*

Proof. The individual rationality constraints and blocking pairs with one unmatched agent give the same inequalities both in the continuous case and the discrete case. Hence, inequalities of the form $c_1 \geq v(f) \geq c_2$ do not change.

For any firm-worker pair (f, w') where $w' = \mu(f')$ the no-blocking condition in the continuous case can be written as $v(f) + v(w') \geq u_f(w') + u_{w'}(f)$. Since $v(f') + v(w') = u_{f'}(w') + u_{w'}(f')$ by Lemma 3 the last inequality is equivalent to $v(f) - v(f') \geq u_f(w') + u_{w'}(f) - u_{f'}(w') - u_{w'}(f') = d_2$. To prove the claim we need to show that $v(f) - v(f') \geq d_2 - 2\Delta$ for any utility imputation in the discrete case.

Assume for contradiction that $v(f) - v(f') < d_2 - 2\Delta$ which is the same as $v(f) + v(w') < u_f(w') + u_{w'}(f) - 2\Delta$ by footnote 14. Let $v^+(f) = \underset{k \in \mathbb{Z}}{min}\{u_f(w') + \Delta k | u_f(w') + \Delta k > v(f)\}$- the smallest real number greater than $v(f)$ which is an integer multiple of Δ away from $u_f(w')$. Define $v^+(w') = \underset{k \in \mathbb{Z}}{min}\{u_{w'}(f) + \Delta k | u_{w'}(f) + \Delta k > v(w')\}$ similarly. We claim that (f, w') form a blocking coalition. Now, we get that $v^+(f) + v^+(w') \leq v(f) + v(w') + 2\Delta < u_f(w') + u_{w'}(f)$. Hence, (f, w') forms a blocking pair (with the appropriate transfer). A contradiction to the assumption.

Analogously, we show that $d_1 + 2\Delta \geq v(f) - v(f')$, which completes the proof. ∎

Proof of Theorem 4. By Lemma 7 every $s' \in Core(\Delta)$ has to be inside C_Δ which implies that there exists $s \in Core$ such that $d(s, s') \leq 2\Delta$. Hence, $\underset{s' \in Core(\Delta)}{sup} \ \underset{s \in Core}{inf} d(s, s') \leq 2\Delta$. Therefore, $\underset{s' \in Core(\Delta)}{sup} \ \underset{s \in Core}{inf} d(s, s') \to 0$ as

$\Delta \to 0$ by the squeeze theorem. To complete the proof we have to show that $\sup_{s\in Core} \inf_{s'\in Core(\Delta)} d(s,s') \to 0$ as $\Delta \to 0$.

Since $Feas(\Delta) \cap Core \subseteq Core(\Delta)$ by Lemma 6, we have

$$\sup_{s\in Core} \inf_{s'\in Core(\Delta)} d(s,s') \leq \sup_{s\in Core} \inf_{s'\in Core\cap Feas(\Delta)} d(s,s').$$

As $\Delta \to 0$, $Feas(\Delta)$ becomes a finer lattice and hence $\sup_{s\in Core} \inf_{s'\in Core\cap Feas(\Delta)} d(s,s') \to 0$ by the full-dimension assumption. Therefore, by the squeeze theorem we conclude that $\sup_{s\in Core} \inf_{s'\in Core(\Delta)} d(s,s') \to 0$ as $\Delta \to 0$ since $\sup_{s\in Core} \inf_{s'\in Core(\Delta)} d(s,s') \geq 0$ for every Δ. ∎

For any finite set S let $|S|$ be the number of elements of S. By the next lemma, we get that the number of discrete core points outside of the continuous core becomes insignificant compared to the number of discrete core points when Δ becomes small.

Lemma 8: *$\frac{|Core(\Delta)-Core|}{|Core(\Delta)|} \to 0$ as $\Delta \to 0$.*

Proof. This follows from the full-dimension assumption. Since $Core(\Delta) \subseteq C_\Delta$ for all Δ by Lemma 7 and $Core(\Delta) \subseteq Feas(\Delta)$, we have $Core(\Delta) - Core \subseteq (C_\Delta \cap Feas(\Delta)) - Core$ and hence $|Core(\Delta) - Core|/|Core(\Delta)| \leq |(C_\Delta \cap Feas(\Delta)) - Core|/|Core(\Delta)|$. As $\Delta \to 0$, $|(C_\Delta \cap Feas(\Delta)) - Core|$ grows at most linearly with Δ^{r-1} where r is the number of matched pairs and hence the dimension of the core. On the other hand, $|Core(\Delta)|$ grows linearly with Δ^r. Therefore, $|(C_\Delta \cap Feas(\Delta)) - Core|/|Core(\Delta)| \to 0$. Hence, by the squeeze theorem we get $|Core(\Delta) - Core|/|Core(\Delta)| \to 0$. ∎

Our last lemma shows that for any agent a, the ratio of the number of discrete core points inside the continuous core which give the agent a utility level bigger than $v_{median}(a)$ and the number of such points which give a utility level smaller than $v_{median}(a)$ converges to one.

Lemma 9: *$\frac{|v\in Core\cap Core(\Delta)\ s.t.\ v(a)\leq v_{median}(a)|}{|v\in Core\cap Core(\Delta)\ s.t.\ v(a)\geq v_{median}(a)|} \to 1$ as $\Delta \to 0$ for all $a \in A$.*

Proof. By construction of v_{median} we know that $\lambda_r\{v \in Core\, s.t.\, v(a) \leq v_{median}(a)\} = \lambda_r\{v \in Core\, s.t.\, v(a) \geq v_{median}(a)\}$. Moreover, as $\Delta \to 0$ we have:

$$\frac{|v \in Core \cap Core(\Delta)\ s.t.\ v(a) \leq v_{median}(a)|}{\lambda_r\{v \in Core\ s.t.\ v(a) \leq v_{median}(a)\}/\Delta^r} \to 1,$$

and

$$\frac{|v \in Core \cap Core(\Delta) s.t. v(a) \geq v_{median}(a)|}{\lambda_r\{v \in Core | v(a) \geq v_{median}(a)\}/\Delta^r} \to 1.$$

Therefore, as $\Delta \to 0$,

$$\frac{|v \in Core \cap Core(\Delta)\ s.t.\ v(a) \leq v_{median}(a)|}{|v \in Core \cap Core(\Delta)\ s.t.\ v(a) \geq v_{median}(a)|} \to 1.$$

∎

Proof of Theorem 3. By Lemma 8 as $\Delta \to 0$ the number of stable utility imputations in $Core(\Delta)$ outside $Core$ becomes insignificant compared to those inside $Core$. By Lemma 9 v_{median} is the median utility imputation for those inside of $Core$ for small enough Δ. Hence, we have that $v_{median}(\Delta) \to v_{median}$. ∎

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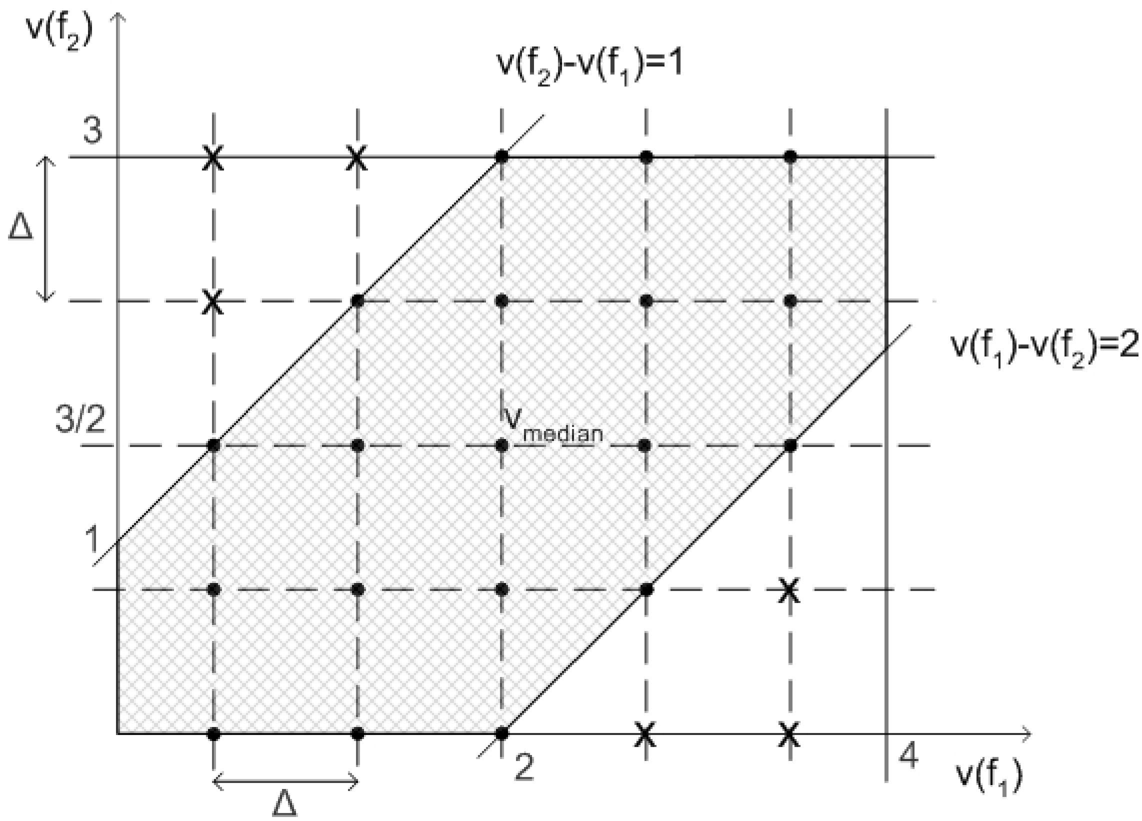

Figure 1: An example demonstrating the continuous core and the discrete core when $\Delta = 3/4$. The continuous core is the shaded area and the discrete core is the union of the nineteen nodes.