NBER WORKING PAPER SERIES

INFLATION UNCERTAINTY AND INTEREST RATES: THEORY AND EMPIRICAL TESTS

Richard Hartman

John H. Makin

Working Paper No. 906

NATIONAL BUREAU OF ECONOMIC RESEARCH
1050 Massachusetts Avenue
Cambridge MA 02138

June 1982

A paper by Levis Kochin (1980) stimulated our interest. Charles Nelson and Bill Strang provided helpful comments. Responsibility for any errors is our own. The research reported here is part of the NBER's research program in Financial Markets and Monetary Economics. Any opinions expressed are those of the authors and not those of the National Bureau of Economic Research.

NBER Working Paper #906
June 1982

Inflation Uncertainty and Interest Rates: Theory and Empirical Tests

ABSTRACT

This paper develops two models, one involving risk neutrality and the other risk aversion, which suggest that inflation uncertainty affects interest rates. Both models give rise to essentially the same interest rate equation for estimation. Empirical evidence supports the hypothesis that inflation uncertainty affects interest rates. Interpreted in terms of the risk neutral model, the empirical results suggest that inflation uncertainty has a negative impact on nominal interest rates and a positive impact on the expected real rate. If the results are interpreted in terms of the risk averse model, inflation uncertainty has a negative impact on nominal interest rates. The expected real rate is not of direct interest in a risk averse world.

The results raise real questions about the use of the Fisherian definition of the real interest rate in situations when there is uncertainty about inflation rates. It is argued that even with risk neutrality the Fisherian definition of the real rate is not the appropriate concept upon which to base economic decisions if inflation uncertainty is present. The appropriate concept is an expected real rate which involves an adjustment for uncertainty. Moreover, if the world is risk averse, the expected real rate is not a relevant concept for economic decisions.

Richard Hartman
John H. Makin
Department of Economics
DK-30
University of Washington
Seattle, WA 98195

(206) 543-5865

I. Introduction

In this paper we examine the relationship between interest rates and inflation uncertainty. We develop two models here, one involving risk neutrality and the other risk aversion, which suggest a theoretical relationship between inflation uncertainty and interest rates. Both models suggest essentially the same reduced form equation which we estimate using quarterly U.S. data for the period 1959-I to 1980-IV. The empirical results support the assertion that inflation uncertainty has an effect on interest rates; the estimated effect is highly significant and enters with the appropriate sign. These results are consistent with the findings of Levi and Makin (1979, 1981).

The theory developed here raises some important points. First, even with risk neutrality, the standard Fisherian definition of the real interest rate is not the relevant concept for economic decisions when the expected rate of inflation is the expected value of a random variable rather than a point expectation; the appropriate concept is an expected real rate which generally includes uncertainty effects. As noted at the beginning of the next section, this point has been made by several authors. Second, with risk aversion, economic decisions depend on higher order moments of probability distributions and the expected real rate which is important in the risk neutral case no longer plays the central role. These two facts raise important questions about the use of the standard Fisherian definition of the real interest rate in situations where there is uncertainty about the future inflation rate.

Section II develops our risk-neutral model and examines its implications. We begin with a simple model of a producer-consumer in a two-period world, then extend the analysis to cover market determination of the nominal and expected real rates of interest, and finally discuss the continuous time version of these rates. Section III examines a simple model with risk aversion. Again we begin with a single producer-consumer and then aggregate to discuss the determination of the equilibrium interest rate. Section IV contains the econometric estimates of the common reduced form equation suggested by both models. Finally, Section V contains a brief summary and some concluding comments.

II. The Risk-Neutral Case

In this section we investigate the relationship between inflation uncertainty and interest rates in a risk-neutral economy.

A central feature of this analysis is an expected real interest rate which plays a critical role in both the investment decision of producers and the borrowing or lending decision of individuals. This expected real rate, which differs in a fundamental way from the standard definition of the real rate associated with the name of Fisher (1930), has been examined by Blejer and Eden (1979), Boonekamp (1978), and Eden (1976) in papers dealing with portfolio choice and the demand for money in discrete time models, and by Fischer (1975, pp. 518-19) in a paper dealing with the demand for indexed bonds in a continuous time model.

With continuous discounting, the standard Fisherian definition of the real rate of interest, r_F, is the nominal rate of interest, i, less the expected rate of inflation, $\bar{\alpha}$: $r_F = i - \bar{\alpha}$. The corresponding definition for discrete time is given by[1] $1 + r_F = (1+i)/(1+\bar{\alpha})$.

However the expected real rate which enters both the production decisions and the consumption or borrowing decisions in this paper is given by

$$1 + r = (1+i)\ E[1/(1+\alpha)]$$

where E is the expectation operator, α is the inflation rate and r is the appropriately defined expected real interest rate. If α is a non-degenerate random variable $(1+i)/(1+\bar{\alpha})$ and $(1+i)\ E[1/(1+\alpha)]$ will differ and the difference may be large. This difference arises from the basic fact that the reciprocal of the expected value of a random variable is not equal to the expected value of the reciprocal. Since $1/(1+\alpha)$ is convex in α, Jensen's inequality implies $r > r_F$. As is clear from Fischer's analysis, the difference does not disappear in continuous time.

We begin by developing a simple model in which risk-neutral individuals make production and consumption and borrowing decisions and show that these decisions are based on an appropriately defined expected real rate of interest. We then investigate the effect of increased inflation uncertainty on the net demand for loans and the resulting effects on the equilibrium expected real and nominal interest rates. The section concludes with a discussion of continuous-time versions of these results.

The Model

Consider an individual consumer in a two-period world. The consumer receives nominal endowments (non-capital income) of W_0 at the beginning of period 0 and W_1 at the beginning of period 1. These can be interpreted as payments for labor services plus transfer payments. Wealth carried over from previous periods is included in W_0. W_1 need not be known with certainty in period 0. We assume $W_1 \geq 0$. The individual can borrow or lend "money" in the initial period at the known nominal interest rate i. Let B denote the individual's net borrowing. "Money" in each period is used to purchase a nonstorable good. In period 0 the good can be either consumed or used as the input to a production process whose output is the good in period 1. For simplicity assume that production is carried out in sole proprietorships. This, of course, implies that borrowing to cover production costs is included in net borrowing, B. Let X be the amount of the period-0 good used in production; the corresponding output in period 1 is F(X). The price of the good in period 0 is 1 and its price in period 1 is P. P is not known with certainty in period 0; P clearly satisfies $P = 1 + \alpha$ where α is the random inflation rate.

Let C_0 and C_1 denote consumption of the good in periods 0 and 1 respectively. The individual's period-0 budget constraint is

$$W_0 + B = C_0 + X. \tag{1}$$

In period 1, the individual faces the constraint

(2) $$W_1 + PF(X) = (1+i)\ B + PC_1.$$

If we were to follow the common practice of identifying risk neutrality with a linear von Neumann-Morgenstern utility function the individual would maximize

$$E\{C_0 + RC_1\}$$

where R is a subjective discount factor reflecting time preference. Since this objective function is linear in B, there is no unique, interior solution. In particular, if $1/R > E(1+i)/P$ the optimal current consumption is unboundedly large, if $1/R < E(1+i)/P$ it is unboundedly small; and if $1/R = E(1+i)/P$ any level is optimal. This is clearly not an adequate solution to the consumer's consumption or borrowing problem, and it suggests either that consumers cannot make these decisions in a risk neutral manner or that this specification of the risk-neutral objective function is inappropriate. The latter point of view is adopted here.

The difficulties just outlined arise because the marginal rate of substitution between consumption in the two periods is constant. To avoid these problems we assume the consumer maximizes

(3) $$U(C_0) + RU(EC_1)$$

where $U' > 0$, $U'' < 0$, and R is a subjective discount factor. This seems to be a reasonable way to specify the objective function of a risk-neutral consumer in a two-period setting. It reflects risk neutrality in that it depends only on the realized or the expected values of those variables of ultimate interest, namely, consumption in the two periods. However it does not require a constant marginal rate of substitution between current consumption and expected future consumption.

From (2) it is clear that

$$EC_1 = E(W_1/P) - E[(1+i)B/P] + F(X). \tag{4}$$

If (1) and (4) are used to eliminate C_0 and EC_1 in (3), the consumer's problem is to choose B and X to maximize

$$g = U(W_0+B-X)+RU\{E(W_1/P)-E[(1+i)B/P]+ F(X)\}.$$

The first-order conditions are $\partial g/\partial B = 0$ which implies

$$U'(C_0) = RU'(EC_1)E[(1+i)/P] \tag{5}$$

and $\partial g/\partial X = 0$ which implies

$$U'(C_0) = RU'(EC_1) F'(X). \tag{6}$$

Optimal B and X are found by solving (5) and (6). Note that once B and X are determined, so also is the optimal C_0 through (1). The resulting C_1 will be determined by (2) once the random variables are observed in period 1.

Consider first the production decision. Since the nominal rate of interest is known in period 0, (5) and (6) imply

$$(7) \qquad F'(X) = (1+i)\, E(1/P),$$

or

$$(8) \qquad \frac{1}{(1+i)E(1/P)}\, F'(X) = 1.$$

The interpretation of (8) is straightforward. The cost of a unit of the input in period 0 is 1. The marginal unit of the input makes an expected real contribution of $F'(X)$ to real wealth in period 1. The discounted value of this contribution is to be equated to the price of the input, and the discount factor is based on $(1+i)E(1/P)$. Note that (7) or (8) essentially gives a Fisher Separation Theorem in that the optimal production-investment decision is independent of consumer preferences. However, the expected real interest rate implicit in (8) is not the standard Fisherian real interest rate. Note also that

$$\frac{1}{(1+i)E(1/P)}\, F'(X) \neq \frac{1}{(1+i)}\, EPF'(X)$$

so it is inappropriate to discount the expected nominal value of the marginal product using the nominal interest rate.

If there are diminishing returns in production, $F''(X) < 0$, and (7) or (8) can be solved for optimal X as a function of $(1+i)E(1/P)$; in essence, the demand for capital is a function of the appropriately defined expected real interest rate.

Consider now the individual's consumption or net borrowing decision. If there are diminishing returns in production, the optimal X can be substituted into (1) and (4), and then (5) can be solved for B as a function of W_0, $E(W_1/P)$, and $(1+i)E(1/P)$. Optimal net borrowing is clearly a function of the appropriately defined expected real interest rate, $(1+i)E(1/P)$. If notation is simplified by letting

$$S = (1+i)E(1/P), \tag{9}$$

the functional relationship for optimal net borrowing can be denoted

$$B = \bar{B}(W_0, E(W_1/P), S). \tag{10}$$

If $F(X)$ exhibits constant returns, $F''(X) \equiv 0$ and there is no demand function for X. However, in this case marginal and average products are equal, and (7) implies that

$$F(X) = X(1+i)E(1/P).$$

If this is substituted into (4), expected consumption in period 1 is

$$EC_1 + E(W_1/P) - (B-X)(1+i)\ E(1/P).$$

There are no unique solutions for B and X under these conditions. This is to be expected since the individual scale of production is not determined under constant returns to scale, and optimal net borrowing will vary as the scale of production varies. Note, however, that B-X can be determined, and optimal C_0 is a function of W_0, $E(W_1/P)$, and S.

The Effects of Inflation Uncertainty and Variation in Expected Inflation on Interest Rates

The notion of increased uncertainty we use is the mean preserving spread analyzed by Rothschild and Stiglitz (1970). We will be making use of the fact that the expected value of a convex function increases as its argument undergoes a mean preserving spread.

We begin by considering the case of constant returns to scale in production. In this case $F'(X)$ is constant, and (7) determines the equilibrium expected real interest rate. Under these conditions the expected real interest rate is constant, and any variation in the distribution of P brings about a change in the equilibrium nominal interest rate to keep $S = (1+i)E(1/P)$ constant.

The effect of increased inflation uncertainty on the nominal interest rate is easily determined here. Since $1/P$ is a convex function, a mean preserving spread in the distribution of P will increase $E(1/P)$. If S is to remain constant, this requires that i fall. Note also that $1/P$ is decreasing in P, and therefore an increase in the mean of P (holding other central moments constant) will decrease $E(1/P)$. Given S, this implies that i must increase.

If there are diminishing returns to scale in production, the market determination of equilibrium interest rates can be investigated using the following simple model. Assume there are n individuals in the economy. They may have different preferences and different endowments, but all are risk neutral in the sense defined above and all have the same (marginal) distribution for P. Let the superscript k index individuals. Equilibrium requires

$$(11) \qquad \sum_{k=1}^{n} \bar{B}^k(W_0^k, E(W_1^k/P), S) \equiv B^*(S, E(W_1^1/P), \ldots, E(W_1^n/P), W_0^1, \ldots, W_0^n) = 0$$

where B^* is the aggregate net demand function for borrowing. Given W_0^k and $E(W_1^k/P)$, $k = 1,2,\ldots,n$, the market can be viewed as determining the equilibrium S. Given the equilibrium S and the distribution of P, the equilibrium nominal interest rate is then determined from (9).

Suppose first that $E(W_1^k/P)$ does not vary as the distribution of P changes for all k, i.e., all period-1 endowments are fully indexed. If there is a unique equilibrium, any variation in the distribution of P must lead to variation in the nominal interest rate, i, in such a way that the expected real interest rate, $S = (1+i)E(1/P)$, remains constant.

If some of the period-1 endowments are not fully indexed, $E(W_1^k/P)$ will vary as the distribution of P changes. This will generally affect the net demand for borrowing and the resulting equilibrium expected real interest rate. To investigate this effect we divide $E(W_1^k/P)$ into two parts, one part being fully indexed and the other not indexed at all. Let

$$(12) \qquad E(W_1^k/P) = w_1^k + E(W_1^{ck}/P)$$

where w_1^k is the real value of the fully indexed part of individual k's period-1 endowment and W_1^{ck} is the nominal value of the nonindexed part. w_1^k is clearly independent of all variation in the distribution of P. Since W_1^{ck} is not indexed at all, it is independent of the distribution of P. Now, W_1^{ck}/P is decreasing and convex in P. Thus, if $W_1^{ck} > 0$, an increase in the mean of P will decrease $E(W_1^k/P)$, and a mean preserving spread in the distribution of P will increase $E(W_1^k/P)$.

It is straightforward but tedious to show that an increase in $E(W_1^k/P)$ increases the net demand for borrowing for the model considered here,

(13) $\partial\bar{B}^k/\partial E\,(W_1^k/P) > 0.$

This, of course, is to be expected since any increase in <u>future</u> real income, *c. p.*, would be expected to increase current net borrowing.

We shall now make the assumption that (11) determines a unique, stable equilibrium. Stability requires that

(14) $\partial B^*/\partial S < 0$

around the equilibrium point. Consider now the effect of variation in $E(W_1^k/P)$ on equilibrium S. The differential of (11) is

$$\sum_{k=1}^{n} \frac{\partial\bar{B}^k}{\partial E(W_1^k/P)}\, dE(W_1^k/P) + \sum_{k=1}^{n} \frac{\partial\bar{B}^k}{\partial S}\, dS \equiv$$

$$\sum_{k=1}^{n} \frac{\partial\bar{B}^k}{\partial E(W_1^k/P)}\, dE(W_1^k/P) + \frac{\partial B^*}{\partial S}\, dS = 0.$$

Given (13) and (14), any increase (decrease) in some or all of the $E(W_1^k/P)$, $k = 1,2,\ldots,n$, will increase (decrease) equilibrium S.

Therefore, so long as $W_1^{ck} \geq 0$ for all k with strict inequality for some k, an increase in the mean of P decreases equilibrium S. This, of course, is equivalent to a decrease in the expected real rate of interest. This result is similar to the Mundell (1963) and Tobin (1965) effects in that a rise in expected (anticipated) inflation depresses the equilibrium expected real rate of interest. Moreover, since a mean preserving spread in P increases $E(W_1^k/P)$, we have that increased inflation uncertainty increases the equilibrium S so long as $W_1^{ck} \geq 0$ for all k with strict inequality for some k.

The results of this subsection can be summarized briefly. For the risk-neutral model considered here, the appropriately defined expected real rate of interest, r = 1-S, will be constant in the cases of constant returns to scale in production or fully indexed future endowments (noncapital income). However, if there are diminishing returns to scale in production, $F'' < 0$, and if some of the individual's have some nonindexed future endowments, then an increase in expected inflation decreases S = 1+r while increased uncertainty about the inflation rate increases S = 1+r. We can denote these effects by writing r as a function of expected inflation, $\bar{\alpha}$, and a shift parameter, v^2, which corresponds to a mean preserving spread in the distribution of P,

$$r = r(\bar{\alpha}, v^2) \tag{15}$$

where $\partial r/\partial\bar{\alpha} \leq 0$ and $\partial r/\partial v^2 \geq 0$. The effects on nominal interest rates follow from (1+r) = S = (1+i) E(1/P) and the fact that 1/P is decreasing and convex. Given S, an increase in expected inflation increases the nominal interest rate, i, while increased uncertainty about the inflation rate decreases i.

Continuous Time

We have argued that the appropriate discrete-time definition of the expected real interest rate satisfies

$$(16) \qquad S = 1 + r = (1+i)E(1/P) = (1+i)\ E(\frac{1}{1+\alpha})$$

where α is the random inflation rate. We now investigate the corresponding relationship for the continuous discounting case. To do this, we divide the period into n subperiods of equal length with compounding carried out in each subperiod and then take the limit as $n \to \infty$. The analysis is complicated by the fact that the inflation rate is random. Once the period is divided into n subperiods, it will be necessary to associate a random inflation rate with each subperiod.

If we divide the period into n subperiods with interest being compounded in each subperiod, and if the growth factor in prices over the k^{th} subperiod is given by $(1 + \alpha_k^{(n)})$ where $\alpha_k^{(n)}$ is a random variable, then (16) is replaced by

$$(17) \qquad (1 + \frac{r}{n})^n = (1 + \frac{i}{n})^n\ E\ \frac{1}{(1+\alpha_1^{(n)})(1+\alpha_2^{(n)})\ldots(1+\alpha_n^{(n)})} \ .$$

We take the $1 + \alpha_k^{(n)}$ to be identically and independently distributed. This is, roughly speaking, the multiplicative equivalent of a stochastic process with independent increments. Let $z_k^{(n)} = 1 + \alpha_k^{(n)}$, $k = 1,2,\ldots,n$. Then,

(18) $$E \frac{1}{z_1^{(n)}} \frac{1}{z_2^{(n)}} \cdots \frac{1}{z_n^{(n)}} = E \frac{1}{z_1^{(n)}} E \frac{1}{z_2^{(n)}} \cdots E \frac{1}{z_n^{(n)}}$$

$$= [E \frac{1}{z_k^{(n)}}]^n$$

since the $z_k^{(n)}$ are independently and identically distributed. It is clear that the results will depend on how the random variables $\alpha_k^{(n)}$ are specified. Since prices cannot be negative, we must have $z_k^{(n)} = 1 + \alpha_k^{(n)} \geq 0$. Two specifications come naturally to mind; one involves letting $z_k^{(n)}$ be log normal while the other lets it have a gamma probability distribution. We consider these in turn.

Let $z_k^{(n)} = (1 + \alpha_k^{(n)})$ be lognormally distributed, i.e., log $(1 + \alpha_k^{(n)})$ is a normal variate with a mean and a variance of, say, $\frac{\mu}{n}$ and $\frac{\sigma^2}{n}$ respectively.[2] (Note that $\frac{\mu}{n}$ and $\frac{\sigma^2}{n}$ are the mean and variance of log $(1 + \alpha_k^{(n)})$ and not of the underlying random variable $\alpha_k^{(n)}$). Note that the mean price change over a subperiod and its variance approach zero as the subperiods get shorter ($n \to \infty$). Thus, we are requiring that prices do not behave too erratically over very short periods of time.

If $z_k^{(n)}$ is a lognormal variate then so also is $1/z_k^{(n)}$, and log $[1/z_k^{(n)}]$ has mean $\frac{-\mu}{n}$ and variance $\frac{\sigma^2}{n}$. Therefore

$$[E[1/z_k^{(n)}]]^n = [e^{\frac{-\mu}{n} + \frac{1}{2}\frac{\sigma^2}{n}}]^n .$$

With the lognormal specification (17) gives

$$(1 + \frac{r}{n})^n = (1 + \frac{i}{n})^n e^{-\mu + \frac{1}{2}\sigma^2}.$$

As $n \to \infty$, this becomes

$$e^r = e^{i - \mu + \frac{1}{2}\sigma^2},$$

so that

$$r = i - \mu + \frac{1}{2}\sigma^2. \tag{19}$$

Recall that the mean, μ, and variance, σ^2, here refer to the mean and variance of log $(1 + \alpha)$ rather than to the mean and variance of α itself.

Suppose now that $z_k^{(n)} = 1 + \alpha_k^{(n)}$ is a gamma variate with mean $1 + (\bar{\alpha}/n)$ and variance v^2/n. Notice that this again implies that the mean price change over the subperiod and its variance both approach zero as $n \to \infty$; we again require that prices do not behave too erratically over very short periods of time. Some straightforward but rather tedious calculations give[3]

$$E \frac{1}{z_k^{(n)}} = \frac{1 + \frac{\bar{\alpha}}{n}}{(1 + \frac{\bar{\alpha}}{n})^2 - \frac{v^2}{n}} \tag{20}$$

$$= \frac{(1 + \frac{\bar{\alpha}}{n})}{\left(1 + \frac{\bar{\alpha} - \frac{1}{2}v^2 + \frac{v}{2}\sqrt{v^2 - 4\bar{\alpha}}}{n}\right)\left(1 + \frac{\bar{\alpha} - \frac{1}{2}v^2 - \frac{v}{2}\sqrt{v^2 - 4\bar{\alpha}}}{n}\right)}$$

Substituting (20) into (18) and that into (17) gives

$$(1+ \frac{r}{n})^n = (1+ \frac{i}{n})^n \frac{(1 + \frac{\overline{\alpha}}{n})^n}{(1+ \frac{\overline{\alpha} - \frac{1}{2}v^2 + \frac{v}{2}\sqrt{v^2 - 4\overline{\alpha}}}{n})^n (1+ \frac{\overline{\alpha} - \frac{1}{2}v^2 - \frac{v}{2}\sqrt{v^2 - 4\overline{\alpha}}}{n})^n} .$$

As $n \to \infty$ this becomes

$$e^r - e^i \frac{e^{\overline{\alpha}}}{e^{\overline{\alpha} - \frac{1}{2}v^2 + \frac{v}{2}\sqrt{v^2 - 4\overline{\alpha}}} \, e^{\overline{\alpha} - \frac{1}{2}v^2 - \frac{v}{2}\sqrt{v^2 - 4\overline{\alpha}}}}$$

which simplifies to

$$e^r = e^{i - \overline{\alpha} + v^2}$$

so that

(21) $$r = i - \overline{\alpha} + v^2.$$

In other words, if the $z_k^{(n)}$ are independent and identical gamma variates, then with continuous discounting the real rate of interest is the nominal rate less the expected rate of inflation plus the variance of the inflation rate. We can identify the shift parameter for the mean preserving spread with the variance, substitute (15) into (21), and rearrange to get

(22) $$i = r(\overline{\alpha},\ v^2) + \overline{\alpha} - v^2$$

where $\partial r / \partial \overline{\alpha} \leq 0$ and $\partial r / \partial v^2 \geq 0$.

III. A Simple Risk Averse Case

In this section we briefly investigate the effect of inflation uncertainty on the interest rate in an economy of risk-averse individuals. Except as noted below, we continue to use the notation of the previous section.

We begin by considering the behavior of a single individual who maximizes

$$E \{U(C_0) + R\ U(C_1)\}$$

where U is a vonNeumann-Morgenstein utility function and R is a subjective discount factor. We will simplify the analysis greatly by making the following two assumptions:

(i) There is no production in the model. Returns from production are simply included in the period-1 endowment; and

(ii) The vonNeumann-Morgenstein utility function is logarithmic,

$$U(C) = \log C.$$

Our justifications of the logarithmic utility function are its relative simplicity and Arrow's (1965, p. 37) argument that the appropriate form of the utility function is logarithmic if we wish to restrict ourselves to the class of functions having constant relative risk aversion.

If there is no production

$$C_0 = W_0 + B.$$

As in equation (12) above, we explicitly recognize that some of the period-1 endowment may be indexed while some of it may not. Real consumption in period-1 is given by

$$C_1 = w_1 + \frac{W_1^c}{P} - (1+i)\,\frac{B}{P}$$

where w_1 is the real value of the fully indexed part of the period-1 endowment and W_1^c is the nominal value of the nonindexed part. Both w_1 and W_1^c may be random, but W_1^c is independent of P.

With the logarithmic specification the first order condition for net borrowing implies

$$\frac{1}{W_0+B} = RE\{\frac{1}{w_1 + \frac{W_1^c}{P} - (1+i)\,\frac{B}{P}}\;\frac{(1+i)}{P}\}$$

or

(23) $$\frac{1}{W_0+B} = RE\{\frac{(1+i)}{Pw_1 + W_1^c - (1+i)B}\}.$$

The optimal B clearly depends on the nominal interest rate, i, on W_0, and on the parameters of the distribution of W_1^c, w_1 and P. The expected real interest rate which played a central role in the risk-neutral model does not enter here since (1+i)/P does not enter as a simple linear term under the expectation sign. This will be true generally for models with risk aversion.

The effects of variation in the distribution of P on the individual's net demand for borrowing are easily determined. The left hand side of (23) is a decreasing function of B; it is graphed in Figure 1.

The right hand side of (23) is an increasing function of B; it is graphed as A in Figure 1. The intersection of these two functions gives optimal net borrowing, $\bar{B}$, in Figure 1. Now, the random variable P appears only on the right hand side of (23). If $w_1 > 0$, $(1+i)/[P w_1 + W_1^c - (1+i)B]$ is decreasing and convex in P. Therefore, if $w_1 > 0$, an increase in the mean of P, c.p., will decrease the right hand side of (23) for any given B; this shifts the curve A down, say, to A". The optimal $\bar{B}$ therefore increases with an increase in the expected inflation rate, $\bar{\alpha}$. Moreover, since the expression inside the expectation sign on the right hand side of (23) is convex in P, a mean preserving spread in the distribution of P will increase the right hand side of (23) for any given B. This shifts the curve A up, say to A', and leads to a decrease in $\bar{B}$. If we denote the individual's net demand for borrowing as a function of the nominal interest rate, i, the expected inflation rate, $\bar{\alpha}$, and a shift parameter, v^2, reflecting a mean preserving spread in the distribution of P,

(24) $\bar{B}\ (i,\ \bar{\alpha},\ v^2).$

We have that $\partial\bar{B}/\partial\bar{\alpha} > 0$ and $\partial\bar{B}/\partial v^2 < 0$ so long as $w_1 > 0$.

We turn now to the determination of the equilibrium interest rate. Assume there are n individual's in the economy. They may have different endowments, but all have the same vonNeumann-Morgenstern utility function and all have the same marginal distribution of P. We let the superscript k index individuals. We assume $w_1^k \geq 0$ for all k with strict inequality for some k.

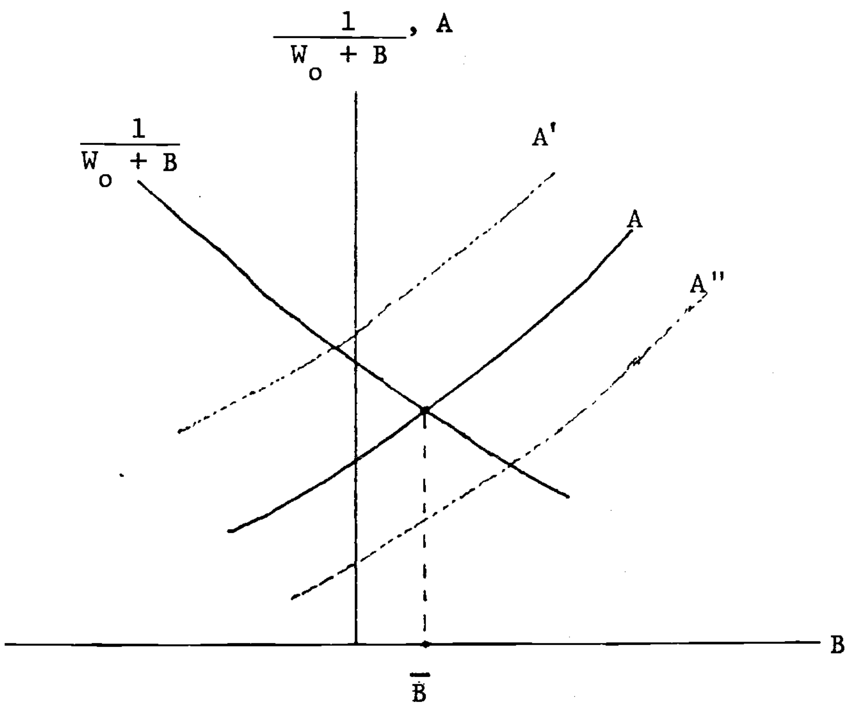

Figure 1

Market equilibrium requires

(25) $$\sum_{k=1}^{n} \bar{B}^k (i, \bar{\alpha}, v^2) \equiv B^* (i, \bar{\alpha}, v^2) = 0.$$

We again assume a unique, stable equilibrium. Stability requires

(26) $$\partial B^*/\partial i < 0,$$

around the equilibrium point. Under these conditions we can solve (25) for equilibrium i in terms of $\bar{\alpha}$ and v^2,

(27) $$i = i(\bar{\alpha}, v^2).$$

Standard operations show that

$$\frac{\partial i}{\partial \bar{\alpha}} = - \frac{\sum_{k=1}^{n} \partial \bar{B}^k / \partial \bar{\alpha}}{\partial B^* / \partial i}$$

and

$$\frac{\partial i}{\partial v^2} = - \frac{\sum_{k=1}^{n} \partial \bar{B}^k / \partial v^2}{\partial B^* / \partial i} \quad .$$

Given (26) and the properties of (24), it is clear that $\partial i / \partial \bar{\alpha} > 0$ and $\partial i / \partial v^2 < 0$ so long as $w_1^k \geq 0$ for all k with strict inequality for some k. In other words, given this specification of the risk-averse model, the equilibrium nominal interest rate increases with expected inflation and decreases with uncertainty about the inflation rate.

IV. Empirical Tests

We have developed two models which suggest that the equilibrium nominal interest rate responds to variations in the expected inflation rate and to uncertainty in the inflation rate.

The risk-neutral model operates through an appropriately defined expected real interest rate which, with the gamma specification, gives rise to equation (21),

(21) $$i = r + \bar{\alpha} - v^2$$

giving the nominal interest rate in terms of the expected real rate and the mean and variance of the inflation rate. If the expected real rate is constant, the variance, v^2, in (21) enters with a coefficient of (-1).

Statistical investigations of the behavior of the real rate in models with point expectations have appeared with increasing frequency since publication of Fama's (1975) provocative article and suggest that it is inappropriate to treat the real rate as a constant, particularly outside of the 1953-1 through 1973-2 sample period investigated by Fama. Nelson and Schwert (1977) argued that Fama's test of the joint hypothesis of rationality of inflation forecasts and constancy of the real rate was not sufficiently powerful and after applying more powerful tests concluded that the data permitted rejection of the constant real rate hypothesis. Mishkin (1981) showed the sensitivity of Fama's tests to sample period. Studies by Levi and Makin (1979, 1981), Tanzi (1980), Mishkin (1981), Makin (1982), Hafer and Hein (1982) and others have found that the real rate may be affected, at least in the short run, by a number of specific variables including measures of the stage of the business cycle, unanticipated changes in the money supply, measures of inflation variability, total borrowing demand relative to GNP and anticipated inflation itself.

The risk-neutral model considered in this paper suggests that the expected real rate may respond negatively to the expected inflation rate and positively to uncertainty about the inflation rate. (See (15) above.) If the expected real rate is not constant, which seems a safe operating hypothesis, and if we maintain the gamma specification of the distribution of P, we have (22) of Section II as a specification of an interest rate equation for the risk neutral model. The other variables which the studies mentioned above have found to be important determinants of the real rate can be viewed as variables from outside our model which shift the aggregate net demand for borrowing and hence affect the equilibrium expected real interest rate.

The expected or anticipated inflation rate may affect the real rate for the reason given in Section II above or because of the Mundell and Tobin effects.

In a model of point expectations, the hypothesized impact of a money surprise on real interest arises from an assumption of "sticky" price adjustment. Money growth above its anticipated level results in an excess supply of money if prices are sticky in the short run, assuming also that surprise money growth does not immediately cause a rise in real income sufficient to absorb excess money supply. Until prices adjust fully to absorb excess money supply, the only alternative is for real interest to fall thereby lowering (cet. par.) nominal interest by an amount sufficient to clear the money market.[4]

The real rate may also be affected by budget deficits. In addition to the possible impact of deficits on anticipated inflation, a measure of the deficit or total borrowing relative to GNP can be viewed as a shift parameter in our function given the aggregate net demand for borrowing. An increase in this ratio corresponds to an upward shift in the net demand function, and, as was shown in Section II, this will tend to increase the expected real rate.

Combining these variables with those in our equation (15) gives rise to an expression for the expected real rate involving expected inflation, $\bar{\alpha}$, a measure of borrowing relative to GNP, d, a measure of surprises in money growth, M, and the variance of the inflation rate or some other measure of uncertainty, v^2. In linearized form we have

$$r = \lambda_0 - \lambda_1 \bar{\alpha} + \lambda_2 d - \lambda_3 M + \lambda_4 v^2,$$

with $\lambda_i > 0$, $i = 0,\ldots,4$. Substituting this in (21) and adding an error term, e, gives the linear equation

$$i = \lambda_0 + (1-\lambda_1)\bar{\alpha} + \lambda_2 d - \lambda_3 m - (1-\lambda_4)v^2 + e. \tag{28}$$

The expected real rate does not play a role in our model of interest rate determination with risk aversion. However, as shown in Section III, the equilibrium nominal rate is an increasing function of expected inflation, $\bar{\alpha}$, and a decreasing function of a measure of inflation uncertainty, v^2, which we can identify with the variance. Moreover, the arguments given above that the measures d and M may act as shift parameters in the aggregate net demand function for borrowing can be applied here too. If we combine these shift parameters with the parameters explicitly recognized in (27), linearize, and introduce an additive error term, e, we have

$$i = \beta_0 + \beta_1 \bar{\alpha} + \beta_2 d - \beta_3 M - \beta_u v^2 + e \tag{29}$$

where $\beta_i > 0$, $i = 1,\ldots,4$. Equations (28) and (29) are not generally distinguishable, and we therefore are not in a position to test which of our two models is appropriate. However, both imply that inflation uncertainty affects the nominal interest rate, a proposition which we now test.

The common equation given by (28) or (29) will be estimated employing the transfer function methodology developed by Box and Jenkins (1970). The methodology has a number of advantages over usual, linear econometric estimation techniques. It is possible to entertain any autoregressive (AR), moving average (MA), or combined ARMA representation of residuals and to estimate it simultaneously with relationships between the endogenous and exogenous variable(s). Usual methodology only allows iterative estimation of a first-order autoregressive process to represent residuals.[5] The transfer function also enables parsimonious representation of possible distributed lag relationships between the endogenous variable and exogenous variables. In addition, transfer function output includes leading and lagging cross correlations between the endogenous and exogenous variables which enable the investigator to check for all possible lagged relationships while also providing a check on possible feedback running from the endogenous variable to exogenous variables. It is worth noting that the transfer function in its simplest representation produces estimation results identical to usual OLS methodology with serially uncorrelated residuals. As is well-known, however, the standard assumptions required for OLS estimation are often violated, and in such cases transfer function technique provides the investigator with a useful and more flexible tool with which to test for empirical relationships.

Overall, the transfer function is reasonably well described as a means of combining time series and structural explanations of behavior of economic variables such as interest rates. By providing maximum flexibility with respect to the form of the ARMA model required to model residuals, the procedure imposes a more stringent test of the existence of a relationship between dependent and independent variables. In effect, significant explanatory variables must explain the residuals from the best ARMA model representing time series behavior of the dependent variable.

The result of estimating the equation given by (28) or (29) over the 1959-I - 1980-IV sample period is reported as equation (1.1) in Table 1. Since the ratio of total borrowing to GNP (d) is not significant even at the 10 percent level, equation (1.1) is reestimated omitting that variable with the result reported as equation (1.2).[6] A number of conclusions emerge. Anticipated inflation produces a positive impact upon nominal interest. The estimated coefficient on $\bar{\alpha}$ is 0.791, significantly below unity. This is consistent with the risk-neutral model where expected inflation affects the expected real rate, either through our (15) or through the Mundell and Tobin effects. It is also consistent with the risk averse model, (27), and the fact that $\partial i/\partial\bar{\alpha} < 0$ as shown in Section III. The money "surprise" variable carries the anticipated negative sign and is significant at the 0.01 level.

The primary object of our investigation, the effect of inflation uncertainty, produces the anticipated negative impact on the nominal interest rate and is significant at the 0.01 level. The value of the coefficient, -0.271, is significantly different from -1. If we adopt the risk-neutral version of the model given by (21), this is consistent with having the expected real rate depend positively on inflation

TABLE 1

ESTIMATION OF THE INTEREST RATE EQUATION

Dependent Variable: (3 Mo. T-Bill Rate)	Constant	Anticipated Inflation[1]	d^2	M^3 Money Surprise	Variance of Inflation[4]	$\bar{R}^2$	F	SE[5]
(1.1)	3.15 (3.92)	0.733 (5.83)	3.77 (1.17)	-0.393 (5.47)	-0.216 (2.07)	0.925	150.78	0.658
(1.2)	3.52 (4.72)	0.791 (6.74)		-0.435 (8.23)	-0.271 (2.78)	0.925	174.87	0.659
(1.3)	3.28 (4.00)	0.666 (5.07)		-0.385 (5.47)		0.920	199.2	0.679

(Sample Period: 1959-I - 1980-IV; t-statistics are in parentheses).

[1]Anticipated inflation is based on Livingston survey data on 6-month inflationary expectations. Interpolation is employed to obtain a quarterly series.

[2]d is defined as total borrowing divided by GNP. See Makin (1982) for further discussion.

[3]The money surprise is measured by residuals from an AR-1, MA-5 model of M1-B growth estimated using quarterly data from 1959-I - 1980-IV. The chi-square test of the hypothesis that residuals are white noise has a significance level of 0.94.

[4]The variance of inflation is measured by the variance across respondents to the Livingston survey on 6-month inflationary expectations. Variance lagged one quarter produced the only significant impact on the dependent variable and that result is reported here.

[5]Both equations (1.1) and (1.2) are estimated as transfer functions. Residuals are modeled by an AR-1, MA2,4 model (all terms significant at the 0.01 level). The significance level of the chi-square test that the first 24 residuals from equation (1.1) are white noise is 0.873; for equation (1.2) it is 0.916.

uncertainty as argued in Section II. Alternatively, it is consistent with the risk-averse model in which the nominal interest rate depends negatively on inflation uncertainty.

The results reported in Table 1 support the hypothesis that inflation uncertainty is an important determinant of interest rates. Within a risk-neutral setting, inflation uncertainty has an impact on both the nominal and the appropriately defined expected real interest rates. With risk-aversion, inflation uncertainty has an impact on the nominal interest rate. Failure to take account of inflation uncertainty may well account for the frequent empirical findings by Gibson (1972), Tanzi (1980), and others that the impact of anticipated inflation on nominal interest rates lies below the level suggested by previous theory. As noted by Levi and Makin (1979) omitting a variable such as v^2 which is positively correlated with anticipated inflation and negatively correlated with nominal interest rates will bias downward the estimated impact of anticipated inflation on nominal interest rates. Such an effect is evident from equation (1.3) in Table 1. Omission of v^2 causes the estimated impact on the nominal interest rate of a unit change in expected inflation to fall from 0.791 to 0.666.

V. Concluding Remarks

We have developed two models, one involving risk-neutrality and the other risk-aversion, which suggest that inflation uncertainty affects interest rates. Since both models give rise to essentially the same interest rate equation for estimation, we cannot distinguish between the two models implying that inflation uncertainty has an impact on nominal interest rates. Our empirical evidence supports the hypothesis that inflation uncertainty affects interest rates. If we interpret the empirical results in terms of the risk-neutral model, inflation uncertainty has a negative impact on nominal interest rates and a positive impact on the expected real rate as implied by our theory in Section II. If we interpret the results in terms of the risk-averse model, inflation uncertainty has a negative impact on nominal interest rates. The expected real rate is not of direct interest in a risk-averse world.

We believe our results raise real questions about the use of the Fisherian definition of the real interest rate in situations when there is uncertainty about inflation rates. We have argued that even with risk-neutrality the Fisherian definition of the real rate is not the appropriate concept upon which to base economic decisions if inflation uncertainty is present. The appropriate concept is an expected real rate which involves an adjustment for uncertainty. Moreover, if the world is risk-averse, the expected real rate is not a relevant concept for economic decisions.

FOOTNOTES

1. To see that this gives rise to the continuous time Fisherian definition, first divide the period into n subperiods of equal length with compounding carried out in each subperiod. This gives rise to

$$(1 + \frac{r_F}{n})^n = \frac{(1 + \frac{i}{n})^n}{(1 + \frac{\bar{\alpha}}{n})^n} .$$

For continuous compounding take the limit as $n \to \infty$ on both sides to get

$$e^{r_F} = e^{i - \bar{\alpha}} .$$

2. See Aitchison and Brown (1963) for a discussion of the properties of the lognormal distribution. Since the product of independent lognormal variates is lognormal, it follows that P is lognormal. Although this specification and Fischer's (1975) both lead to the price level being lognormally distributed, they really are not directly comparable since Fischer assumes the inflation rate is normally distributed.

3. See Parzen (1960, pp. 160-65, 220) for a discussion of the gamma probability distribution and of the mathematics used in this deviation. If we drop superscripts and subscripts, the density function of z is

$$f(z) = \frac{\lambda}{\Gamma(\rho)} (\lambda z)^{\rho-1} e^{-\lambda z} \quad (z > 0, \ \rho > 0)$$

where $\Gamma(\rho)$ is a gamma function. The mean of the distribution is ρ/λ and the variance is ρ/λ^2. Since $\rho/\lambda = 1 + (\bar{\alpha}/n)$ and $\rho/\lambda^2 = v^2/n$, we have $\lambda = [1 + (\bar{\alpha}/n)]/(v^2/n)$ and $\rho = [1 + (\bar{\alpha}/n)]^2/(v^2/n)$. Now

$$E(1/z) = \int_0^\infty \frac{1}{z} \frac{\lambda}{\Gamma(\rho)} (\lambda z)^{\rho-1} e^{-\lambda z} dz = \frac{\lambda^\rho}{\Gamma(\rho)} \int_0^\infty z^{\rho-2} e^{-\lambda z} dz$$

$$= \frac{\lambda^\rho}{\Gamma(\rho)} \frac{\Gamma(\rho-1)}{\lambda^{\rho-1}} = \frac{\lambda}{\rho-1} .$$

Therefore,

$$E \frac{1}{z} = \frac{1 + \frac{\bar{\alpha}}{n}}{(1 + \frac{\bar{\alpha}}{n})^2 - \frac{v^2}{n}} .$$

4. Some investigators, see Grossman (1981) and Engel and Frankel (1982), have argued the reverse. Unanticipated money growth, they claim, will cause markets to anticipate tightening by the Federal Reserve which in turn will cause interest rates to rise in anticipation of the tightening. This could be termed an "expectations" effect which operates in the opposite direction upon interest rates as does the "liquidity" effect from unanticipated money growth.

Two observations are in order on the expectations effect. First, it is very short-lived. Grossman found it operative over a two-hour period from one hour before Friday's money supply annoucement to one hour after. Frankel, testing for the same effect to persist until the following Monday did not detect a statistically significant effect. It appears that the expectations effect is operative only on an intra-day basis while over a longer period such as a quarter. Makin's (1982) results strongly support dominance of the liquidity effect.

Second, it must be remembered that the expectations effect, which suggests a change in policy, is likely conditional upon the position of the actual money supply relative to its target when a "surprise" occurs. If money supply is below target, a positive surprise may produce no change (or no expected change) in policy while the reverse would hold given money supply above target and a positive surprise.

5. For a more thorough discussion of implications of improperly modeling residuals see Hendry (1977).

6. Total borrowing/GNP may serve as a crude proxy for variance of inflation. The correlation coefficiency between it and σ^2 is 0.41.

REFERENCES

Aichison, J. and J.A.C. Brown (1963). The Lognormal Distribution, Cambridge University Press.

Arrow, Kenneth J. (1965). Aspects of the Theory of Risk-Bearing, Helsinki; Yrjö Johnsson Säätiö.

Blejer, Mario I. and Benjamin Eden (1979). "A Note on the Specification of the Fisher Equation under Inflation Uncertainty," Economics Letters 3: 249-255.

Boonekamp, C.F.J. (1978). "Inflation, Hedging, and the Demand for Money," American Economic Review LXVIII (December): 821-833.

Box, G.E.P. and G.M. Jenkins (1970). Time Series Analysis, Forecasting and Control, Holden-Day, Inc., San Francisco.

Eden, B. (1976). "On the Specification of the Demand for Money: The Real Rate of Return Versus the Rate of Inflation," Journal of Political Economy LXXXIV (December): 1353-59.

Engel, C. and J. Frankel (1982). "Why Money Announcements Move Interest Rates: An Answer from the Foreign Exchange Market," Mimeographed, Institute of Business and Economic Research, University of California, Berkeley, (February).

Fama, Eugene F. (1975). "Short-Term Interest Rates as Predictors of Inflation," American Economic Review (June): 269-82.

____ (1976). "Inflation Uncertainty and Expected Returns on Treasury Bills," Journal of Political Economy (June): 427-48.

Feldstein, Martin (1976). "Inflation, Income Taxes, and the Rate of Interest: A Theoretical Analysis," American Economic Review (December): 809-20.

Fischer, S. (1975). "The Demand for Indexed Bonds," Journal of Political Economy 83, No. 3 (June): 509-534.

Fisher, Irving (1930). The Theory of Interest, New York: Macmillan.

Gibson, W.E. (1972). "Interest Rates and Inflationary Expectations," American Economic Review 62, No. 5 (December): 854-865.

Grossman, J. (1981). "The 'Rationality' on Money Supply Expectations and the Short-Run Response of Interest Rates to Monetary Surprises," Journal of Money, Credit and Banking (November): 409-24.

Hafer, R.W. and S.E. Hein (1982). "Monetary Policy and Short-Term Real Rates of Interest," Federal Reserve Bank of St. Louis Review (March): 13-19.

Kochin, Levis (1980). "Interest Rates and Uncertain Future Price Levels: Irving Fisher's Money Illusion," University of Washington.

Levi, Maurice D. and John H. Makin (1978). "Anticipated Inflation and Interest Rates: Further Interpretation of Findings on the Fisher Equation," American Economic Review (December): 801-12.

____ (1979). "Fisher, Phillips, Friedman and the Measured Impact of Inflation on Interest," Journal of Finance (March): 35-52.

Makin, John H. (1982). "Effects of Inflation Control Programs on Expected Real Interest Rates," International Monetary Fund: Staff Papers (June).

Mishkin, Frederic S. (1981). "The Real Interest Rate: An Empirical Investigation," Carnegie-Rochester Conference Series on Public Policy 15 (Autumn): 151-200.

Mundell, Robert A. (1963). "Inflation and Real Interest," Journal of Political Economy (June): 280-83.

Nelson, Charles R. and G. William Schwert (1977). "On Testing the Hypothesis that the Real Rate of Interest is Constant," American Economic Review (June): 478-86.

Parzen, Emanuel (1960). Modern Probability Theory and Its Applications, Wiley.

Peek, Joe (1981). "Interest Rates, Income Taxes, and Anticipated Inflation," Mimeographed, Department of Economics, Boston College, (July).

Rothschild, M. and J.E. Stiglitz (1970). "Increasing Risk: I.A. Definition," Journal of Economic Theory II (September): 225-43.

Tanzi, Vito (1976). "Inflation, Indexation and Interest Income Taxation," Banca Nazionale del Lavoro, Quarterly Review 29 (March): 54-76.

____ (1977). "Inflation and the Incidence of Income Taxes on Interest Income: Some Results for the United States, 1972-74," International Monetary Fund Staff Papers 24 (July): 500-513.

____ (1980). "Inflationary Expectations, Economic Activity, Taxes, and Interest RAtes," American Economic Review 70 (March): 12-21.

____ (1980a). Inflation and the Personal Income Tax, Cambridge: Cambridge University Press, 1980.

Tobin, James (1965). "Money and Economic Growth," Econometrica (October): 671-84.

Zellner, A. and F. Palm (1974). "Time Series Analysis and Simultaneous Equation Econometric Models," Journal of Econometrics 2: 17-54.