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POISSON-GAUSSIAN PROCESSES AND THE BOND MARKETS

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ABSTRACT

That interest rates move in a discontinuous manner is no surprise to participants in the bond markets. This paper proposes and estimates a class of Poisson-Gaussian processes that allow for jumps in interest rates. Estimation is undertaken using exact continuous-time and discrete-time estimators. Analytical derivations of the characteristic functions, moments and density functions of jump-diffusion stochastic process are developed and employed in empirical estimation. These derivations are general enough to accommodate any jump distribution. We find that jump processes capture empirical features of the data which would not be captured by diffusion models. The models in the paper enable an assessment of the impact of Fed activity and day-of-week effects on the stochastic process for interest rates. There is strong evidence that existing diffusion models would be well-enhanced by jump processes.

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"Natura non facit saltum", - **Nature does not jump.**
Alfred Marshall, title page, *Principles of Economics*, 1890.

1. INTRODUCTION

This paper examines the role of jump-enhanced diffusions (i.e., Poisson-Gaussian processes) in modelling the term structure of interest rates. Theoretical work on jump-diffusion term structure models exists,[1] and is of recent origin, but little in the way of empirical examination of these models has been undertaken so far. This paper (i) develops jump-diffusion analytics for a wide class of models, and (ii) empirically examines the value of these models.

Motivation: Why should we expect jumps to be a satisfactory modelling device? Stylized facts from the bond markets suggest that jump behavior is ubiquitous. Exogenous interventions in the markets by the Federal Reserve causes jumps. Supply shocks are another factor, as regular debt refundings inject sufficient volume to magnify price effects. Demand shocks such as market behavior at Treasury auctions often result in jumps, as do economic news announcements. As Merton [36] emphasizes, routine trading information releases are well depicted by smooth changes in interest rates, yet bursts of information are often reflected in price behavior as jumps. Jump effects tend to be prevalent in regulated "intervention" environments such as the interest rate and foreign exchange markets.

Raw statistical evidence is strongly suggestive of jumps. Interest rate volatility is very high at the short end of the term structure,[2] and changes in interest rates demonstrate considerable skewness and kurtosis. Poisson-Gaussian processes[3] can flexibly accommodate a wide range of skewness and kurtosis effects. Kurtosis can substantially affect the pricing of derivative securities. Table 1 provides summary statistics for the short rate of interest. The presence of leptokurtosis in interest rate changes in undeniable and makes a strong case for jump models.[4]

The extent of the volatility "smile", symptomatic of excess kurtosis (fat tails) in the conditional distribution of changes in interest rates, cannot theoretically be sustained by Gaussian models. One way to model the excess kurtosis is by means of Poisson-Gaussian processes. Other approaches which capture leptokurtosis are stochastic volatility models or simpler time-varying volatility models, such as ARCH processes. The paper examines these various alternative models. The

[1] See Ahn and Thompson [1], Attari [4], Das and Foresi [23], Babbs and Webber [6], Backus, Foresi and Wu [7], Baz and Das [11], Das [25][26], Chacko, Heston [30], Naik and Lee [37], Burnetas and Ritchken [14], Shirakawa [38] for a range of theoretical models.

[2] Coleman, Fisher and Ibbotson [19] find that in the 1980s, the standard deviation of monthly changes in the short (1-month) rate was 128 basis points.

[3] We will use the term 'Gaussian' to denote the smooth component of interest rate behavior, and the term 'jump' or 'Poisson shock' to denote the discontinuous component of the interest rate process.

[4] See Backus, Foresi and Wu [7] for an excellent exposition of why jumps may better explain the high degree of curvature in yield curves. More on this towards the end of the paper.

degree of conditional leptokurtosis varies with the time interval between data observations (see Das and Sundaram [24]), and jump-diffusion models allow for parameter choices which match conditional skewness and kurtosis at varying maturities. Chan, Karolyi, Longstaff and Sanders [18] found that interest rates display level dependent volatility to an extent not accounted for by existing theoretical models. Recent work by Brenner, Harjes and Kroner [15] and Koedijk, Nissen, Schotman and Wolff [32] provides strong evidence that in addition to level dependence, time varying volatility (i.e. ARCH) models provide a better empirical fit. In this article, we explore simple versions of these models with jumps, and find improvements in fit. We essentially conclude that mixed models with stochastic volatility and jumps are predicated. We briefly summarize the results of the paper, theoretical and empirical.

Analytics: The following are developed. First, characteristic functions for a range of jump-diffusion stochastic processes (irrespective of jump distribution) are derived, thereby obtaining the primary tool for pricing and statistical analysis of our models. Second, probability density functions are obtained for estimation by maximum-likelihood and for derivative security pricing. Third, the first four moments for jump-diffusion stochastic processes are calculated in closed form so as to enable method of moments estimation methods. Fourth, analytical expressions for bond prices are derived. Hence, the paper offers a comprehensive set of tools for the application of jump-diffusion processes to term structure models. These methods do not rely on specific choices of the jump distribution, but apply to any jump distribution with finite moments.

Empirical Work: The content of the paper is as follows. First, exact estimation of specific jump-diffusion models is possible in continuous time, and is undertaken, evidencing a good fit for this class of models. Second, a more easily implementable and analogous discrete time method is used to integrate ARCH type models with jump-diffusions, and estimation results show that the best models we consider are those that contain features of jumps as well as time varying volatility. Third, the flexibility of the estimation approach is exploited by enhancing the model for day of the week effects, wherein we find that jumps are most likely on Wednesdays and Thursdays, probably on account of option expiry effects. Fourth, the model is enhanced to make jumps dependent on Federal Reserve activity, and we find that the two-day meetings of the Federal Open Market Committee appear to have an information effect. Fifth, estimation is also undertaken using the analytically derived moments, and the presence of jumps is confirmed in a general model, where no jump distribution is imposed a priori. Sixth, an examination of the structure of empirical moments confirms that they would not be generated by diffusion processes alone, indicating strongly that jumps be added to diffusion-based term structure models.

The discussion so far begs the question: should we eschew diffusion processes in favor of jump models of the term structure? Our empirical results in the paper show that jumps are a necessary addition to existing diffusion models. We show that the jump process accounts for a large part of the total variation in interest rates, and that the patterns of higher-order moments cannot be generated by diffusion models alone, even if they are multidimensional diffusions. Therefore, rather than view jump models as competing with the best diffusion models, we demonstrate the strong complimentarity of these two stochastic process choices in modelling the term structure.

Thus, this article provides a comprehensive toolkit for the application of jump-diffusion methods to term structure models. The detailed content of the paper comprises two sections. Section 2 provides analytics, and Section 3 contains the empirical implementation. We conclude in Section 4.

2. ANALYTICAL METHODOLOGY

This section deals with the derivation of the analytics required for maximum-likelihood and method of moments estimation, as well as the analytics of the term structure..

Estimating mean-reverting interest rate processes with jumps entails complications beyond those encountered for processes without mean reversion. There are two main reasons for this:

- For interest rates, no common modelling approach seems to be adopted, and a wide variety of stochastic processes are used, where often, transition density functions are unavailable in closed form.[5]
- Mean reversion substantially complicates the derivation of the conditional transition density function, used for maximum-likelihood estimation. With the lognormal form used for stocks, the exact time at which a jump occurs in any time interval does not matter in the determination of the transition density function, while with interest rate processes, the presence of mean reversion is important, as it affects the drift of the process differentially, depending on where in time the jump occurs.

In this paper, a generalized derivation of the probability function and the moments of the jump-diffusion process surmounts these issues in a framework where estimation is undertaken with the exact densities from the continuous time stochastic process.

The plan for this section is as follows. The stochastic process for the jump-diffusion model is presented, followed by a derivation of the characteristic function. This provides two important by-products: (i) the conditional moments of the process, in particular the kurtosis, and (ii) the transition density functions.

2.1. **The Stochastic Process.** The following is the mean reverting process for interest rates employed in this paper:

$$dr = k(\theta - r)dt + vdz + J\ d\pi(h) \tag{2.1}$$

where θ is a central tendency parameter for the interest rate r, which reverts at rate k. Therefore the interest rate evolves with mean-reverting drift and two random terms, one a diffusion and the other a Poisson process embodying a random jump J. The variance coefficient of the diffusion is v^2 and the arrival of jumps is governed

[5]On the other hand, for equities, the generally accepted process for stock prices is a lognormal diffusion with jumps, which has well known properties. This makes the derivation of the density function quite straightforward since the stochastic differential equation for the stock price process is easily solved.

by a Poisson process π with arrival frequency parameter h, which denotes the number of jumps per year. The jump size J can be completely general and may be a constant or drawn from a probability distribution. The diffusion and Poisson processes are independent of each other, and each of them is independent of J as well.

2.2. **The Characteristic Function.** Assessing the impact of jumps on pricing interest rate dependent securities requires an analysis of the probability distribution of a jump-diffusion interest rate process, and the moments of this distribution. The characteristic function of the jump-diffusion process offers the raw material with which to derive the density functions as well as the moments.

Assume that we are at time $t = 0$, and that we are looking ahead to time $t = T$. We are interested in the distribution of $r(T)$ given the current value of the interest rate $r(0) \equiv r_0 = r$. In order to derive the T-interval characteristic function $F(r, T; s)$ for the process (2.1), (s is the characteristic function parameter) we solve its Kolmogorov backward equation (KBE) subject to the boundary condition that

$$F(r, T = 0; s) = \exp(isr). \tag{2.2}$$

where $i = \sqrt{-1}$. The backward equation is

$$0 = \frac{\partial F}{\partial r} k(\theta - r) + \frac{1}{2} \frac{\partial^2 F}{\partial r^2} v^2 - \frac{\partial F}{\partial T} + hE\left[F(r + J) - F(r)\right]. \tag{2.3}$$

The solution (derived in Appendix A) is provided below:

$$\begin{aligned} F(r, T; s) &= \exp\left[A(T; s) + rB(T; s)\right] \\ A(T; s) &= \int \left(k\theta B(T; s) + \frac{1}{2} v^2 B(T; s)^2 + hE\left[e^{JB(T;s)} - 1\right] \right) dT \\ B(T; s) &= is \exp\left(-kT\right) \end{aligned}$$

Given the characteristic function, we can obtain the moments and the probability density functions for any choice of jump distribution.

For the special case where the jump size is distributed Bernoulli-exponential (with Bernoulli parameter ψ for the sign of the jump and exponential distribution parameter α for the absolute size of the jump), the characteristic function of the probability distribution of the interest rate of $r(t + \tau)$ conditional on $r(t)$ is derived in closed form (see [23]):

$$F(r(t), \tau; s) = \exp[\hat{A}(\tau; s) - \hat{B}(\tau; s) r(t)],$$

where

$$\begin{aligned} \hat{A}(\tau; s) &= isk\theta \left(\frac{1 - e^{-k\tau}}{k} \right) - s^2 v^2 \left(\frac{1 - e^{-2k\tau}}{4k} \right) \\ &\quad + \frac{ih(1 - 2\psi)}{k} \left[\text{Arctan}\left(\frac{s}{\alpha} e^{-k\tau} \right) - \text{Arctan}\left(\frac{s}{\alpha} \right) \right] \\ &\quad + \frac{h}{2k} \ln \left(\frac{\alpha^2 + s^2 e^{-2k\tau}}{\alpha^2 + s^2} \right), \\ \hat{B}(\tau; s) &= -is \exp(-k\tau). \end{aligned}$$

The closed-form characteristic function above exploits the fact that we can solve the partial-differential difference equation for the characteristic function. No other solutions are currently known to exist for interest rate processes.

2.3. **The Moments.** The moments of the jump-diffusion process offer valuable insights. First, the behavior of options prices may be inferred from a study of the moments. Second, the moments are easily used in method of moments estimation models. In this subsection, the derivation of the moments incorporates two innovations: (i) moments are obtained for any jump distribution, and (ii) the moments are derived without necessarily obtaining the characteristic function in closed form.

To obtain the moments, we differentiate the characteristic function successively with respect to s and then find the value of the derivative when $s = 0$. Let μ_n denote the nth moment, and F_n be the nth derivative of F with respect to s, i.e. $F_n = \frac{\partial^n F}{\partial s^n}$. Then

$$\mu_n = \frac{1}{i^n}\left[F_n \mid s = 0\right].$$

Likewise $E[J^n]$ denotes the nth moment of the jump shock. The first four moments (as derived in Appendix B) are:

$$\begin{aligned}
\mu_1 &= \left(\theta + \frac{hE[J]}{k}\right)(1 - e^{-kT}) + re^{-kT} \\
\mu_2 &= \frac{v^2 + hE\left[J^2\right]}{2k}(1 - e^{-2kT}) + \mu_1^2 \\
\mu_3 &= hE[J^3]\left(\frac{1 - e^{-3kT}}{3k}\right) \\
&\quad +3\mu_1\left(v^2 + hE[J^2]\right)\left(\frac{1 - e^{-2kT}}{2k}\right) + \mu_1^3 \\
\mu_4 &= hE[J^4]\left(\frac{1 - e^{-4kT}}{4k}\right) + 3\left(\left(v^2 + hE[J^2]\right)\left(\frac{1 - e^{-2kT}}{2k}\right)\right)^2 \\
&\quad +4\mu_1 hE[J^3]\left(\frac{1 - e^{-3kT}}{3k}\right) \\
&\quad +6\mu_1^2\left(\left(v^2 + hE[J^2]\right)\left(\frac{1 - e^{-2kT}}{2k}\right)\right) + \mu_1^4
\end{aligned}$$

Any jump distribution where the moments are known and finite is admissible, since we only need the values of $E[J^n], n = 1, 2, 3, 4$.

2.4. **Density Functions.** The estimation of Poisson-Gaussian processes in continuous time requires the conditional transition probability density of the jump-diffusion process. This is derived via Fourier inversion of the characteristic function. If t denotes today, and τ denotes the time interval, such that the horizon $T = t + \tau$, then Fourier inversion of the characteristic function $F(r(t), \tau; s)$ provides the necessary transition density function $f(r(t), \tau)$, i.e.

$$f[r(t + \tau) \mid r(t)] = \frac{1}{\pi}\int_0^\infty \text{Re}[\exp(-isr(t + \tau))F(r(t), \tau; s)]\, ds$$

Therefore the transition probability function is obtained by numerical integration over the characteristic function. Estimation is carried out by maximum-likelihood, using a discrete time series of interest rates $r(t), t = 0...T$. If the time interval between observations is Δ, then the log-likelihood function being maximized is:

$$L = \max_{k,\theta,v,h,\{E[J^n]\},\forall n} \sum_{t=0}^{T-1} \log(f\,[r(t+\Delta) \mid r(t)])$$

i.e.

$$\max_{k,\theta,v,h,\{E[J^n]\},\forall n} \sum_{t=0}^{T-1} \log\left(\frac{1}{\pi}\int_0^\infty \mathrm{Re}[\exp(-isr(t+\Delta))F(r(t),\Delta;s)]\,ds\right) \tag{2.4}$$

Fourier inversion is achieved by computing the integral in equation (2.4) numerically using a quadrature routine. Numerical second derivatives of the likelihood function evaluated at the optimal parameter set $\Omega^* = [k^*, \theta^*, v^*, h^*, \{E[J^n]\}^*]$ provide the Hessian Matrix: $Y = \frac{\partial^2 L}{\partial\Omega\partial\Omega'}$. The standard errors are then computed from the Hessian as $\sqrt{\mathrm{diag}(-Y^{-1})}$. These standard errors are employed in obtaining the required T-statistics, and the results of the continuous time estimation are presented subsequently in Section 3.

2.5. **Analytical Jump-diffusion Models for Bond Pricing.** Equation (2.1) states the statistical process for the interest rate. However, the pricing of interest rate sensitive securities is undertaken by translating this statistical process to a risk-neutral one. Assume that an equivalent martingale measure exists such that the risk-neutral (drift adjusted) interest rate process is

$$dr = [k(\theta - r) - \lambda v]\,dt + v\,dz + J\,d\pi(h^*) \tag{2.5}$$

where λ is the unit price of diffusion risk and $h^* = h(1 - \lambda_J)$ is the risk-neutral hazard rate where λ_J is the parameter implementing the price of jump risk. The adjustment in equation (2.5) required two risk adjustments: for diffusion risk via the price of risk λ, and for jump risk via the risk adjustment λ_J. The pricing of interest rate sensitive securities is undertaken for this risk-neutral interest rate process.

Denote present time by the variable t, the maturity date of a bond by T, and the time to maturity $\tau = T - t$. Define the price of a zero-coupon bond to be $P(r,\tau)$; we assume that all factors determining the value of this bond are captured in equation (2.5) above. The solution for the bond price under a jump-diffusion model is given by (the proof is given in Appendix C):

$$\begin{aligned}
P(r,\tau) &= e^{A(\tau)+rB(\tau)} \\
B(\tau) &= \frac{e^{-k\tau}-1}{k} \\
A(\tau) &= \int \left((k\theta - \lambda v)B(\tau) + \frac{1}{2}v^2 B(\tau)^2 + q[B(\tau)]\right) d\tau, \quad A(0) = 0 \\
q[B(\tau)] &= h^* E(e^{JB(\tau)} - 1) \\
h^* &= h(1 - \lambda_J)
\end{aligned}$$

The expression $q[B(\tau)]$ accommodates any jump distribution provided the moments are finite. In general, the integral equation for $A(\tau)$ (which is the solution to an ordinary differential equation) does not always admit a closed-form. Specifically, even in the simplest cases, when J is a constant or is distributed Gaussian, no closed-form solution is achievable. Das and Foresi [23] find that in the special case of a jump with a sign based on a Bernoulli distribution and a size based on an exponential distribution, it is possible to obtain a closed-form solution. Chacko [16],[17] extends the Das-Foresi model to incorporate stochastic volatility and stochastic mean as well by exploiting the facile properties of the Bernoulli-exponential form. When the Bernoulli-exponential form is not availed of, the ordinary differential equation for $A(\tau)$ is solved numerically by Runge-Kutta methods.

An important aspect of this solution is the fact that the yields $\left[= \frac{-1}{\tau} \ln (P[\tau])\right]$ are 'affine', i.e. linear functions of the short rate. This is a useful property for estimation purposes.

This concludes the development of all the analytics needed for estimating and pricing jump-diffusion based term structure derivative securities. The next section deals with the application of these methods to bond market data.

3. ESTIMATION

The analytics from Section 2 are applied to daily data on the Fed funds rate for the period January 1988 to December 1997. The total number of observations is 2609. The data is from the Federal Reserve web site and is plotted in Figure 1. The descriptive statistics for the data are in Table 1. An examination of the

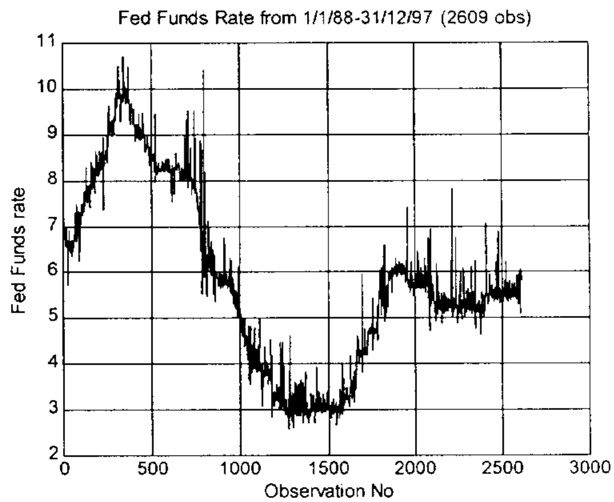

FIGURE 1

data reveals that changes in interest rates evidence a very high degree of kurtosis. a stylized fact that predicates the use of a jump model. Over the entire 10 year period, rates have quickly risen to a peak of 10% and then fallen to a low of 3%, finally stabilizing at a 6% level.

Our estimation exercise uses (i) continuous time estimators, (ii) discrete approximation estimators, and (iii) method of moments techniques. Our jump-diffusion

TABLE 1. **Descriptive Statistics**

The following table presents descriptive statistics for the Fed Funds rate over the period January 1988 to December 1997. The data is daily in frequency. The statistics reported are for the interest rate level (r) and the change in interest rates (dr).

Statistic	r	dr
Mean	5.8100	-0.0005
Standard Deviation	1.9558	0.2899
Skewness	0.3032	0.3950
Excess Kurtosis	-0.8304	19.8667
Minimum	2.58	-2.70
Maximum	10.71	2.83

TABLE 2. **Continuous-Time Estimation**

This table presents the results of the continuous-time jump-diffusion model. Estimation is undertaken using maximum-likelihood where the transition density function is obtain by numerical Fourier inversion at each point in time, Optimization of the likelihood function is then undertaken numerically over the numerically obtained probability density. The parameter estimates for the continuous-time model are reported below.

Parameter	Estimate	T-Statistic
k	0.6521	2.8089
θ	0.0173	0.9553
v	0.0146	24.5698
h	118.87	11.0819
ψ	0.5411	22.7500
α	365.62	16.6437
Log-likelihood	12541.11	

models are extended for ARCH effects. They allow for mean-reversion in jump processes, and also test for the impact of Federal Reserve actions and day-of-the-week effects.

3.1. **Continuous-Time Estimation.** The process was estimated using continuous time transition density functions for the jump-diffusion process. The log-likelihood function in equation 2.4 is used for the estimation. Since the Vasicek model is nested within the jump-diffusion framework of this paper, a comparison of log-likelihoods reveals the improvement in fit obtained by adopting a jump-diffusion model. The results are provided in Table 2. The model finds a large number of jumps in the data, seen in parameter h. The average size of each jump is given by $\frac{1}{\alpha} = 1/365.62$, i.e. 27 basis points. From the descriptive statistics it is known that mild positive skewness exists, evidenced by the parameter $\psi = 0.54$. Therefore, the results (i) confirm that the jump parameters are statistically significant, and (ii) that the jump comprises a reasonable component of the stochastic variation in interest rates.

We now conduct a simple analysis to check for consistency of the estimation scheme with the theoretical model. In Section 2.3 we derived the moments of the model, and found that the conditional mean of the jump-diffusion process was equal to $\left(\theta + \frac{hE[J]}{k}\right)\left(1 - e^{-kT}\right) + re^{-kT}$. The unconditional mean may be derived as the limit when $T \longrightarrow \infty$, which yields $\theta + \frac{hE[J]}{k}$. Since the jump size is distributed exponential, the average size is $\frac{1}{\alpha}$. The expected value of the jump then is $\psi E(J) + (1-\psi)E(-J) = \psi\frac{1}{\alpha} + (1-\psi)(-\frac{1}{\alpha}) = 0.00022194$, using the parameters in Table 2. Thus, using the other parameters, we obtain that $\theta + \frac{hE[J]}{k} = 0.0578$, i.e. 5.78% which is almost equal to the mean of the data given in Table 1. It is important to note that under the jump-diffusion model, θ is no longer the unconditional mean of the data; it needs to be replaced with the expression $\theta + \frac{hE(J)}{k}$ above.

3.2. **Estimation using Discrete-Time Approximations.** Estimation using the continuous-time method of the previous section is an exceedingly intensive numerical process. It requires numerical optimization over a density function that is itself obtained by numerical Fourier inversion. In this section a simpler discrete-time approach allows us to estimate a model where the jumps are normally distributed.

We estimate the Poisson-Gaussian interest rate model using a Bernoulli approximation first introduced in Ball & Torous [9]. The assumption being made here is that in each time interval either only one jump occurs or no jump occurs. This is tenable for short frequency data, and may be debatable for data at longer frequencies. Since we use daily data, this approximation is justifiable. As Ball and Torous found, it makes the estimation procedure highly tractable, stable and convergent. Since the limit of the Bernoulli process is governed by a Poisson distribution, we can approximate the likelihood function for the Poisson-Gaussian model using a Bernoulli mixture of the normal distributions governing the diffusion and jump shocks.[6] In discrete time, we express the process in equation (2.1) as follows:

$$\Delta r = k(\theta - r)\,\Delta t + v\,\Delta z + J(\mu, \gamma^2)\,\Delta\pi(q) \tag{3.2}$$

where v^2 is the annualized variance of the Gaussian shock, and Δz is a standard normal shock term. $J(\mu, \gamma^2)$ is the jump shock, which is normally distributed with mean μ and variance γ^2. $\Delta\pi(q)$ is the discrete-time Poisson increment, approximated by a Bernoulli distribution with parameter $q = h\,\Delta t + O(\Delta t)$. Then, the transition probabilities for the interest rate following a Poisson-Gaussian process are written as (for $s > t$)

$$\begin{aligned} f[r(s) \mid r(t)] &= q \exp\left(\frac{-(r(s) - r(t) - k(\theta - r(t))\,\Delta t - \mu)^2}{2(v^2\,\Delta t + \gamma^2)}\right) \frac{1}{\sqrt{2\pi(v^2\,\Delta t + \gamma^2)}} \\ &\quad + (1-q)\exp\left(\frac{-(r(s) - r(t) - k(\theta - r(t))\,\Delta t)^2}{2v^2\,\Delta t}\right) \frac{1}{\sqrt{2\pi v^2\,\Delta t}} \end{aligned} \tag{3.3}$$

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[6] The Bernoulli approximation is achieved as follows: Define the indicator variable $Y_i = 1$ if a jump occurs, else $Y_i = 0$ for all $i \ldots N$, and where $\Delta t = T/N$, for the time series spanning T.

$$\begin{aligned} \Pr[Y_i = 0] &= 1 - h\,\Delta t + O(\Delta t) \\ \Pr[Y_i = 1] &= h\,\Delta t + O(\Delta t) \\ \Pr[Y_i > 1] &= O(\Delta t) \end{aligned} \tag{3.1}$$

Let $M = \sum_{i=1}^{N} Y_i$. M is distributed Binomial being the sum of independent Bernoulli variables. For x occurrences,

$$Pr[M = x] = {}^N C_x (hT/N)^x (1 - hT/N)^{N-x}, \ \forall x$$

$$\lim_{N \to \infty} Pr[M = x] = \frac{e^{-hT}(hT)^x}{x!}$$

Therefore, it is clear that the Bernoulli approximation converges to the appropriate Poisson density.

TABLE 3. Basic Poisson-Gaussian Estimation

We present results for the estimation of pure-Gaussian, Poisson-Gaussian, ARCH-Poisson-Gaussian and ARCH-Gaussian processes on daily data covering the period January 1988 to December 1997. The total number of observations is 2609. Estimation is carried out using maximum-likelihood incorporating the transition density function in equation (3.3). The discretized ARCH-Poisson-Gaussian process estimated is specified as follows

$$\begin{aligned} \Delta r &= k(\theta - r)\,\Delta t + v\,\Delta z + J(\mu, \gamma^2)\,\Delta\pi(q) \\ v^2_{t+\Delta t} &= a_0 + a_1[\Delta r_t - E(\Delta r_t)]^2 \end{aligned}$$

The other processes are special cases of the one above. T-statistics are presented below the parameter estimates. The variable q, the probability of a jump in the interval Δt is analogous to the continuous time parameter h for jump arrival intensity, by the relation $q \approx h\Delta t$.

Parameter	Pure Gaussian	Poisson Gaussian	ARCH-Poisson Gaussian	ARCH Gaussian
k	2.8832	0.8542	0.5771	1.2810
	3.68	2.26	2.02	4.66
θ	0.0576	0.0330	0.0346	0.0974
	10.91	2.57	2.50	9.78
v	0.0466	0.0173		
	111.01	24.01		
a_0			0.0001	0.0008
			17.65	66.48
a_1			127.0201	232.07
			13.92	29.85
μ		0.0004	0.0017	
		1.38	5.66	
γ		0.0058	0.0045	
		24.50	16.60	
q		0.2162	0.1564	
		17.91	13.14	
Log-Likelihood	13938.13	14890.90	15197.67	14509.50

where $q = h\,\Delta t + O(\Delta t)$. This approximates the true Poisson-Gaussian density with a mixture of normal distributions. Estimation involves the following maximization:

$$\max_{[k,\theta,v,\mu,\gamma^2,q]} \sum_{t=1}^{T} (\log(f[r(s) \mid r(t)]))$$

Maximum likelihood estimation results are presented in Table 3. In order to compare different processes for the short rate, we estimated four nested models on the data set. Using data from different sampling frequencies enables us to examine whether the stochastic process used is sensitive to this criterion. The models estimated are (i) a pure-Gaussian model ($h = 0$), (ii) the Poisson-Gaussian model of equation (2.1), (iii) an ARCH-Poisson-Gaussian model, which consists of the Poisson-Gaussian model with the variance of the Gaussian component following an ARCH(1) process,[7] and (iv) a pure ARCH-Gaussian model. This parallels to

[7] This is done by modelling the variance of the Gaussian process as follows:

$$v(s + \Delta t)^2 = a_0 + a_1[r(s) - E(r(s))]^2$$

and estimating the parameters a_0, a_1.

a large extent the analyses carried out by Jorion [31] for the equity and foreign exchange markets.

Since the ARCH-Poisson-Gaussian model subsumes the other three models, likelihood ratio tests may be applied to compare nested models. Comparison of nested log-likelihoods via a χ^2 statistic with degrees of freedom equal to the difference in the number of parameters between two models reveals in Tables 3 that the ARCH-Poisson-Gaussian model outperforms the rest. The Poisson-Gaussian process fits the data significantly better than the pure-Gaussian one. Whereas the Poisson-Gaussian and ARCH-Gaussian models are not nested, the likelihood for the Poisson-Gaussian model is greater, suggesting that Poisson-Gaussian processes provide a better fit than ARCH volatility models.[8]

A comparison of the pure-Gaussian model and the Poisson-Gaussian model reveals a sharp drop in Gaussian volatility (v) when jumps are introduced into a pure-Gaussian model. For example, in Table 3, the Gaussian volatility drops to one-third its prior level suggesting that jumps account for a substantial component of volatility.

Once again, the parameter θ appears downward biased, but is actually so because of skewness from the jump. The unconditional mean of the interest rate under the discretized process is given by $\theta + h\mu = \theta + q\mu/\Delta$, and computations using the values in Table 3 arrive at a value of 0.0557 or 5.57%, once again close to the mean value in Table 1.

In the Poisson-Gaussian model (Table 3) we find that $q = 0.2162$, which under our Bernoulli model is simply the probability of a jump on any day. Thus, we find that jumps occur once every five days over our sample period. In contrast, the ARCH-Poisson-Gaussian model provides a jump probability of only 0.1564, evidence of the fact that stochastic volatility will account for some of the jumps. We conclude by noting that pure-Gaussian models do not capture the features of the data. Moreover, Poisson-Gaussian and ARCH-Gaussian models as well fall short of the efficacy of the ARCH-Poisson-Gaussian model. This has implications that theoretical work be driven in the direction of a combined ARCH-Poisson-Gaussian model.

Observe that the coefficient of mean reversion drops from 2.88 to 0.85 when jumps are added to the diffusion model. This may imply that jumps provide a source of mean reversion. This happens when the skewness of the jump distribution depends on the level of the interest rate in such a way as to induce mean reversion i.e. there is a greater chance of a positive jump at low interest rate levels, and a higher chance of a negative jump at high interest rate levels. Thus, we may find that the jump size distribution is positively skewed at low levels of r and negatively skewed at high levels of r. This can be modelled by allowing the mean of the jump size to depend on the level of r. For example, we may use a specification such as $\mu_t = \alpha_0 + \alpha_1(\theta - r_t)$. When $\alpha_1 > 0$, we obtain mean reversion of the short rate through the jump component of the process.

Table 4 reports the results of the time-varying mean reverting model when jumps inject mean reversion. The mean reversion in the process is now attributable to

[8] Application of the Akaike Information Criterion (not reported), where the likelihood is adjusted downwards by the number of parameters, provides evidence of this.

TABLE 4. **Estimation of the Time Varying Jump Means Model**

We present results for the estimation of the ARCH-Poisson-Gaussian model allowing for time variation in the mean of the jump size. This enables assessment of the mean reversion effects of the jump process. Estimation is carried out using maximum-likelihood incorporating the transition density function in equation (3.3). The process estimated is specified in the following equations:

$$\begin{aligned} \Delta r &= k(\theta - r)\,\Delta t + v\,\Delta z + J(\mu, \gamma^2)\,\Delta\pi(q) \\ v^2_{t+\Delta t} &= a_0 + a_1[\Delta r_t - E(\Delta r_t)]^2 \\ \mu_t &= \alpha_0 + \alpha_1(\theta - r_t) \end{aligned}$$

T-statistics are presented below the parameter estimates. The χ^2 statistic is computed for twice the difference between the mean reverting jump model and the constant mean jump model (where $\alpha_0 = \mu$ and $\alpha_1 = 0$). The degrees of freedom used is one, being the difference in the number of parameters between the two models. The variable q, the probability of a jump in the interval Δt is analogous to the continuous time parameter h for jump arrival intensity, by the relation $q \approx h\Delta t$.

Model:	Jump-diffusion		ARCH-jump	
Parameter	Estimate	T-stat	Estimate	T-stat
k	0.6336	1.64	0.5023	1.72
θ	0.0233	1.04	0.0304	1.69
v	0.0173	24.04		
a_0			0.0001	17.65
a_1			126.6914	13.90
q	0.2163	17.91	0.1567	13.14
α_0	0.0018	1.46	0.0022	3.13
α_1	0.0414	3.04	0.0202	1.44
γ	0.0057	23.16	0.0045	16.03
Log-Likelihood	14895.65		15198.82	
Log-L for constant μ	14890.90		15197.67	
P-Val for $\chi^2(1) =$	0.0021		0.1294	

both the drift term and the jump term. Since jump arrivals are uncertain, the rate of mean reversion is now time-varying, and the drift in the interest rate becomes stochastic. Ait-Sahalia [2] and Stanton [39] demonstrate that the drift term displays non-linear behavior, which may be partially explained if jumps inject 'extra' mean reversion at interest rates far away from the long run mean of the short rate. In fact these papers find that the mean reversion pull is far stronger when the interest rate lies outside the range 4%-17%, which is consistent with the phenomenon suggested here. We extend our empirical model to estimate the parameters (α_0, α_1). We estimated the Poisson-Gaussian and ARCH-Poisson-Gaussian model with time-varying jump means (Table 4). Table 4 can be compared with Table 3. Notice that the coefficient of mean reversion k is lower, as the mean reverting component has been redistributed partly to the jump component of the process. The T-statistic for α_1 is significant for the jump model indicating that the mean of the jump process is time-varying. However, when an ARCH effect is added to the model, the time-varying drift coefficient becomes insignificant. The joint evidence of these two models appears to suggest that different specifications of the volatility and jump may result in a linear drift model. We explore this issue in greater detail in a later subsection.

TABLE 5. **Jump Estimation with Day of the Week Effects**

The table presents results of the estimation of a jump-diffusion model when the jump arrival intensity is assumed to be affected by the day of the week. The jump intensity follows a linear model

$$q_t = \lambda_0 + \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 + \lambda_4 d_4$$

where $d_i, i = 1, 2, 3, 4$ are dummy variables for Monday, Tuesday, Wednesday and Thursday.

Parameter	Estimate	T-Statistic
k	0.7960	2.09
θ	0.0259	1.60
v	0.0171	24.38
λ_0	0.1222	6.17
λ_1	0.0413	1.34
λ_2	0.0147	0.54
λ_3	0.2523	6.85
λ_4	0.1777	5.46
μ	0.0004	1.52
γ	0.0057	24.94
Log-likelihood	14932.74	

3.3. **Day of the Week Effects.** In this section we examine whether jumps are more likely to occur on specific days of the week, by introducing a modification to make the arrival intensity of jumps depend on the day of the week. There are several reasons which make jumps more likely on some days of the week rather than others. For example, jumps would be more likely on Mondays since the release of pent up information over the weekend may drive up the possibility of a large change in interest rates. Likewise, option expiry may inject jumps into the behavior of interest rates, and this would be likely on Wednesdays and Thursdays. Jumps may also occur on Fridays when last minute trading may create excess volatility.

By using dummy variables for each day of the week, we assume a linear model for the arrival intensity of jumps in the short rate of interest:

$$\lambda_t = \lambda_0 + \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 + \lambda_4 d_4$$

where λ_0 is the arrival probability of a jump if the day is Friday, and $\lambda_i, i = 1, 2, 3, 4$ is the incremental arrival intensity of jumps over Friday's level when the day of the week is Monday, Tuesday, Wednesday and Thursday respectively. $d_i, i = 1, 2, 3, 4$ are dummy data variables indicating the day of the week for Monday, Tuesday, Wednesday and Thursday respectively. Estimation was conducted over the two models containing jumps, i.e. (i) the jump-diffusion model and (ii) the ARCH-jump-diffusion model. The results of the estimation are presented in Table 5. Intuitive results emanate from this analysis. The likelihood of jumps is highest on Fridays, but jumps are also likely on Wednesdays and Thursdays, when information from options expiry is released. This lends credence to the proposition that jumps are caused by large bursts of information being released into the market. Once again, there is no evidence of skewness, but kurtosis exists. The jump tends to be of the order of 50 basis points.

3.4. **Federal Reserve Activity.** Jumps may arise from intervention by the Federal Reserve in the bond markets. The Federal Open Market Committee (FOMC) meets periodically, and informs their open market desk of the range they wish to establish for the Fed Funds rate. Short rates tend to track this rate rather closely. Existing models do not account for Federal Reserve activity. It is possible that these meetings form an important information event. If so, a model that accounts for this will prove to be superior for traders. In this section, we enhance our jump model by making the jump intensity depend on the FOMC meeting. By examining the impact of the meeting on the jump probability we can ascertain whether the meeting is a significant information event.

The FOMC meets 8 times each year. There are two types of meetings of the FOMC: one-day meetings and two-day meetings. There are usually 2 two-day meetings and 6 one-day meetings every year. Our sample over the ten-year period consists of 58 one-day meetings and 22 two-day meetings. In total there were 80 meetings, i.e. one every 6-7 weeks. The first and fourth meetings every calendar year are two-day events. They begin at 2:30 pm on the first day, continuing at 9:00 am the following day. The one-day meetings always begin at 9:00 am. All meetings begin on Tuesdays.

At these meetings, the FOMC examines information about the economy and decides on whether to undertake open market operations in the dollar or other currencies. They also determine the level of short-term rates. The usual issues relating to the economic outlook are considered: consumer spending, non-farm payroll, industrial production, retail sales, real business fixed investment, nominal deficit, consumer price inflation, currency rates, money supply (M2,M3), and housing activity. At the two-day meetings additional policy directives are issued. In particular, these relate to domestic open market operations, authorization of foreign bank limits for foreign currency operations, foreign currency directives, and procedural instructions with reference to foreign currency operations. We find that the two-day meetings appear to have a greater information impact than one-day meetings.

In addition to foreign currency directives, the Fed also undertakes other distinct activity at the two-day meetings. By (the Humphrey-Hawkins) law, the Fed must report to Congress twice a year on monetary policy, i.e. in February and July. The two-day meetings are the setting for the discussions on monetary policy as well. The FOMC thus votes on the range of growth rates of M2,M3 and the debt levels it expects to see. Thus, two-day meetings tend to evidence more forward-looking discussions than usually occur at one-day meetings. However, these votes are not announced immediately, and only get reported in minutes two weeks after the meeting. Thus, it is not clear that this activity of the Fed in any way forms an information event. However, we do find that the two-day meetings seem to impact parameter estimation, in contrast to the one-day meetings.

To begin, we carry out a few simple regressions to ascertain if the volatility of interest rates is in any way related to information released at FOMC meetings. This is done by regressing the squared change in interest rates on interest rate level and a dummy variable for the FOMC meeting. The regression equation is as follows:

$$[r_{t+1} - r_t]^2 = a + br_t + cf_{t+1} + e_{t+1}$$

where f_t is the dummy variable indicating the FOMC meeting. It may take four different forms as described in Table 6 below. Since some of the meetings last 2 days, combinations are possible. First, we assign a dummy variable which is the

TABLE 6. **FOMC Meeting impact: Linear Regressions**

We examine via simple regressions whether the FOMC meeting results in a information surprise. The regression is

$$[r_{t+1} - r_t]^2 = a + br_t + cf_{t+1} + e_{t+1} \tag{3.4}$$

where f_t is the dummy variable for the FOMC meeting. T-statistics are presented below the parameter estimates.

Dummy variable (f_t)	a	b	c	R^2
1st day all meetings	0.0373	0.0078	0.0441	0.0019
	1.55	1.98	0.99	
1st day, 1 day meetings	0.0388	0.0078	-0.0120	0.0016
	1.61	1.99	-0.27	
1st day, 2 day meetings	0.0377	0.0075	0.3178	0.0070
	1.57	1.91	3.79	
2nd day, 1 day meetings	0.0381	0.0078	0.0723	0.0018
	1.58	1.98	0.86	

first day of all meetings. As can be seen, this has little impact on volatility, and hence provides evidence of no unexpected information. A similar result holds when we examine only 1 day meetings. However, when we set the dummy variable to be the first day of a 2-day meeting, the coefficient comes in strongly positive. This indicates that there may be a significant information release on the first day of the 2-day FOMC meetings. We also examined whether the information impact occured on the second day of the 2-day meeting and found little effect. Thus, if there is an information effect, it occurs on the first day of the two-day meeting. Table 6 summarizes the regression results.

This informal regression proxies for the possible impact of the FOMC meeting on interest rate changes. We now turn to the examination of whether the probability of a jump is linked in any way to the FOMC meetings. We achieve this using a modification of our Poisson-Gaussian estimation model depicted in equation (3.3). In the estimation equation (3.3), we specify that the arrival probability of a jump, denoted by the parameter q, be a function of the Fed meetings (f_t). It is possible that jumps in the interest rate are caused by Fed actions, and then the information on meetings would determine the probability of a jump taking place. Thus we specify

$$q_t = \lambda_0 + \lambda_1 f_t$$

The equation above accommodates a base level of jump probability λ_0, augmented by a Fed dependent attribute, λ_1. For the ARCH-diffusion model, we investigate whether the Fed meetings have an impact on conditional volatility by specifying the ARCH equation with an additional coefficient $a_{\{.\}}$ on the Fed event, i.e. the variance will be $a_0 + a_1\epsilon_t^2 + a_{\{.\}}f_t$. First, we examine the one-day meetings only. The results are presented in Table 7. The 1-day meetings appear to have very little impact on the usual levels of jump probability, as seen in the jump-diffusion model. The parameter λ_1 is not significant. And in fact, the ARCH model evidences a decrease in volatility when a one-day FOMC meeting takes place. We now examine the information impact of the 2-day meetings in Table 8. As in the basic regression in Table 6, this dummy variable proves to be significant, i.e. it increases the probability of a jump. This probability more than doubles im magnitude. In the ARCH model,

TABLE 7. **FOMC Meeting impact: One-day meetings**

We examine via ARCH and jump models whether the FOMC meeting results in a information surprise. The jump model is extended by $q_t = \lambda_0 + \lambda_{1day} f_t$ where f_t is the dummy variable for the FOMC meeting. The ARCH model is written as $a_0 + a_1 \epsilon_t + a_{1day} f_t$.

Model:	ARCH-diffusion		Jump-diffusion	
Parameter	Estimate	T-stat	Estimate	T-stat
k	1.2563	4.55	0.8625	2.28
θ	0.0971	9.58	0.0336	2.67
v			0.0173	24.03
a_0	0.0008	65.73		
a_1	232.9621	29.67		
a_{1day}	-0.0003	-4.22		
λ_0			0.2138	17.53
λ_{1day}			0.0803	1.17
μ			0.0004	1.36
γ			0.0058	24.48
Log-likelihood	14511.06		14891.68	

TABLE 8. **FOMC Meeting impact: Two-day meetings**

We examine via ARCH and jump models whether the FOMC meeting results in a information surprise. The jump model is extended by $q_t = \lambda_0 + \lambda_{2day} f_t$ where f_t is the dummy variable for the FOMC meeting. The ARCH model is written as $a_0 + a_1 \epsilon_t + a_{2day} f_t$.

Model:	ARCH-diffusion		Jump-diffusion	
Parameter	Estimate	T-stat	Estimate	T-stat
k	1.2915	4.75	0.8574	2.26
θ	0.0976	9.92	0.0334	2.64
v			0.0173	24.02
a_0	0.0008	66.45		
a_1	232.1430	30.02		
a_{2day}	0.0009	1.82		
λ_0			0.2139	17.78
λ_{2day}			0.3394	2.09
μ			0.0004	1.36
γ			0.0057	24.50
Log-likelihood	14511.91		14893.85	

the FOMC meeting appears to be of little consequence. Finally, we put both 1 day and 2 day meetings together in one model and ascertain the results in Table 9. The results here are an amalgamation of those from the prior two tables. The two-day meetings result in a sharp increase in the possibility of a jump. The one-day meetings in fact seem to predicate a reduction in conditional volatility. One might speculate that the two-day meetings do result in information surprises, whereas the one-day meetings confirm the market's forecasts.

3.5. **The Pervasiveness of the Non-linear Drift.** In recent papers, Ait-Sahalia [2], Conley, Hansen, Luttmer and Scheinkman [20], and Stanton [39] have found the drift of the short rate to be non-linear in the lagged interest rate. This may simply be an outcome of the specification of the stochastic process as a diffusion.

TABLE 9. **FOMC Meeting impact: One-day and Two-day meetings**

We examine via ARCH and jump models whether the FOMC meeting results in a information surprise. The jump model is extended by $q_t = \lambda_0 + \lambda_{1day} f_{1t} + \lambda_{2day} f_{2t}$ where f_{1t}, f_{2t} are the dummy variables for the FOMC meetings. The ARCH model is written as $a_0 + a_1 \epsilon_t + a_{1day} f_{1t} + a_{2day} f_{2t}$.

Model:	ARCH-diffusion		Jump-diffusion	
Parameter	Estimate	T-stat	Estimate	T-stat
k	1.2677	4.63	0.8660	2.26
θ	0.0973	9.72	0.0340	2.64
v			0.0173	24.04
a_0	0.0008	65.66		
a_1	232.9507	29.82		
a_{1day}	-0.0003	-4.09		
a_{2day}	0.0009	1.81		
λ_0			0.2114	17.39
λ_{1day}			0.0832	1.21
λ_{2day}			0.3422	2.11
μ			0.0003	1.34
γ			0.0057	24.48
Log-likelihood	14513.39		14894.69	

Introducing jumps into the specification may well render the drift linear in interest rates. We explore this aspect in this section.

We estimate four models allowing for non-linear drift terms: (i) a pure-diffusion model, (ii) a jump-diffusion model, (iii) an ARCH-diffusion model and (iv) an ARCH-jump-diffusion model. The general econometric specification is as follows:

$$dr_t = \left[k(\theta - r_t) + \alpha_2 r_t^2 + \alpha_3/r_t\right] dt + v_t dz_t + J(\mu, \gamma^2) d\pi(h)$$
$$v_{t+1} = a_0 + a_1 \left[dr_t - E(dr_t)\right]^2$$

The critical parameters are (α_2, α_3). They examine whether the drift is a function of squared interest rates or inversely related to interest rate levels. If any of these parameters is significantly different from zero, it means that the drift term is non-linear. The results are presented in Table 10.

The ARCH diffusion model failed to converge. It is evident from the table that the introduction of the jump does diminish the size of the non-linear coefficients (α_2, α_3). There is also a reduction in the level of significance. In fact the non-linearity parameters are significant at the 95% level but not at the 99% level once the jump model is introduced. Hence, it is possible that the jumps do make the model linear in drift. However, from the results here, this is not a strong conclusion, though it is suggestive. It is important to note that the introduction of non-linearity in the drift does not eliminate the statistical significance of the jump process.

We plot in Figure 2 the graph of the drift term for the pure-diffusion model and the jump-diffusion model. The dotted line shows the drift in the pure-diffusion model for interest rates varying from 1% to 12%. The full line depicts the jump-diffusion model. The non-linearity diminishes with the introduction of jumps in the extreme ranges of the graph.

It is also likely that the jump model with time-varying jump means may resolve the non-linear drift issue. Since it appears that jumps tend to be positively skewed

Table 10. Model Estimation with non-linear drift

This table presents the results of the estimation model where the drift term is non-linear. The model specification is as follows:

$$\begin{aligned} \Delta r &= [k(\theta - r) + \alpha_2 r + \alpha_3/r]\,\Delta t + v\,\Delta z + J(\mu,\gamma^2)\,\Delta\pi(q) \\ v^2_{t+\Delta t} &= a_0 + a_1[\Delta r_t - E(\Delta r_t)]^2 \end{aligned}$$

Model:	Pure-diffusion		Jump-diffusion		ARCH-jump	
Parameter	Estimate	T-stat	Estimate	T-stat	Estimate	T-stat
k	-81.6448	-4.67	-25.6544	-2.08	-20.0436	-2.09
θ	0.0546	36.86	0.0580	18.88	0.0554	23.33
α_2	-475.9293	-4.90	-142.9902	-2.02	-115.6292	-2.10
α_3	0.0769	4.57	0.0264	2.38	0.0189	2.20
v	0.0465	108.47	0.0173	23.96		
a_0					0.0001	17.65
a_1					127.0719	13.95
$q = hdt$			0.2162	17.90	0.1553	13.07
μ			0.0004	1.43	0.0017	5.69
γ			0.0058	24.46	0.0045	16.55
Log-likelihood	13944.11		14894.29		15200.17	

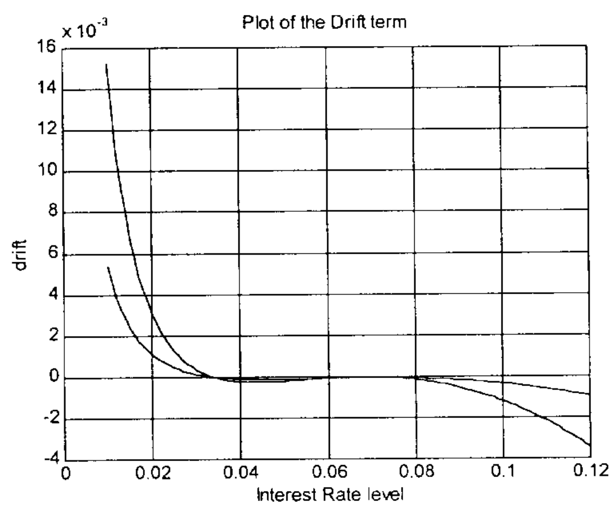

FIGURE 2

at lower interest rates, and negatively skewed at high rates (see Table 4), explicitly accounting for this fact may result in the drift becoming linear in the short rate. Therefore, we rerun the estimation carried out in Table 4 allowing for a non-linear drift term. The stochastic process specification is as follows:

$$\begin{aligned} dr_t &= \left[k\left(\theta - r_t\right) + \alpha_2 r_t^2 + \alpha_3/r_t\right] dt + v_t dz_t + J(\mu_t, \gamma^2) d\pi(h) \\ v_{t+1} &= a_0 + a_1\left[dr_t - E(dr_t)\right]^2 \\ \mu_t &= \alpha_0 + \alpha_1\left(\theta - r_t\right) \end{aligned}$$

The results are presented in Table 11. However, from Table 11, it is clear that

TABLE 11. **Estimation of the Time Varying Jump Means Model with non-linear drift**

We present results for the estimation of the Poisson-Gaussian model allowing for time variation in the mean of the jump size when the drift term is non-linear. This enables assessment of the mean reversion effects of the jump process, and its impact on the drift. Estimation is carried out using maximum-likelihood incorporating the transition density function in equation (3.3). The process estimated is specified in the following equations:

$$\begin{aligned}\Delta r &= [k(\theta - r) + \alpha_2 r^2 + \alpha_3/r]\,\Delta t + v\,\Delta z + J(\mu,\gamma^2)\,\Delta\pi(q)\\ \mu_t &= \alpha_0 + \alpha_1(\theta - r_t)\end{aligned}$$

T-statistics are presented below the parameter estimates. The variable q, the probability of a jump in the interval Δt is analogous to the continuous time parameter h for jump arrival intensity, by the relation $q \approx h\Delta t$.

Model:	Jump-diffusion	
Parameter	Estimate	T-stat
k	-26.9835	-2.18
θ	0.0577	20.27
α_2	-149.6366	-2.11
α_3	0.0272	2.45
v	0.0173	23.98
q	0.2163	17.90
α_0	0.0004	1.23
α_1	0.0419	3.07
γ	0.0057	23.15
Log-Likelihood	14899.13	

the mean-reversion introduced by the jump process is not sufficient to rule out the non-linearity in the drift term. One may conclude that while jump processes may ameliorate the non-linearity in the drift, it is still a feature that appears robust to enhanced specifications such as that introduced in this paper.

3.6. **Estimation using the Method of Moments.** The method of moments has the advantage that the jump distribution can be modelled quite generally, and the estimation scheme is easy to implement. Empirical estimation for the method of moments is undertaken using the standard Hansen [28] efficient generalized method of moments estimation approach. We estimated two models: (i) a pure diffusion model and (ii) the jump-diffusion model.

First, the pure diffusion model was estimated. All first four moments of the distribution were used for the estimation so as to be consistent with the jump-diffusion model. As in the paper by Chan, Karolyi, Longstaff and Sanders [18], the instruments used here are a constant, and the lagged value of the short rate. For the pure diffusion model, the moments used are a special case of the moments in Section 2.3, where the jump variables are eliminated from the moment expressions

TABLE 12. **Method of Moments Estimation**

This table presents results for method of moments estimation, using four raw moment conditions. The instruments used are a constant and once-lagged values of the short rate. The table presents estimates for a pure-diffusion model and a jump-diffusion model.

Model:	Pure-diffusion		Jump-diffusion	
Parameter	Estimate	T-stat	Estimate	T-stat
k	2.6880	3.69	3.0992	4.09
θ'	0.0593	10.82	0.0580	12.22
v'	0.0387	11.13	0.0447	10.82
$hE(J^3)$			1.0688×10^5	2.04
$hE(J^4)$			-2.08×10^7	-1.16
H	29.89		12.97	

by setting the jump intensity parameter $h = 0$.

$$
\begin{aligned}
\mu_1 &= \theta\left(1 - e^{-kT}\right) + re^{-kT} \\
\mu_2 &= \frac{v^2}{2k}(1 - e^{-2kT}) + \mu_1^2 \\
\mu_3 &= 3\mu_1 v^2\left(\frac{1 - e^{-2kT}}{2k}\right) + \mu_1^3 \\
\mu_4 &= 3\left(v^2\left(\frac{1 - e^{-2kT}}{2k}\right)\right)^2 \\
&\quad +6\mu_1^2\left(v^2\left(\frac{1 - e^{-2kT}}{2k}\right)\right) + \mu_1^4
\end{aligned}
$$

The results are presented in Table 12. The moment conditions may be summarized as

$$
E\left(\begin{bmatrix} \mu_1 - r_t \\ \mu_2 - r_t^2 \\ \mu_3 - r_t^3 \\ \mu_4 - r_t^4 \end{bmatrix} \otimes \begin{pmatrix} 1 \\ r_{t-1} \end{pmatrix}\right) = 0
$$

We minimize the usual objective function (denoted H). In contrast to Chan, Karolyi, Longstaff and Sanders, we use the exact continuous time moments rather than moments from a discretization of the short rate process. The jump-diffusion model is also estimated. One of the difficulties with the method of moments is that some parameters are not identifiable separately from the others. In the case of this model, the first jump moment $E[J]$ enters only as a sum with θ in the first moment. In addition, the second jump moment $E[J^2]$ always enters as a sum with v^2 in the second, third and fourth moments, and hence, is not separately identified. The values of these two variables are subsumed into θ and v^2 respectively. We relabel these parameters $\theta' = \theta + \frac{hE(J)}{k}$ and $v'^2 = v^2 + hE(J^2)$. Also, we estimate the composites $hE(J^3)$, and $hE(J^4)$ since $E(J^3), E(J^4)$ do not appear except as multiplied by h. The reparameterized moments from Section 2.3 are as follows:

$$\begin{aligned}
\mu_1 &= \theta'\left(1-e^{-kT}\right)+re^{-kT} \\
\mu_2 &= \frac{v'^2}{2k}(1-e^{-2kT})+\mu_1^2 \\
\mu_3 &= hE[J^3]\left(\frac{1-e^{-3kT}}{3k}\right) \\
&\quad +3\mu_1 v'^2\left(\frac{1-e^{-2kT}}{2k}\right)+\mu_1^3 \\
\mu_4 &= hE[J^4]\left(\frac{1-e^{-4kT}}{4k}\right)+3\left(v'^2\left(\frac{1-e^{-2kT}}{2k}\right)\right)^2 \\
&\quad +4\mu_1 hE[J^3]\left(\frac{1-e^{-3kT}}{3k}\right) \\
&\quad +6\mu_1^2\left(v'^2\left(\frac{1-e^{-2kT}}{2k}\right)\right)+\mu_1^4
\end{aligned}$$

Estimation results in Table 12 indicate a better fit for the jump-diffusion model versus the pure diffusion model, though neither model offers a very good statistical match. The difference in the objective functions (tested by χ^2 statistics) between the two models is significant. Specializations of the GMM approach used here may be achieved by simply choosing varied distributions for the jump. We have chosen here to retain a general form for the jump distribution.

Going beyond the method of moments analysis, it is instructive to examine the empirical moments over different data intervals. While the estimation results in Table 12 dealt with data only at daily intervals, we can use the theoretical results in Section 2.3 over multiple intervals with a view to understand whether the jump model is a-priori justified. Define the time interval between observations in the data as T. From Section 2.3, the variance of the jump-diffusion process is:

$$\mu_2-\mu_1^2=\frac{v^2+hE\left[J^2\right]}{2k}(1-e^{-2kT}).$$

The skewness is

$$\begin{aligned}
\text{Skewness} &= \frac{E(J-\mu_1)^3}{(\mu_2-\mu_1^2)^{3/2}} \\
&= \frac{2\sqrt{2k}e^{-kT}\left(1+e^{kT}+e^{2kT}\right)hE(J^3)}{3\left(1+e^{kT}\right)(v^2+hE(J^2))\sqrt{(1-e^{-2kT})(v^2+hE(J^2))}}.
\end{aligned}$$

If $h=0$, then the skewness is zero. The kurtosis of the process is:

$$\begin{aligned}
\text{Kurtosis} &= \frac{E(J-\mu_1)^4}{(\mu_2-\mu_1^2)^2} \\
&= \frac{(e^{2kT}-1)(3h^2E(J^2)^2+6hv^2E(J^2)+3v^4)+khE(J^4)(e^{2kT}+1)}{(e^{2kT}-1)(v^2+hE(J^2))^2}
\end{aligned}$$

When $h=0$, i.e. no jumps, the kurtosis is 3, which is the normal level. The conditional skewness and kurtosis are depicted in the Figures 3 and 4 for various intervals. The primary feature of these plots is that conditional skewness and

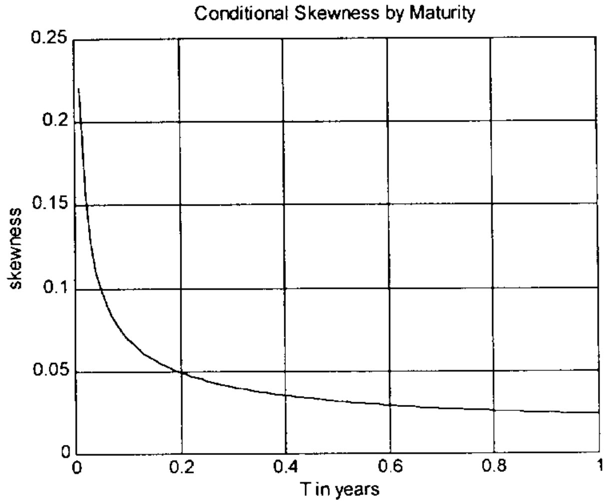

FIGURE 3

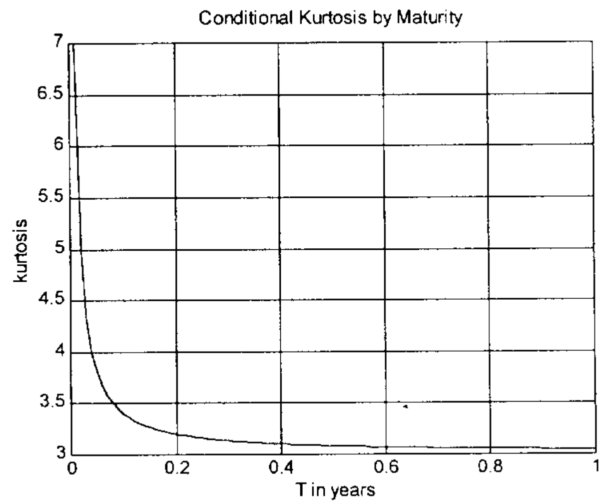

FIGURE 4

kurtosis decline monotonically as T increases. We will use this theoretical property shortly.

As a first check we compute the moments of the conditional distribution of interest rates using the estimated parameters for the jump-diffusion model in Table 3. Given the estimated values for the jump distribution (μ, γ^2), we can compute the following values: $E(J) = \mu, E(J^2) = \mu^2 + \gamma^2, E(J^3) = \mu^3 + 3\mu\gamma^2$, and $E(J^4) = \mu^4 + 6\mu^2\gamma^2 + 3\gamma^4$. Since our data is daily, the horizon T is $\frac{1}{262} = 3.8462 \times 10^{-3}$, given the number of trading days in a year. In order to make a rough comparison, the moments of the change in interest rates (dr) in Table 1 will correspond to

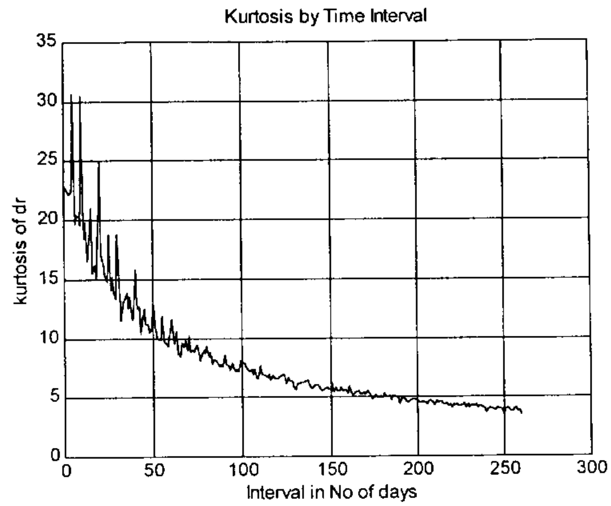

FIGURE 5

the computed moments at $T = 1/262$. In fact they correspond well. The values are (values from Table 1 are in brackets): standard deviation=0.0029 (0.0029), skewness=0.3553 (0.3950), and kurtosis=13.36 (19.86).

We now use the moments to understand the differences between the diffusion-based class of models and the jump-diffusion class. We first note that for any n-factor pure diffusion model, as the time interval for sampling the process goes to zero, i.e. $T \downarrow 0$, the conditional skewness and kurtosis also goes to zero. For example, in the case of a stochastic volatility diffusion model, when the time interval is very small, the volatility of volatility has little time to achieve any play, and so the higher moments are negligible. As T increases, these moments kick in, and skewness and kurtosis increase in magnitude. As T becomes very large, the multivariate diffusion model starts returning to being asymptotically Gaussian, with the result that the skewness and kurtosis revert to normal values. Thus, the graph for skewness and kurtosis tends to be hump-shaped, beginning at normal values for small T, then increasing with T, and finally declining back to normal.

On the other hand, in a jump-diffusion model, since at small T, there is still a chance that a large jump will take place, the possibility of a large outlier in comparison to normal variance is very high. This makes for substantial conditional skewness and kurtosis at short maturities. As T increases, the magnitude of the jump in relation to the diffusion shock decreases, and so skewness and kurtosis revert to normal values. Thus, the graph of skewness and kurtosis decline monotonically with T, as can be seen from Figures 3 and 4.

Therefore, an examination of the behavior of the kurtosis of the time series of changes in interest rates (dr) offers a simple way to check if the raw data itself suggests a jump model. If kurtosis declines monotonically with T, then it suggests that a jump process is required, since that feature would not be possible with a diffusion model, no matter how many factors it contained. The plot in Figure 5 depicts the kurtosis for interest rate changes where the time interval between observations varies from 1 day to 260 days. The plot has been generated by intervalling the data for n days, where $n = 1, 2, ...260$. When $n > 1$, the data set yields more than one intervalled time series; for example, when $n = 2$, we have two series, each 2 days apart. The reported kurtosis is the average of the kurtosis of each series. This eliminates to some extent any day-of-week effects that might affect the graph. These day-of-week effects still exists as may be seen from the jaggedness of the plot. However, the monotonic decline in kurtosis is unmistakeable.

Since the empirical kurtosis declines monotonically, as predicted by the theoretical moments from the jump-diffusion model, it confirms two aspects of term-structure models already identified previously in the empirical section: (i) that jumps exist, since the declining kurtosis plot would not arise from a pure-diffusion model alone, unless it were mixed with a jump process, and (ii) in the case of a mixed jump-diffusion model, a declining plot would only arise if jumps constituted a substantial component of the variation in the interest rate sample path. This, as we have seen from the results in Table 3, is certainly the case.

Therefore, the analysis of empirical moments adds conclusive evidence to the maximum-likelihood estimation results in confirming the presence of jumps in the data.

4. CONCLUDING COMMENTS

The objective of this paper is twofold. One, we develop technical methods for jump-diffusion term structure models. Two, we examine whether it is worthwhile to enhance existing diffusion models with jump processes.

The methodological innovations of the paper are as follows. (i) We derived the characteristic function for a general jump-diffusion stochastic process for the short rate. The process admits any jump distribution, and hence allows a wide range of possible specifications. This provides the raw material for further analysis. (ii) From the characteristic function we obtained the conditional moments for the short rate. These are useful in examining properties of the higher-order moments and distinguishing jump processes from diffusion models. (iii) We also derived the transition probability function, which is useful in carrying out maximum-likelihood estimation of the model. (iv) Finally, we derived an analytical expression for bond prices in the model.

The methodology is useful in examining the efficacy of jump processes for interest rate models. The evidence appears overwhelming. First, enhancement of the diffusion model with jumps resulted in a significant improvement in fit. Second, the jump model lends itself easily to extended analysis of the impact of information variables, such as the meetings of the Fed Open Market Committee. We found mild evidence that the two-day meetings of the Fed were in fact information revealing to the market. Third, we were able to use the jump model to examine day-of-week effects, and found these to be quite significant. Wednesdays and Thursdays evidence a much higher likelihood of jumps than other days of the week. This is likely to be the case since option expiry effects may result in sharp market movements. Fourth, recent research has found that the drift term in the stochastic process for interest rates appears to be non-linear. We found that this may be because of an incomplete specification of the random variation in the stochastic process. The addition of a jump process substantially diminishes the extent of non-linearity. In addition, when an ARCH model is superimposed, this provides an even greater reduction.

Finally, it is pertinent to ask whether jump processes do better than diffusion processes in modelling interest rates. We have certainly made a case for the enhancement of diffusion models with jump processes. To say that jump models do better than the best diffusion models would be going too far. For one, the literature is unclear as to what the 'best' diffusion model is. And two, the empirical work here clearly suggests an amalgamation of stochastic volatility cum jump-diffusion models. This paper provides in modest fashion, a comprehensive set of methods for jump processes in interest rate modelling, as well as a detailed empirical examination of the term structure using these techniques.

It is worthwhile suggesting further avenues of research, which would benefit from the framework of this paper. First, the issue of what information releases cause jumps is an open question. Locating jumps in the data and associating them with market events is one way of addressing this question. Second, a question of importance is whether Fed actions are endogenous or exogenous to the interest rate markets. This aspect is a strong determinant in the choice of the modelling framework (see Balduzzi, Bertola and Foresi [8]). Third, the shape of the term structure is also a function of whether agents price jump risk or not. Using the pricing equations in this study, cross sectional estimation of parameters would shed light on this question.[9] This would also address the issue of whether the expectations hypothesis (see CIR [21]) holds in a Poisson-Gaussian world. Fourth, rather than model jumps in the level of the interest rate, modelling jumps in the mean and volatility of the short rate is an alternate approach (see Naik & Lee [37]). Fifth, in a recent innovation, Heston [30] employs a gamma process as an alternative to the Poisson-Gaussian framework. A comparison of this model with the one in this paper would be insightful. Sixth, this work may be related to the work of Brandt and Santa-Clara [13], who develop a method of estimation using simulated transition density functions. Their work relates to diffusions only, and hence may be extended to jump-diffusions and then confirmed using the closed-form results of this paper. Finally, examining very short frequency intra-day data may reveal better the possible causes of jumps in bond yields. Eurodollar yields have also been suggested as a better benchmark for tests of this sort (see Duffee [27]). This paper leaves this rich menu of research projects for future work.

Appendix A. Deriving the Characteristic Function

The Kolmogorov backward equation (KBE) is

$$0 = \frac{\partial F}{\partial r} k(\theta - r) + \frac{1}{2}\frac{\partial^2 F}{\partial r^2} v^2 - \frac{\partial F}{\partial T} + hE\left[F(r+J) - F(r)\right]. \tag{A.1}$$

We guess a solution to this equation of the form

$$F(r, T; s) = \exp\left[A(T; s) + rB(T; s)\right]. \tag{A.2}$$

[9]See Bates [10] for a study of this issue in the foreign exchange markets.

Taking derivatives (notation for a derivative in subscripts) we obtain:

$$\begin{aligned} F_r &= BF \\ F_{rr} &= B^2 F \\ F_T &= F\left(A_T + rB_T\right). \end{aligned}$$

Rewriting equation (A.1) with the posited solution and rearranging gives

$$0 = r\left[-kB - B_T\right] + \left(k\theta B + \frac{1}{2}v^2 B^2 - A_T + hE\left[e^{JB} - 1\right]\right)$$

where $E(.)$ stands for the expectations operator over the probability distribution for J. Since $r = 0$ everywhere, the expressions is square brackets in the equation above must equal zero, and therefore provide us two ordinary differential equations which may be solved subject to appropriate boundary conditions. These conditions follow from equations (A.2) and (2.2) and are

$$\begin{aligned} A(T &= 0; s) = 0 \\ B(T &= 0; s) = is. \end{aligned}$$

The solution for the first ODE is

$$B(T; s) = is \exp\left(-kT\right)$$

and the solution for the second ODE is expressable as an integral after application of the boundary condition (though not in closed form):

$$A(T; s) = \int \left(k\theta B(T; s) + \frac{1}{2}v^2 B(T; s)^2 + hE\left[e^{JB(T;s)} - 1\right]\right) dT.$$

♠

Appendix B. Deriving Moments from the Characteristic Function

To obtain the moments, differentiate the characteristic function successively with respect to s. Let μ_n denote the nth moment, and F_n be the nth derivative of F with respect to s, i.e. $F_n = \frac{\partial F}{\partial s}$. Then

$$\mu_n = \frac{1}{i^n} F_n(s = 0).$$

Likewise let A_n, B_n be the nth derivatives of A and B respectively with respect to s. First let us compute the A_n. Substituting for B in A, we can write A as

$$A(T; s) = \int \left(k\theta i s e^{-kT} - \frac{1}{2}v^2 s^2 e^{-2kT} + hE\left[e^{Jise^{-kT}} - 1\right]\right) dT.$$

Then,

$$\begin{aligned}
\frac{dA}{ds} &= \int \left[k\theta i e^{-kT} - v^2 s e^{-2kT} + hie^{-kT} E\left[J e^{Jise^{-kT}}\right]\right] dT \\
\frac{d^2A}{ds^2} &= \int \left[-v^2 e^{-2kT} - he^{-2kT} E\left[J^2 e^{Jise^{-kT}}\right]\right] dT \\
\frac{d^3A}{ds^3} &= \int \left[-ihe^{-3kT} E\left[J^3 e^{Jise^{-kT}}\right]\right] dT \\
\frac{d^4A}{ds^4} &= \int \left[-ihe^{-4kT} E\left[J^4 e^{Jise^{-kT}}\right]\right] dT.
\end{aligned}$$

which makes use of the fact that the integral is bounded and the expectation $E(.)$ is also bounded. We can also compute the derivatives of A evaluated at $s = 0$, which are:

$$\begin{aligned}
\left(\frac{dA}{ds}\right)_{s=0} &= \int i\left[k\theta e^{-kT} + he^{-kT} E\left[J\right]\right] dT \\
&= i\left(-\theta e^{-kT} - \frac{h}{k} E[J] e^{-kT}\right) + c_1
\end{aligned}$$

Using the fact that $A(T = 0; s) = 0$, we get that $c_1 = \theta + \frac{h}{k}E[J]$, which when subsitututed back gives us:

$$\left(\frac{dA}{ds}\right)_{s=0} = i\left(\left(\theta + \frac{h}{k}E[J]\right)\left(1 - e^{-kT}\right)\right).$$

In like fashion, we can obtain the other derivatives evaluated at $s = 0$:

$$\begin{aligned}
\left(\frac{d^2A}{ds^2}\right)_{s=0} &= -\frac{v^2 + hE\left[J^2\right]}{2k}\left(1 - e^{-2kT}\right) \\
\left(\frac{d^3A}{ds^3}\right)_{s=0} &= -ihE[J^3]\left(\frac{1 - e^{-3kT}}{3k}\right) \\
\left(\frac{d^4A}{ds^4}\right)_{s=0} &= hE[J^4]\left(\frac{1 - e^{-4kT}}{4k}\right)
\end{aligned}$$

and the derivatives of B with respect to s:

$$\begin{aligned}
\frac{dB}{ds} &= ie^{-kT} \\
\frac{d^2B}{ds^2} &= \frac{d^3B}{ds^3} = \frac{d^4B}{ds^4} = 0.
\end{aligned}$$

We can write the intermediate value

$$\left(\frac{dA}{ds} + r\frac{dB}{ds}\right)_{s=0} = i\left(\left(\theta + \frac{hE[J]}{k}\right)\left(1 - e^{-kT}\right) + re^{-kT}\right) = i\mu_1$$

We can now evaluate the moments for the distribution of the interest rate r which are:

$$\mu_n = \frac{1}{i^n} F_n(s=0).$$

The first moment is

$$\begin{aligned}\mu_1 &= \frac{1}{i}\left(\frac{dF}{ds}\right)_{s=0} \\ &= \frac{1}{i}\left(\frac{dA}{ds} + r\frac{dB}{ds}\right)_{s=0}\end{aligned}$$

Since $A(s=0) = 0$ and $B(s=0) = 0$, and $\left[\frac{dA}{ds}\right]_{s=0}$ is given above, the above expression evaluates to:

(B.1)
$$\begin{aligned}\mu_1 &= \frac{1}{i}\left(\left(\frac{dA}{ds}\right)_{s=0} + r\frac{dB}{ds}\right) \\ &= \left(\theta + \frac{hE[J]}{k}\right)(1 - e^{-kT}) + re^{-kT}.\end{aligned}$$

Likewise, we can compute the other moments as well. The second moment is:

$$\begin{aligned}\mu_2 &= \frac{1}{i^2}\left(\frac{d^2F}{ds^2}\right)_{s=0} \\ &= \frac{1}{i^2}\left(e^{A+rB}\left[\frac{d^2A}{ds^2} + \left(\frac{dA}{ds} + r\frac{dB}{ds}\right)^2\right]\right)_{s=0} \\ &= -\left(\frac{d^2A}{ds^2}\right)_{s=0} - \left[\left(\frac{dA}{ds} + r\frac{dB}{ds}\right)^2\right]_{s=0} \\ &= \frac{v^2 + hE\left[J^2\right]}{2k}(1 - e^{-2kT}) \\ &\quad + \left(\left(\theta + \frac{hE[J]}{k}\right)(1 - e^{-kT}) + re^{-kT}\right)^2 \\ &= \frac{v^2 + hE\left[J^2\right]}{2k}(1 - e^{-2kT}) + \mu_1^2.\end{aligned}$$

The third moment is:

$$\begin{aligned}\mu_3 &= \frac{1}{i^3}\left(\frac{d^3A}{ds^3} + 3\frac{d^2A}{ds^2}\left(\frac{dA}{ds} + r\frac{dB}{ds}\right) + \left(\frac{dA}{ds} + r\frac{dB}{ds}\right)^3\right)_{s=0} \\ &= hE[J^3]\left(\frac{1 - e^{-3kT}}{3k}\right) \\ &\quad + 3\mu_1\left(v^2 + hE[J^2]\right)\left(\frac{1 - e^{-2kT}}{2k}\right) + \mu_1^3.\end{aligned}$$

And finally, the fourth moment is:

$$\begin{aligned}
\mu_4 &= \frac{1}{i^4}[\frac{d^4A}{ds^4} + 3\left(\frac{d^2A}{ds^2}\right)^2 + 4\frac{d^3A}{ds^3}\left(\frac{dA}{ds} + r\frac{dB}{ds}\right) \\
&\quad +6\frac{d^2A}{ds^2}\left(\frac{dA}{ds} + r\frac{dB}{ds}\right)^2 + \left(\frac{dA}{ds} + r\frac{dB}{ds}\right)^4]_{s=0} \\
&= hE[J^4]\left(\frac{1-e^{-4kT}}{4k}\right) + 3\left((v^2 + hE[J^2])\left(\frac{1-e^{-2kT}}{2k}\right)\right)^2 \\
&\quad +4\mu_1 hE[J^3]\left(\frac{1-e^{-3kT}}{3k}\right) \\
&\quad +6\mu_1^2\left((v^2 + hE[J^2])\left(\frac{1-e^{-2kT}}{2k}\right)\right) + \mu_1^4
\end{aligned}$$

Now, these four moments may be used to carry out method of moments estimation. Notice that this is possible to do for any jump distribution where the moments are known since we need the values of $E[J^n], n = 1, 2, 3, 4$. ♠

Appendix C. Derivation of the Bond Pricing Equation

Denote present time by the variable t, the time point at the maturity of a bond by T, and the time to maturity $\tau = T - t$. Define the price of a zero-coupon bond to be $P(r, \tau)$; we assume that all factors determining the value of this bond are captured in equation (2.1) above.

Using an extended version of Ito's Lemma (see for example, Kushner [33], Merton [35] for details), the differential process for $P(r, t)$ is:

$$\begin{aligned}
dP &= \left(k(\theta - r)P_r + P_t + \frac{1}{2}v^2 P_{rr}\right) dt + vP_r dz \\
&\quad + ([P(r + J, \tau) - P(r, \tau)] \mid J)\, d\pi
\end{aligned} \tag{C.1}$$

where the subscripts denote the relevant derivatives.

In equilibrium, the risk adjusted return on all bonds must be the same. The market price per unit of diffusion risk (λ) for the bond is proportional to the volatility (v) of the interest rate. We shall assume that the price of jump risk (λ_J) modifies the arrival rate of the jump i.e. the risk-neutral arrival rate is $h^* = h(1 - \lambda_J)$.[10] Standard partial equilibrium arguments (see Vasicek [40]) lead to the following arbitrage-free pricing partial differential-difference equation:

$$\begin{aligned}
0 &= (k(\theta - r) - \lambda v)\, P_r - P_\tau + \frac{1}{2}v^2 P_{rr} - rP \\
&\quad +h^*\, E\left[(P(r + J, \tau) - P(r, \tau))\right]
\end{aligned} \tag{C.2}$$

where τ has been substituted for t. Equation (C.2) is the partial differential-difference equation (PDDE) describing the behavior of the bond price. We solve this subject to the following boundary condition:

$$P(r, \tau = 0) = 1 \tag{C.3}$$

[10] For a derivation of the prices of risk in an equilibrium framework see Das and Foresi [23]. We assume here that under the changes made to the statistical process, there exists a risk-neutral martingale measure under which interest rate derivative securities may be valued.

The solution is obtained by positing that the functional form of the bond price is given by $P(r,\tau) = \exp[A(\tau) + rB(\tau)]$. Taking the appropriate derivatives and inserting them into the PDDE (C.2) above results in the following separable equation:

$$\begin{aligned} 0 &= [k(\theta - r) - \lambda v]B - A_\tau - rB_\tau + \frac{1}{2}v^2B^2 - r \\ &\quad + \left[h^* E(e^{JB} - 1)\right] \end{aligned} \tag{C.4}$$

This can be restated as:

$$\begin{aligned} 0 &= r\left[-kB - B_\tau - 1\right] \\ &\quad + \left(k\theta B - \lambda vB - A_\tau + \frac{1}{2}v^2B^2 + q\right) \end{aligned} \tag{C.5}$$

where

$$q = h^* E(e^{JB} - 1)$$

The elements in square brackets in equation (C.5) form two ordinary differential equations in τ only and are easily solved to provide the solution to the entire PDE. The two ODEs are as follows:

$$\begin{aligned} 0 &= kB + B_\tau - 1 \\ 0 &= k\theta B - \lambda vB - A_\tau + \frac{1}{2}v^2B^2 + q \end{aligned} \tag{C.6}$$

We solve the first ODE subject to the boundary condition that $B(\tau = 0) = 0$ to obtain the expression for $B(\tau) = \frac{e^{-k\tau}-1}{k}$. Using this result in the second ODE subject to the boundary condition that $A(\tau = 0) = 0$ gives the solution for $A(\tau)$. We obtain $A(\tau)$ as the solution to the following integral equation:

$$A(\tau) = \int \left((k\theta - \lambda v)B(\tau) + \frac{1}{2}v^2B(\tau)^2 + q[B(\tau)]\right) d\tau$$

In general, the second ODE does not always admit a closed-form solution. Specifically, even in the simplest case, when J is a constant, no closed-form solution is achievable. Also, when J follows a Gaussian distribution, no closed-form solution results. Das and Foresi [23] find that in the special case of a jump with a sign based on a Bernoulli distribution and a size based on an exponential distribution, it is possible to obtain a closed-form solution. ♠

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