ELSEVIER

European Journal of Operational Research 136 (2002) 635–654

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/dsw

Stochastics and Statistics

Balancedness of infrastructure cost games

Henk Norde [a,*], Vito Fragnelli [b], Ignacio García-Jurado [c], Fioravante Patrone [d], Stef Tijs [a]

[a] *Faculty of Economics and Business Administration, Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, Netherlands*

[b] *University of Piemonte Orientale, Corso Borsalino 54, 15100 Alessandria, Italy*

[c] *Faculty of Mathematics, Department of Statistics and OR, University of Santiago de Compostela, 15771 Santiago de Compostela, Spain*

[d] *Department of Mathematics, Via Dodecaneso 35, University of Genoa, 16146 Genoa, Italy*

Received 4 August 1999; accepted 9 February 2001

Abstract

In this paper we study the class of infrastructure cost games. A game in this class models the infrastructure costs (both building and maintenance) produced when a set of users of different types makes use of a certain infrastructure, which may consist of several facilities. Special attention is paid to one facility infrastructure cost games. Such games are modeled as the sum of an airport game and a maintenance cost game. It turns out that the core and nucleolus of these games are very closely related to the core and nucleolus of an associated generalized airport game. Furthermore we provide necessary and sufficient conditions under which an infrastructure cost game is balanced.

Keywords: Game theory; Cost allocation; Balancedness

1. Introduction

The main motivation for this paper is a cost allocation problem arising from the reorganization of the railway sector in Europe. Application of the European Community (EC) directive 440/91 involves the separation between infrastructure management and transport operations. In this situation two main problems arise. One is to allocate the track capacity among the various operators. This issue has been treated, for instance, in Nilsson (1995), Brewer and Plott (1996) and Bassanini and Nastasi (1997). The second problem is to study how the infrastructure costs must be allocated to the operators through a fair fees system. We devote the present paper to approach this second problem from a game theoretical point of

* Corresponding author. Tel.: +31-13-466-3245; fax: +31-13-466-3280.
E-mail address: h.norde@kub.nl (H. Norde).

0377-2217/02/$ - see front matter
PII: S0377-2217(01)00073-X

view. Tijs and Driessen (1986) and Young (1994) provide a survey on game theoretical applications in cost allocation problems.

Consider a railway infrastructure, that is used by different types of trains (fast trains, local trains, freight trains) belonging to several operators, and consider the problem of dividing the infrastructure costs among these trains. Clearly it is a problem of joint cost allocation. To settle the question, one can see the infrastructure as consisting of several facilities (tracks, signaling system, stations, etcetera). Trains of different type need these facilities at different levels of sophistication: fast trains need a track and signaling system of a higher quality than local trains, for which instead station services are more important (in particular "small" stations). Furthermore, infrastructure costs can be seen as the sum of "building" costs and "maintenance" costs. If we consider only building costs, especially in the case of a single facility, we are facing a problem similar to the so-called "airport game" (see, for instance, Littlechild and Owen, 1973 and Dubey, 1982). For what concerns maintenance costs, it is a reasonable first-order approximation to assume that they depend on the type of trains, and are proportional both to the building costs and to the number of users.

Similar considerations extend to related problems: for example the costs for a bridge, to be used by small and big cars. There are building costs, that depend on the type of bridge needed (one for small cars only or one to be used by big cars too) and maintenance costs, that depend on the type of bridge but also on the type of car using the bridge. Moreover these costs are proportional to the number of vehicles using the bridge. Another related problem arises when some community has to buy a set of glasses: the "building" costs depend on the kind of glasses, while the maintenance costs can be considered as proportional to the number of glasses (the proportionality coefficient could be related with the probability of breaking a glass during some given unit of time).

In order to model the infrastructure costs we use the class of infrastructure cost games, which has been introduced by Fragnelli et al. (2000). In this paper we analyze these games from the point of view of the core (exploiting the special structure of these games, we get a minimal collection of conditions that are equivalent to balancedness). In Section 2 we give some preliminary definitions and results on airport games and generalized airport games (a new class of games which turns out to be closely related to that of one facility infrastructure cost games). In Section 3 we focus on one facility infrastructure cost games (being the sum of an airport game and a maintenance cost game). In particular we study under what conditions such a game is balanced. Moreover, we provide simple formulas for solutions based on the concept of egalitarianism (see Dutta and Ray, 1989) and on the nucleolus. These solutions always select a core element if the game is balanced. Section 4 is devoted to the study of infrastructure cost games, where the infrastructure consists of two or more facilities. Again, we analyze under what conditions such a game is balanced.

Notation. Throughout this paper capital letters with one subscript are used to denote partial sums, e.g. $B_j = b_1 + \cdots + b_j$, $N_i = n_1 + \cdots + n_i$, $\hat{B}_k = \hat{b}_1 + \cdots + \hat{b}_k$, etcetera. For a vector $(x_i)_{i\in N}$ and an $S \subseteq N$ we will write $x(S)$ instead of $\sum_{i\in S} x_i$. For a $k \in \mathbb{N}$ we denote the set $\{1,\ldots,k\}$ by K.

2. Airport games

First we recall the definition of an airport game (see Littlechild and Owen, 1973 and Littlechild, 1975).

Definition 2.1. Suppose we are given k non-empty groups of players $g_1,\ldots,g_k$ with $n_1,\ldots,n_k$ players respectively and k non-negative numbers $b_1,\ldots,b_k$. The *airport game* corresponding to $g_1,\ldots,g_k$ and $b_1,\ldots,b_k$ is the cooperative (cost) game $\langle N,c\rangle$ with $N = \bigcup_{i=1}^k g_i$ and cost function c defined by

$$c(S) = B_{j(S)} \ \left(= b_1 + \cdots + b_{j(S)}\right)$$

for every $S \subseteq N$, where $j(S) = \max\{j : S \cap g_j \neq \emptyset\}$. We denote by $B(g_1, \ldots, g_k)$ the set of all airport games with groups of players $g_1, \ldots, g_k$.

Airport games are cost games for the building of one facility (a landing strip) where the wishes of the coalitions are linearly ordered. Coalitions desiring a more sophisticated facility (a larger landing strip) have to pay at least as much as coalitions desiring a less sophisticated facility (a smaller landing strip). If we drop this monotonicity condition we get the class of generalized airport games.

Definition 2.2. Suppose we are given k non-empty groups of players $g_1, \ldots, g_k$ with $n_1, \ldots, n_k$ players respectively and k real numbers $b_1, \ldots, b_k$ such that $0 \leqslant B_l \leqslant B_k$ for every $l \in \{1, \ldots, k-1\}$. The *generalized airport game* corresponding to $g_1, \ldots, g_k$ and $b_1, \ldots, b_k$ is defined in the same way as an airport game corresponding to $g_1, \ldots, g_k$ and $b_1, \ldots, b_k$.

Airport games are known to be concave. Consequently, the Shapley value of such a game provides a core element. In fact this core element is symmetric, i.e., treats players belonging to the same group in the same way.

Definition 2.3. Let $\langle N, c\rangle$ be a cooperative game and let $g_1, \ldots, g_k$ be a partition of N. The *symmetric core* of $\langle N, c\rangle$ corresponding to $g_1, \ldots, g_k$ is the collection

$$C^{\mathrm{sym}}(c) := \{(x_i)_{i \in N} \in C(c) : x_i = x_{i'} \text{ for every } i, i' \in N \text{ with } j(i) = j(i')\}.$$

Here $C(c) := \{(x_i)_{i \in N} \in \mathbb{R}^N : x(S) \leqslant c(S) \text{ for every } S \subseteq N, x(N) = c(N)\}$ denotes the core of $\langle N, c\rangle$ and $j(i)$ denotes the index of the group to which player i belongs. For a symmetric core element $((x_1)_{i \in g_1}, \ldots, (x_k)_{i \in g_k})$ we will write briefly $(x_1, \ldots, x_k)$.

In the following proposition we provide a complete description of the symmetric core of a generalized airport game.

Proposition 2.1. *Let $\langle N, c\rangle$ be a generalized airport game corresponding to $g_1, \ldots, g_k$ and $b_1, \ldots, b_k$. Then $(x_1, \ldots, x_k) \in C^{\mathrm{sym}}(c)$ iff*

$$\begin{aligned} & x_i \geqslant 0 \quad \text{for every } i \in K, \\ & n_1 x_1 + \cdots + n_i x_i \leqslant B_i \quad \text{for every } i \in \{1, \ldots, k-1\}, \\ & n_1 x_1 + \cdots + n_k x_k = B_k. \end{aligned} \tag{1}$$

Proof. "$\Rightarrow$". Suppose $x = (x_1, \ldots, x_k) \in C^{\mathrm{sym}}(c)$. Then for every $i \in K$ we have $n_i x_i = x(g_i) = x(N) - x(N \setminus g_i) = c(N) - x(N \setminus g_i) \geqslant c(N) - c(N \setminus g_i) \geqslant 0$ which implies $x_i \geqslant 0$. The other conditions in (1) are implied by $x(S) \leqslant c(S)$ for $S = \bigcup_{l=1}^{i} g_l$ with $i \in \{1, \ldots, k-1\}$ and $x(N) = c(N)$.

"$\Leftarrow$". Suppose that $(x_1, \ldots, x_k)$ satisfies the conditions (1). Take $S \subseteq N$ and let $s_j := |S \cap g_j|$ for every $j \in K$. Then $x(S) = \sum_{l=1}^{j(S)} s_l x_l \leqslant \sum_{l=1}^{j(S)} n_l x_l \leqslant B_{j(S)} = c(S)$. Clearly, the last condition in (1) guarantees efficiency. □

As a consequence of Proposition 2.1 we get that generalized airport games are balanced.

Corollary 2.1. *Every generalized airport game is balanced.*

Proof. Let $\langle N, c\rangle$ be a generalized airport game corresponding to $g_1, \ldots, g_k$ and $b_1, \ldots, b_k$. One easily verifies that $(x_1, \ldots, x_k)$ defined by $x_i := 0$ for every $i \in \{1, \ldots, k-1\}$ and $x_k := B_k/n_k$ satisfies conditions (1). □

The following proposition provides a relation between generalized airport games and airport games. With every generalized airport game we associate an airport game in the following way: join a group with negative marginal costs with its preceding group and repeat this procedure as long as necessary. Eventually we will end up with an airport game whose symmetric core is closely related to the symmetric core of the generalized airport game we started with.

Proposition 2.2. *Let $\langle N, c\rangle$ be a generalized airport game corresponding to $g_1, \ldots, g_k$ and $b_1, \ldots, b_k$ and let $i \in \{2, \ldots, k\}$ be such that $b_i < 0$. Let $\langle N, c'\rangle$ be the generalized airport game corresponding to $g_1, \ldots, g_{i-2}, g_{i-1} \cup g_i, g_{i+1}, \ldots, g_k$ and $b_1, \ldots, b_{i-2}, b_{i-1} + b_i, b_{i+1}, \ldots, b_k$. Then we have*
(i) *if $(x_1, \ldots, x_k) \in C^{\text{sym}}(c)$ then $(x_1, \ldots, x_{i-2}, y, x_{i+1}, \ldots, x_k) \in C^{\text{sym}}(c')$, where $y = (n_{i-1}x_{i-1} + n_i x_i)/(n_{i-1} + n_i)$;*
(ii) *if $(x_1, \ldots x_{i-2}, y, x_{i+1}, \ldots, x_k) \in C^{\text{sym}}(c')$ then $(x_1, \ldots, x_k) \in C^{\text{sym}}(c)$ for every non-negative x_{i-1} and x_i satisfying $n_{i-1}x_{i-1} + n_i x_i = (n_{i-1} + n_i)y$.*

Proof. The result is an immediate consequence of Proposition 2.1 for both games $\langle N, c\rangle$ and $\langle N, c'\rangle$ and the fact that, for $\langle N, c\rangle$, the condition $n_1x_1 + \cdots + n_{i-1}x_{i-1} \leqslant B_{i-1}$ can be skipped, since it is implied by the other inequalities. □

Example 2.1. Let $\langle N, c\rangle$ be the generalized airport game corresponding to g_1, g_2, g_3, g_4 and $b_1 = 7, b_2 = 2, b_3 = -3, b_4 = 4$. Application of Proposition 2.2 (twice) yields that we can compute the symmetric core of $\langle N, c\rangle$ by considering the airport game $\langle N, c'\rangle$ corresponding to $g_1 \cup g_2 \cup g_3, g_4$ and $c'_1 = 7 + 2 + (-3) = 6$ and $c'_2 = 4$. For every $(y_1, y_2) \in C^{\text{sym}}(c')$ (i.e. for every (y_1, y_2) satisfying $y_1 \geqslant 0, y_2 \geqslant 0, (n_1 + n_2 + n_3)y_1 \leqslant 6, (n_1 + n_2 + n_3)y_1 + n_4 y_2 = 10$), every vector (x_1, x_2, x_3, x_4) satisfying $x_i \geqslant 0$ for every $i \in \{1, \ldots, 4\}$, $n_1x_1 + n_2x_2 + n_3x_3 = (n_1 + n_2 + n_3)y_1$ and $x_4 = y_2$ is a symmetric core element of $\langle N, c\rangle$. In this way we obtain all symmetric core elements of $\langle N, c\rangle$.

3. One facility infrastructure cost games

In airport games costs for the building of one facility (the landing strip) are modeled. Now we consider the maintenance costs of this facility, which lead to the class of maintenance cost games. Maintenance cost games have been introduced and studied more widely in Fragnelli et al. (2000). Basic assumptions are that maintenance costs depend on the type of user and are increasing with the degree of sophistication of the facility and that each group contributes to the cost an amount which is proportional to its size.

Definition 3.1. Suppose we are given k non-empty groups of players $g_1, \ldots, g_k$ with $n_1, \ldots, n_k$ players respectively and $k(k+1)/2$ non-negative numbers $\{A_{ij}\}_{i,j \in K, i \leqslant j}$, with $A_{ij} \leqslant A_{ij'}$ for every $i \leqslant j \leqslant j'$. The *maintenance cost game* corresponding to $g_1, \ldots, g_k$ and $\{A_{ij}\}_{i,j \in K, i \leqslant j}$ is the cooperative (cost) game $\langle N, c\rangle$ with $N = \bigcup_{i=1}^k g_i$ and cost function c defined by

$$c(S) = \sum_{i=1}^{j(S)} |S \cap g_i| A_{ij(S)} \tag{2}$$

for every $S \subseteq N$. We denote by $M(g_1, \ldots, g_k)$ the set of all maintenance cost games with groups of players $g_1, \ldots, g_k$.

The following interpretation can be given to formula (2). Suppose that coalition S faces the problem of estimating the maintenance costs that are due to the fact that all players in S have used some facility. Since all players in S want to use this facility again it has to be restored up to level $j(S)$. However, players in S can be of different types, i.e., belong to different groups, and hence cause different maintenance costs. If a player in S belongs to group g_i (with $i \leqslant j(S)$) then the maintenance costs, caused by this player, are $A_{ij(S)}$. Summing up the costs of all players in S yields formula (2). So, the number A_{ij} represents the maintenance costs, caused by one player in group g_i, if the facility is going to be restored up to level j. Observe that the higher the level of restoration of the facility is (i.e. the higher j is) the higher the maintenance costs are.

Cost games which take both building and maintenance costs into account are infrastructure cost games. In Fragnelli et al. (2000) these games are introduced and a simple expression of the Shapley value of such a game is provided.

Definition 3.2. A *one facility infrastructure cost game* with groups of players $g_1, \ldots, g_k$ is the cooperative (cost) game $\langle N, c \rangle$ with $N = \bigcup_{i=1}^{k} g_i$ and cost function $c = c' + c''$ such that $\langle N, c' \rangle \in B(g_1, \ldots, g_k)$ and $\langle N, c'' \rangle \in M(g_1, \ldots, g_k)$. An *infrastructure cost game* with groups of players $g_1, \ldots, g_k$ is the cooperative (cost) game $\langle N, c \rangle$ with $N = \bigcup_{i=1}^{k} g_i$ and cost function $c = c^1 + \cdots + c^l$ such that, for every $r \in \{1, \ldots, l\}$, $\langle N, c^r \rangle$ is a one facility infrastructure cost game with groups of players $g_{\pi^r(1)}, \ldots, g_{\pi^r(k)}$, where π^r is a permutation of $K(= \{1, \ldots, k\})$.

In infrastructure cost games both building and maintenance costs are taken into account. From an applicational point of view one could for example think of the building of a freeway. First, a loan is taken in order to construct the freeway and later the users have to pay the loan together with the maintenance costs. Another example that fits in our model is the maintenance of a railway infrastructure. The infrastructure has been built many years ago and the building costs have already been paid for. So, the maintenance costs are the only costs to be taken into account. However, these maintenance costs can be decomposed into a fixed part, i.e., a part that does not depend upon the number of users, and a variable part that depends on the number of users. In this situation the fixed costs correspond to an airport game, and the variable costs to a maintenance cost game. See for example Fragnelli et al. (2000) for a realistic example.

From the definition above we see that a one facility infrastructure cost game is the sum of an airport game and a maintenance cost game with the same groups of players ordered in the same way. An infrastructure cost game is the sum of a finite set of one facility infrastructure cost games with the same groups of players but, perhaps, ordered in a different way. Because of this, it is not true that every infrastructure cost game is a one facility infrastructure cost game.

In this section we focus on one facility infrastructure cost games. In Section 4 we study infrastructure cost games, where the infrastructure consists of two or more facilities. First we provide a characterization of the balanced one facility infrastructure cost games.

Proposition 3.1. *Let $\langle N, c \rangle$ be the one facility infrastructure cost game with groups $g_1, \ldots, g_k$, and non-negative numbers $\{b_i\}_{i \in K}$ and $\{A_{ij}\}_{i,j \in K, i \leqslant j}$. Then $\langle N, c \rangle$ is balanced iff*

$$\sum_{i=1}^{j} n_i (A_{ik} - A_{ij}) \leqslant B_j \tag{3}$$

for every $j \in \{1, \ldots, k-1\}$.

Proof. "$\Rightarrow$". Suppose that $\langle N, c \rangle$ is balanced. Let $j \in \{1, \ldots, k-1\}$. The collection $\{\bigcup_{l=1}^{j} g_l, \bigcup_{l=j+1}^{k} g_l\}$ is a balanced collection. Hence $c(N) \leqslant c(\bigcup_{l=1}^{j} g_l) + c(\bigcup_{l=j+1}^{k} g_l)$ or, equivalently,

$$\sum_{i=1}^{j} n_i(A_{ik} - A_{ij}) \leqslant B_j.$$

So the conditions (3) are satisfied.

"⇐". Suppose that the conditions (3) are satisfied. Define the symmetric allocation $x = (x_i)_{i \in K}$ by

$$x_i := \begin{cases} A_{ik} & \text{if } i \in \{1, \ldots, k-1\}, \\ A_{kk} + (B_k/n_k) & \text{if } i = k. \end{cases}$$

We will show that $x \in C^{\text{sym}}(c)$. First note that $x(N) = c(N)$. Take $S \subseteq N$ and let $s_j := |S \cap g_j|$ for every $j \in K$. If $j(S) = k$ then

$$x(S) = s_k \frac{B_k}{n_k} + \sum_{i=1}^{k} s_i A_{ik} \leqslant B_k + \sum_{i=1}^{k} s_i A_{ik} = c(S).$$

If $l := j(S) < k$ then

$$\begin{aligned} x(S) &= \sum_{i=1}^{l} s_i A_{ik} \\ &= \sum_{i=1}^{l} s_i A_{il} + \sum_{i=1}^{l} s_i (A_{ik} - A_{il}) \\ &\leqslant \sum_{i=1}^{l} s_i A_{il} + \sum_{i=1}^{l} n_i (A_{ik} - A_{il}) \\ &\leqslant \sum_{i=1}^{l} s_i A_{il} + B_l \\ &= c(S). \qquad \square \end{aligned}$$

Condition (3) is obtained by considering minimal balanced collections which correspond to "splits" of N into two groups of the following kind: $g_1 \cup \cdots \cup g_j$ and $g_{j+1} \cup \cdots \cup g_k$. The interpretation of these conditions is the following: the maintenance costs that the players in $g_1 \cup \cdots \cup g_j$ have to pay for the level of the "needs" of the other players should be less than or equal to the building costs for the facility at the level needed by these groups themselves. In the following proposition we present a complete description of the symmetric core in case the one facility infrastructure cost game is balanced.

Proposition 3.2. *Let $\langle N, c \rangle$ be the one facility infrastructure cost game with groups $g_1, \ldots, g_k$, and non-negative numbers $\{b_i\}_{i \in K}$ and $\{A_{ij}\}_{i,j \in K, i \leqslant j}$. Then $(x_1, \ldots, x_k) \in C^{\text{sym}}(\hat{c})$ iff*

$$\begin{aligned} & x_j \geqslant A_{jk} \quad \textit{for every } j \in K, \\ & \sum_{i=1}^{j} n_i x_i \leqslant B_j + \sum_{i=1}^{j} n_i A_{ij} \quad \textit{for every } j \in \{1, \ldots, k-1\}, \\ & \sum_{i=1}^{k} n_i x_i = B_k + \sum_{i=1}^{k} n_i A_{ik}. \end{aligned} \tag{4}$$

Proof. "⇒". Suppose $x = (x_1, \ldots, x_k) \in C^{\text{sym}}(c)$. Then for every $j \in K$ we have $n_j x_j = x(g_j) = x(N) - x(N \setminus g_j) = c(N) - x(N \setminus g_j) \geqslant c(N) - c(N \setminus g_j) \geqslant n_j A_{jk}$ which implies $x_j \geqslant A_{jk}$. The other conditions in (4) are implied by $x(S) \leqslant c(S)$ for $S = \bigcup_{i=1}^{j} g_i$ with $j \in \{1, \ldots, k-1\}$ and $x(N) = c(N)$.

"$\Leftarrow$". Suppose that $(x_1,\ldots,x_k)$ satisfies the conditions (4). Take $S \subseteq N$ and let $s_j := |S \cap g_j|$ for every $j \in K$. Then

$$\begin{aligned} x(S) - c(S) &= \sum_{i=1}^{j(S)} s_i x_i - \left(B_{j(S)} + \sum_{i=1}^{j(S)} s_i A_{ij(S)} \right) \\ &= \sum_{i=1}^{j(S)} s_i (x_i - A_{ij(S)}) - B_{j(S)} \\ &\leqslant \sum_{i=1}^{j(S)} n_i (x_i - A_{ij(S)}) - B_{j(S)} \\ &\leqslant 0. \end{aligned}$$

The first inequality follows from the fact that $x_i \geqslant A_{ik} \geqslant A_{ij(S)}$ for every $i \in \{1,\ldots,j(S)\}$ and the second inequality from conditions (4). □

In the following proposition we show that the symmetric core of a one facility infrastructure cost game is in fact the shifted symmetric core of a corresponding generalized airport game.

Proposition 3.3. *Let $\langle N, c\rangle$ be the one facility infrastructure cost game with groups $g_1,\ldots,g_k$, and non-negative numbers $\{b_i\}_{i\in K}$ and $\{A_{ij}\}_{i,j\in K, i\leqslant j}$. Suppose that $\langle N, c\rangle$ is balanced, i.e., that conditions (3) are satisfied. Let $\langle N, \hat{c}\rangle$ be the generalized airport game corresponding to $g_1,\ldots,g_k$ and $\hat{b}_1,\ldots,\hat{b}_k$, where the numbers $\hat{b}_1,\ldots,\hat{b}_k$ are given by*

$$\begin{array}{lcl} \hat{b}_1 & = & B_1 - n_1(A_{1k} - A_{11}) \\ \hat{b}_1 + \hat{b}_2 & = & B_2 - \sum_{i=1}^{2} n_i(A_{ik} - A_{i2}) \\ \vdots & & \vdots \\ \hat{b}_1 + \hat{b}_2 + \cdots + \hat{b}_{k-1} & = & B_{k-1} - \sum_{i=1}^{k-1} n_i(A_{ik} - A_{i(k-1)}) \\ \hat{b}_1 + \hat{b}_2 + \cdots + \hat{b}_{k-1} + \hat{b}_k & = & B_k. \end{array}$$

Let $\hat{x} \in \mathbb{R}^k$ and define $x \in \mathbb{R}^k$ by

$$x_i := \hat{x}_i + A_{ik}$$

for every $i \in K$. Then we have $x \in C^{\mathrm{sym}}(c)$ iff $\hat{x} \in C^{\mathrm{sym}}(\hat{c})$.

Proof. The proof follows directly from Propositions 2.1 and 3.2. □

The interpretation of Proposition 3.3 is the following. In order to determine the symmetric core of a one facility infrastructure cost game the infrastructure costs should be divided in the following way. Let every player pay the maintenance costs which occur if he uses the facility and this facility is going to be restored up to the highest level of sophistication. Now divide the building costs by considering any symmetric core allocation of the generalized airport game in which the costs for $g_1 \cup \cdots \cup g_j$ ($j \in \{1,\ldots,k-1\}$) are the costs for building the facility at the level they wish minus the extra amount of maintenance costs they already paid for the level of the needs of the other players.

For airport games the egalitarian solution (see Dutta and Ray, 1989) and the nucleolus (see, e.g., Littlechild, 1974) provide core elements and are easily computable. Their formulas can be extended in a straightforward way to generalized airport games, which leads to two solutions for one facility infrastructure cost games. These solutions are described in Definitions 3.3 and 3.4.

Definition 3.3. Let $\langle N, c\rangle$ be a balanced one facility infrastructure cost game with groups $g_1, \ldots, g_k$, and non-negative numbers $\{b_i\}_{i\in K}$ and $\{A_{ij}\}_{i,j\in K, i\leqslant j}$. Let the numbers $\hat{b}_1, \ldots, \hat{b}_k$ be defined as in Proposition 3.3. Let the vector $(y_1, \ldots, y_k)$ be defined recursively by

$$y_1 := \min\left\{\frac{\hat{B}_j}{N_j} : 1 \leqslant j \leqslant k\right\},$$

$$y_i := \min\left\{\frac{\hat{B}_j - (n_1 y_1 + \cdots + n_{i-1} y_{i-1})}{N_j - N_{i-1}} : i \leqslant j \leqslant k\right\}$$

for every $i \in \{2, \ldots, k\}$,

i.e., the vector $(y_1, \ldots, y_k)$ is the egalitarian solution of the generalized airport game corresponding to $g_1, \ldots, g_k$ and $\hat{b}_1, \ldots, \hat{b}_k$. Define the allocation $\Phi^1(c) = (\Phi^1_1(c), \ldots, \Phi^1_k(c))$ by $\Phi^1_i(c) := A_{ik} + y_i$ for every $i \in K$.

Proposition 3.4. *Let $\langle N, c\rangle$ be the one facility infrastructure cost game with groups $g_1, \ldots, g_k$, and non-negative numbers $\{b_i\}_{i\in K}$ and $\{A_{ij}\}_{i,j\in K, i\leqslant j}$. Suppose that $\langle N, c\rangle$ is balanced. Then $\Phi^1(c) \in C^{\text{sym}}(c)$.*

Proof. One easily verifies that, according to Proposition 2.1, the vector $(y_1, \ldots, y_k)$ is a symmetric core element of the generalized airport game $\langle N, \hat{c}\rangle$, corresponding to the one facility infrastructure cost game $\langle N, c\rangle$. By Proposition 3.3 we infer that $\Phi^1(c) \in C^{\text{sym}}(c)$. □

Definition 3.4. Let $\langle N, c\rangle$ be a balanced one facility infrastructure cost game with groups $g_1, \ldots, g_k$, and non-negative numbers $\{b_i\}_{i\in K}$ and $\{A_{ij}\}_{i,j\in K, i\leqslant j}$. Let the numbers $\hat{b}_1, \ldots, \hat{b}_k$ be defined as in Proposition 3.3. Let the vector $(z_1, \ldots, z_k)$ be defined recursively by

$$z_1 := \min\left\{\frac{\hat{B}_j}{W_j} : 1 \leqslant j \leqslant k\right\},$$

$$z_i := \min\left\{\frac{\hat{B}_j - (n_1 z_1 + \cdots + n_{i-1} z_{i-1})}{W_j - N_{i-1}} : i \leqslant j \leqslant k\right\}$$

for every $i \in \{2, \ldots, k\}$,

where $W_j := N_j + 1$ for every $j \in \{1, \ldots, k-1\}$ and $W_k := N_k$. Define the allocation $\Phi^2(c) = (\Phi^2_1(c), \ldots, \Phi^2_k(c))$ by $\Phi^2_i(c) := A_{ik} + z_i$ for every $i \in K$.

Proposition 3.5. *Let $\langle N, c\rangle$ be the one facility infrastructure cost game with groups $g_1, \ldots, g_k$, and non-negative numbers $\{b_i\}_{i\in K}$ and $\{A_{ij}\}_{i,j\in K, i\leqslant j}$. Suppose that $\langle N, c\rangle$ is balanced. Then $\Phi^2(c) \in C^{\text{sym}}(c)$. Moreover, $\Phi^2(c)$ is the nucleolus of $\langle N, c\rangle$.*

Proof. First of all we will show that $0 \leqslant z_1 \leqslant z_2 \leqslant \cdots \leqslant z_k$. According to Proposition 3.1 balancedness of $\langle N, c\rangle$ implies that $\hat{B}_j \geqslant 0$ for every $j \in \{1, \ldots, k\}$. Therefore, $z_1 \geqslant 0$. If $z_{i+1} < z_i$ for some $i \in \{1, \ldots, k-1\}$ there is a $j \in \{i+1, \ldots, k\}$ with

$$z_{i+1} = \frac{\hat{B}_j - (n_1 z_1 + \cdots + n_i z_i)}{W_j - N_i} < z_i.$$

Hence $\hat{B}_j - (n_1 z_1 + \cdots + n_i z_i) < (W_j - N_i) z_i$ which is equivalent to $\hat{B}_j - (n_1 z_1 + \cdots + n_{i-1} z_{i-1}) < (W_j - N_{i-1}) z_i$. Consequently

$$z_i > \frac{\hat{B}_j - (n_1 z_1 + \cdots + n_{i-1} z_{i-1})}{W_j - N_{i-1}},$$

contradicting the definition of z_i. Hence $0 \leqslant z_1 \leqslant z_2 \leqslant \cdots \leqslant z_k$.

Let, for every allocation $(x_i)_{i \in N}$ and every coalition $S \subseteq N$, $e(S,x) := c(S) - x(S)$ be the excess of coalition S with respect to allocation x, and let $E(x) := (e(S,x))_{S \subseteq N}$ be the vector of excesses. Let θ be the map that assigns to every $y \in \mathbb{R}^{2^N}$ the vector that orders all coordinates of y in a non-decreasing way. Let $\succeq_{\mathrm{L}}$ be the lexicographical order on $\mathbb{R}^{2^n}$. In order to show that $w := \Phi^2(c) \in C^{\mathrm{sym}}(c)$ it is sufficient to show that w is the nucleolus of $\langle N, c \rangle$. However, since $\langle N, c \rangle$ is balanced, it is even sufficient to show that w is the prenucleolus of $\langle N, c \rangle$. Therefore, we have to show that for every efficient allocation $(x_i)_{i \in N}$ with $\theta(E(x)) \succeq_{\mathrm{L}} \theta(E(w))$ we have $x = w$.

So, suppose that $(x_i)_{i \in N}$ is an efficient allocation such that $\theta(E(x)) \succeq_{\mathrm{L}} \theta(E(w))$. In order to show that $x = w$ it suffices to show for every $l \in \{0, \ldots, k-1\}$ that the following statement is true: if $x_i = w_i$ for every $i \in \bigcup_{j=1}^{l} g_j$ (which is true by convention for $l = 0$) then $x_i = w_i$ for every $i \in g_{l+1}$. So, let $l \in \{0, \ldots, k-1\}$ and assume that $x_i = w_i$ for every $i \in \bigcup_{j=1}^{l} g_j$.

First, note that

$$e(S,x) = c(S) - x(S) = c(S) - w(S) = e(S,w) \tag{5}$$

for every $S \subseteq \bigcup_{j=1}^{l} g_j$, and

$$\begin{aligned} e(S,x) &= c(S) - x(S) \\ &= c(S) - (x(N) - x(N \setminus S)) \\ &= c(S) - (w(N) - w(N \setminus S)) \\ &= c(S) - w(S) \\ &= e(S,w) \end{aligned} \tag{6}$$

for every $S \subseteq N$ with $N \setminus S \subseteq \bigcup_{j=1}^{l} g_j$. Define now

$$\mathscr{S} = \left\{ S \subseteq N : \text{there exist } i_1, i_2 \in \bigcup_{j=l+1}^{k} g_j \text{ with } i_1 \in S \text{ and } i_2 \notin S \right\}.$$

If $\mathscr{S} = \emptyset$ then for every $S \subseteq N$ we have either $S \subseteq \bigcup_{j=1}^{l} g_j$ or $N \setminus S \subseteq \bigcup_{j=1}^{l} g_j$ and hence, according to (5) and (6), $e(S,x) = e(S,w)$. Therefore $x = w$. So, we may assume that $\mathscr{S} \neq \emptyset$, which implies $|\bigcup_{j=l+1}^{k} g_j| \geqslant 2$. So,

$$l + 1 < k \text{ or } n_k \geqslant 2. \tag{7}$$

First we will show that

$$\min_{S \in \mathscr{S}} e(S,w) = z_{l+1}. \tag{8}$$

Therefore, let $S \in \mathscr{S}$. If $j(S) < k$ (but $j(S) \geqslant l+1$ since $S \cap (\bigcup_{j=l+1}^{k} g_j) \neq \emptyset$) we have

$$\begin{aligned} e(S,w) &= c(S) - w(S) \\ &= B_{j(S)} + \sum_{i=1}^{j(S)} s_i A_{ij(S)} - \left(\sum_{i=1}^{j(S)} s_i A_{ik} + \sum_{i=1}^{j(S)} s_i z_i \right) \\ &= B_{j(S)} - \sum_{i=1}^{j(S)} s_i (A_{ik} - A_{ij(S)}) - \sum_{i=1}^{j(S)} s_i z_i \\ &\geqslant B_{j(S)} - \sum_{i=1}^{j(S)} n_i (A_{ik} - A_{ij(S)}) - \sum_{i=1}^{j(S)} n_i z_i \\ &= \hat{B}_{j(S)} - \sum_{i=1}^{j(S)-1} n_i z_i - n_{j(S)} z_{j(S)} \\ &\geqslant (W_{j(S)} - N_{j(S)-1}) z_{j(S)} - n_{j(S)} z_{j(S)} \\ &\geqslant z_{j(S)} \\ &\geqslant z_{l+1}. \end{aligned}$$

At the first inequality we used the fact that $A_{ik} \geqslant A_{ij(S)}$ and $z_i \geqslant 0$ for every $i \in \{1, \ldots, j(S)\}$. The second inequality is a consequence of the definition of the number $z_{j(S)}$.

If $j(S) = k$ then let $j \in \{l+1, \ldots, k\}$ be such that $(N \setminus S) \cap g_j \neq \emptyset$. Then

$$\begin{aligned} e(S,w) &= c(S) - w(S) \\ &= B_k + \sum_{i=1}^{k} s_i A_{ik} - \left(\sum_{i=1}^{k} s_i A_{ik} + \sum_{i=1}^{k} s_i z_i \right) \\ &= B_k - \sum_{i=1}^{k} s_i z_i \\ &= \sum_{i=1}^{k} n_i z_i - \sum_{i=1}^{k} s_i z_i \\ &= \sum_{i=1}^{k} (n_i - s_i) z_i \\ &\geqslant z_j \\ &\geqslant z_{l+1}. \end{aligned}$$

At the fourth equality we used the fact that $\sum_{i=1}^{k} n_i z_i = B_k$ and at the first inequality that $n_j \geqslant s_j + 1$. Moreover, for every $i \in g_{l+1}$ we have $N \setminus \{i\} \in \mathscr{S}$ (as a consequence of the fact that $|\bigcup_{j=l+1}^{k} g_j| \geqslant 2$). If $S = N \setminus \{i\}$ then the two inequalities above are satisfied by equality. Hence $e(N \setminus \{i\}, w) = z_{l+1}$. This establishes (8).

Since $\theta(E(x)) \succeq_{\mathrm{L}} \theta(E(w))$ and $e(S,x) = e(S,w)$ for every $S \subseteq N$ with $S \notin \mathscr{S}$ we find $e(S,x) \geqslant z_{l+1}$ for every $S \in \mathscr{S}$. Let j^* be the smallest index in $\{l+1, \ldots, k\}$ such that

$$z_{l+1} = \frac{\hat{B}_{j^*} - \sum_{i=1}^{l} n_i z_i}{W_{j^*} - N_l}.$$

In order to show that $x_i = w_i$ for every $i \in g_{l+1}$ we distinguish between two cases.

Case (i): $j^* < k$. Let $S = \bigcup_{j=1}^{j^*} g_j$ and $\hat{S} = \bigcup_{j=l+1}^{j^*} g_j$. Then $S \setminus \hat{S} = \bigcup_{j=1}^{l} g_j$, so $x(S \setminus \hat{S}) = w(S \setminus \hat{S})$. Moreover, we have $x(N) = c(N)$ and hence

$$\begin{aligned}
(W_{j^*} - N_l)z_{l+1} &= \left(\sum_{j=l+1}^{j^*} n_j + 1\right) z_{l+1} \\
&= \left(\sum_{j=l+1}^{j^*} n_j\right) z_{l+1} + z_{l+1} \\
&\leqslant \sum_{i \in \hat{S}} e(N \setminus \{i\}, x) + e(S, x) \\
&= \sum_{i \in \hat{S}} (c(N \setminus \{i\}) - x(N \setminus \{i\})) + c(S) - x(S) \\
&= \sum_{i \in \hat{S}} (c(N \setminus \{i\}) - x(N)) + c(S) - x\left(S \setminus \hat{S}\right) \\
&= \sum_{i \in \hat{S}} (c(N \setminus \{i\}) - c(N)) + c(S) - w\left(S \setminus \hat{S}\right) \\
&= -\sum_{j=l+1}^{j^*} n_j A_{jk} + B_{j^*} + \sum_{j=1}^{j^*} n_j A_{jj^*} - \left(\sum_{j=1}^{l} n_j A_{jk} + \sum_{j=1}^{l} n_j z_j\right) \\
&= B_{j^*} - \sum_{j=1}^{j^*} n_j \left(A_{jk} - A_{jj^*}\right) - \sum_{j=1}^{l} n_j z_j \\
&= \hat{B}_{j^*} - \sum_{j=1}^{l} n_j z_j \\
&= (W_{j^*} - N_l) z_{l+1}.
\end{aligned}$$

At the sixth equality we used the fact that for every $j \in \{l+1, \ldots, j^*\}$ and $i \in g_j$ we have $c(N) - c(N \setminus \{i\}) = A_{jk}$. This implies that $e(N \setminus \{i\}, x) = z_{l+1}$ for every $i \in \hat{S}$. In particular we get $e(N \setminus \{i\}, x) = z_{l+1} = e(N \setminus \{i\}, w)$ for every $i \in g_{l+1}$, which implies $x_i = w_i$ for every $i \in g_{l+1}$.

Case (ii): $j^* = k$. First of all we show that $n_k \geqslant 2$. According to (7), we may assume $l + 1 < k$. Since $\hat{B}_k = B_k \geqslant \hat{B}_{k-1}$ we get

$$z_{l+1} = \frac{\hat{B}_k - \sum_{i=1}^{l} n_i z_i}{W_k - N_l} < \frac{\hat{B}_{k-1} - \sum_{i=1}^{l} n_i z_i}{W_{k-1} - N_l} \leqslant \frac{\hat{B}_k - \sum_{i=1}^{l} n_i z_i}{W_{k-1} - N_l},$$

where the strict inequality is due to the minimality assumption upon j^*. Consequently $W_k > W_{k-1}$ or $N_k > N_{k-1} + 1$. Hence $n_k \geqslant 2$. Let $\hat{S} = \bigcup_{j=l+1}^{k} g_j$. Then $x(N \setminus \hat{S}) = w(N \setminus \hat{S})$ by assumption. For every $i \in \hat{S}$ we have $N \setminus \{i\} \in \mathscr{S}$. Using moreover $x(N) = c(N)$ we find

$$
\begin{aligned}
(W_k - N_l)z_{l+1} &= (N_k - N_l)z_{l+1} \\
&\leqslant \sum_{i \in \hat{S}} e(N \setminus \{i\}, x) \\
&= \sum_{i \in \hat{S}} (c(N \setminus \{i\}) - x(N \setminus \{i\})) \\
&= \sum_{i \in \hat{S}} (c(N \setminus \{i\}) - x(N)) + x\left(\hat{S}\right) \\
&= \sum_{i \in \hat{S}} (c(N \setminus \{i\}) - c(N)) + x(N) - x\left(N \setminus \hat{S}\right) \\
&= -\sum_{j=l+1}^{k} n_j A_{jk} + c(N) - w\left(N \setminus \hat{S}\right) \\
&= B_k + \sum_{j=1}^{l} n_j A_{jk} - \left(\sum_{j=1}^{l} n_j A_{jk} + \sum_{j=1}^{l} n_j z_j\right) \\
&= B_k - \sum_{j=1}^{l} n_j z_j \\
&= \hat{B}_k - \sum_{j=1}^{l} n_j z_j \\
&= (W_k - N_l)z_{l+1}.
\end{aligned}
$$

Consequently $e(N \setminus \{i\}, x) = z_{l+1}$ for every $i \in \hat{S}$. In the same way as for case (i) we derive that $x_i = w_i$ for every $i \in g_{l+1}$. □

4. Infrastructure cost games

In this section we consider infrastructure cost games, i.e., cost games which are dealing with the building costs and maintenance costs of an arbitrary number m of facilities, where no special requirements upon the ordering of the wishes of the coalitions for the several facilities will be made.

So, suppose we are given an infrastructure cost game $\langle N, c\rangle$ with groups of players $g_1, \ldots, g_k$. Let $c = c^1 + \cdots + c^m$ be such that, for every $l \in M := \{1, \ldots, m\}$, $\langle N, c^l\rangle$ is a one facility infrastructure cost game with groups of players $g_{\pi^l(1)}, \ldots, g_{\pi^l(k)}$, where π^l is some permutation of $K(= \{1, \ldots, k\})$. Let, moreover, $(b_i^l)_{i \in K}$ and $(A_{ij}^l)_{i,j \in K, i \leqslant j}$ be the non-negative numbers which define the one facility infrastructure game $\langle N, c^l\rangle$ and let $n_i^l := n_{\pi^l(i)}$ be the number of players in the group, ranked at the ith place for facility l. Define $j^l(L) := \max\{j : \pi^l(j) \in L\}$ for every $L \subseteq K$ and $l \in M$. So, $j^l(L)$ measures the level of sophistication of facility l, desired by the union of the groups in L (instead of $j^l(\{i\})$ we write $j^l(i)$). A group $j \in K$ is said to be *dominated* by the collection of groups L if $j^l(L \cup \{j\}) = j^l(L)$ for every $l \in M$, i.e., j is dominated by L if the needs of j are covered by the needs of L. The collection L is *complete* if there is no $j \in K \setminus L$ such that j is dominated by L. Let $L \subseteq K$ and define $M_L := \{l \in M : j^l(L) < k\}$, i.e., M_L is the collection of facilities for which the level of sophistication, desired by the groups in L, is not maximal. Suppose $M_L \neq \emptyset$. Define for every $l \in M_L$ the collection $R_L^l := \{\pi^l(j) : j > j^l(L)\}$, denoting the collection of groups whose wishes with respect to facility l exceed the wishes of the groups in L. Two facilities l and l' in M_L are called *directly connected* if $R_L^l \cap R_L^{l'} \neq \emptyset$, i.e., there is at least one group whose wishes with respect to both facilities l and l' exceed the wishes of the groups in L. Two facilities l and l' are called *indirectly connected* if there is a sequence $l = l_0, \ldots, l_s = l'$ in M_L such that l_{t-1} and l_t are directly connected for every $t \in \{1, \ldots, s\}$. A complete collection of groups L is called *essential* if $M_L \neq \emptyset$ and every pair of facilities in M_L is directly or indirectly connected.

Example 4.1. Consider the infrastructure cost game, dealing with the building and maintenance costs of four facilities, where the ordering of the wishes of the groups for the several facilities are given by

facility 1	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
facility 2	g_6	g_4	g_7	g_1	g_8	g_5	g_2	g_3
facility 3	g_7	g_3	g_5	g_2	g_8	g_1	g_4	g_6
facility 4	g_4	g_8	g_6	g_2	g_5	g_7	g_1	g_3.

Consider the complete collection $L = \{4\}$. Then $M_L = \{1, 2, 3, 4\}$, $R_L^1 = \{5, 6, 7, 8\}$, $R_L^2 = \{1, 2, 3, 5, 7, 8\}$, $R_L^3 = \{6\}$ and $R_L^4 = \{1, 2, 3, 5, 6, 7, 8\}$, and the corresponding graph on M_L is given in Fig. 1. This graph consists of one component, so $L = \{4\}$ is essential.

Now consider $L = \{1, 2, 5, 7\}$. Then $M_L = \{1, 2, 3, 4\}$, $R_L^1 = \{8\}$, $R_L^2 = \{3\}$, $R_L^3 = \{4, 6\}$ and $R_L^4 = \{3\}$ and the corresponding graph on M_L is given in Fig. 2. This graph consists of three components, so $L = \{1, 2, 5, 7\}$ is not essential.

If an infrastructure cost game is the sum of balanced one facility infrastructure cost games then clearly this game is balanced. The following example shows that the converse statement is not true. For balancedness of an infrastructure cost game the balancedness of the corresponding one facility infrastructure cost games is not required.

Example 4.2. Consider the infrastructure cost game $\langle N, c \rangle$, dealing with the building and maintenance costs of two facilities, where the ordering of the wishes of the three groups involved for the facilities are given by

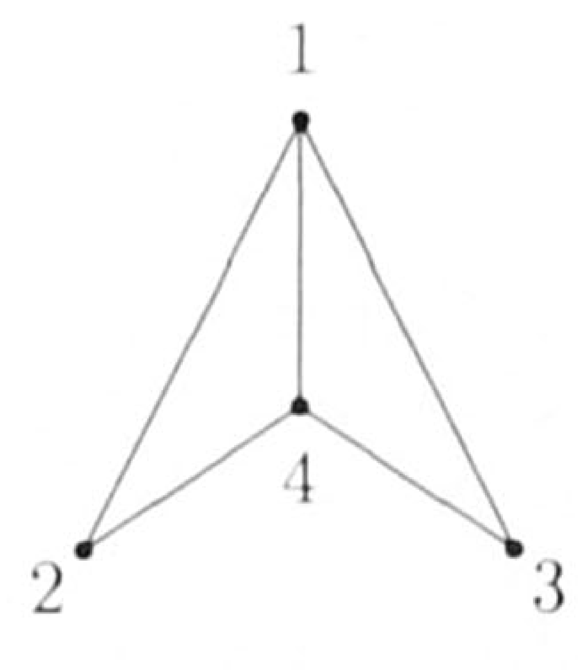

Fig. 1. The graph on M_L for $L = \{4\}$.

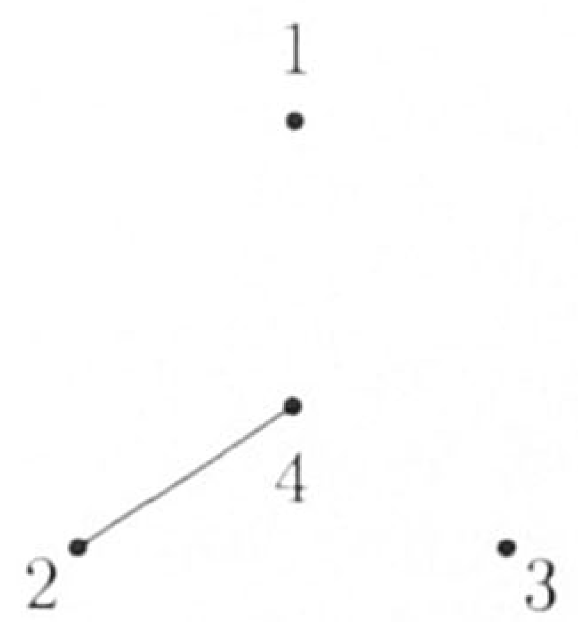

Fig. 2. The graph on M_L for $L = \{1, 2, 5, 7\}$.

facility 1 g_1 g_2 g_3

facility 2 g_2 g_1 g_3.

Suppose that every group has precisely one player, say $g_1 = \{1\}$, $g_2 = \{2\}$, and $g_3 = \{3\}$. The one facility infrastructure cost games $\langle N, c^1 \rangle$ and $\langle N, c^2 \rangle$ are defined by the numbers

$$\begin{array}{lll} b_1^1 = b_1^2 = 1 & b_2^1 = b_2^2 = 9 & b_3^1 = b_3^2 = 1 \\ A_{11}^1 = A_{11}^2 = 1 & A_{12}^1 = A_{12}^2 = 2 & A_{13}^1 = A_{13}^2 = 3 \\ & A_{22}^1 = A_{22}^2 = 2 & A_{23}^1 = A_{23}^2 = 3 \\ & & A_{33}^1 = A_{33}^2 = 3. \end{array}$$

One easily verifies that $c^1(1) = 2$, $c^1(2) = 12$, $c^1(3) = c^1(12) = 14$, $c^1(13) = c^1(23) = 17$, and $c^1(123) = 20$. Since $c^1(123) > c^1(1) + c^1(23)$ we conclude that $\langle N, c^1 \rangle$ is not balanced. Moreover, we have $c^2(1) = 12$, $c^2(2) = 2$, $c^2(3) = c^2(12) = 14$, $c^2(13) = c^2(23) = 17$, and $c^2(123) = 20$. From $c^2(123) > c^2(2) + c^2(13)$ we infer that $\langle N, c^2 \rangle$ is not balanced. The game $\langle N, c \rangle$ is specified by the data $c(1) = 14$, $c(2) = 14$, $c(3) = c(12) = 28$, $c(13) = c(23) = 34$, and $c(123) = 40$. One easily verifies that $(6, 6, 28)$ is a core element of $\langle N, c \rangle$, so $\langle N, c \rangle$ is balanced.

In the following proposition we show that, in order to get a symmetric core element of an infrastructure cost game, it is sufficient that every player pays at least his maintenance costs and that every coalition, which is the union of an essential collection of groups, is satisfied.

Proposition 4.1. *Let $\langle N, c \rangle$ be an infrastructure cost game as defined above. Then $(x_1, \dots, x_k) \in C^{\mathrm{sym}}(c)$ iff*

$$\begin{aligned} & x_i \geqslant \sum_{l \in M} A^l_{j^l(i)k} \quad \text{for every } i \in K, \\ & \sum_{j \in L} n_j x_j \leqslant c(\cup\{g_j : j \in L\}) \quad \text{for every essential collection } L, \\ & \sum_{j \in K} n_j x_j = c(N). \end{aligned} \tag{9}$$

Proof. "$\Rightarrow$". Suppose $(x_1, \dots, x_k) \in C^{\mathrm{sym}}(c)$. Then for every $i \in K$ we have $n_i x_i = x(g_i) = x(N) - x(N \setminus g_i) = c(N) - x(N \setminus g_i) \geqslant c(N) - c(N \setminus g_i) \geqslant n_i(A^1_{j^1(i)k} + \dots + A^m_{j^m(i)k})$ which implies $x_i \geqslant A^1_{j^1(i)k} + \dots + A^m_{j^m(i)k}$. The other conditions in (9) are implied by $x(S) \leqslant c(S)$ for every coalition S, which is the union of an essential collection of groups, and $x(N) = c(N)$.

"$\Leftarrow$". Suppose that $(x_1, \dots, x_k)$ satisfies the conditions (9). Define $e(S, x) := c(S) - x(S)$ for every $S \subseteq N$. We have to show that every coalition S is satisfied, i.e., $e(S, x) \geqslant 0$. Denote by $\mathscr{B}_1$ the set of coalitions, which are the union of an essential collection of groups, by $\mathscr{B}_2$ the set of coalitions, which are the union of a complete collection of groups, by $\mathscr{B}_3$ the set of coalitions, which are the union of an arbitrary collection of groups, and by $\mathscr{B}_4$ the collection of all coalitions. Clearly, $\mathscr{B}_1 \subseteq \mathscr{B}_2 \subseteq \mathscr{B}_3 \subseteq \mathscr{B}_4$. By assumption every coalition in $\mathscr{B}_1$ is satisfied. In order to show that every coalition in $\mathscr{B}_4$ is satisfied, we will first show that every coalition in $\mathscr{B}_2$ is satisfied (step 1), secondly, that every coalition in $\mathscr{B}_3$ is satisfied (step 2), and finally that every coalition in $\mathscr{B}_4$ is satisfied (step 3).

Step 1: Let $S \in \mathscr{B}_2$, i.e., $S = \cup\{g_i : i \in L\}$, where L is a complete collection of groups. If L is essential, then S is satisfied by assumption. Otherwise, there is at least one pair of facilities in M_{L} which are not indirectly connected. Let M' be a connected component in M_{L}, $L' := \cup\{R^s_{\mathrm{L}} : s \in M'\}$ and $S' := \cup\{g_i : i \in L'\}$. Let, moreover, $L'' := L \cup L'$ and $S'' := \cup\{g_i : i \in L''\} = S \cup S'$. For every $i \in M'$ we have $j^i(L'') = k = j^i(K)$ and $j^i(K \setminus L') = j^i(L)$ and for every $i \in M_{\mathrm{L}} \setminus M'$ we have $j^i(L'') = j^i(L)$ and $j^i(K \setminus L') = j^i(K)$. Hence, L'' is a complete collection of groups such that $M_{L''}$ has one component less than M_L and $K \setminus L'$ is an essential collection. Moreover, we have

$$\begin{aligned} c^i(S) - c^i(S'') - c^i(N \setminus S') + c^i(N) &= \sum_{j=1}^{j^i(L)} s_j A^i_{jj^i(L)} - \sum_{j=1}^{k} s''_j A^i_{jk} - \sum_{j=1}^{j^i(L)} (n^i_j - s'_j) A^i_{jj^i(L)} + \sum_{j=1}^{k} n^i_j A^i_{jk} \\ &= \sum_{j=1}^{k} (n^i_j - s''_j) A^i_{jk} - \sum_{j=1}^{j^i(L)} (n^i_j - s'_j - s_j) A^i_{jj^i(L)} \\ &= \sum_{j=1}^{k} (n^i_j - s''_j) A^i_{jk} - \sum_{j=1}^{j^i(L)} (n^i_j - s''_j) A^i_{jj^i(L)} \geqslant 0 \end{aligned}$$

for every $i \in M'$ (where $s_j := |S \cap g_{\pi^i(j)}|$, etc.), and

$$\begin{aligned} c^i(S) - c^i(S'') - c^i(N \setminus S') + c^i(N) &= \sum_{j=1}^{j^i(L)} s_j A^i_{jj^i(L)} - \left(\sum_{j=1}^{j^i(L)} s''_j A^i_{jj^i(L)} \right) - \sum_{j=1}^{k} (n^i_j - s'_j) A^i_{jk} + \sum_{j=1}^{k} n^i_j A^i_{jk} \\ &= \sum_{j=1}^{k} s'_j A^i_{jk} - \sum_{j=1}^{j^i(L)} s'_j A^i_{jj^i(L)} \geqslant 0 \end{aligned}$$

for every $i \in M \setminus M'$. Summation of these inequalities over all $i \in M$ gives

$$c(S) - c(S'') - c(N \setminus S') + c(N) \geqslant 0,$$

which is equivalent to $c(S) \geqslant c(S'') + c(N \setminus S') - c(N)$. Hence

$$\begin{aligned} e(S,x) = c(S) - x(S) &\geqslant c(S'') + c(N \setminus S') - c(N) - x(S) \\ &= c(S'') + c(N \setminus S') - x(N) - x(S) \\ &= c(S'') - x(S'') + c(N \setminus S') - x(N \setminus S') \\ &= e(S'',x) + e(N \setminus S',x) \\ &\geqslant e(S'',x), \end{aligned}$$

where the last inequality is due to the fact that $N \setminus S' \in \mathscr{B}_1$. So, coalition S is (weakly) more satisfied than coalition S'', which is the union of the complete collection of groups L'' with $M_{L''}$ having one component less than M_L. In this way we can prove by induction to the number of components in M_L that every coalition, which is the union of a complete collection of groups, is satisfied. The induction base is provided by the assumption that every coalition, which is the union of an essential collection of groups, i.e., a complete collection L such that M_L has one component, is satisfied.

Step 2: Let $S \in \mathscr{B}_3$, i.e., $S = \cup\{g_i : i \in L\}$, where L is some collection of groups. If L is complete, then $S \in \mathscr{B}_2$, and we have already shown in step 1 that S is satisfied. Otherwise, there is a group $j \in K \setminus L$, which is dominated by L. Define $L' := L \cup \{j\}$ and $S' := \cup\{g_i : i \in L'\} = S \cup g_j$. Since $j^i(L') = j^i(L)$ for every $i \in M$ we have

$$\begin{aligned} c(S') - c(S) &= n_j (A^1_{j^1(j)j^1(L)} + \cdots + A^m_{j^m(j)j^m(L)}) \\ &\leqslant n_j (A^1_{j^1(j)k} + \cdots + A^m_{j^m(j)k}) \\ &\leqslant n_j x_j \\ &= x(S') - x(S). \end{aligned}$$

Hence $e(S,x) \geqslant e(S',x)$. So, coalition S is (weakly) more satisfied than coalition S', which is the union of the collection of groups L', which admits one dominated group less than the collection of groups L does. So, by induction to the number of dominated groups we can prove that every coalition, which is the union of some collection of groups, is satisfied.

Step 3: Let $S \in \mathscr{B}_4$, i.e., S is a coalition which is not necessarily the union of a collection of groups. If $i \in N \setminus S$ is a player which belongs to a group g_j which is already present in S (i.e. $S \cap g_j \neq \emptyset$) then $c(S \cup \{i\}) - c(S) \leqslant (A^1_{j^1(j)k} + \cdots + A^m_{j^m(j)k}) \leqslant x_j = x(S \cup \{i\}) - x(S)$ and hence $e(S,x) \geqslant e(S \cup \{i\},x)$. So, defining $\tilde{S} \in \mathscr{B}_3$ as the coalition which is the "completion" of S, i.e., the union of the groups which are already present in S, we get $e(S,x) \geqslant e(\tilde{S},x) \geqslant 0$, which finishes the proof. □

In Proposition 4.1 it is shown that an infrastructure cost game $\langle N,c\rangle$ is balanced if and only if system (9) has a solution. If $m = 1$ this system is equivalent to system (4), since the essential collections are precisely the collections $\{\pi^1(1),\ldots,\pi^1(i)\}$ with $i \in \{1,\ldots,k-1\}$. Moreover, it follows from Propositions 3.1 and 3.2 that system (4) has a solution if and only if conditions (3) are satisfied.

Also for $m = 2$ it is possible to provide a collection of conditions on $\langle N,c\rangle$, which turn out to be equivalent to the solvability of system (9) (and hence to the balancedness of $\langle N,c\rangle$). In order to do so we assume, without loss of generality, that $\pi^1(i) = i$ for every $i \in K$ and we write $\pi = \pi^2$. Moreover, let $n'_i := n_{\pi(i)}$ for every $i \in K$, so n'_i is the number of players in $g_{\pi(i)}$, which is ranked at place i for the second facility. A group g_i is *dominated* by another group g_j if g_i precedes g_j for both facilities, i.e., $i < j$ and $j^2(i) < j^2(j)$. Let $K_\mathrm{d} \subseteq K$ denote the collection of dominated groups and $K_\mathrm{u} = K \setminus K_\mathrm{d}$ the collection of undominated groups. The ordering of the groups in K_u for the first facility is precisely the reverse of the ordering of these groups for the second facility. To see this write $K_\mathrm{u} = \{j_1,\ldots,j_s\}$ with $j_1 < \cdots < j_s$ for the collection of undominated groups. If the first group in K_u for facility 2 is not g_{j_s}, then this first group is dominated by g_{j_s} which gives a contradiction. So, the first group in K_u for facility 2 is g_{j_s}. If the second group of K_u for facility 2 is not $g_{j_{s-1}}$, then this second group is dominated by $g_{j_{s-1}}$ which gives a contradiction. So, the second group in K_u for facility 2 is $g_{j_{s-1}}$, and so on.

We provide a collection of conditions which turn out to be necessary and sufficient for the balancedness of $\langle N,c\rangle$. These conditions can be divided into two groups: the conditions of "type 1" and the conditions of "type 2".

Conditions of type 1. These conditions are defined in the following way. Let $L \subseteq K$ be a complete collection of groups such that there is a $k^* \in K$ with $j^1(k^*) > j^1(L)$ and $j^2(k^*) > j^2(L)$. So, the needs of group g_{k^*} exceed the needs of any of the groups in L, i.e., $k^* \in R^1_L \cap R^2_L$. So, the facilities 1 and 2 are directly connected and hence L is essential. Define $S := \cup\{g_j : j \in L\}$ and consider the partition $\{S, N \setminus S\}$, which will be called a type 1 split. Since this partition is a balanced collection the condition $c(N) \leqslant c(S) + c(N \setminus S)$ is a necessary condition for the game $\langle N,c\rangle$ to have a non-empty core. This inequality is equivalent to

$$\sum_{i\in L} n_i\left(A^1_{ik} - A^1_{ij^1(L)} + A^2_{j^2(i)k} - A^2_{j^2(i)j^2(L)}\right) \leqslant B^1_{j^1(L)} + B^2_{j^2(L)} \tag{10}$$

or

$$\sum_{i\in L} n_i\left(A^1_{ik} + A^2_{j^2(i)k}\right) \leqslant c(S). \tag{11}$$

The interpretation of inequality (10) is that the maintenance costs that the players in $\cup\{g_j : j \in L\}$ have to pay for the two facilities for the level of the needs of the other players should be less than or equal to the building costs for both facilities at the level needed by these groups themselves (cf. inequalities (3)).

Conditions of type 2. These conditions are defined in the following way. Let $l_1, l_2 \in \{1,\ldots,k-1\}$ and let $L_1 = \{1,\ldots,l_1\}, L_2 = \{\pi(1),\ldots,\pi(l_2)\}$. Suppose that l_1 and l_2 are such that $L_1 \cup L_2 = K$, i.e., the union of the first l_1 groups for facility 1 and the first l_2 groups for facility 2 is K. Let $S_1 := \cup\{g_j : j \in L_1\}$, $S_2 := \cup\{g_j : j \in L_2\}$ and $S_3 := \cup\{g_j : j \in K \setminus (L_1 \cap L_2)\}$. Then $\{S_1, S_2, S_3\}$ is a balanced collection (with weights 1/2 for every coalition) and hence $2c(N) \leqslant c(S_1) + c(S_2) + c(S_3)$. This last inequality is equivalent to

$$\sum_{i \in L_1} n_i \left(A^1_{ik} - A^1_{il_1}\right) + \sum_{i \in L_2} n_i \left(A^2_{j^2(i)k} - A^2_{j^2(i)l_2}\right) \leqslant B^1_{l_1} + B^2_{l_2}, \tag{12}$$

or

$$B^1_k + B^2_k \leqslant c(S_1) - \sum_{i \in L_1} n_i \left(A^1_{ik} + A^2_{j^2(i)k}\right) + c(S_2) - \sum_{i \in L_2} n_i \left(A^1_{ik} + A^2_{j^2(i)k}\right). \tag{13}$$

The interpretation of inequality (12) is the following: the maintenance costs that the groups in L_1 have to pay for the level of the needs with respect to facility 1 of the other players plus the maintenance costs that the groups in L_2 have to pay for the level of the needs with respect to facility 2 of the other players should be less than or equal to the building costs of facility 1 for the level of the needs of the groups in L_1 plus the building costs of facility 2 for the level of the needs of the groups in L_2.

Example 4.3. Let $\langle N, c\rangle$ be the sum of two one facility infrastructure cost games where the ordering of the wishes of the groups for both facilities are given by

facility 1	$g_1\ g_2\ g_3\ g_4\ g_5$
facility 2	$g_1\ g_5\ g_4\ g_2\ g_3$.

So the permutation π is given by $\pi(1) = 1$, $\pi(2) = 5$, $\pi(3) = 4$, $\pi(4) = 2$ and $\pi(5) = 3$. Then the collection of dominated groups is $K_d = \{1, 2\}$ and the collection of undominated groups is $K_u = \{3, 4, 5\}$. Note that the ordering of the groups g_3, g_4, g_5 for the first facility is the reverse of the ordering of these groups for the second facility. The collections $\{1\}$ and $\{1, 2\}$ provide the following necessary type 1 conditions for the game $\langle N, c\rangle$ to have a non-empty core:

$$n_1(A^1_{1,1} + A^2_{1,1}) \leqslant B^1_1 + B^2_1,$$
$$n_1(A^1_{1,2} + A^2_{1,4}) + n_2(A^1_{2,2} + A^2_{4,4}) \leqslant B^1_2 + B^2_4,$$

where $A^l_{i,j}$ is a short-hand notation for $A^l_{i5} - A^l_{ij}$. The following conditions are the type 2 conditions for $\langle N, c\rangle$ to be balanced:

$$n_1 A^1_{1,3} + n_2 A^1_{2,3} + n_3 A^1_{3,3} + n_1 A^2_{1,3} + n_5 A^2_{2,3} + n_4 A^2_{3,3} \leqslant B^1_3 + B^2_3,$$
$$n_1 A^1_{1,3} + n_2 A^1_{2,3} + n_3 A^1_{3,3} + n_1 A^2_{1,4} + n_5 A^2_{2,4} + n_4 A^2_{3,4} + n_2 A^2_{4,4} \leqslant B^1_3 + B^2_4,$$
$$n_1 A^1_{1,4} + n_2 A^1_{2,4} + n_3 A^1_{3,4} + n_4 A^1_{4,4} + n_1 A^2_{1,2} + n_5 A^2_{2,2} \leqslant B^1_4 + B^2_2,$$
$$n_1 A^1_{1,4} + n_2 A^1_{2,4} + n_3 A^1_{3,4} + n_4 A^1_{4,4} + n_1 A^2_{1,3} + n_5 A^2_{2,3} + n_4 A^2_{3,3} \leqslant B^1_4 + B^2_3,$$
$$n_1 A^1_{1,4} + n_2 A^1_{2,4} + n_3 A^1_{3,4} + n_4 A^1_{4,4} + n_1 A^2_{1,4} + n_5 A^2_{2,4} + n_4 A^2_{3,4} + n_2 A^2_{4,4} \leqslant B^1_4 + B^2_4.$$

In order to prove that the conditions (10) and (12) are necessary and sufficient for the balancedness of $\langle N, c\rangle$ we need the following lemma.

Lemma 4.1. *Let $s \in \mathbb{N}, s \geqslant 2$ and $a_1, \ldots, a_{s-1}, b_1, \ldots, b_{s-1}, c$ be non-negative numbers. Then the following system*

$$\left\{\begin{array}{cccccccccccccc}
z_1 & & & & & & & & & & & \leqslant & a_1 \\
z_1 & + & z_2 & & & & & & & & & \leqslant & a_2 \\
z_1 & + & z_2 & + & z_3 & & & & & & & \leqslant & a_3 \\
\vdots & & \vdots & & \vdots & & \ddots & & & & & \vdots & \vdots \\
z_1 & + & z_2 & + & z_3 & + & \cdots & + & z_{s-1} & & & \leqslant & a_{s-1} \\
 & & z_2 & + & z_3 & + & \cdots & + & z_{s-1} & + & z_s & \leqslant & b_{s-1} \\
 & & & & z_3 & + & \cdots & + & z_{s-1} & + & z_s & \leqslant & b_{s-2} \\
 & & & & & & \ddots & & \vdots & & \vdots & \vdots & \vdots \\
 & & & & & & & & z_{s-1} & + & z_s & \leqslant & b_2 \\
 & & & & & & & & & & z_s & \leqslant & b_1 \\
z_1 & + & z_2 & + & z_3 & + & \cdots & + & z_{s-1} & + & z_s & = & c
\end{array}\right. \tag{14}$$

has a non-negative solution iff $a_i + b_j \geqslant c$ for every $i, j \in \{1, \ldots, s-1\}$ with $i + j \geqslant s$.

Proof. "$\Rightarrow$". Suppose system (14) has a non-negative solution $z = (z_1, \ldots, z_s)$. Then, for every $i, j \in \{1, \ldots, s-1\}$ with $i + j \geqslant s$ we have $c = z_1 + \cdots + z_s \leqslant (z_1 + \cdots + z_i) + (z_{s-j+1} + \cdots + z_s) \leqslant a_i + b_j$.

"$\Leftarrow$". The proof of this implication is by induction to s. For $s = 2$ we have to check whether the system

$$\left\{\begin{array}{lcl}
z_1 & \leqslant & a_1, \\
z_2 & \leqslant & b_1, \\
z_1 + z_2 & = & c
\end{array}\right.$$

has a non-negative solution, provided that a_1, b_1, c are non-negative numbers satisfying $a_1 + b_1 \geqslant c$. One easily verifies this.

Suppose the implication has been proved for $s = k$ and suppose we are given system (14) with $s = k + 1$, where the non-negative numbers $a_1, \ldots, a_{s-1}$, $b_1, \ldots, b_{s-1}$, c are such that $a_i + b_j \geqslant c$ for every $i, j \in \{1, \ldots, s-1\}$ with $i + j \geqslant s$. If $c \leqslant \min\{a_1, \ldots, a_{s-1}\}$ one easily verifies that $z = (c, 0, \ldots, 0)$ is a solution of (14). If $c > \min\{a_1, \ldots, a_{s-1}\}$ define $z_1 := \min\{a_1, \ldots, a_{s-1}\} = a_{i^*}$ for some $i^* \in \{1, \ldots, s-1\}$. Then $(z_1, \ldots, z_s)$ is a solution of (14) iff $(z_2, \ldots, z_s)$ is a solution of

$$\left\{\begin{array}{cccccccccccccc}
z_2 & & & & & & & & & & & \leqslant & a'_1 \\
z_2 & + & z_3 & & & & & & & & & \leqslant & a'_2 \\
z_2 & + & z_3 & + & z_4 & & & & & & & \leqslant & a'_3 \\
\vdots & & \vdots & & \vdots & & \ddots & & & & & \vdots & \vdots \\
z_2 & + & z_3 & + & z_4 & + & \cdots & + & z_{s-1} & & & \leqslant & a'_{s-2} \\
z_2 & + & z_3 & + & z_4 & + & \cdots & + & z_{s-1} & + & z_s & \leqslant & b_{s-1} \\
 & & z_3 & + & z_4 & + & \cdots & + & z_{s-1} & + & z_s & \leqslant & b_{s-2} \\
 & & & & z_4 & + & \cdots & + & z_{s-1} & + & z_s & \leqslant & b_{s-3} \\
 & & & & & & \ddots & & \vdots & & \vdots & \vdots & \vdots \\
 & & & & & & & & z_{s-1} & + & z_s & \leqslant & b_2 \\
 & & & & & & & & & & z_s & \leqslant & b_1 \\
z_2 & + & z_3 & + & z_4 & + & \cdots & + & z_{s-1} & + & z_s & = & c',
\end{array}\right. \tag{15}$$

where $a'_i = a_{i+1} - a_{i^*} \geqslant 0$ for every $i \in \{1, \ldots, s-2\}$ and $c' = c - a_{i^*} \geqslant 0$. Since $i^* + s - 1 \geqslant s$ we have $a_{i^*} + b_{s-1} \geqslant c$. Hence $c' \leqslant b_{s-1}$. So, in (15), $z_2 + \cdots + z_s \leqslant b_{s-1}$ is redundant and can be omitted. Now, for every $i, j \in \{1, \ldots, s-2\}$ with $i + j \geqslant s - 1$ we have $a'_i + b_j = a_{i+1} + b_j - a_{i^*} \geqslant c - a_{i^*} = c'$. By induction hypothesis we may conclude that (15) has a non-negative solution and hence (14) has a non-negative solution. □

Now we can prove the following proposition.

Proposition 4.2. *Let $\langle N,c\rangle$ be an infrastructure game with two facilities. Then $\langle N,c\rangle$ is balanced iff inequalities* (10) *and* (12) *are satisfied.*

Proof. "⇒". This implication is straightforward since the inequalities (10) and (12) are obtained by writing down the core conditions corresponding to special balanced collections.

"⇐". Suppose $\langle N,c\rangle$ is such that the inequalities (10) and (12) are satisfied. We will distinguish between two cases: (i) $k=\pi(k)$ and (ii) $k\neq\pi(k)$.

Case (i): $k=\pi(k)$, i.e., the last group for the first facility is the same as the last group for the second facility. Define the vector $x=(x_1,\dots,x_k)$ by

$$x_i=\begin{cases}A^1_{ik}+A^2_{j^2(i)k} & \text{if } i\in\{1,\dots,k-1\},\\ A^1_{kk}+A^2_{kk}+(B^1_k+B^2_k)/n_k & \text{if } i=k.\end{cases}$$

So, players belonging to a group i which is not the last group only have to pay the maintenance costs $A^1_{ik}+A^2_{j^2(i)k}$, whereas the players belonging to the last group are also dividing the building costs $B^1_k+B^2_k$. We will show that x is a symmetric core element of $\langle N,c\rangle$. According to Proposition 4.1, it is sufficient to show that every coalition S, which is the union of an essential collection L of groups, is satisfied, i.e., $e(S,x):=c(S)-x(S)\geqslant 0$. So, let L be an essential collection of groups and $S:=\cup\{g_j:j\in L\}$. Then $(\pi(k)=)k\notin L$ because otherwise $M_L=\emptyset$. So, every group in L is dominated by group g_k and hence $x(S)=\sum_{i\in L}n_i(A^1_{ik}+A^2_{j^2(i)k})$. Since $\{S,N\setminus S\}$ is type 1 split we have, according to (11), $c(S)\geqslant x(S)$, and hence $e(S,x)\geqslant 0$.

Case (ii): $k\neq\pi(k)$, i.e., the last group of the first facility is not the last group of the second facility. Write $K_u=\{r_1,\dots,r_s\}$ with $\pi(k)=r_1<\dots<r_s=k$ for the collection of undominated groups. Write $t_j=\pi^{-1}(r_j)$ for every $j\in\{1,\dots,s\}$, then $t_s<\dots<t_1$. Define for every $i\in K$ the coalitions $S_i:=\cup\{g_j:1\leqslant j\leqslant i\}$ and $T_i:=\cup\{g_{\pi(j)}:1\leqslant j\leqslant i\}$, i.e., S_i is the union of the first i groups of the first facility and T_i is the union of the first i groups of the second facility. Define, moreover, for every $j\in\{1,\dots,s-1\}$ the numbers a_j and b_j by

$$a_j:=\min\left\{c(S_i)-\sum_{l=1}^{i}n_l(A^1_{lk}+A^2_{j^2(l)k}):r_j\leqslant i<r_{j+1}\right\},$$

$$b_j:=\min\left\{c(T_i)-\sum_{l=1}^{i}n_{\pi(l)}(A^1_{\pi(l)k}+A^2_{lk}):t_{s-j+1}\leqslant i<t_{s-j}\right\}$$

and finally, define $c:=B^1_k+B^2_k$. Let $i,j\in\{1,\dots,s-1\}$ be such that $i+j\geqslant s$. There is some $p\in\{r_i,\dots,r_{i+1}-1\}$ and $q\in\{t_{s-j+1},\dots,t_{s-j}-1\}$ with $a_i=c(S_p)-\sum_{l=1}^{p}n_l(A^1_{lk}+A^2_{j^2(l)k})$ and $b_j=c(T_q)-\sum_{l=1}^{q}n_{\pi(l)}(A^1_{\pi(l)k}+A^2_{lk})$. Since $\{1,\dots,p\}$ contains the undominated groups $r_1,\dots,r_i$ and the groups dominated by them and $\{\pi(1),\dots,\pi(q)\}$ contains the undominated groups $r_{s-j+1},\dots,r_k$ and the groups dominated by them, we have $\{1,\dots,p\}\cup\{\pi(1),\dots,\pi(q)\}=K$. According to (13), we have $a_i+b_j=c(S_p)-\sum_{l=1}^{p}n_l(A^1_{lk}+A^2_{j^2(l)k})+c(T_q)-\sum_{l=1}^{q}n_{\pi(l)}(A^1_{\pi(l)k}+A^2_{lk})\geqslant B^1_k+B^2_k=c$. Moreover, we have for every $i\in\{1,\dots,s-1\}$ that $a_i\geqslant c-b_{s-i}\geqslant 0$ and $b_i\geqslant c-a_{s-i}\geqslant 0$. According to Lemma 4.1 system (14) has a non-negative solution $(z_1,\dots,z_s)$. We claim that $(x_1,\dots,x_k)$ defined by

$$\begin{cases} x_i := A^1_{ik} + A^2_{j^2(i)k} & \text{if } i \in K_d, \\ x_i := A^1_{ik} + A^2_{j^2(i)k} + \frac{z_l}{n_l} & \text{if } i = r_l \text{ with } l \in \{1,\dots,s\} \end{cases} \tag{16}$$

is a symmetric core element of $\langle N, c\rangle$. Players belonging to dominated groups only have to pay the maintenance costs $A^1_{ik} + A^2_{j^2(i)k}$, whereas the players belonging to the undominated groups are also dividing the building costs $B^1_k + B^2_k$. According to Proposition 4.1 we only have to show that every coalition, which is the union of an essential collection of groups, is satisfied. So, let L be an essential collection and $S := \cup\{g_j : j \in L\}$. If $M_L = \{1, 2\}$, then there is a group $j \in R^1_L \cap R^2_L$ which dominates all the groups in L. So, $x(S) = \sum_{l\in L} n_l(A^1_{lk} + A^2_{j^2(l)k})$. Since $\{S, N \setminus S\}$ is type 1 split we have, according to (11), $c(S) - x(S) \geqslant 0$, and hence $e(S, x) \geqslant 0$. If $M_L = \{1\}$ then $S = S_i$ for some $i \in \{r_1 \dots, r_s - 1\}$. Let $i^* \in \{1, \dots, s-1\}$ be such that $i \in \{r_{i^*}, \dots, r_{i^*+1} - 1\}$. Then $x(S) = \sum_{l=1}^{i} n_l(A^1_{lk} + A^2_{j^2(l)k}) + z_1 + \dots + z_{i^*} \leqslant \sum_{l=1}^{i} n_l(A^1_{lk} + A^2_{j^2(l)k}) + a_{i^*} \leqslant c(S_i) = c(S)$ and hence $e(S, x) \geqslant 0$. If $M_L = \{2\}$ the satisfaction of S is proved in a similar fashion. □

The proof of Proposition 4.2 provides the basis for an algorithm that finds a core element of an infrastructure cost game with two facilities. First of all the maintenance costs have to be divided by allocating to every group g_i its own maintenance costs $A^1_{ik} + A^2_{j^2(i)k}$. Secondly, the building costs are distributed among the undominated groups only by looking for a solution of system (14) with the a_i's, b_j's and c as defined in the proof of Proposition 4.2. The proof of Lemma 4.1 indicates how such a solution can be found recursively in a maximum of $s - 1$ steps.

Acknowledgements

The financial support of the EuROPE-TRIP (European Railway Optimisation Planning Environment-Transportation Railways Integrated Planning) research project is gratefully acknowledged. We acknowledge Michael Maschler and an anonymous referee for their helpful suggestions and remarks.

References

Bassanini, A., Nastasi, A., 1997. A market based model for railroad capacity allocation. Research Report RR-97.08, Dipartimento di Informatica, Sistemi e Produzione, Università degli Studi di Roma Tor Vergata.

Brewer, P.J., Plott, C.R., 1996. A Binary Conflict Ascending Price (BICAP) mechanism for the decentralized allocation of the right to use railroad tracks. International Journal of Industrial Organization 14, 857–886.

Dubey, P., 1982. The Shapley value as aircraft landing fees revisited. Management Science 28, 869–874.

Dutta, B., Ray, D., 1989. A concept of egalitarianism under participation constraints. Econometrica 57, 615–635.

Fragnelli, V., García-Jurado, I., Norde, H., Patrone, F., Tijs, S., 2000. How to share railways infrastructure costs?. In: Patrone, F., García-Jurado, I., Tijs, S. (Eds.), Game Practice: Contributions from Applied Game Theory. Kluwer Academic Publishers, Dordrecht, pp. 91–101.

Littlechild, S.G., 1974. A simple expression for the nucleolus in a special case. International Journal of Game Theory 3, 21–29.

Littlechild, S.G., 1975. Common costs, fixed charges, clubs and games. Review of Economic Studies 42, 117–124.

Littlechild, S., Owen, G., 1973. A simple expression for the Shapley value in a special case. Management Science 20, 370–372.

Nilsson, J.E., 1995. Allocation of track capacity. CTS Working Paper 1995:1, Centre for Research in Transportation and Society, Dalarna University College, Sweden.

Tijs, S.H., Driessen, T.S.H., 1986. Game theory and cost allocation problems. Management Science 32, 1015–1028.

Young, H.P., 1994. Cost allocation. In: Aumann, R.J., Hart, S. (Eds.), Handbook of Game Theory, vol. 2. North-Holland, Amsterdam, pp. 1193–1235.