

# The maximum edge biclique problem is NP-complete

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## **Abstract**

We prove that the maximum edge biclique problem in bipartite graphs is NP-complete.

**Key words:** Complexity, bipartite graphs, biclique

# 1 Introduction

Let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ . A pair of two disjoint subsets  $A$  and  $B$  of  $V$  is called a *biclique* if  $\{a, b\} \in E$  for all  $a \in A$  and  $b \in B$ . Thus the edges  $\{a, b\}$  form a complete bipartite subgraph of  $G$  (which is not necessarily an induced subgraph if  $G$  is not bipartite). A biclique  $\{A, B\}$  clearly has  $|A| + |B|$  vertices and  $|A| * |B|$  edges. In this note we restrict ourselves to case when  $G$  is bipartite. The two colour classes of  $G$  will be denoted by  $V_1$  and  $V_2$ , so  $V = V_1 \cup V_2$ .

Already in the book of Garey and Johnson [2, GT24] the complexity of deciding whether or not a bipartite graph contains a biclique of a certain size is discussed. If the requirement is that  $|A| = |B| = K$  for some integer  $K$  (this is called the *balanced complete bipartite subgraph problem* or *balanced biclique problem*), then the problem is NP-complete. If however the requirement is that  $|A| + |B| \geq K$  (the *maximum vertex biclique problem*), the problem can be solved in polynomial time via the matching algorithm. The complexity of the maximum vertex biclique problem for general graphs depends on the precise definition of a biclique in this case. With the above definition the problem is solvable in polynomial time since there is a one to one correspondence between bicliques in the bipartite double<sup>1</sup> of the graph and bicliques in the graph itself (see also [4]). If one defines a biclique as an induced complete bipartite subgraph (so  $A$  and  $B$  are independent sets

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<sup>1</sup>The bipartite double of a graph with adjacency matrix  $A$  is the bipartite graph with adjacency matrix  $\begin{bmatrix} O & A \\ A & O \end{bmatrix}$ .

in  $G$ ), then the maximum vertex biclique problem for general graphs is NP-complete (see [8]). A natural third variant is the so-called *maximum edge biclique problem (MBP)* where the requirement is that  $|A| * |B| \geq K$ . Up to now the complexity of this problem was still undecided.

In various papers the complexity of MBP is mentioned and guessed to be NP-complete ([1, 4, 3, 7]. In [1] some applications of MBP are discussed and it is shown that the weighted version of MBP is NP-complete. Furthermore the authors show that four variants of MBP are NP-complete. Using different techniques Hochbaum [4], Haemers [3] and Pasechnik [7] derive upper bounds for the maximum number of edges in a biclique. Hochbaum [4] presents a 2-approximation algorithm for the minimum number of edges needed to be removed so that the remainder is a biclique based on an LP-relaxation. Inspired by the work of Lovász on the Shannon capacity of a graph ([6]), Haemers [3] and Pasechnik [7] derive similar inequalities for the maximum biclique problem. Pasechnik uses semidefinite programming techniques whereas Haemers uses eigenvalue techniques.

In the next section we prove that indeed MBP is NP-complete. The reduction used is inspired by the reduction that is used to prove the NP-completeness of the balanced biclique problem (see [5]). As a consequence MBP is also NP-complete for general graphs.

## 2 The reduction

We define MBP as follows:

**Maximum edge biclique problem (MBP):** Given a bipartite graph  $G = (V_1 \cup V_2, E)$  and a positive integer  $K$ , does  $G$  contain a biclique with at least  $K$  edges?

**Theorem 1** *MBP is NP-complete.*

**Proof:** We shall reduce CLIQUE to MBP. This reduction is a modification of the reduction from CLIQUE to BALANCED COMPLETE BIPARTITE SUBGRAPH referred to in [2, GT24] and published in [5].

Let  $G = (V, E)$  and  $K$  provide an instance of CLIQUE. Without loss of generality we may assume that  $K = \frac{1}{2}|V|$ .

Now construct an instance  $G' = (V_1 \cup V_2, E')$ ,  $K'$  of MBP as follows: Let

$$V_1 = V,$$

$$V_2 = E \cup W,$$

where  $W$  is a set of  $\frac{1}{2}K^2 - K$  new elements.

$$E' = \{\{v, e\} : v \in V; e \in E; v \notin e\} \cup \{\{v, w\} : v \in V; w \in W\}$$

$$K' = K^3 - \frac{3}{2}K^2$$

This construction can clearly be performed in polynomial time. Suppose  $G$  has a clique  $C$  of size  $K$ . Take  $A := V - C$  and  $B := W \cup \{\{c, d\} : c, d \in C; c \neq d\}$ . Then  $\{A, B\}$  is a biclique in  $G'$  with  $|A| * |B| = K * (\frac{1}{2}K^2 - K + \frac{1}{2}K(K - 1)) = K^3 - \frac{3}{2}K^2$  edges. So if  $G$  has a clique of size  $K$  then  $G'$  has a biclique with  $K'$  edges.

Now suppose  $G$  has no clique of size  $K$ . Let  $\{A, B\}$  be a biclique of  $G'$  with  $A \subseteq V_1$  and  $B \subseteq V_2$ . We shall finish the proof by showing that  $|A| * |B| < K'$  in this case. Without loss of generality  $W \subseteq B$ . Let  $a := |A|$  and  $b := |B| - |W|$ .

The  $b$  elements of  $B \cap E$  correspond with edges in  $G$  whose endpoints are not in  $A$ . There are  $2K - a$  vertices of  $G$  that are not in  $A$  so  $b \leq \frac{1}{2}(2K - a)(2K - a - 1)$ , with equality if and only if  $V - A$  is a clique with edge set  $B \cap E$ .

We consider two cases:

1. Suppose  $a > K$ , so  $|V - A| = K - c$  with  $c := a - K$  (So  $0 < c \leq K$ ).  
Then  $b \leq \frac{1}{2}(K - c)(K - c - 1)$ , so

$$|A| * |B| \leq [K + c] * \left[ \frac{1}{2}K^2 - K + \frac{1}{2}(K - c)(K - c - 1) \right]$$

This reduces to

$$|A| * |B| - (K^3 - \frac{3}{2}K^2) \leq \frac{1}{2}c(c^2 - (K - 1)c - 2K)$$

Now  $c^2 - (K - 1)c - 2K$  is negative for  $0 \leq c \leq K$ , so  $|A| * |B| < K'$  for  $0 < c \leq K$ .

2. Suppose  $a \leq K$ , so  $|V - A| = K + c$  with  $c := K - a$  (So  $0 \leq c \leq K$ ).  
Since  $G$  has no cliques with  $K$  vertices, the number of edges in the subgraph of  $G$  induced by  $V - A$ , and consequently  $b$ , is strictly less than  $\frac{1}{2}(K + c)(K + c - 1) - c$ . This leads to

$$|A| * |B| < [K - c] * \left[ \frac{1}{2}K^2 - K + \frac{1}{2}(K + c)(K + c - 1) - c \right]$$

which reduces to

$$|A| * |B| - (K^3 - \frac{3}{2}K^2) < \frac{1}{2}c^2(-c + 3 - K)$$

Since we may assume that  $K \geq 4$ , the right hand side is negative for  $1 \leq c \leq K$  and zero for  $c = 0$ . So  $|A| * |B| < K'$ .

□

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