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NOTE

A Characterization of Ordinal Potential Games

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This note characterizes ordinal potential games by the absence of weak improvement cycles and an order condition on the strategy space. This order condition is automatically satisfied if the strategy space is countable. *Journal of Economic Literature* Classification Number: C72.

1. INTRODUCTION

Monderer and Shapley (1996) introduce several classes of potential games. A common feature of these classes is the existence of a real-valued function on the strategy space that incorporates the strategic possibilities of all players simultaneously. In their paper, Monderer and Shapley (1996) distinguish between exact and ordinal potential games. As an example of an exact potential game, consider the two-person game in Fig. 1a, where the first player chooses either T or B, and the second player simultaneously and independently chooses either L or R. The numbers in the corresponding cells are the payoffs to players 1 and 2, respectively. Also, consider the real-valued function on the strategy space given in Fig. 1b. Note that the change in the payoff to a unilaterally deviating player exactly equals the corresponding change in the value of this function. For instance, if the second player deviates from (T, L) to (T, R), his payoff increases by one, just like the function in Fig. 1b. This function is therefore called an exact potential of the game. Exact potential games are characterized in Monderer and Shapley (1996) by the property that the changes in payoff to deviating players along a cycle sum to zero, where a cycle in the

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	L	R
T	0, 2	− 1, 3
B	1, 0	0, 1

a

	L	R
T	0	1
B	1	2

b

FIGURE 1

strategy space is a closed sequence of strategy combinations in which players unilaterally deviate from one point to the next. The game in Fig. 2a is an example of an ordinal potential game. Consider the function in Fig. 2b and note that the sign of the change in the payoff to a unilaterally deviating player exactly matches the sign of the corresponding change in this function. For instance, if the second player deviates from (T, L) to (T, R), his payoff increases, just like the function in Fig. 2b. Since deviating from (T, R) to (B, R) does not change player 1's payoff, the value of the function remains the same. For this reason, this function is called an ordinal potential of the game.

Monderer and Shapley do not give a characterization of ordinal potential games. The class of finite ordinal potential games was characterized in Voorneveld (1996) through the absence of weak improvement cycles, i.e., cycles along which a unilaterally deviating player never incurs a lower payoff and at least one such player increases his payoff. The necessity of this condition is immediate, since a potential function would never decrease along a weak improvement cycle, but increases at least once. This gives a contradiction, because a cycle ends up where it started. Proving sufficiency is harder. In this note we characterize the total class of ordinal potential games. It turns out that countable ordinal potential games are still characterized by the absence of weak improvement cycles, but that for uncountable ordinal potential games an additional order condition on the strategy space is required.

The organization of this note is as follows: Definitions and some preliminary results are given in Section 2. In Section 3 we provide a full characterization of ordinal potential games. In Section 4 we indicate that

	L	R
T	0, 2	0, 3
B	1, 0	0, 1

a

	L	R
T	0	2
B	1	2

b

FIGURE 2

the absence of weak improvement cycles characterizes ordinal potential games with a countable strategy space, but not necessarily ordinal potential games in which the strategy space is uncountable.

2. DEFINITIONS AND PRELIMINARY RESULTS

A *strategic game* is a tuple $G = \langle N, (X^i)_{i \in N}, (u^i)_{i \in N} \rangle$, where N is the player set; for each $i \in N$ the set of player i's strategies is X^i, and $u^i\colon \prod_{i \in N} X^i \to \mathbb{R}$ is player i's payoff function.

For brevity, we define $X = \prod_{i \in N} X^i$ and for $i \in N$: $X^{-i} = \prod_{j \in N \setminus \{i\}} X^j$. Let $x \in X$ and $i \in N$. With a slight abuse of notation, we sometimes denote $x = (x^i, x^{-i})$.

Let $G = \langle N, (X^i)_{i \in N}, (u^i)_{i \in N} \rangle$ be a strategic game. A *path* in the strategy space X is a sequence $(x_1, x_2, \ldots)$ of elements $x_k \in X$ such that for all $k = 1, 2, \ldots$ the strategy combinations x_k and x_{k+1} differ in exactly one, say the $i(k)$th, coordinate. A path is *nondeteriorating* if $u^{i(k)}(x_k) \leq u^{i(k)}(x_{k+1})$ for all $k = 1, 2, \ldots$. A finite path $(x_1, \ldots, x_m)$ is called a *weak improvement cycle* if it is nondeteriorating, $x_1 = x_m$, and $u^{i(k)}(x_k) < u^{i(k)}(x_{k+1})$ for some $k \in \{1, \ldots, m-1\}$.

Define a binary relation $\lhd$ on the strategy space X as follows: $x \lhd y$ if there exists a nondeteriorating path from x to y. The binary relation $\approx$ on X is defined by $x \approx y$ if $x \lhd y$ and $y \lhd x$.

By checking reflexivity, symmetry, and transitivity, one sees that the binary relation $\approx$ is an equivalence relation. Denote the equivalence class of $x \in X$ with respect to $\approx$ by $[x]$, i.e., $[x] = \{y \in X \mid y \approx x\}$, and define a binary relation $\prec$ on the set $X_{\approx}$ of equivalence classes as follows: $[x] \prec [y]$ if $[x] \neq [y]$ and $x \lhd y$. To show that this relation is well-defined, observe that the choice of representatives in the equivalence classes is of no concern:

$$\forall x, \tilde{x}, y, \tilde{y} \in X \text{ with } x \approx \tilde{x} \text{ and } y \approx \tilde{y}: \qquad x \lhd y \Leftrightarrow \tilde{x} \lhd \tilde{y}.$$

Note, moreover, that the relation $\prec$ on $X_{\approx}$ is irreflexive and transitive. The equivalence relation $\approx$ plays an important role in the characterization of ordinal potential games.

DEFINITION 1 (Monderer and Shapley, 1996). A strategic game $G = \langle N, (X^i)_{i \in N}, (u^i)_{i \in N} \rangle$ is an *ordinal potential game* if there exists a function $P\colon X \to \mathbb{R}$ such that

$$\forall i \in N, \forall x^{-i} \in X^{-i}, \forall x^i, y^i \in X^i:$$

$$u^i(x^i, x^{-i}) > u^i(y^i, x^{-i}) \Leftrightarrow P(x^i, x^{-i}) > P(y^i, x^{-i}).$$

The function P is called an *(ordinal) potential* of the game G.

In other words, if P is an ordinal potential function for G, the sign of the change in payoff to a unilaterally deviating player matches the sign of the change in the value of P.

A necessary condition for the existence of an ordinal potential function is the absence of weak improvement cycles.

LEMMA 2.1. *Let $G = \langle N, (X^i)_{i \in N}, (u^i)_{i \in N} \rangle$ be a strategic game. If G is an ordinal potential game, then X contains no weak improvement cycles.*

Proof. Assume G is an ordinal potential game and suppose that $(x_1, \dots, x_m)$ is a weak improvement cycle. By definition, $u^{i(k)}(x_k) \leq u^{i(k)}(x_{k+1})$ for all $k \in \{1, \dots, m-1\}$ with strict inequality for at least one such k. But then $P(x_k) \leq P(x_{k+1})$ for all and strict inequality for at least one $k \in \{1, \dots, m-1\}$, implying $P(x_1) < P(x_m) = P(x_1)$, a contradiction. ■

In Section 4, we will show that the converse of Lemma 2.1 is true if $(X_{\approx}, \prec)$ is properly ordered.

DEFINITION 2. Consider a tuple $(A, \prec)$ consisting of a set A and an irreflexive and transitive binary relation $\prec$. $(A, \prec)$ is *properly ordered* if there exists a function $F\colon A \to \mathbb{R}$ that preserves the order $\prec$:

$$\forall x, y \in A: \qquad x \prec y \Rightarrow F(x) < F(y).$$

Properly ordered sets are a key topic of study in utility theory. Not every tuple $(A, \prec)$ with $\prec$ irreflexive and transitive is properly ordered. A familiar example is the lexicographic order on $\mathbb{R}^2$. See, e.g., Fishburn (1979) for more details. However, if the set A is countable, i.e., if A is finite or if there exists a bijection between A and $\mathbb{N}$, then $(A, \prec)$ is properly ordered.

LEMMA 2.2. *Let A be a countable set and $\prec$ be a binary relation on A that is irreflexive and transitive. Then $(A, \prec)$ is properly ordered.*

Proof. Since A is countable, we can label its elements and write $A = \{x_1, x_2, \dots\}$. For $k \in \mathbb{N}$ define $A_k = \{x_1, \dots, x_k\}$. We define $F\colon A \to \mathbb{R}$ by an inductive argument. Define $F(x_1) = 0$. Let $k \in \mathbb{N}$ and assume F has already been defined on A_k such that

$$\forall x, y \in A_k: \qquad x \prec y \Rightarrow F(x) < F(y). \tag{1}$$

We extend F to A_{k+1}. Define

$$L_k = \{x \in A_k \mid x \prec x_{k+1}\}$$
$$U_k = \{x \in A_k \mid x_{k+1} \prec x\}.$$

If $L_k \neq \varnothing$, take $l = \max_{z \in L_k} F(z)$ and $x \in \arg\max_{z \in L_k} F(z)$. If $U_k \neq \varnothing$, take $u = \min_{z \in U_k} F(z)$ and $y \in \arg\min_{z \in U_k} F(z)$. If both L_k and U_k are nonempty, then $x \prec x_{k+1}$ and $x_{k+1} \prec y$ imply $x \prec y$ by transitivity; so given that F satisfies (1), $l = F(x) < F(y) = u$.

- If $L_k = \varnothing$ and $U_k = \varnothing$, take $F(x_{k+1}) = 0$.
- If $L_k = \varnothing$ and $U_k \neq \varnothing$, take $F(x_{k+1}) \in (-\infty, u)$.
- If $L_k \neq \varnothing$ and $U_k = \varnothing$, take $F(x_{k+1}) \in (l, \infty)$.
- If $L_k \neq \varnothing$ and $U_k \neq \varnothing$, take $F(x_{k+1}) \in (l, u)$.

Note that F is now correctly defined on A_{k+1}:

$$\forall x, y \in A_{k+1}: \qquad x \prec y \Rightarrow F(x) < F(y).$$

It follows that by proceeding in this way we find a function F on A as in the theorem. ■

In Example 4.1, we will give an example of a game in which $(X_{\approx}, \prec)$ is not properly ordered. A sufficient condition for an uncountable set $(A, \prec)$ to be properly ordered is the existence of a countable subset B of A such that if $x \prec z$, $x \notin B$, $z \notin B$, there exists a $y \in B$ such that $x \prec y$, $y \prec z$. Such a set B is $\prec$*-order dense* in A.

LEMMA 2.3. *Let A be a set and $\prec$ a binary relation on A that is irreflexive and transitive. If there exists a countable subset of A that is $\prec$-order dense in A, then $(A, \prec)$ is properly ordered.*

Proof. This is a corollary of Theorem 3.2 in Fishburn (1979). ■

3. CHARACTERIZATION OF ORDINAL POTENTIAL GAMES

This section contains a characterization of ordinal potential games. The absence of weak improvement cycles was a necessary condition. If $(X_{\approx}, \prec)$ is properly ordered, this is also a sufficient condition.

THEOREM 3.1. *A strategic game $G = \langle N, (X^i)_{i \in N}, (u^i)_{i \in N} \rangle$ is an ordinal potential game if and only if the following two conditions are satisfied:*

1. *X contains no weak improvement cycles;*
2. *$(X_{\approx}, \prec)$ is properly ordered.*

Proof. $(\Rightarrow)$ Assume P is an ordinal potential for G. X contains no weak improvement cycles by Lemma 2.1. Define $F: X_{\approx} \to \mathbb{R}$ by taking for all $[x] \in X_{\approx} : F([x]) = P(x)$. To see that F is well-defined, let $y, z \in [x]$. Since $y \approx z$ there is a nondeteriorating path from y to z and vice versa. But since the game has no weak improvement cycles, all changes in the payoff to the deviating players along these paths must be zero: $P(y) = P(z)$.

Now take $[x], [y] \in X_{\approx}$ with $[x] \prec [y]$. Since $x \lhd y$, there is a nondeteriorating path from x to y, so $P(x) \leq P(y)$. Moreover, since x and y are in different equivalence classes, some player must have gained from deviating along this path: $P(x) < P(y)$. Hence $F([x]) < F([y])$.

($\Leftarrow$) Assume that the two conditions hold. Since $(X_{\approx}, \prec)$ is properly ordered, there exists a function $F\colon X_{\approx} \to \mathbb{R}$ that preserves the order $\prec$. Define $P\colon X \to \mathbb{R}$ by $P(x) = F([x])$ for all $x \in X$. Let $i \in N$, $x^{-i} \in X^{-i}$, and $x^i, y^i \in X^i$.

- If $u^i(x^i, x^{-i}) - u^i(y^i, x^{-i}) > 0$, then $(y^i, x^{-i}) \lhd (x^i, x^{-i})$, and by the absence of weak improvement cycles, not $(x^i, x^{-i}) \lhd (y^i, x^{-i})$. Hence $[(y^i, x^{-i})] \prec [(x^i, x^{-i})]$, which implies $P(x^i, x^{-i}) - P(y^i, x^{-i}) = F([(x^i, x^{-i})]) - F([(y^i, x^{-i})]) > 0$.

- Assume $P(x^i, x^{-i}) - P(y^i, x^{-i}) > 0$. Then $[(x^i, x^{-i})] \neq [(y^i, x^{-i})]$, so $u^i(x^i, x^{-i}) \neq u^i(y^i, x^{-i})$. If $u^i(x^i, x^{-i}) < u^i(y^i, x^{-i})$, then $(x^i, x^{-i}) \lhd (y^i, x^{-i})$, and hence $[(x^i, x^{-i})] \prec [(y^i, x^{-i})]$. But then $P(x^i, x^{-i}) - P(y^i, x^{-i}) = F([(x^i, x^{-i})]) - F([(y^i, x^{-i})]) < 0$, a contradiction. Hence $u^i(x^i, x^{-i}) - u^i(y^i, x^{-i}) > 0$.

Conclude that P is an ordinal potential for the game G. ∎

The first condition in Theorem 3.1 involving cycles closely resembles a characterization of exact potential games in Monderer and Shapley (1996). A strategic game is an exact potential game if and only if the payoff changes to deviating players along a cycle sum to zero. In fact, it suffices to look at cycles involving only four deviations. The next example indicates that the absence of weak improvement cycles involving four deviations only is not sufficient to characterize ordinal potential games.

EXAMPLE 3.1. Suppose P is an ordinal potential of the game in Fig. 3. Then P has to satisfy $P(T, L) > P(T, R) = P(M, R) = P(M, M) = P(B, M) = P(B, L) = P(T, L)$, which is clearly impossible: this is not an ordinal potential game. It is easy to check, however, that the order condition is satisfied and that there are no weak improvement cycles involving exactly four deviations.

	L	M	R
T	0, 1	1, 2	0, 0
M	1, 1	0, 0	0, 0
B	0, 0	0, 0	1, 1

FIGURE 3

4. COUNTABLE AND UNCOUNTABLE GAMES

Lemmas 2.2 and 2.3 give sufficient conditions for $(X_{\approx}, \prec)$ to be properly ordered. A consequence of Lemma 2.2 is that a game G with a countable strategy space X is an ordinal potential game if and only if it contains no weak improvement cycles. The strategy space X is countable if the set N of players is finite and every player $i \in N$ has a countable set X^i of strategies.

THEOREM 4.1. *Let $G = \langle N, (X^i)_{i \in N}, (u^i)_{i \in N} \rangle$ be a strategic game. If X is countable, then G is an ordinal potential game if and only if X contains no weak improvement cycles.*

Proof. If X is countable, $X_{\approx}$ is countable. According to Lemma 2.2, $(X_{\approx}, \prec)$ is properly ordered, so the result now follows from Theorem 3.1. ■

Theorem 4.1 generalizes the analogous result from Voorneveld (1996) for finite games. The mixed extension of a finite ordinal potential game may not be an ordinal potential game, as shown in Sela (1992).

A consequence of Lemma 2.3 is that if $(X_{\approx}, \prec)$ contains a countable $\prec$-order dense subset, then the absence of weak improvement cycles is once again enough to characterize ordinal potential games.

THEOREM 4.2. *Let $G = \langle N, (X^i)_{i \in N}, (u^i)_{i \in N} \rangle$ be a strategic game. If $(X_{\approx}, \prec)$ contains a countable $\prec$-order dense subset, then G is an ordinal potential game if and only if X contains no weak improvement cycles.*

Proof. By Lemma 2.3, $(X_{\approx}, \prec)$ is properly ordered. The result follows from Theorem 3.1. ■

This section is concluded with an example of a game with an uncountable strategy space in which no weak improvement cycles exist, but which is not an ordinal potential game since $(X_{\approx}, \prec)$ is not properly ordered.

EXAMPLE 4.1. Consider the two-player game G with $X^1 = \{0, 1\}$, $X^2 = \mathbb{R}$, and payoff functions defined by

$$u^1(x, y) = \begin{cases} x & \text{if } y \in \mathbb{Q} \\ -x & \text{if } y \notin \mathbb{Q} \end{cases} \quad \text{and}$$

$$u^2(x, y) = y \text{ for all } (x, y) \in \{0, 1\} \times \mathbb{R}.$$

This game has no weak improvement cycles, since every weak improvement cycle trivially has to include deviations by at least two players. But if the second player deviates once and improves his payoff, he has to return to his initial strategy eventually, thereby reducing his payoff.

We show that this game nevertheless is not an ordinal potential game. Suppose, to the contrary, that P is an ordinal potential for G. We show that this implies the existence of an injective function f from the uncountable set $\mathbb{R} \setminus \mathbb{Q}$ to the countable set $\mathbb{Q}$, a contradiction.

For each $y \in \mathbb{R} \setminus \mathbb{Q}$, $u^1(0, y) = 0 > -1 = u^1(1, y)$, so $P(0, y) > P(1, y)$. Fix $f(y) \in [P(1, y), P(0, y)] \cap \mathbb{Q}$. In order to show that $f\colon \mathbb{R} \setminus \mathbb{Q} \to \mathbb{Q}$ is injective, let $x, z \in \mathbb{R} \setminus \mathbb{Q}$, $x < z$. Then there exists a number $y \in (x, z) \cap \mathbb{Q}$. However,

$$\begin{cases} u^2(0, x) < u^2(0, y) \\ u^1(0, y) < u^1(1, y) \\ u^2(1, y) < u^2(1, z) \end{cases} \Rightarrow \begin{cases} P(0, x) < P(0, y) \\ \qquad\quad\; < P(1, y) \\ \qquad\quad\; < P(1, z). \end{cases}$$

Since $f(x) \in [P(1, x), P(0, x)]$ and $f(z) \in [P(1, z), P(0, z)]$, it follows that $f(x) < f(z)$. So f is injective, a contradiction.

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