Optimization Methods and Software
Vol. 20, No. 1, February 2005, 99–113

Taylor & Francis
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Limiting behavior of the central path in semidefinite optimization

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(Received 17 June 2002; revised 30 June 2003; in final form 20 May 2004)

This paper is dedicated to Professor Yury Evtushenko in honor of his 65th birthday

It was recently shown by [Halická *et al.* (2002). On the convergence of the central path in semidefinite optimization. *SIAM J. Optimization*, 12(4), 1090–1099] that, unlike in linear optimization, the central path in semidefinite optimization (SDO) does not converge to the analytic center of the optimal set in general. In this article, we analyze the limiting behavior of the central path to explain this unexpected phenomenon. This is done by deriving a new necessary and sufficient condition for strict complementarity. We subsequently show that, in the absence of strict complementarity, the central path converges to the analytic center of a certain subset of the optimal set. We further derive sufficient conditions under which this subset coincides with the optimal set, i.e. sufficient conditions for the convergence of the central path to the analytic center of the optimal set. Finally, we show that convex quadratically constrained quadratic optimization problems, when rewritten as SDO problems, satisfy these sufficient conditions. Several examples are given to illustrate the possible convergence behavior.

Keywords: Semidefinite optimization; Linear optimization; Interior-point method; Central path; Analytic center

AMS Subject Classifications: 90C51; 90C22

1. Introduction

We first formulate semidefinite programs and recall the definition of the central path and some of its properties.

We denote by S^n the space of all real symmetric $n \times n$ matrices and for any $M, N \in S^n$, we define

$$M \cdot N = \text{trace}(MN) = \sum_{i,j} m_{ij} n_{ij}.$$

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Optimization Methods and Software
ISSN 1055-6788 print/ISSN 1029-4937 online
http://www.tandf.co.uk/journals
DOI: 10.1080/10556780410001727718

The convex cones of symmetric positive semidefinite matrices and positive definite matrices will be denoted by S^n_+ and S^n_{++}, respectively; $X \succeq 0$ and $X \succ 0$ mean that a symmetric matrix X is positive semidefinite and positive definite, respectively.

We will consider the following primal–dual pair of semidefinite programs in the standard form

$$(P): \quad \min_{X \in S^n} \{C \cdot X \colon A^i \cdot X = b_i \ (i = 1, \dots, m), X \succeq 0\},$$

$$(D): \quad \max_{y \in \mathbb{R}^m, Z \in S^n} \left\{ b^{\mathrm{T}} y \colon \sum_{i=1}^{m} A^i y_i + Z = C, Z \succeq 0 \right\},$$

where $A^i \in S^n$ $(i = 1, \dots, m)$ and $C \in S^n$, $b \in \mathbb{R}^m$. The solutions X and (y, Z) will be referred to as feasible solutions as they satisfy the primal and dual constraints, respectively. The primal and dual feasible sets will be denoted by $\mathcal{P}$ and $\mathcal{D}$, respectively, and $\mathcal{P}^*$ and $\mathcal{D}^*$ will denote the respective optimal sets. We make the following two standard assumptions throughout the article.

ASSUMPTION 1.1 *$A^i (i = 1, \dots, m)$ are linearly independent.*

ASSUMPTION 1.2 *There exist $X^0 \in \mathcal{P}$ and $(Z^0, y^0) \in \mathcal{D}$ such that $X^0 \succ 0$ and $Z^0 \succ 0$.*

It is well known that under our assumptions both $\mathcal{P}^*$ and $\mathcal{D}^*$ are non-empty and bounded. Moreover, $(X, Z) \in (\mathcal{P} \times \mathcal{D})$ is an optimal solution pair if and only if $X \cdot Z = 0$, or equivalently $XZ = 0$. The last equation is a complementarity condition. A strictly complementary solution is defined as an optimal solution pair (X, Z) satisfying the rank condition: $\operatorname{rank}(X) + \operatorname{rank}(Z) = n$. Contrary to the case for linear optimization (LO), the existence of a strictly complementary solution is not generally ensured in semidefinite optimization (SDO). A pair of optimal solutions $(X, Z) \in \mathcal{P}^* \times \mathcal{D}^*$ is called a *maximally complementary solution pair* if it maximizes $\operatorname{rank}(X) + \operatorname{rank}(Z)$ over all optimal solution pairs. The set of maximally complementary solutions coincides with the relative interior of $(\mathcal{P}^* \times \mathcal{D}^*)$.

In order to define the central path, we consider the centering system

$$\begin{aligned} A^i \cdot X = b_i, \quad & X \succeq 0 \ (i = 1, \dots, m) \\ \sum_{i=1}^{m} A^i y_i + Z = C, \quad & Z \succeq 0 \\ XZ = \mu I, & \end{aligned} \tag{1}$$

where I is the identity matrix and $\mu \geq 0$ is a parameter. It is easy to see that for $\mu = 0$, equation (1) forms the necessary and sufficient conditions for optimality, and hence may have multiple solutions. On the other hand, it is well known that for $\mu > 0$, the system (1) has a unique solution, denoted by $(X(\mu), Z(\mu), y(\mu))$. Similarly as for LO, this solution is seen as the parametric representation of an analytic curve (the *central path*) in terms of the parameter μ.

It has been shown that the central path for SDO shares many properties with the central path for LO. First, the basic property was established that the central path, when restricted to $0 < \mu \leq \bar{\mu}$ for some $\bar{\mu} > 0$, is bounded and thus has limit points in the optimal set as $\mu \downarrow 0$ [1,2]. Then, it was shown that the limit points are maximally complementary optimal solutions [1,3]. Finally, it was claimed in ref. [3] that any limit point of the central path for $\mu \downarrow 0$ can be characterized as the analytic center of the optimal set. As the analytic center is defined uniquely, this claim implies the convergence of the central path to an optimal solution. However, a counterexample was presented in ref. [4], which shows that the central path need not converge to the analytic center if the SDO program does not have a strictly complementary solution.

This fact has reopened the question of characterization of limit points of the central path in SDO. Moreover, it made it necessary to re-examine proofs of the convergence of the central path, which do not rely on the analytic center characterization of limit points.

In ref. [5], Kojima *et al.* established the convergence of the central path for the monotone linear complementarity problem using a triangularization approach from algebraic geometry. In ref. [6], Kojima *et al.* mentioned, without giving details, that a similar approach can be used to prove convergence of the central path for the monotone semidefinite linear complementarity problem (SLCP), which includes SDO as a special case.

Convergence of the central path for a class of convex SDO problems that includes SDO was proved in ref. [7].

A simpler, direct proof of the convergence of the central path for SDO was given in ref. [4].

All these proofs use 'existence' results from real algebraic geometry and do not yield any characterization of the limit point.

The counterexample from ref. [4] shows that the central path may have very unexpected behavior, if no strictly complementary solution exists. Nevertheless, under the assumption of strict complementarity, the central path in SDO has the same properties as the central path in LO: it converges to the analytic center of the optimal set [2,8], the distance of the μ-center on the central path to the optimal set is $O(\mu)$ [2], and the central path can be analytically extended beyond $\mu = 0$, which implies that the derivatives of the central path are bounded at the limit point [9].

In this article, we first give a new characterization of strict complementarity. This allows us to show that, in the absence of strict complementarity, the central path converges to the analytic center of a certain subset of the optimal set. Then, we derive conditions under which this subset coincides with the optimal set, and hence the central path converges to the analytic center of the optimal set. Finally, we show that the convex quadratically constraint quadratic program, when rewritten as an SDO program, satisfies these sufficient conditions.

2. Preliminaries

As mentioned earlier, the central path $(X(\mu), Z(\mu), y(\mu))$ converges as $\mu \downarrow 0$ and its limit point (X^*, Z^*, y^*) is a maximally complementary optimal solution. This result forms the basis for our subsequent analysis. Denote

$$|B| := \text{rank } X^* \quad \text{and} \quad |N| := \text{rank } Z^*.$$

Obviously, $|B| + |N| \leq n$. As X^* and Z^* commute, they can be simultaneously diagonalized. Hence, without loss of generality (applying an orthonormal transformation of problem data, if necessary), we can assume that

$$X^* = \begin{bmatrix} X_B^* & 0 \\ 0 & 0 \end{bmatrix}, \quad Z^* = \begin{bmatrix} 0 & 0 \\ 0 & Z_N^* \end{bmatrix},$$

where $X_B^* \in S_{++}^{|B|}$ and $Z_N^* \in S_{++}^{|N|}$ are diagonal. Moreover, each optimal solution pair $(\hat{X}, \hat{Z})$ is of the form

$$\hat{X} = \begin{bmatrix} \hat{X}_B & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{Z} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{Z}_N \end{bmatrix},$$

where $\hat{X}_B \in S_+^{|B|}$ and $\hat{Z}_N \in S_+^{|N|}$ (The arguments for this claim are given, e.g., in ref. [4].). In this notation, the condition of existence of a strictly complementary solution is equivalent to $|B| + |N| = n$, and non-existence with $|B| + |N| < n$.

Given an index set $T \subset \{1, \ldots, n\}$, let $\bar{T}$ denote its complement. In what follows, we consider the two possible partitions of any $n \times n$ matrix M, corresponding to the above optimal partitions for the optimal X^* and Z^*, namely,

$$M = \begin{bmatrix} M_B & M_{B\bar{B}} \\ M_{\bar{B}B} & M_{\bar{B}} \end{bmatrix} \text{ (primal partition)}, \quad M = \begin{bmatrix} M_{\bar{N}} & M_{\bar{N}N} \\ M_{N\bar{N}} & M_N \end{bmatrix} \text{ (dual partition)}. \tag{2}$$

Here M_B and $M_{\bar{N}}$ are, respectively, the $|B| \times |B|$ and $(n - |N|) \times (n - |N|)$ matrices and hence if no strictly complementary solution exists, they are of different dimensions. We denote by $\mathcal{I} = \{B, B\bar{B}, \bar{B}B, \bar{B}\}$ and $\mathcal{J} = \{\bar{N}, \bar{N}N, N\bar{N}, \bar{N}\}$ the index sets corresponding to the optimal partitions of X^* and Z^*, respectively. If we refer, for example, to all the blocks of the primal partition of M except M_B, we will write $M_i (i \in \mathcal{I} - B)$. Using this notation, the first two equations in equation (1) can be rewritten as

$$\begin{aligned} \sum_{j \in \mathcal{I}} A_j^i \cdot X_j(\mu) = b_i \quad (i = 1, \ldots, m), \\ \sum_{i=1}^{m} A_j^i y_i(\mu) + Z_j(\mu) = C_j \quad (j \in \mathcal{J}). \end{aligned} \tag{3}$$

Owing to the optimality conditions [equation (1), where $\mu = 0$], the optimal sets $\mathcal{P}^*$ and $\mathcal{D}^*$ can be characterized, respectively, by using the primal and dual optimal partitions:

$$\mathcal{P}^* = \{X\colon A_B^i \cdot X_B = b_i \ (i = 1, \ldots, m), X_B \in S_+^{|B|}, X_k = 0 \ (k \in \mathcal{I} - B)\}, \tag{4}$$

$$\mathcal{D}^* = \left\{(Z, y)\colon \sum_{i=1}^{m} A_N^i y_i + Z_N = C_N, Z_N \in S_+^{|N|}, \right. \\ \left. \sum_{i=1}^{m} A_k^i y_i = C_k, Z_k = 0 \ (k \in \mathcal{J} - N) \right\}. \tag{5}$$

This description of the optimal sets allows to identify the relative interiors of $\mathcal{P}^*$ and $\mathcal{D}^*$, respectively, as $\text{ri}(\mathcal{P}^*) = \{X \in \mathcal{P}^*\colon X_B \in S_{++}^{|B|}\}$ and $\text{ri}(\mathcal{D}^*) = \{(Z, y) \in \mathcal{D}^*\colon Z_N \in S_{++}^{|N|}\}$.

The analytic centers of these sets are defined as follows: $X^a \in \mathcal{P}^*$ is the analytic center of $\mathcal{P}^*$ if

$$X_B^a = \arg \max_{X_B \in S_{++}^{|B|}} \{\ln \det X_B\colon A_B^i \cdot X_B = b_i \ (i = 1, \ldots, m)\}, \tag{6}$$

and $(y^a, Z^a) \in \mathcal{D}^*$ is the analytic center of $\mathcal{D}^*$ if

$$(y^a, Z_N^a) = \arg \max_{y \in \mathbb{R}^m, Z_N \in S_{++}^{|N|}} \\ \left\{\ln \det Z_N\colon \sum_{i=1}^{m} A_N^i y_i + Z_N = C_N, \sum_{i=1}^{m} A_k^i y_i = C_k \ (k \in \mathcal{J} - N)\right\}. \tag{7}$$

By the Lagrange rule, the analytic center of $\mathcal{P}^*$ can be characterized as $X^a \in \mathcal{P}^*$ that satisfies the condition $(X_B^a)^{-1} - \sum_{i=1}^m u^i A_B^i = 0$, for some u^i $(i = 1, \ldots, m)$ or, equivalently, $(X_B^a)^{-1} \in \mathcal{R}(A_B^{\mathrm{T}})$. Here $\mathcal{R}(A_B^{\mathrm{T}})$ is the range of the linear operator A_B^{T} (A_B^{T} associates $y \in \mathbb{R}^m$ with $\sum_{i=1}^m A_B^i y_i$). The analytic center of $\mathcal{D}^*$ can be characterized similarly.

Remark The descriptions (4) and (5) of the optimal sets allow us to identify the necessary and sufficient conditions for the uniqueness of the optimal solutions. In fact, the primal optimal solution is uniquely defined if and only if the matrices A_B^i $(i = 1, \ldots, m)$ span the space $S^{|B|}$. Similarly, the dual optimal solution is uniquely defined if and only if the matrices

$$\begin{bmatrix} A_{\bar{N}}^i & A_{\bar{N}N}^i \\ A_{N\bar{N}}^i & 0 \end{bmatrix}, \quad i = 1, \ldots, m$$

are linearly independent.[1] These conditions coincide with the concepts of weak dual and weak primal non-degeneracy, respectively, as introduced by Yildirim [10]. Obviously, if weak primal (dual) non-degeneracy holds, then the dual (primal) central path converges to the unique solution, which is the analytic center of $\mathcal{D}^*(\mathcal{P}^*)$. In what follows, we are interested in the case when weak primal or dual non-degeneracy does not hold.

3. Characterization of the limit point of the central path

We first identify a property of the central path, which holds if and only if a strictly complementary solution exists. In other words, we give an alternative characterization of strict complementarity. To this aim, we introduce

$$\tilde{Z}_B(\mu) := \left(\frac{1}{\mu}\right) Z_B(\mu) \quad \text{and} \quad \tilde{X}_N(\mu) := \left(\frac{1}{\mu}\right) X_N(\mu), \tag{8}$$

and we study the limiting behavior of $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$.

THEOREM 3.1 *Both $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ converge as $\mu \downarrow 0$, and the limit matrices $\tilde{Z}_B^* := \lim_{\mu\downarrow 0} \tilde{Z}_B(\mu)$ and $\tilde{X}_N^* := \lim_{\mu\downarrow 0} \tilde{X}_N(\mu)$ are positive definite. Moreover, $|B| + |N| = n$ (strict complementarity holds) if and only if $(\tilde{Z}_B^*)^{-1} = X_B^*$ and $(\tilde{X}_N^*)^{-1} = Z_N^*$.*

Proof We first note that $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ are bounded as $\mu \downarrow 0$. This follows from

$$X_B^* \cdot \tilde{Z}_B(\mu) + Z_N^* \cdot \tilde{X}_N(\mu) = n, \tag{9}$$

which is implied by the identity $(X^* - X(\mu)) \cdot (Z^* - Z(\mu)) = 0$. Now, the proof of convergence for $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ follows the same pattern as the proof of convergence for the central path in ref. [4]. The latter proof exploited the fact that the centering system of conditions defines a semialgebraic set. However, if we substitute $Z_B = \mu\tilde{Z}_B$ and $X_N = \mu\tilde{X}_N$ into the centering conditions, the new system of conditions defines a semialgebraic set as well. Hence, the same procedure as in the proof of Theorem A.3 in ref. [4] yields the convergence $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$. This means that the limit points $\tilde{Z}_B^*$ and $\tilde{X}_N^*$ exist and are positive semidefinite.

[1] This observation was made thanks to an e-mail discussion with Yildirim.

We now prove that $\tilde{Z}_B^*$ and $\tilde{X}_N^*$ are positive definite. To this end, we use both the primal and dual partition of the equation $X(\mu)Z(\mu) = \mu I$ to obtain the following two pairs of equations:

$$X_B(\mu)Z_B(\mu) + X_{B\bar{B}}(\mu)Z_{\bar{B}B}(\mu) = \mu I_B, \quad X_B(\mu)Z_{B\bar{B}}(\mu) + X_{B\bar{B}}(\mu)Z_{\bar{B}}(\mu) = 0, \quad (10)$$

$$X_N(\mu)Z_N(\mu) + X_{N\bar{N}}(\mu)Z_{\bar{N}N}(\mu) = \mu I_N, \quad X_{\bar{N}}(\mu)Z_{\bar{N}N}(\mu) + X_{\bar{N}N}(\mu)Z_N(\mu) = 0. \quad (11)$$

Expressing $X_{B\bar{B}}(\mu)$ and $Z_{\bar{N}N}(\mu)$ from the second equation of each pair and substituting this to the first ones, we obtain

$$\tilde{Z}_B(\mu) = X_B^{-1}(\mu) + G_B(\mu), \quad \tilde{X}_N(\mu) = Z_N^{-1}(\mu) + G_N(\mu), \quad (12)$$

where

$$G_B(\mu) := \left(\frac{1}{\mu}\right) Z_{B\bar{B}}(\mu)Z_{\bar{B}}^{-1}(\mu)Z_{\bar{B}B}(\mu), \quad G_N(\mu) := \left(\frac{1}{\mu}\right) X_{N\bar{N}}(\mu)X_{\bar{N}}^{-1}(\mu)X_{\bar{N}N}(\mu). \quad (13)$$

As $\tilde{Z}_B(\mu)$, $\tilde{X}_N(\mu)$, $X_B^{-1}(\mu)$ and $Z_N^{-1}(\mu)$ converge as $\mu \downarrow 0$, we obtain that $G_B(\mu)$ and $G_N(\mu)$ converge to G_B^* and G_N^* (say). Moreover, as $(X_B^*)^{-1}$ and $(Z_N^*)^{-1}$ are positive definite, $\tilde{Z}_B^*$ and $\tilde{X}_N^*$ are positive definite as well.

Now, we prove the last part of the theorem. Substituting equation (12) into equation (9) and letting $\mu \to 0$, we obtain

$$X_B^* \cdot (X_B^*)^{-1} + Z_N^* \cdot (Z_N^*)^{-1} + X_B^* \cdot G_B^* + Z_N^* \cdot G_N^* = n.$$

It is easy to see that the first two terms in the last expression are equal to $|B| + |N|$ and the other two are non-negative. Hence $|B| + |N| = n$ if and only if the latter two terms vanish that are equivalent to $G_B^* = 0$ and $G_N^* = 0$, i.e., $\tilde{Z}_B^* = (X_B^*)^{-1}$ and $\tilde{X}_N^* = (Z_N^*)^{-1}$, by equation (12). ■

Hence the theorem states that both the matrices $X_B^* - (\tilde{Z}_B^*)^{-1}$ and $Z_N^* - (\tilde{X}_N^*)^{-1}$ vanish if and only if there exists a strictly complementary solution. In Lemma 3.1, we derive another interpretation for these matrices, which relate them to Schur complements of $X_{\bar{B}}$ and $Z_{\bar{N}}$, respectively. To this end, we define

$$F_B(\mu) := X_{B\bar{B}}(\mu)X_{\bar{B}}(\mu)^{-1}X_{\bar{B}B}(\mu) \quad \text{and} \quad F_N(\mu) := Z_{N\bar{N}}(\mu)Z_{\bar{N}}(\mu)^{-1}Z_{\bar{N}N}(\mu), \quad (14)$$

and note that $X_B(\mu) - F_B(\mu)$ is the Schur complement of $X_{\bar{B}}(\mu)$ in $X(\mu)$ and $Z_N(\mu) - F_N(\mu)$ is the Schur complement of $Z_{\bar{N}}(\mu)$ in $Z(\mu)$.

LEMMA 3.2 *$F_B(\mu)$ and $F_N(\mu)$ given by equation* (14) *converge as $\mu \downarrow 0$ and*

$$F_B^* := \lim_{\mu\downarrow 0} F_B(\mu) = X_B^* - (\tilde{Z}_B^*)^{-1} \in S_+^{|B|} \quad \text{and} \quad F_N^* := \lim_{\mu\downarrow 0} F_N(\mu) = Z_N^* - (\tilde{X}_N^*)^{-1} \in S_+^{|N|}. \quad (15)$$

Proof Applying the formula for the inverse of a block matrix [see, e.g., ref. 11] to $X(\mu)$, and using equation (15), we obtain $(X^{-1}(\mu))_B = (X_B(\mu) - F_B(\mu))^{-1}$. Combining this with $(X^{-1}(\mu))_B = \tilde{Z}_B(\mu)$ implied by $X(\mu)Z(\mu) = \mu I$, and taking the inversion yields $F_B(\mu) = X_B(\mu) - \tilde{Z}_B^{-1}(\mu)$. Analogously, we obtain $F_N(\mu) = Z_N(\mu) - \tilde{X}_N^{-1}(\mu)$. Now, the lemma follows from Theorem 3.1, where the convergence of $\tilde{Z}_B(\mu)$ and $\tilde{X}_N(\mu)$ to positive definite matrices is ensured. Positive semidefiniteness of F_B^* and F_N^* follows from equation (15). ■

From Lemma 3.1 and Theorem 3.1 we obtain the following corollary.

COROLLARY 3.1 *Defining F_B^* and F_N^* as in equation (15), one has*

(i) $X_B^* - F_B^* \succ 0$ *and* $Z_N^* - F_N^* \succ 0$,
(ii) $|B| + |N| = n$ *if and only if* $F_B^* = 0$ *and* $F_N^* = 0$.

We can define subsets of the primal and dual optimal sets in terms of F_B^* and F_N^* as follows:

$$\mathcal{P}_{F^*} := \{X : A_B^i \cdot X_B = b^i \ (i = 1, \ldots, m),\ X_B - F_B^* \in S_+^{|B|},\ X_k = 0\ (k \in \mathcal{I} - B)\},$$

$$\mathcal{D}_{F^*} := \left\{(Z, y) : \sum_{i=1}^m A_N^i y_i + Z_N = C_N,\ Z_N - F_N^* \in S_+^{|N|},\ \sum_{i=1}^m A_k^i y_i = C_k,\right.$$
$$\left. Z_k = 0\ (k \in \mathcal{J} - N)\right\}.$$

It is easy to see that $\mathcal{P}_{F^*} \subseteq \mathcal{P}^*$ and $\mathcal{D}_{F^*} \subseteq \mathcal{D}^*$. Moreover, both $\mathcal{P}_{F^*}$ and $\mathcal{D}_{F^*}$ are non-empty as $X^* \in \mathcal{P}_{F^*}$ and $(Z^*, y^*) \in \mathcal{D}_{F^*}$. The relative interiors can be described as

$$\text{ri}(\mathcal{P}_{F^*}) = \{X \in \mathcal{P}_{F^*} : X_B - F_B^* \in S_{++}^{|B|}\} \quad \text{and} \quad \text{ri}(\mathcal{D}_{F^*}) = \{(Z, y) \in \mathcal{D}_{F^*} : Z_N - F_N^* \in S_{++}^{|B|}\}$$

and due to Corollary 3.1 (i), the analytic centers of these set are well defined as the unique solutions to the following problems

$$\max_{X_B - F_B^* \in S_{++}^{|B|}} \{\ln\det(X_B - F_B^*) : A_B^i \cdot X_B = b_i \ (i = 1, \ldots, m)\}, \tag{16}$$

$$\max_{y \in \mathbf{R}^m, Z_N - F_N^* \in S_{++}^{|N|}} \left\{\ln\det(Z_N - F_N^*) : \sum_{i=1}^m A_N^i y_i + Z_N = C_N,\right.$$
$$\left. \sum_{i=1}^m A_k^i y_i = C_k \ (k \in \mathcal{J} - N)\right\}. \tag{17}$$

THEOREM 3.2 *The limit point (X^*, Z^*, y^*) of the central path and the corresponding matrices F_B^* and F_N^* have the following properties:*

(i) X_B^* *is the analytic center of* $\mathcal{P}_{F^*}$ *and*
(ii) (Z^*, y^*) *is the analytic center of* $\mathcal{D}_{F^*}$.

If there exists a strictly complementary solution, then $\mathcal{P}_{F^*} = \mathcal{P}^*$ and $\mathcal{D}_{F^*} = \mathcal{D}^*$.

Proof The last part of the theorem follows from Corollary 3.1 (ii). We only prove (i) as the proof of (ii) follows the same pattern. To this end, it suffices to prove that $(X_B^* - F_B^*)^{-1} \in \mathcal{R}(A_B^T)$, as there exist $u^i\,(i = 1, \ldots, m)$ such that $(X_B^* - F_B^*)^{-1} - \sum_{i=1}^m u^i A_B^i = 0$, and hence X^* satisfies the sufficient condition for the optimal solution to equation (16).

Applying the primal optimal partition to the dual feasibility condition, we obtain

$$\sum_{i=1}^m A_B^i y^* = C_B \quad \text{and} \quad \sum_{i=1}^m A_B^i y(\mu) + Z_B(\mu) = C_B,$$

which implies that $\tilde{Z}_B(\mu) \in \mathcal{R}(A_B^{\mathrm{T}})$ for all $\mu > 0$. As the space $\mathcal{R}(A_B^{\mathrm{T}})$ is closed, and $\tilde{Z}_B(\mu)$ converges to $\tilde{Z}_B^*$ by Theorem 3.1, we have $\tilde{Z}_B^* \in \mathcal{R}(A_B^{\mathrm{T}})$. Now by Lemma 3.1 we have $(X_B^* - F_B^*)^{-1} = \tilde{Z}_B^* \in \mathcal{R}(A_B^{\mathrm{T}})$, which completes the proof. ■

4. Examples

We provide two simple examples for which the central paths can be expressed in closed form. Both examples do not have strictly complementary solutions and have multiple dual optimal solutions. For both examples, we construct the corresponding sets $\mathcal{D}_{F^*}$. For the first example, $\mathcal{D}_{F^*} \neq \mathcal{D}^*$ and the analytic center of $\mathcal{D}_{F^*}$ does not coincide with the analytic center of $\mathcal{D}^*$. Hence, the dual central path does not converge to the analytic center of the dual optimal set.[2] For the second example, $\mathcal{D}_{F^*} = \mathcal{D}^*$ and thus the central path converges to the analytic center of the optimal set.

Example 1 Let $n = 4, m = 3, b = [0\ 0\ -1]^{\mathrm{T}}$ and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The dual problem maximizes $-y_3$ such that

$$Z = \begin{bmatrix} y_3 & 0 & y_2 & 0 \\ 0 & y_2 & 0 & 0 \\ y_2 & 0 & 1 - y_1 & 0 \\ 0 & 0 & 0 & y_1 \end{bmatrix} \succeq 0.$$

The optimal solutions are $y_2 = y_3 = 0$ and $1 - y_1 \geq 0, y_1 \geq 0$. The analytic center is $y_1 = 1/2$. The primal problem minimizes X_{33} such that

$$X = \begin{bmatrix} 1 & X_{12} & -\frac{1}{2}X_{22} & X_{14} \\ X_{12} & X_{22} & X_{23} & X_{24} \\ -\frac{1}{2}X_{22} & X_{23} & X_{33} & X_{34} \\ X_{14} & X_{24} & X_{34} & X_{33} \end{bmatrix} \succeq 0. \tag{18}$$

[2] An example for which the central path does not converge to the analytic center was first described in ref. [5]. The example presented here is much simpler – its central path can be analytically computed – and it allows geometric insight into limiting behavior of the central path (see section 7).

The unique optimal solution is $X_{11} = 1$ and the other components of X are 0. It can easily be verified that the central path is $y_1(\mu) = 3/5$, $y_2(\mu) = \sqrt{(3/10)\mu}$, $y_3 = (3/2)\mu$, $X_{33}(\mu) = (5/2)\mu$, $X_{22}(\mu) = \sqrt{(10/3)\mu}$, and the other components of X in equation (18) are 0 along the central path. Hence, the dual central path does not converge to the analytic center of the dual optimal set, as $y_1^* = 3/5$. However, for this example, we have

$$Z_{\bar{N}} = \begin{bmatrix} y_3 & 0 \\ 0 & y_2 \end{bmatrix}, \quad Z_{\bar{N}N} = \begin{bmatrix} y_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_N = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{y_2^2}{y_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{1}{5} \end{bmatrix} = F_N^*,$$

and thus $y_1^* = 3/5$ is the analytic center of $\mathcal{D}_{F^*}$, where

$$\mathcal{D}_{F^*} = \left\{ (y, Z) \in \mathcal{D}^*: \begin{bmatrix} 1 - y_1 & 0 \\ 0 & y_1 - \dfrac{1}{5} \end{bmatrix} \succeq 0 \right\}.$$

Example 2 Let $n = 5, m = 3, b = [0\ 0\ -1]^T$ and

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The dual problem maximizes $-y_3$ such that

$$Z = \begin{bmatrix} y_3 & 0 & y_2 & 0 & 0 \\ 0 & y_2 & 0 & 0 & 0 \\ y_2 & 0 & 1 - y_1 & 0 & 0 \\ 0 & 0 & 0 & y_1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \succeq 0.$$

The optimal solutions are $y_2 = y_3 = 0$ and $1 - y_1 \geq 0, y_1 \geq 0$. The analytic center is $y_1 = 1/2$. The primal problem minimizes $X_{33} + X_{55}$ such that

$$X = \begin{bmatrix} 1 & X_{12} & -\dfrac{1}{2}X_{22} & X_{14} & X_{15} \\ X_{12} & X_{22} & X_{23} & X_{24} & X_{25} \\ -\dfrac{1}{2}X_{22} & X_{23} & X_{33} & X_{34} & X_{35} \\ X_{14} & X_{24} & X_{34} & X_{33} & X_{45} \\ X_{15} & X_{25} & X_{35} & X_{45} & X_{55} \end{bmatrix} \succeq 0. \tag{19}$$

The unique optimal solution is $X_{11} = 1$ and the other components of X are 0. It can be easily computed that the primal–dual central path is given by $y_1(\mu) = 1/2$, $y_2(\mu) = \sqrt{\mu/2}$, $y_3 = (3/2)\mu$, $X_{55}(\mu) = (3/2)\mu$, $X_{33}(\mu) = 2\mu$, $X_{22}(\mu) = 2\sqrt{\mu/2}$, and the other components of X in equation (19) are 0 along the central path. Hence, the dual central path converges to the analytic center of the dual optimal set. For this example, we have

$$Z_{\bar{N}} = \begin{bmatrix} y_3 & 0 \\ 0 & y_2 \end{bmatrix}, \quad Z_{\bar{N}N} = \begin{bmatrix} y_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{y_2^2}{y_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = F_N^*.$$

We observe that

$$Z_N = \begin{bmatrix} 1-y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Z_N - F_N^* = \begin{bmatrix} 1-y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}.$$

This means that although $F_N^* \neq 0$, the dual central path converges to the analytic center. This is caused by the fact that the only non-zero element of F_N^* is at the position where Z_N has a fixed element (i.e., equal to 1), and hence the optimal set (being constrained by $Z_N \succ 0$) is the same as the set $\mathcal{D}_{F^*}$ (constrained by $Z_N - F_N^* \succ 0$). This observation will be generalized in section. 5.

5. SDO Problems with a special block-diagonal structure

In this section, we describe a class of SDO problems where the central path converges to the analytic center of the optimal set. We assume the data matrices to be in the block diagonal form:

$$A^i = \begin{bmatrix} A_U^i & 0 \\ 0 & A_V^i \end{bmatrix}, \quad (i = 1, \ldots, m), \quad C = \begin{bmatrix} C_U & 0 \\ 0 & C_V \end{bmatrix}, \qquad (20)$$
$$A_U^i, C_U \in S^s, \quad \text{for some } s \leq n.$$

Denote $Z_U(y) = C_U - \sum_{i=1}^m A_U^i y_i$ and $Z_V(y) = C_V - \sum_{i=1}^m A_V^i y_i$. It is easy to see that each dual feasible solution (Z, y) is of the block diagonal form with positive semidefinite matrices $Z_U(y)$ and $Z_V(y)$ on the diagonal. In what follows, we will make the following assumption.

ASSUMPTION 5.1 *There exists $Z_U^* \succeq 0$ such that each dual optimal solution (Z, y) satisfies $Z_U(y) = Z_U^*$. Moreover, there exists an optimal solution (Z, y) for which $Z_V(y) \succ 0$.*

This assumption ensures that the corresponding optimal set can be characterized as

$$\mathcal{D}^* = \{(Z, y): Z_U(y) = Z_U^*, \; Z_V(y) \succeq 0\}.$$

THEOREM 5.1 *Let* SDO *be of the form* (20) *and satisfy Assumptions* 1.1, 1.2, *and* 5.1. *Then* $\mathcal{D}^* = \mathcal{D}_{F^*}$, *and hence the dual central path* $(y(\mu), Z(\mu))$ *converges to the analytic center of* $\mathcal{D}^*$.

Proof Without loss of generality, we may assume that Z_U is partitioned as

$$Z_U = \begin{bmatrix} Z_{U1} & Z_{U2} \\ Z_{U2}^{\mathrm{T}} & Z_{U3} \end{bmatrix},$$

where at any optimal solution (Z, y), it is $Z_{U1}(y) = 0$, $Z_{U2}(y) = 0$, and $Z_{U3}(y) = Z_{U3}^* \succ 0$. This and Assumption 5.1 imply that the components $Z_{\bar{N}}$, $Z_{\bar{N}N}$, and Z_N of the optimal partition of Z are

$$Z_N = \begin{bmatrix} Z_{U3} & 0 \\ 0 & Z_V \end{bmatrix}, \quad Z_{\bar{N}} = Z_{U1}, \quad \text{and} \quad Z_{\bar{N}N} = \begin{bmatrix} Z_{U2} & 0 \end{bmatrix}. \tag{21}$$

This implies that the optimal set is

$$\begin{aligned} \mathcal{D}^* &= \{(Z, y) \colon Z_{\bar{N}}(y) = 0, Z_{\bar{N}N}(y) = 0, Z_N(y) \succeq 0\} \\ &= \{(Z, y) \colon Z_{U1}(y) = 0, Z_{U2}(y) = 0, Z_{U3}(y) = Z_{U3}^*, Z_V(y) \succeq 0\}. \end{aligned}$$

Substituting equation (21) into equation (14) we obtain the formula for $F_N(\mu)$, where by Lemma 3.1, $F_N(\mu)$ converges. This means we have

$$F_N(\mu) = \begin{bmatrix} Z_{U2}^{\mathrm{T}}(\mu) Z_{U1}^{-1}(\mu) Z_{U2}(\mu) & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} F_{U3}^* & 0 \\ 0 & 0 \end{bmatrix} = F_N^*.$$

As $Z_{U3}(y) = Z_{U3}^*$ at any optimal solution y and $Z_{U3}^* - F_{U3}^* \succ 0$ by Corollary 3.1, we obtain that the condition $Z_N(y) - F_N^* \succeq 0$ is equivalent with $Z_V(y) \succeq 0$. This means

$$\begin{aligned} \mathcal{D}_{F^*} &= \{(Z, y) \colon Z_{\bar{N}}(y) = 0, Z_{\bar{N}N}(y) = 0, Z_N(y) - F_N^* \succeq 0\} \\ &= \{(Z, y) \colon Z_{U1}(y) = 0, Z_{U2}(y) = 0, Z_{U3}(y) = Z_{U3}^*, Z_V(y) \succeq 0\} = \mathcal{D}^* \end{aligned}$$

and the theorem is proved. ■

6. Convex quadratically constrained quadratic optimization

In this section, we show that a convex quadratically constrained quadratic program can be viewed as an SDO problem of the form (20) with Assumption 5.1, and hence the central path converges to the analytic center of the optimal set.

Consider the general convex program

$$(C) \quad \min_{y \in C} \{f_0(y) \colon f_i(y) \le 0 (i = 1, \dots, L)\},$$

where C is an open convex subset of $\mathbb{R}^n$, and $f_i \colon R^n \to R$ $(i = 0, 1, \dots, L)$ are convex and smooth on C. Let C^* be the set of optimal solutions which we assume to be nonempty and bounded.

The following result was first shown in Monteiro and Zhou [12] for the case where the f_is are analytic functions defined over the whole $\mathbb{R}^n$ and was later generalized by McCormick and Witzgall [13] to the form stated in the following theorem. (The more general result is more appropriate for purposes of our discussion in the section 5.) Some related results are given in refs. [14,15].

THEOREM 6.1 *If the convex program (C) meets the Slater condition and the functions f_i are weakly analytic,*[3] *then the central path of (C) converges to the (unique) analytic center of the optimal set.*

Therefore, it follows from Theorem 6.1 that the central path converges to the analytic center of the optimal set when the f_is are convex quadratic functions. Here our goal is, therefore, only to show that the assumptions in Theorem 5.1 are indeed met for some interesting sub-classes of SDO problems.

Consider the convex program (C), where all functions f_i $(i = 0, \dots, L)$ are convex quadratic, i.e.

$$f_i(y) = y^{\mathrm{T}} P_i y - q_i^{\mathrm{T}} y - r_i, \quad P_i \succeq 0. \tag{22}$$

Such a program is called a convex quadratically constrained quadratic program and we will denote it as (CQ). We first prove an auxiliary result about the set of optimal solutions C^* of (CQ). By $I(y)$, we denote the index set of inequalities, which are active at an optimal point y.

LEMMA 6.1 *Let $y^* \in \mathrm{ri}(C^*)$. Then for each $i \in I(y^*) \cup \{0\}$, both $P_i^{1/2} y$ and $q_i^{\mathrm{T}} y$ are constant on C^*.*

Proof The following result was proved in ref. [16]. Let $\mathcal{S}^*$ be the optimal set for $\min_{y \in \mathcal{S}} f_0(y)$, where $f_0(y)$ is of the form (22) and $\mathcal{S}$ is convex; then $P_0 y$ and $q_0^{\mathrm{T}} y$ are constant on $\mathcal{S}^*$. Hence, the $i = 0$ (objective function) case of the lemma follows from the result [16]. Now, let $i \in I(y^*)$ where $y^* \in \mathrm{ri}(C^*)$ and consider the program:

$$\min_{y \in C^*} f_i(y). \tag{23}$$

Now, the lemma for $i \in I(y^*)$ follows by application of the aforementioned result in ref. [16] to equation (23), provided $f_i(y)$ is constant on the whole C^*. However, the fact that $f_i(y) = 0$ for each $y \in C^*$ follows from convexity of C^* and $f_i(y)$, and the assumption $y^* \in \mathrm{ri}(C^*)$. The proof of this fact is straightforward and hence omitted. ■

It is well known that (CQ) with equation (22) can be equivalently reformulated as the convex quadratically constrained program:

$$(Q): \quad \max_{t,y} \{-t \colon f_0(y) \le t, \ f_i(y) \le 0 \ (i = 1, \dots, L)\},$$

where f_i $(i = 0, \dots, L)$ are given by equation (22). As shown in ref. [17], this problem can be rewritten as the SDO problem

$$(\mathrm{SDQ}): \max_{t,y} \{-t \colon Z_0(y, t) \succeq 0, \ Z_i(y) \succeq 0 \ (i = 1, \dots, L)\},$$

[3] A function is called weakly analytic if it has the following property: if it is constant on some line segment, then it is defined and constant on the entire line containing this line segment.

where

$$Z_0(y,t) := \begin{bmatrix} I & P_0^{1/2}y \\ y^{\mathrm{T}}P_0^{1/2} & q_0^{\mathrm{T}}y + r_0 + t \end{bmatrix}, \quad Z_i(y) := \begin{bmatrix} I & P_i^{1/2}y \\ y^{\mathrm{T}}P_i^{1/2} & q_i^{\mathrm{T}}y + r_i \end{bmatrix} \quad (i = 1, \dots, L).$$

This is an SDO program in the standard dual form (D), if we define Z as the block diagonal matrix with Z_i $(i = 0, \dots, L)$ as diagonal blocks. The condition $Z_0(y,t) \succeq 0$ corresponds to $f_0(y) \leq t$, and the conditions $Z_i(y) \succeq 0, i = 1, \dots, L$ correspond to the constraints $f_i(y) \leq 0$.

THEOREM 6.2 *Let* (SDQ) *satisfy Assumptions* 1.1 *and* 1.2. *Then it also satisfies the other assumptions of Theorem* 5.1. *Hence* $\mathcal{D}^* = \mathcal{D}_{F^*}$, *and the dual central path* $(y(\mu), Z(\mu))$ *converges to the analytic center of* $\mathcal{D}^*$.

Proof Applying Corollary 6.1 to Z_0, we can see that both $P_0^{1/2}y$ and $r_0^{\mathrm{T}}y$ are constant on $\mathcal{D}^*$. Moreover, t is also constant, and hence the entire block Z_0 can be considered as a part of the block Z_U from Assumption 5.1. Now, we consider a block Z_i for some $i \in \{1, \dots, L\}$. If there exists a $y^* \in \mathcal{D}^*$ such that $Z_i(y^*) \succ 0$, then this block is considered to be a part of Z_V. In other case, $Z_i(y)$ is singular on $\mathcal{D}^*$, which means that the corresponding inequality $f_i(y) \leq 0$ is active at each $y^* \in \mathrm{ri}(\mathcal{D}^*)$, and hence $i \in I(y^*)$, and we can apply Lemma 6.1. We obtain that all components of Z_i are constant on $\mathcal{D}^*$ and thus the entire block Z_i, for which $i \in I(y^*)$, can be considered to be a part of Z_U.

As each block Z_i, $i = 0, \dots, L$ was added either to Z_U or to Z_V, the problem is of the form (20) and satisfies Assumption 5.1. Thus, the dual central path converges to the analytic center of $\mathcal{D}^*$. ∎

7. Concluding remarks

We have shown in this article that, in the absence of strict complementarity, the central path converges to the analytic center of a certain subset of the optimal set. Unfortunately, the description of this subset does not depend only on the problem data and the optimal (block) partition, and therefore does not give a nice geometrical characterization of the limit point. It would be interesting to understand whether such a characterization is possible.

We have also described a class of SDO problems with block-diagonal structure, where the central path converges to the analytic center of the optimal set. Convex, quadratically constrained quadratic programs, when formulated as SDO problems, fall in this class.

We conclude with some detailed remarks about the examples we have considered.

- Regarding the SDO reformulation (SDQ) of convex quadratic optimization problems, we have proved the convergence of the central path to the analytic center of the optimal set for the program in the standard dual form, but this result does not say anything about the situation for the corresponding dual counterpart in the standard primal form. In fact, the dual need not correspond to a quadratic program, and its central path need not converge to the analytic center of its optimal set. To see this, consider Example 1, where in the primal formulation we neglect the variables that vanish along the central path. Then the feasibility constraints can be described by one convex quadratic constraint $1/4x_{22}^2 - x_{33} \leq 0$, and two linear constraints $-x_{22} \leq 0$ and $-x_{33} \leq 0$. Hence, the primal problem corresponds to a convex quadratic program. However, the corresponding dual problem does not correspond to a convex quadratic program, and the central path does not converge to the analytic center, as shown in section 5.

- The standard dual form of Example 1 is closely related to the non-linear problem:

$$\max_{y \in C} \left\{ -y_3 \colon \frac{y_2^2}{y_1} - y_3 \leq 0, y_1 \geq 0, 1 - y_1 \geq 0, y_2 \geq 0 \right\}, \tag{24}$$

  where C is the open half-space where $y_1 > 0$. The optimal set and central path of the standard dual form of Example 1 coincides with that of problem (24). In particular, the respective central paths do not converge to the analytic centers of the respective optimal sets. It is easy to show that all the functions in equation (24) are convex on C. However, the function $y_2^2/y_1 - y_3$ is not a weakly analytic function, and this is the reason why Theorem 6.1 does not apply here. In other words, it shows that the 'weakly analytic' requirement in Theorem 6.1 cannot simply be dropped.
- The simplicity of Example 1 allows us a geometric insight, as the central path can be interpreted as a set of analytic centers of the level sets of the duality gap [see, e.g., ref. 18]. For positive μ, the hyperbolic inequality of Example 1 'pushes' the analytic center of the level set away from $y_1 = 0$, i.e., it has an influence on the description of the analytic center of the level set. However, at $\mu = 0$ this inequality disappears, and, therefore, has no influence on the description of the analytic center of the optimal set. Intuitively, this is the reason why the central path does not converge to the analytic center of the optimal set for this example.

  One might suspect from this example that any appearance of a hyperbolic constraint courses a shift of the limit point away from the analytic center. However, this is not true as shown in the following example.

Example 3

$$\max \left\{ -y_3 \colon \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} \succeq 0, \begin{bmatrix} y_3 & y_2 \\ y_2 & (1 - y_1) \end{bmatrix} \succeq 0, y_2 \geq 0 \right\}.$$

It can be shown easily that the central path for this problem converges to the analytic center ($y_1^* = 1/2$), and that $\mathcal{D}_{F^*} \neq \mathcal{D}^*$.

While the present article was being refereed, new papers appeared treating the central path in SDO. In refs. [19–21] the limiting behavior of weighted central paths is analyzed under strict complementarity. In ref. [22], a characterization of the limit point of the central path is provided as being a solution of certain convex program; however, the uniqueness of this program is not ensured in the paper. In ref. [23], the limiting behavior of the central path is analyzed under the assumption that the distance of the non-strictly complementary components of the central path to the optimal set is $O(\sqrt{\mu})$. Some deep results on analyticity of the central path are established there, which imply the convergence of the central path and provide a characterization of the limit point.

Acknowledgement

The research of M. Halická was supported in part by VEGA grants 1/9154/02 and 1/1000/04 of Slovak Scientific Grant Agency.

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